# Game of Thrones

Analysis and Predictions for Season 8

Tim Sudijono  Lisa Yang  Shiv Nadimpalli

BMCM 2017: Problem 1

# Contents

# 1 Summary

*Will the people of Westeros survive the next Long Night, when the army of the dead descends upon the lands of the living?* Mathematics says that they will, and we assure the people of Westeros that in fact, the threat is not as large as it seems.

Given the raw bloodthirst of 100,000 undead would have accumulated over millenia, we assume that the White Walkers, supernatural creatures whose only weakness is an obsidian-like substance called dragonglass, and their army will head straight to Winterfell, and then push south directly towards the capital of Westeros, King's Landing.

To develop a strategy for the different armies scattered around Westeros, we first create a model of a battle between the White Walkers and humans. We then use this model to simulate battles with varying parameters such as the size of the human armies and how much dragonglass-coated weapons are available. The model allows us to see how many soldiers, with a given fraction of them wielding dragonglass-coated weapons, it takes to ensure a victory over the White Walkers. We discover that if the reserves of dragonglass are unlimited, then we can guarantee a win for humanity against the dead, while using only half the total manpower of the Seven Kingdoms. Our model also predicts the intuitive fact that the more dragonglass we have, the fewer soldiers we need to be victorious.

Given the predictions from our model, we next establish the distances and times it takes for armies, including those of the White Walkers, to travel to the first city of interest, Winterfell. Using a back-of-the-envelope calculation, we find that amount of time is enough so that the human armies can mine enough dragonglass to equip a large enough number of people to entirely defeat the army of the dead at Winterfell.

We recommend that in order to achieve this solution, the civilians in the northern parts of the continent should flee south. The armies at Dragonstone, the Iron Islands, and Winterfell, three landmarks in Westeros, should assemble at the same city by the time the army of the dead reaches Winterfell, for a decisive victory.

While the people of Westeros are quite likely to survive the threat of the undead, whether or not they survive the various threats from among the living remains up to them.

## 2 Introduction

Game of Thrones, the hit TV series based on George R.R. Martin's epic fantasy series "A Song of Ice and Fire", just concluded its seventh season. Taking place on the fictional continent of Westeros, the series primarily centers around a dynastic war among several families for the control of Westeros, alongside the rising threat of supernatural beings from Westeros's vast, frozen north. These supernatural beings, called White Walkers, not only possess superhuman strength, but also have the ability to reanimate dead humans into zombie-like non-sentient creatures called Wights. Armed with 1,000 White Walkers and 100,000 Wights, the army of the dead brings down an 8,000-year-old ice wall known simply as "The Wall". The Walkers start moving southwards from The Wall, with the objective of the of capturing Winterfell, the northern stronghold of the kingdom, and then King's Landing, the capital of Westeros. These White Walkers are resistant to fire, and are hard to defeat with ordinary melee weapons, but are known to be vulnerable to an obsidian-like material known as dragonglass. However, all the dragonglass in Westeros is initially present on an island in the south-east of Westeros called Dragonstone.

In our report, we provide a deterministic model for battles between armies of White Walkers and human armies, and then use this battle model to construct a strategy that allows humanity to combat the spread of the White Walkers through Westeros. The inspiration for our battle model was the SIR model in epidemiology which computes the theoretical number of people infected with a contagious illness in a closed population over time. Analogizing the given situation, we can think of the army of the White Walkers as the pathogen, and the human armies as the immune response of the host organism. Our battles are parametrized by sizes of the respective armies, the rate at which Walkers can reanimate dead humans, and the dependence of Wights on Walkers. We construct a system of linear ordinary differential equations reflecting these assumptions, and provide several simulations of battles under different conditions, in order to showcase the richness and flexibility of our model. All of this is discussed in detail in §3 & §4.

Our model makes several assumptions, which are discussed in §3 & §4, especially about distances between cities in Westeros and the time it takes for transportation between locations using Medieval modes of transportation. The main assumption made by our model is that the Walkers move south with the goal of first reaching Winterfell, and then move further south to capture King's Landing. We assume that these beings are sentient, but are not military tacticians or strategists. We then discuss our model in §4, and provide several strategies for humanity to win against the White Walkers in §5. Assuming the White Walker army moves at a slower speed than humans, and assuming that they choose to not split up and target various strongholds of the kingdom, we find that if the reserves of dragonglass are unlimited, then we can guarantee a win for humanity against the dead, while using only half the total manpower of the kingdom.

An analysis of our model is provided in §6, where we suggest possible modifications and future work in §7. Our simulation code and references are also provided.

# 3 Assumptions

In addition to those below, we make several other assumptions regarding the interactions of White Walkers, Wights, and Humans, which are most relevant to the battle model we develop in the section; hence we discuss those assumptions in § 4.1. We also note that the assumptions made in §3.7 is a vital component of our model, and is discussed further in § 5.1.

## 3.1 Geographic layout

The map of the Game of Thrones universe is modelled as a discrete grid, with each cell scaling to a 300 miles $\times$ 300 miles region of land. Distances between locations were calculated with reference to the known distance between Winterfell and The Wall (Castle Black):


| | | | | | | |
|---|---|---|---|---|---|---|
| | | **The Wall** | | | | |
| | | | | | | |
| | | | | | | |
| | | | **W** | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| **I** | | **C** | | | | |
| | | | | **K** | | **D** |
| | | | | | | |

| Symbol | Location |
|---|---|
| **W** | Winterfell |
| **I** | Iron Islands |
| **C** | Casterly Rock |
| **K** | King's Landing |
| **D** | Dragonstone |
| ▭ | Ocean |

**Scale:** ▭ = 300 miles.

Figure 1: Game of Thrones Map

The map in Fig. 1 naturally informs the best way to move between cities in our universe. In particular, we observe that the fastest way to move from Dragonstone to Winterfell is to sail for some period to as close to Winterfell as possible, then march the remaining distance.

### 3.2 Transportation

| | |
|---|---|
| Army (by foot) | 8 miles/day |
| Army (by horse) | 15 miles/day |
| Ship | 150 miles/day |
| Supply Train | 30 miles/day |
| White Walkers | 7 miles/day |
| Wights | 4 miles/day |

We assume Game of Thrones to be set in a universe at the technological level of medieval Europe, and hence, we estimate the speeds of various modes of transportation as above.[6] Furthermore, we assume that the White Walkers are able to move at speeds comparable to humans, whereas Wights, which are reanimated humans, move at a speed significantly slower than humans.[2] Assuming that the White Walkers have little incentive to venture further than their army, we can assume their mobility is restricted to that of the Wight army.

### 3.3 Population numbers of humans, White Walkers, and Wights

The distribution of human armies is known: 80,000 at Dragonstone, 8,000 at Casterly Rock, 26,500 at Winterfell, 80,000 at King's Landing, and 30,000 at Iron Islands. In accordance with [1], we estimate the number of White Walkers to be 1,000, and the number of Wights to be 100,000. Non-military inhabitants of the Game of Thrones universe are ignored in our model: as the White Walkers' start at The Wall and are moving south, civilians can be evacuated to the southern parts of the continent. This not only seems like a natural response to the White Walker threat, but also ensures that the number of Wights does not grow at an unreasonable rate: the White Walkers could poach remaining civilians so as to increase the size of their Wight army.

### 3.4 Complete information

The war between the White Walkers and humans is modelled as a game with complete information, and thus, knowledge about other players is available to all participants. This means that humans are aware of abilities, strategies, and "types" of White Walkers, and vice versa.

### 3.5 White Walker-dependence of Wights

As the Wights are reanimated by a White Walker, our model place a natural relation between a White Walker and its Wights: the death of a White Walker results in the instant death of all its Wights. Therefore, as there are about 100 Wights per White Walker, killing a White Walker, on average, will result in the death of about 100 Wights as well. Moreover, we assume that when a Wight is killed, it cannot be revived.

### 3.6 White Walker strategy

Our model assumes that the primary objective of the White Walkers is to take over the continent, and hence assume that they can only move south. Thus, the White Walkers and their forces are not allowed to retreat north. Therefore, the first stop of the White Walkers is taken as Winterfell. While there are two equally valid ways to treat the spread of the White Walkers, as below, we treat only the first case in our report, and discuss how to approach the second problem in future work.

1. The White Walkers follow a naive strategy and simply move southward with their full force.
2. The White Walkers play optimally, splitting up and moving southwards to different locations, converging at King's Landing.

### 3.7 Dragonglass distribution and weaponization

We assume that all the dragonglass in the Game of Thrones Universe is initially located at Dragonstone. We assume that the reserves of dragonglass are unlimited unless otherwise stated, as this seems to be a natural assumption to make; the case when the reserves of dragonglass are limited is considered in §5.3.

Inherent is the assumption that a dragonglass-coated weapon is enough to kill a White Walker. We also assume that it is easier to kill a Wight with a dragonglass-coated weapon as opposed to a regular melee weapon. We assume here that around 10 ounces worth of dragonglass is enough to equip a soldier with a coated weapon and a small dagger. This allows us to estimate how many weapons can be made in a given amount of time, a task which we consider in §6, the analysis of our model.

### 3.8 New White Walkers

The Night King, the ruler of the White Walkers and the Wights, can transform living human infants into White Walkers. However, on the time scale of the following series of events, this phenomenon adds little to none to the strength of the White Walkers, and hence is ignored by our model.

# 4 Modelling battle outcomes

To inform the humans' strategy on how to allocate their armies in mining dragonglass and fighting the White Walkers, we must first address the following subproblem:

*Under what conditions is it favorable for the human army to battle the White Walkers*?

We can recast this problem in a framework that is similar to an SIR model in epidemiology. In an SIR model, the spread of a non-lethal disease throughout a population is modeled by considering populations of susceptible, infected, and recovered individuals; these populations change over time according to the following assumptions:[3]

- The total population remains fixed, and is comprised of three disjoint groups: susceptible, infected, and recovered individuals.
- The susceptible population moves to the infected group at a rate proportional to the size of the infected population, and the size of the susceptible population itself. This assumption reflects the idea that the more infected individuals there are, the faster people are infected.
- The infected population can only recover from the disease, at some fixed rate.
- A recovered individual cannot become infected again.

If we let $s, i, r$ denote the *proportions* of the populations of susceptible, infected, and recovered individuals, and $N$ be the entire population, these three assumptions lead naturally to the following first order system of ordinary differential equations[3]:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -bs(t)i(t)$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = bs(t)i(t) - ki(t)$$
$$\frac{\mathrm{d}r}{\mathrm{d}t} = ki(t)$$

where $b, k$ are parameters reflecting the infection rate of a disease and the recovery rate. Even more complex variants of the SIR model have been proposed, modeling cases where there are deaths of infected individuals involved, or subsets of the population with immunity.[4]

## 4.1 Battle model

To model the outcome of a battle between the White Walkers and human armies, we adopt an approach similar to the spirit of the SIR model, considering populations of humans, dead humans, White Walkers, and Wights, which we denote by the variables $H, D, WW, W$ respectively. We derive a system of first order ODEs that model the changes in these populations over the course of a battle. As simplifications of actual warfare, we assume that the human armies are more or less homogeneous in terms of skill, along with the Wight and White Walker armies. Furthermore we assume all combatants are melee. As mentioned previously, we factor in the assumption that killing a White Walker kills 100 Wights, since they are the creatures that reanimated those dead beings; we further assume that dead wights cannot be reanimated. We give some estimates for the relative strengths of Wights, White Walkers, and humans below.

### 4.1.1 Modeling $H$.

To model the change in the number of humans, we note that the change is only a function from the rate of fatalities during the battle; all of these dead individuals are then moved into the dead human group, whose population is denoted by $D$. We model the number of humans killed in an instant as proportional to the number of wights $W$ times the strength ratio between Wights and humans, plus the number of White Walkers $WW$ times the strength ratio between White Walkers and humans. Hence we can write:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -[C_{H:W}W + C_{H:WW}WW]$$

where it remains to model the strength ratios of Wights and White Walkers to humans, denoted by $C_{H:W}, C_{H:WW}$ respectively.

To model the strength ratios, we say that the strength of a wight to that of a human is comparable, but dependent on the proportion of soldiers with dragonglass weapons in the human army, which we denote by $p$. We estimate the ratio to be $1 : C_1(1+p)$, so that 1 human can kill $C_1(1+p)$ Wights, and vice versa. We might think of wights being on the average much stronger than normal soldiers, but accounting for outliers, the heroes of the series who have killed hundreds of wights, we say estimate this ratio to be near 1, or lower. The intuition that this scales with $p$ is that dragonglass is an effective weapon against Wights, as mentioned in the problem statement.

On the other hand, the strength of 1 White Walker is comparable to $C_2$ humans; i.e. 1 White Walker is expected to kill $C_2$ humans, and it takes on average $C_2$ humans to kill a White Walker. By the lore of Game of Thrones, we observe that White Walkers have the battle capabilities of at least 5 men, as discussed in the description of the problem, although the value seems much higher based on evidence of supernatural strength.[5] For our simulations that follow, we took $C_1 = 0.5, C_2 = 20$. However, note that these constants can be regarded as parameters, and it is simple to redo the analysis that follows with different strength parameters.

Finally, we propose that in a battle, not all combatants are fighting at the same time; it can be envisioned that in war, there is a line of fighting that only involves a fraction of the participants on both sides. It is usually not the case that the armies are 'well mixed': every person on the battle field is is engaged in melee combat with another opponent. We call the proportion of combatants engaged in fighting $r$, and hence the number of human casualties is scaled by the rate $r$, since only a proportion of $H$ is in battle. We can then model the change in the population of humans as:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -r\big(\frac{W}{C_1(1+p)} + C_2WW\big)$$

We call this constant $r$ the mixing rate.

### 4.1.2 Modeling $D$.

The population of dead humans only increases at the rate at which humans are killed, and decreases at the rate that corpses are reanimated into wights. We assume that the reanimation of corpses into wights is proportional to the number of dead humans at a given moment, multiplied by how many White Walkers are remaining - we can think of the resurrection power of the White Walkers as linear in $WW$. Hence we can model

$$\frac{\mathrm{d}D}{\mathrm{d}t} = -\frac{\mathrm{d}H}{\mathrm{d}t} - D \cdot \rho \cdot \frac{WW}{WW_0},$$

where $WW_0$ is the initial population of White Walkers.

#### 4.1.3 Modeling $WW$.

To model the population of White Walkers, we note that White Walkers can only be killed by soldiers wielding dragonglass; we say that the rate at which they are killed is proportional to the number human soldiers wielding dragonglass, times the strength ratio between the two groups. Also noting that the White Walkers are leaders of their army, we can assume there is an incentive for them not to be killed. We account for this by saying that the more Wights there are, the safer the White Walkers are, while if there are no wights, the White Walkers have no protective advantage. We model this last independence via an inverse logarithm function. Concretely, we have:

$$\frac{\mathrm{d}WW}{\mathrm{d}t} = -r\frac{H \cdot p}{C_{H:WW}} \frac{1}{1 + \log_{10}(1 + W)}$$

Here, $r$ denotes the 'mixing' rate parameter we discussed previously. As mentioned in the Assumptions section, we assume that $WW$ cannot increase, even in light of the fact that infants can be transformed into White Walkers.

#### 4.1.4 Modeling $W$.

To model the change in the population of Wights, we note that changes in the number of Wights are influenced by the following parameters:

- The number of White Walkers, owing to the White Walker–Wight dependence as discussed in § 3.5.

  The death of a White Walker results in the instant death of the (on average) a 100 White Walkers, and hence we can write:

  $$\frac{\mathrm{d}W}{\mathrm{d}t} \propto 100 \cdot \frac{\mathrm{d}WW}{\mathrm{d}t}.$$

  We take the proportionality constant to be 1 in the above equation.

- Say reanimation function $g$ gives the number of new Wights reanimated per unit time. This would depend on the number of dead humans ($D$), the reanimation rate of one White Walker ($\rho$), and the number of White Walkers ($WW$). We assume that $g(D, \rho)$ is directly proportional to the number of dead humans ($D$), and the reanimation rate $\rho$. Furthermore, the number of Dead Humans that are reanimated will depend on the number of remaining White Walkers, relative to the initial number of White Walkers ($WW_0$). Hence, we can say

  $$g(D, \rho) = D \cdot \rho \cdot \frac{WW}{WW_0}.$$

- The number of Humans ($H$), and the strength ratio of humans to Wights ($C_{H:WW} = \frac{1}{C_1}$). As discussed in §4.1.1, this rate is given by:

$$\text{Rate of Wights killed by humans} = -r \cdot C_1(1+p) \cdot H.$$

  The above expression captures our assumption that it is easier to kill a Wight with dragonglass, and that the number of Wights killed by humans also depends on the degree to which both the armies mix.

We note that the killed Wights cannot be reanimated, and thus, putting the above expressions together, we have:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = 100\frac{\mathrm{d}WW}{\mathrm{d}t} + D \cdot \rho \cdot \frac{WW}{WW_0} - r \cdot C_1(1+p) \cdot H.$$

## 4.2 Simulations

To showcase our model, we run simulations for battles using various values of $p, \rho$ in particular. We highlight particular cases of interest, the edge cases where $p, \rho$ are extreme values, that show that the model accords with our intuition. For the following, we set the initial populations of the battle participants to be $WW_0 = 1,000$ , $W_0 = 100,000, D_0 = 0, H_0 = N$. The point is to vary $N$ to discover the numbers of human soldiers needed to win a battle against the White Walkers. The choices for the other values are informed by speculations on the size of the White Walkers' army.[1] Hence our complete set of ordinary differential equations can be written:

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= -r\Big(\frac{W}{C_1(1+p)} + C_2 WW\Big) && \ldots(1)\\
\frac{\mathrm{d}D}{\mathrm{d}t} &= -\frac{\mathrm{d}H}{\mathrm{d}t} - D \cdot \rho \cdot \frac{WW}{1000} && \ldots(2)\\
\frac{\mathrm{d}WW}{\mathrm{d}t} &= -r\frac{H \cdot p}{C_2}\frac{1}{1+\log_{10}(1+W)} && \ldots(3)\\
\frac{\mathrm{d}W}{\mathrm{d}t} &= -r \cdot C_1(1+p) \cdot H + 100\frac{\mathrm{d}WW}{\mathrm{d}t} + D \cdot \rho \cdot \frac{WW}{1000} && \ldots(4)
\end{aligned}$$

Note that we are given that each variable $H, D, WW, W$ is positive. For our simulations, we have chosen the parameters:

- $r = 0.8$, the mixing rate, denoting the proportion of both armies engaged in battle.
- $\rho$, the reanimation rate, denoting what proportion of the dead humans are resurrected. We set $\rho = 1$ unless otherwise noted.
- $C_1 = 0.5$, $C_2 = 20$ as discussed previously.

To simulate this system of ODEs, we employed Euler's method with a step size of $t = \frac{1}{10}$ using Python[7]; we took $t$ here to be on the scale of hours. In global scales, we did not consider lengths of battles into account, but only the results.

### 4.2.1 Worst-case scenario: $\rho = 1, p = 0$

As a first check of our model, we simulate the worst case possible for the human army: there are no dragonglass weapons, and all the dead are reincarnated into wights at each time step. Intuitively, a human army of any size would still be defeated, since there is no method of killing White Walkers without dragonglass. Our simulations confirm this suspicion. We plot this case for values of $N$, the size of the human army to be $N = 50000, 100000, 200000, 400000$ below. Note that for each of Figures $2, 3, 4$ the number of humans always goes to zero. We see that initially there is always a dip in the population of Wights, accounted by the humans fighting them, but as more humans die and are converted to Wights, the Wight population increases while the Dead population increases first and then decreases. Meanwhile, the White Walker population remains steady at 1000, though this is difficult to see.

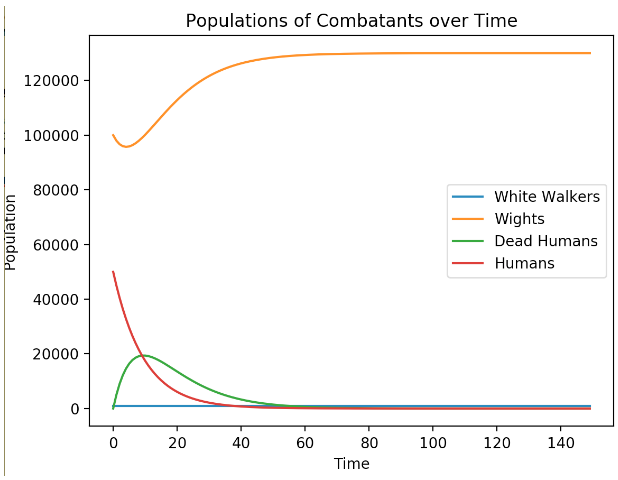


Figure 2: $N = 50,000$ soldiers.

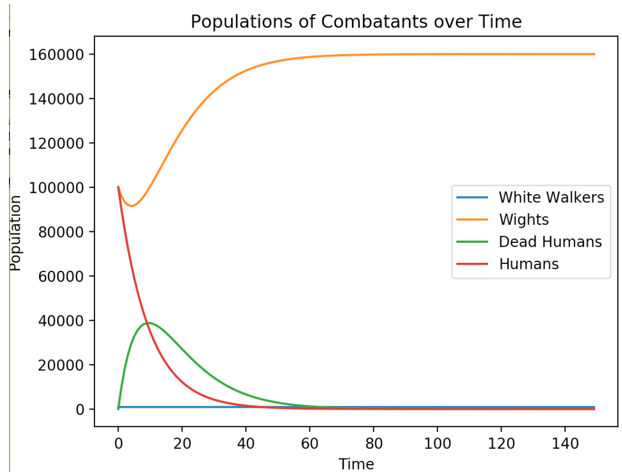


Figure 3: $N = 100,000$ soldiers.

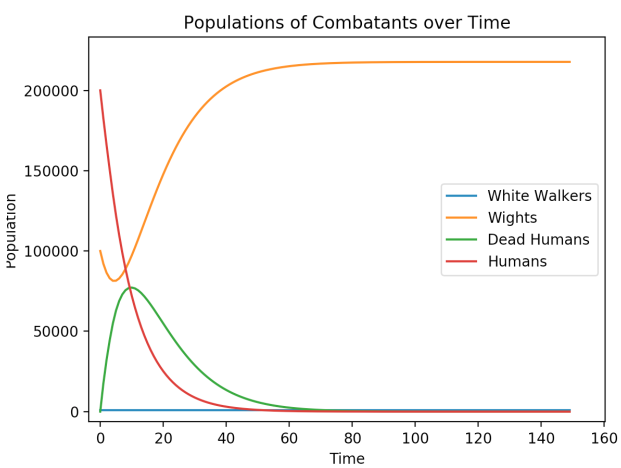


Figure 4: $N = 200,000$ soldiers.

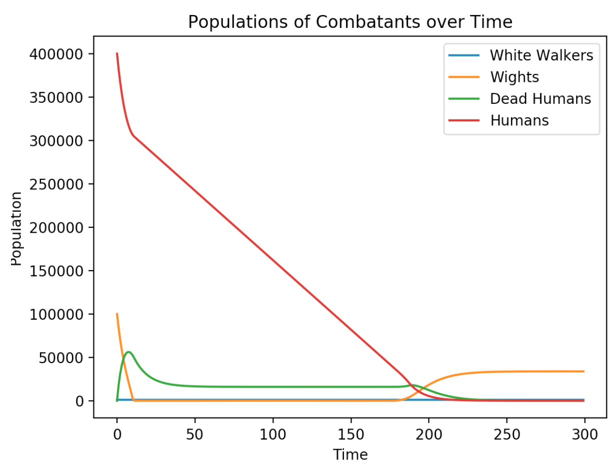


Figure 5: $N = 400,000$ soldiers.

As an example of the richness of our model, consider the same parameters with $N = 400,000$ human soldiers, which corresponds to Figure 5 above. We observe that initially, there are large enough masses of humans to overwhelm the number of wights; the number of wights (and humans) drops steeply until the Wight population is 0. After this period, the human population decreases more slowly given that they are only being killed off by the White Walkers, but there are still enough humans to eliminate newly reanimated wights as soon as they spawn. However, since the White Walkers are impervious to non-dragonglass weapons, the human population decreases to zero until at some point, there are not enough humans to prevent the dead human population from being reanimated into wights. In the end, only the White Walkers are left, with a small population of Wights resurrected from the most recently deceased humans.

#### 4.2.2 Best-case scenario: $\rho = 0, p = 1$

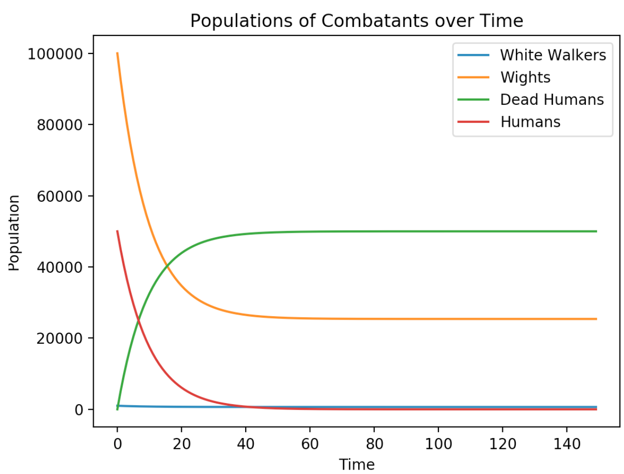


Figure 6: N = 50,000 soldiers.

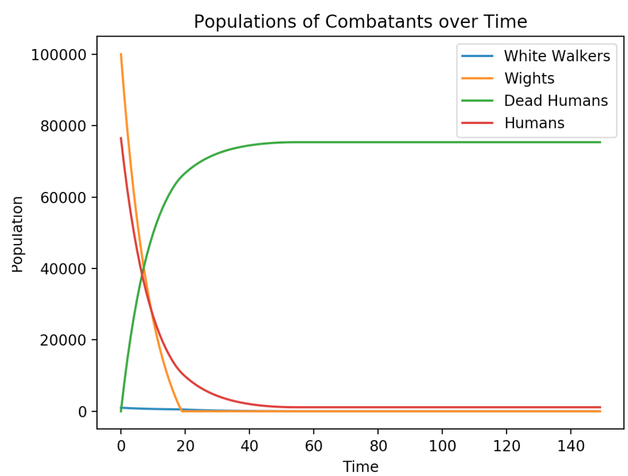


Figure 7: N = 76,500 soldiers.

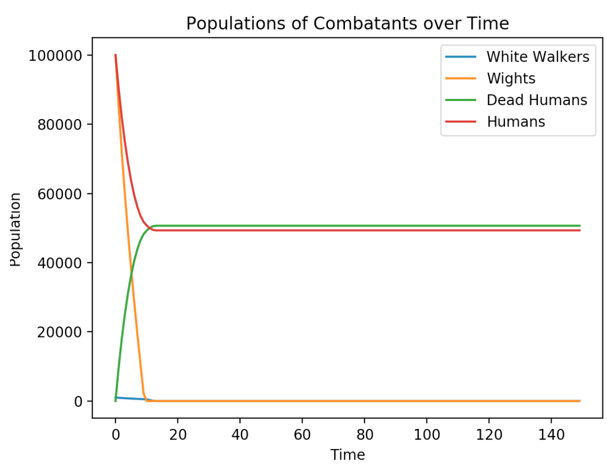


Figure 8: N = 100,000 soldiers.

Conversely, we consider the best case possible for the human army to engage, when the entire army is equipped with dragonglass, and the reanimation rate is 0. Indeed, we see that it is possible for the humans to win with not too many soldiers.

In our simulations, we found that there exists a critical value of the size of the human army, $N = 76,500$, above which the war was always won by humans. This is depicted in Figure 7 above. Note that the fact that the population of humans is steady after a period of time indicates that there are no Wights or White Walkers left; in other words, humans have won the battle. The number of remaining human soldiers is very small, but the number of White Walkers and Wights is zero. For values of $N$ below 76,500, we see that it is impossible for humans to win against the army of the White Walkers. Figure 6 depicts this simulation with $N = 50,000$. Note that the fact that the population of Wights remains constant after some period indicates that there are no humans left, and the dead humans are finished reanimating. This indicates that the White Walkers have won the battle. While we cannot prove that this critical point exists, intuition and the simulations seem to show that this is true: for example, as we see in Figure 8, when the size of the army is 100,000 soldiers, we rapidly are able to kill all the White Walkers and the Wights, but for any value lower than $76,500$, the human army always loses.

### 4.2.3 Intermediate cases

To further explore our model and showcase the flexibility of our model, we consider a more two intermediate cases:

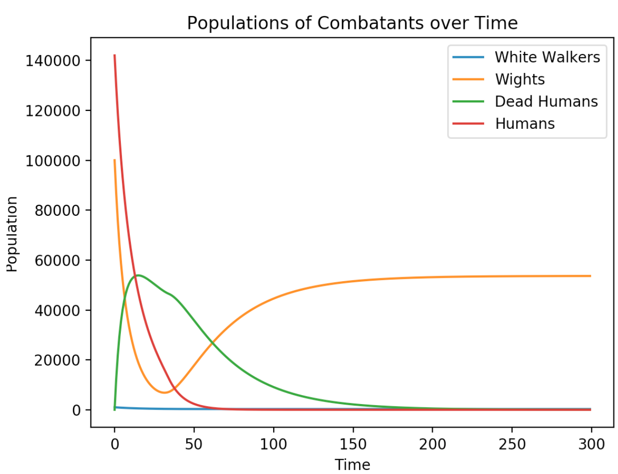


Figure 9: N = 142,000 soldiers.

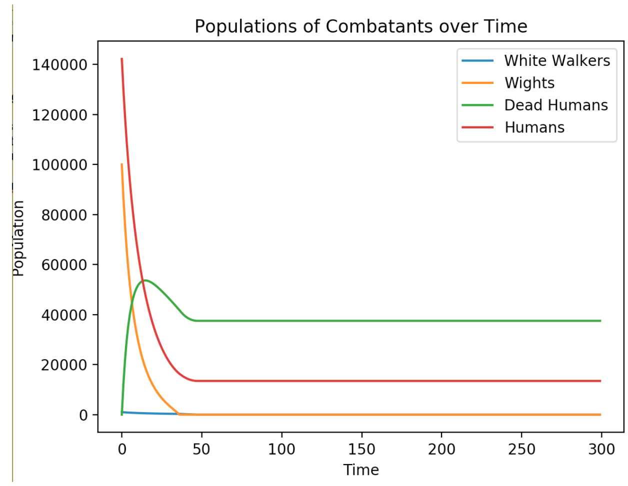


Figure 10: N = 142,200 soldiers.

In both of the above scenarios, we have $p = 0.5$, and $\rho = 1$. Thus, half the humans are equipped with dragonglass-coated weapons, and the reanimation rate is still 1. We note some interesting behaviour here: with $N = 142,000$, we see that the humans lose to the White Walkers (Figure 9), but a small increase of just 200 soldiers allows humans to win, as seen in Figure 10. Both these cases agree with our intuition, but our ODE model exhibits the same interesting critical-point behaviour here.

# 5 Choosing Optimal Strategies

Given our model for predicting battle outcomes, it suffices to develop a policy for winning the war against the White Walkers. We make observations on the distances between important cities in Westeros, and the times it takes to transport soldiers to and from these locations, via a grid we previously established, and the auxiliary information provided in the Transportation section earlier in our paper. We also provide estimates on how long to mine dragonglass and craft weapons from them, along with the time before the White Walkers descend on Winterfell.

## 5.1 Dragonglass Allocation

We assume 10 oz. of dragonglass is sufficient to coat a weapon that will be lethal to White Walkers. At the technological level of medieval Europe, we assume that iron-tools are prevalent, that the swing of a pick-axe mines 1 oz. of dragonglass, and that a worker can swing a pick-axe every 10 seconds. This means one worker can mine 360 oz. of dragonglass an hour, and hence, 4320 oz. of dragonglass a day.

These parameters can be varied to reflect more realistic situations more closely.

## 5.2 Recommendations

We make the following recommendations *under the assumption that there exist unlimited reserves of Dragonglass*. We know that the White Walkers will reach Winterfell from Castle Black in approximately 150 days, with the assumptions made on distances and travel speeds in §3. In view of this, we suggest the following actions:

- Evacuate all civilians and non-military personnel to the southern regions of the continent.
- Mobilize the army at the Iron Islands and move them to Winterfell by ship. According to §3, this should take less than 10 days.
- Allocate approximately 2000 people to mine dragonglass 12 hours/day for 100 days. At the rate discussed in the previous section, this will result in approximately 8,640,000 ounces of dragonglass, which is more than sufficient to coat 432,000 weapons, assuming 10 oz. per weapon.
- Mobilize the army at Dragonstone and move them to Winterfell: this journey will take 6 days of travel by sea, and 40 days of travel by land. We also ship our mined dragonglass along with this army.
- Pool together the three armies at Winterfell, and equip each soldier with a dragonglass-coated weapon.

Thus, at the end of approximately 150 days, we should have 136,500 dragonglass-wielding soldiers at Winterfell, and thus, by our simulations in §4.2.2, it is easy to see that the human armies will be victorious in the battle against the White Walker army; we also include a simulation below that demonstrates when $N = 136,500, p = 1, \rho = 1$, that the human armies will win.

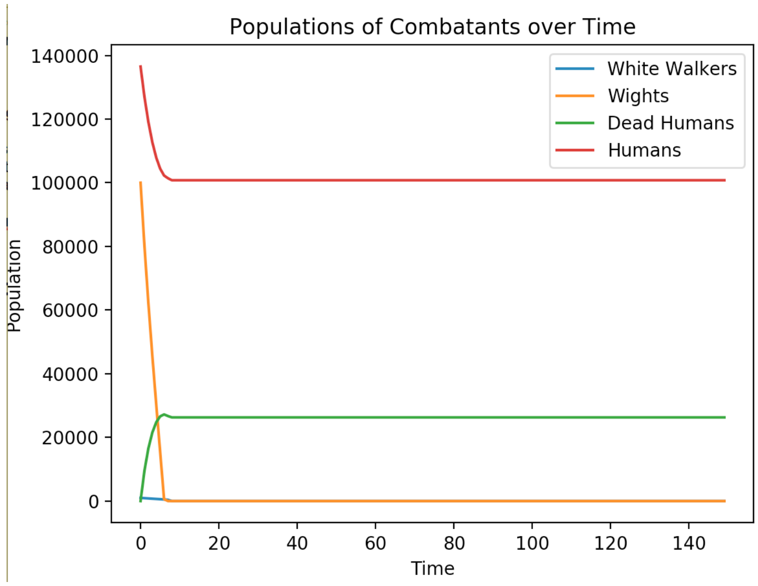


Figure 11: The final battle

### 5.3 The Worst Case

While the above model shows that we have all the necessary resources to be victorious, even at Winterfell, one might ask for the worst possible situation for Westeros: the case the amount of dragonglass is capped. In this case, given we have amassed all the armies of Westeros for a total of 224,500 soldiers, what is the minimum proportion of soldiers we need to equip with dragonglass weapons to be victorious? Note in this analysis that it is barely just possible for all armies to assemble in Winterfell before the White Walkers' army reaches the city, given each army travels by sea as much as possible. Simulating battles for various values of $p$, we find that roughly 17.5% of the army needs to be equipped, as shown below.

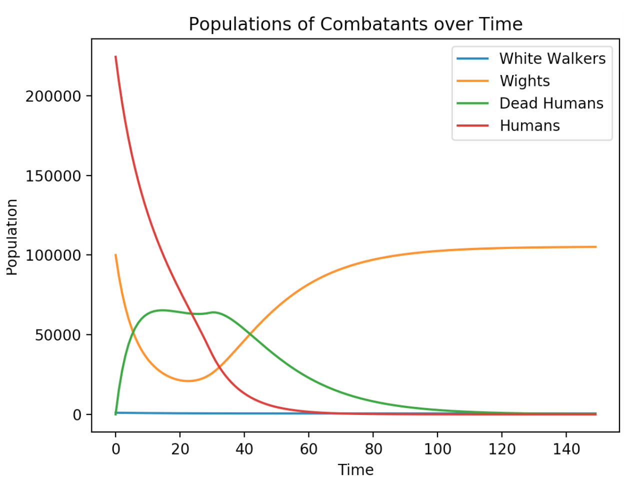


Figure 12: $p = 0.17$

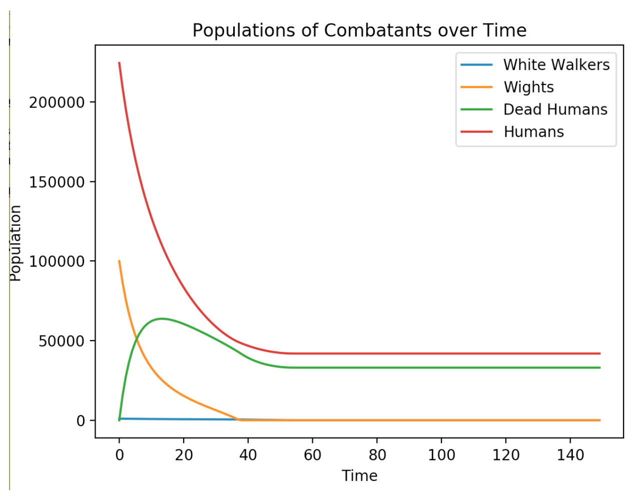


Figure 13: $p = 0.175$

Now by our estimate of 10 ounces of dragonglass per soldier, this is roughly $\boxed{380,000}$ ounces of dragonglass that needs to be mined. We can conclude that if there is not enough dragonglass on Dragonstone to supply this much weaponry, then Westeros should be advised to flee the continent, find additional reinforcements, or find other sources of dragonglass.

# 6 Analysis and Conclusion

## 6.1 Model advantages

The battle model we created, adapted from the classic SIR model of epidemiology, offers great flexibility and accuracy in modeling battle outcomes. We are able to include parameters for the initial populations of humans, Wights, and White Walkers along with their relative strengths, the proportion of dragonglass in the human armies, the proportions of each army engaged in battle, as well as reanimation rates of dead humans. This flexibility allows us to simulate battles under almost all conditions, which allows us to analyze best strategies for both the humans and White Walkers. Note that the model is capable of handling more information: we could have modeled the interactions between populations in a different manner, by substituting different functions with more desirable properties. For example, we could have modeled the reanimation rates of dead humans not as a linear function of the dead population, but as a logarithmic function, or otherwise. In a different manner, if we took into account that the human armies have dragons, we could raise the average human to wight strength ratio to reflect this new information. In this sense, the versatility of our model transforms the given problem into a problem of game theory and operations research, as we saw before.

While one might criticize our model as too complicated, the beauty is that the ability to set different parameters for the quantities we discussed above allows us to set most parameters to be some constant that is reflected in the data we are given. We can then analyze how varying only a couple parameters effects the outcomes of battles: in our case, we mainly analyzed the effects of $p$, the proportion of dragonglass, $\rho$, the reanimation rate of dead humans, and the number of human participants determined the outcome of a battle.

One last choice we made was modelling the map of Westeros as a discrete grid. While this does not allow for the richness of battles and movement outside of cities, it simplifies our analysis of where to allocate our armies.

## 6.2 Model disadvantages

While the battle model is extremely flexible in the parameters we provided and correctly reflects many qualitative observations that we made, the model is inflexible in the sense that it is deterministic. The outcome of a real battle is not determined by the sizes of the two opposing armies and how they interact in a fixed manner. Rather, warfare is inherently stochastic. In this viewpoint, we could have reinterpreted our model probabilistically, perhaps changing the populations of humans, Wights, and White Walkers in some random manner that still reflects the assumptions we made above. While this model, which can be interpreted as a system of stochastic differential equations, would allow us to give measures of the chances of victory—which more accurately reflects reality—it is also more difficult to analyze, and requires Monte Carlo simulation to give estimates of these probabilities. This would be far more computationally intensive than our analysis given here. We leave the design and implementation of the stochastic scheme for future work.

A small thing to note about our model is the existence of a critical point of humans such that any army with greater size wins battles, whereas any army with fewer numbers always loses. This extreme change in behavior of solutions is a fascinating artifact of our ODE model, but it might be desirable to not have this behavior; in this way, a stochastic model is more likely to eliminate behavior of this form.

### 6.3 Conclusion

This report outlines a strategy for the inhabitants of the Game of Thrones universe to face the threat of the White Walkers: in the simplest scenario where the White Walkers choose simply push south, by allocating enough resources to rapidly mine and mobilize the reserves of dragonglass, the battle model described in the earlier section combined with estimates of how much dragonglass can be mined and distributed predicts that humanity will survive. We also offer a worst-case analysis as described in §5.3. In the event of the White Walkers playing optimally, we could suitably modify our ODE-model and the grid map, as, unlike a totally simplified graphical model, our grid model does not ignore the geography of the continent, while remaining tractable in terms of complexity as opposed to a model that does not simplify the original map.

# 7 Future Work

Aside from the stochastic version of our model that could have been implemented, we also point out that the problem of adopting best strategies and allocating armies to both fighting and dragonglass collection tasks could be developed further. The main task would be to analyze the necessary numbers of soldiers to win given any proportion of dragonglass, and any inputs for the initial populations of humans, White Walkers, and Wights, via our battle model. Using this information, we could do a more in depth analysis of what strategies the White Walkers would take; in particular, we could analyze how best to respond when the White Walkers split their army into parts, the case of our analysis we did not account for more complicated strategies could be adopted by the White Walkers - for example, had human populations - both soldiers and civilian populations - not been able to communicate or coordinate perfectly, the White Walkers could have stayed longer different areas or visited different cities with the goal of massing wight armies, while slowly depleting the human armies. This corresponds to a game of imperfect information, where the humans do not play perfectly; in this case we also put into consideration the civilian populations of Westeros, which we omitted in our current model.

In another direction, we could have modelled the White Walkers as a non-sentient mass with the goal of expanding and conquering every part of Westeros; under this model, we would also have needed to solve the problem of allocating armies to fight and mine dragonglass, but the method for modelling the White Walker spread in this case seems to be markedly different from the approach we take here.

# A Code

```
import matplotlib.pyplot as plt
import numpy as np

# Euler's method for approximating ODEs is
# used to simulate a battle.
def simulate2(p, rho, A, B, y0, timespan, rate):
# y0 = [WW,W,D,H] initial populations as described in sec. 4.
# A = ratio in human to Wight strength.
# B = ratio in human to White Walker strength.
# rate = mixing rate of the two armies.
# p = proportion of human soldiers with dragonglass.
# rho = reanimation rate of Wights by White Walkers.

    time  = 0
    WW,W,D,H = [0]*timespan,[0]*timespan,[0]*timespan,[0]*timespan

    WW[0]  = y0[0]
    W[0]  = y0[1]
    D[0]  = y0[2]
    H[0]  = y0[3]

    t = 1/10
    #simulate this procedure
    for i in range(1,timespan):
        dHdt = t*min(rate*(W[i-1]/(A*(1+p)) + B*WW[i-1]),H[i-1])

        H[i] = H[i-1] - dHdt
        D[i] = max(0,D[i-1] + dHdt - t*D[i-1]*rho*WW[i-1]/1000 )
        W[i] = max(0, W[i-1] - t*100*(rate*H[i-1]*p/B)*(1/(1 + np.log10(1 + W[i-1]))) +
                t*D[i-1]*rho*WW[i-1]/1000 - t*rate*(A*(1+p))*H[i-1] )
        WW[i] = max(0,WW[i-1] -t*(rate*H[i-1]*p/B) * (1/(1 + np.log10(1 + W[i-1]))) )

    timeperiod = np.linspace(0,timespan-1,timespan)

    plt.plot(timeperiod,WW,label = 'White Walkers')
    plt.plot(timeperiod,W,label = 'Wights')
    plt.plot(timeperiod,D,label = 'Dead humans')
    plt.plot(timeperiod,H,label = 'Humans')
    plt.legend()
    plt.show()

    print(WW)
    print(W)
```

## B References


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