Thrones

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1 Nontechnical Summary

Winter is coming to Westeros, along with an army of undead.Word has just reached Winterfell, that 1000 white walkers and 100,000 wights have just breached the wall at Castle Black and are on their way towards the city. Having fought the undead before, humans know that the white walkers are many times stronger than humans and can only be killed with dragonglass. They also know that wights are weaker than humans and can be killed by dragonglass, fire, and dismemberment. Moreover, walkers can convert human corpses into new wights. Luckily, neither walkers nor wights can travel across water.

Taking all of this into account, our goal was to create a war model that reasonably simulates battle with our while walker opponents while also somewhat capturing the complex political relationships between. We wish to analyze under what conditions humans can survive the white walker onslaught, as well as how various cities defecting (refusing to send troops) would affect the overall outcome.

We first constructed a continuous model that predicts the outcome of the battle depending on the starting number of troops on each side. We then used this model to simulate a larger war with multiple battles. Factors we considered in our simulations include the threshold of retreat, the number of reinforcements each city would send in an attack, and the possible paths the walkers can take to reach King's Landing, among other things. After many simulations, we were able to come up with general "win conditions" for humans against the three possible ways the walkers can attack.

In summary, our central finding is that the ideal battle strategy requires a satisfactorily high level of reinforcements from other cities as well as a conservative retreating threshold (10-20%). If no cities cooperate, no troops are sent and humans will simply not have the manpower to fight the white walker horde. On the other hand, if humans all rush in and fight to the death from the first battle, they will generally lose even with a high number of forces. Thus, the human forces must be both unified and tactical in order to avoid annihilation by white walkers.

2 Introduction

A thousand White Walkers and one hundred-thousand Wights begin at Castle Black and are headed toward Winterfell. Barring defeat to humans at Wintefell, the White Walkers will march south to Casterly Rock and/or/only King's Landing. Human victory comes when King's Landing is free of White Walkers. Human defeat occurs when no humans remain in King's Landing. Since many of the troops, battle tactics and interactions are fictitious, reasonable parameters were determined for our battle model using our knowledge from the books/show and YouTube battle scenes. The prey/predator relationship of White Walkers, Wights and Humans are as follows:

1. **Humans and White Walkers**
   a. Walkers can only be killed by dragonglass. Yet even then, walkers are significantly stronger than humans.
   b. Walkers convert dead humans, who are not dismembered, into Wights.
2. **Humans and Walkers**
   a. Wights can only be killed by dragonglass, fire or dismemberment. However, as rotting, undead entities, wights are weaker than the average human soldier.
3. **White Walkers and Wights**
   a. Each walker is in "possession" of an even proportion of the total number of wights. When a walker dies, its share of wights instantly die as well.

Human troops are dispersed among 5 cities as follows: 26,500 in Winterfell, 8,000 in Casterly Rock, 30,000 in Iron Islands, 80,000 in Dragonstone and 80,000 in King's Landing. Our goal is to determine what levels of cooperation and what types of strategies result in human victory.

Our project is comprised of two main parts: a battle component and a politics component:

- The battle model relies on several parameters, including the proportion of humans wielding dragonglass, the priority humans place on killing walkers over wights, and the "retreat threshold" when humans will retreat, among other things. The model attempts to incorporate assumptions about the nature of medieval warfare.
- The politics component of the project mainly focuses on "cooperation coefficients," which determine each city's willingness to send troops to help other cities, as well as a rallying factor that drives humans to further unite as more human cities are taken over.

3 Model

The skeleton of our model is based upon the information given in the prompt. The outcome of the war is determined by using the battle model to simulate key battles that would occur depending on the walkers' strategies and the level of cooperation between human cities.

3.1 Battle

Our battle model is partially derived from Lanchester's Linear Law, the de facto model used to model "ancient" (strictly melee) combat.[1] Lanchester's Linear Law, however, does not completely account for the complexities of our battle situation. As the enemy consists of two distinct beings, walkers and wights, different attack powers and weaknesses are needed for different classes of enemies - attack power is a proxy for how effective one class of combatant is against another. Wights and walkers are also affected differently by different weapons. Finally, we needed to take into account the dependence of wights on walkers. Wights are continuously summoned by walkers; but, when a walker dies, all wights "possessed" by the walker die as well. The details of this relationship are discussed in the assumptions.

Let W, ω, and H represent the populations of walkers, wights and humans, respectively. Our battle model is then the following:

$$\frac{dW}{dt} = -\delta_{dW}(fD)H \tag{1}$$

$$\frac{d\omega}{dt} = -(\delta_{d\omega}D(1-f) + \delta_f F + \delta_b B)H - \delta_{dW}(fD)\frac{H\omega}{W} + \lambda_{H\omega}(\delta_W W + \delta_\omega \omega) \tag{2}$$

$$\frac{dH}{dt} = -\delta_W W - \delta_\omega \omega \tag{3}$$

We defined the parameters and their values as such:

$\delta_{dW} = 0.03$: attack power of human with dragonglass against a walker per unit time

$\delta_{d\omega} = 1$: attack power of human with dragonglass against a wight per unit time

$\delta_f = 1.5$: attack power of human with fire against a wight per unit time

$\delta_b = 0.8$: attack power of human with standard weapon against a wight per unit time

$\lambda_{H\omega} = 0.5$: rate at which walkers convert dead humans to wights *during* battle

$f = 0.25$: the priority dragonglass-wielders place on attacking walkers instead of wights

$D = 0.05$: proportion of humans fighting with dragonglass

$F = 0.2$: proportion of humans fighting with fire

$B = 0.75$: proportion of humans fighting with standard weapons

3.1.1 Battle Assumptions

1. No Siege Warfare

Upon initial thought, we considered siege warfare to be a considerable advantage for humans. However, a deeper look into the nature of walkers and wights revealed otherwise. In Medieval warfare, troops laying siege to a city utilized the tactic of starving those within the city out.[1] As walkers do not need food to survive, it should be in the best interest of the humans to instantly leave the city to fight the walkers, rather than wait in the city for any period of time and lose resources. Thus, we arrived at the assumption that siege warfare would never be in the best interest of humans.

2. Relative Strengths of White Walkers and Wights to Humans

When calculating the relative strengths of walkers and wights to humans, our only source of information is really things like lore from the books and battle scenes from the TV show. However, as we want to consider the strength of the *average* soldier, we kept in mind that the writers would prioritize the survival of the main characters and thus skew the relative strengths in favor of the humans. Accounting for such bias, we made the assumption that an average soldier would have an attack power of 0.03 and 1 against walkers and wights, respectively. In other words, in the time it would take roughly 33 human soldiers with dragonglass to kill one walker, 1 human soldier with dragonglass would have killed one wight. On the other hand, we took into account the overwhelming strengths of the walkers (i.e. the ability to generate enough torque to pierce dragon scales, plot armor, etc.) and the general swarming attack patterns of wights. Since the wights swarm their opponents, it would take more than one wight to defeat an average human. We concluded that walkers would have an attack power of 10 and wights would have an attack power of 0.7.

3. The Attack Power of Humans Against Wights with Fire and Sharp Weapons

Using video[2], we determined that the attack power of humans using fire (δ_f) would be the proportion of wights killed using fire to the total of wights killed. The same method was used to determine the attack power of humans using blunt weapons.(δ_b)

4. Conversion Rate of Humans to Wights During Battle

A key power of white walkers is their ability to convert dead humans into wights. We determined that during a battle, due to the ongoing battle and the accessibility of dead human bodies, walkers could only convert ($\lambda_{H\omega}$) of newly dead humans to wights at any given time.

5. Conversion Rate of Humans to Wights After Battle

Maintaining consistency with the previous assumption, white walkers should convert a higher rate of dead humans after the battle, as sections of both the show and the book depict white walkers raising all dead corpses after a battle. We made this constant ($\lambda_{H\omega} = 0.9$) to account for some humans being dismembered or otherwise unsuitable for conversion after death. Notice that this equation is not accounted for in the battle model as this technically occurs after a battle is over.

6. The Proportion of Humans With Dragonglass, Fire and Blunt Weapons

Due to the slow speed of white walkers relative to humans (discussed later), we assume humans will have enough time to distribute dragonglass to all troops before the first engagement with walkers. Using the video from the third assumption, we determined the proportion of people with dragonglass, fire and blunt weapons would be similar, but slightly less than, the distribution of the given weapons to the individuals in the video, as the characters in the video represented an above-average group of troops. Such video analysis determined value for D, F and B.

7. Advantages Gained Through Tactics, Terrain, and Other Extraneous Factors

We assumed that there neither the humans nor White Walkers had any advantage with regard to tactics, terrain or extraneous factors.

8. The Number of Wights Dying With a Walker is Proportional to the Walker Population

In correlation to the Game of Thrones world, a number of Wights possessed by a White Walker will all instantly die when that given White Walker is killed.[3] We assume that the number of Wights every White Walker possesses at any given time is uniformly distributed: equal to the number of Wights divided by the number of White Walkers ($\frac{\omega}{W}$).

3.2 War

We now consider the more complicated issue of simulating war. The single most important simplifying assumption in our simulation model is that humans are so fast relative to white walkers that human travel time is irrelevant when it comes to distributing resources. Our second major simplifying assumption is that battles only happen in cities; humans have some kind of information about white walker movements (ravens, sentries,etc.) and can move to various cities beforehand to prepare for a white walker defense. Given these assumptions (as well as some others), we analyze the significance of two factors in the outcome of the war.

The first factor is what we call the cooperation coefficient vector and represents the percentage of troops each city is willing to contribute to a particular battle.We define the vector $C \in \mathbb{R}^5$ and $N \in \mathbb{R}^3$. We have

$$C = [c_w, c_i, c_c, c_k, c_d], \quad N = [0.75, 1, 1.25] \tag{4}$$

C is the vector containing all of the cooperation coefficients of the cities: c_w, c_i, c_c, c_k, and c_d are the coefficients for Winterfell, Iron Islands, Casterly Rock, King's Landing, and Dragonstone respectively. N is a vector of parameters that we created in order to account for a "rallying effect" - that is, the idea that cities will be more cooperative as more cities fall to white walkers in order to save themselves from mutual destruction. We leave N as fixed for our purposes here, but it is certainly something that we have liked to test further given more time. Then,we calculate the cooperation of a city j in the following way for $n_i \in N$ and $c_j \in C$.

$$C_{ij} = \frac{c_j}{n_i}, \qquad i \in \{1, 2, 3\}, j \in \{w, i, c, k, d\} \tag{5}$$

where i is the number of cities still controlled by the humans (cities that have not fallen) and j is the city sending the troops. Then, our calculation of the number of troops each city contributes to a battle at city s can be written as a function:

$$T(s) = \begin{cases} C_{ij} & \text{if } s \neq j \\ P_j & \text{if } s = j \end{cases}$$

Where P_j is the population of city j. This just has the effect of "forcing" troops at a given city to fight if that city is the one that white walkers attack.

Then, given this framework, we consider the cooperation coefficient vector a proxy for humans strategy: humans decide how much they wish to cooperate with the rest of the states.

The second factor is what we call the retreat threshold. From research as well as common sense, we can say that commanders will generally retreat their troops upon reaching some loss threshold instead of fighting until total annihilation. In this particular situation retreating intuitively seems like it should be an integral part of the human strategy for a few reasons:

- Retreating utilizes the greater speed and organization of human armies, allowing them to "maneuver" around the white walker onslaught.
- Retreating helps control the number of human casualties, which in turn lessens the number of wights white walkers can create.
- Given the supernatural nature of the enemy, it makes even more sense for soldiers to lose composure and flee after sustaining heavy losses.

Then, let us define r as the retreat threshold. We then maintain that if for time t,

$$H_t < rH_0$$

holds - where H_0 is the initial population that fought in the battle and H_t is the remaining population at time t - the human army will stop fighting and start retreating. Retreating troops are evenly distributed to all "alive" cities, cities that have not already been overrun by white walkers. Our justification for this is again the assumption that humans are much faster than walkers and so should be able to reorganize fairly effectively after a retreat.

3.2.1 War Assumptions

1. The Travel Speed of Humans is Much Greater than that of Walkers and Wights

Figure 1: A map tracking the distance traveled by Jon Snow (human) and the White Walker army in the same time period. The green line represents Jon Snow and the red line represents the walkers.[5]

From Figure 1 alone, it already seems like the speed of humans makes that of walkers negligible. However, we were skeptical about using this figure alone to generalize the moving speed of large armies due the fact that much of Jon's travel was by boat or by dragon and due to the show's time consistency generally being a point of contention with later seasons.

A stronger argument for the relative speed of white walkers to humans is that, both in the book/show story and in our fictional problem here, the humans should presumably be able to reach Winterfell from locations like King's Landing and Dragonstone in the time it takes the white walker horde to travel from Castle Black to Winterfell. Rough calculations of distances from Figure 1 show that the distance from King's Landing to Winterfell (1480km) is around 3 times the distance from Castle Black to Winterfell(580 km).Thus, a human army must be at least 3 times as fast as the white walker horde for the storyline to make any sense - this is our baseline assumption.

Given this assumption, humans should always be able to reorganize themselves in time to respond to white walkers before every battle. This follows from the fact that the distance between King's Landing and Casterly Rock is relatively short (880km), so the human forces would only need to scout the white walkers - who must follow the road - at some point around 300 meters outside of their castle. Additionally, assuming that ravens travel at speeds similar to homing pigeon, they should be able to cover distances of 500-600km in less than a day.[6] We thus assume that human's have some way to track white walker movements, but this assumption seems fairly reasonable (e.g. humans could have sentry posts at Harrenhal and Riverrun, which are both more than 300 meters away from Casterly Rock and King's Landing, respectively).

This assumption is critical to our model, because it implies that humans can allocate troops and resources (dragonglass) effectively instantaneously between white walker attacks. Then, by assuming that the relative travel time of humans to be much greater than that of walkers, we do not need to incorporate travel time into our models.

2. Battles only Happen at Cities

We also maintain that humans will only fight at cities. Reasons for this include human armies' preferences for familiar terrain, desire to have more preparation time, desire to fight around strategic objectives, etc.

3. The First Battle is at Winterfell

As the prompt states the assume walkers can only travel on roads, it should be reasonable to assume that they do not simply pass by Winterfell the first battle then occurs at Winterfell.

4. The Different Human Cities will Not All be Unified

It intuitively seems like the most efficient strategy for the humans is to send all possible troops to Winterfell; however, we felt that this would be extremely in-feasible due to politics arising between different factions. Thus, we justify our use of a cooperation coefficient vector to analyze how humans fare with differing levels of cooperation.

5. Casualties From Retreating

Preliminary research into medieval warfare, as well as common sense, suggest that a significant number of troops was lost during retreat. But, considering the relatively slow speed of walkers to humans, we determined that 10 percent of a human force was killed when retreating.

6. The Retreating Population from a Fallen City is Evenly Distributed to other Cities

Given that our model has no information about specific city-to-city political alliances (which we acknowledge is one of its major weaknesses), the most sensible way to distribute retreating troops from a fallen city (i.e. troops from Winterfell after the defeat) is just uniformly among remaining cities.

7. "Rallying Factor" based on number of fallen cities

We thought that the use of a 'rallying factor' in the form of N is appropriate as one of the main themes in the storyline at this point is the need for cooperation in the face of a greater foe. This also obviously aligns with rational action for the different states, as white walker domination is extremely unbeneficial to all human parties.

Notice that due to the simplicity of this situation - there are only 5 cities and 2 of them are islands (and so cannot be attacked by walkers) - the only three cities that matter are Winterfell, Casterly Rock, and King's Landing. Thus, the fixed N we use is a 3-dimensional vector.

We set N to the specific values defined above, because after parameter testing we felt that the current choice reasonably scaled the cooperation coefficient without having too great an effect. This relationship is captured by equation (5) above.

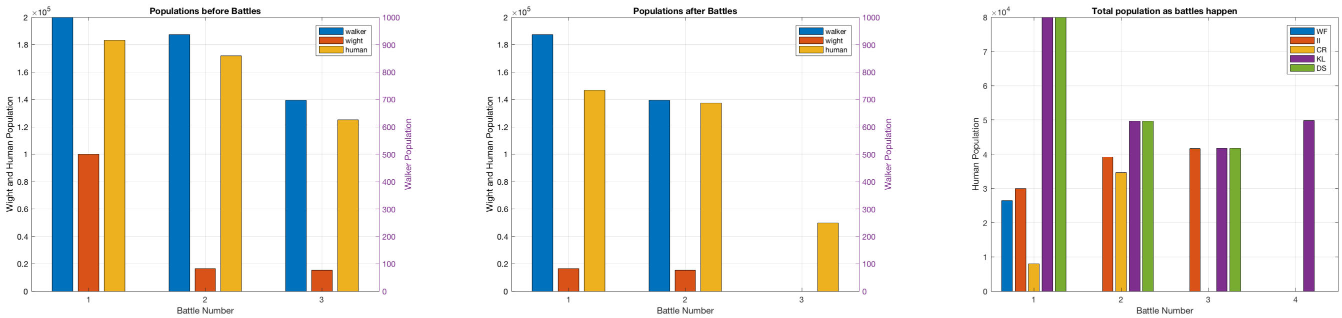
4 Results

4.1 Population Analysis

The two possible outcomes for the war are either a human win or a white walker win. Figure 2 tracks the population when humans win the war. We can see here that the battle model is roughly accurate in tracking the populations of the enemies and the humans given a retreat threshold. Our definition of the humans winning is the annihilation of the walkers while maintaining a population at King's Landing. Figures 2.a and 2.b show that after the last battle at King's Landing there are no walkers or wights, while figure 2.c shows that there are humans alive at King's Landing. Moreover, we see that all of the human population is at King's Landing by the end of the last battle. This shows that our C_{ij} (equation 5) functions as expected. That is, when all the other cities on the mainland have fallen and the walkers are advancing towards King's Landing, it would be in all of the cities' best interests to fight in the last battle.

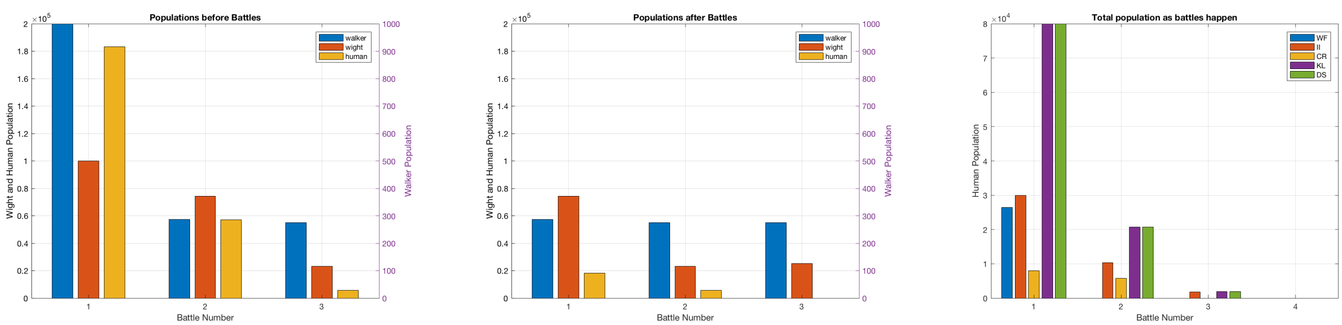
For the losing outcome, there are two possible conditions that result in the humans losing. Figure 3 portrays the case when the retreat threshold is very low and the cooperation is very high. This would be the cases when the armies fight to near annihilation at every battle. This would be a horrible strategy for the humans and result in the rapid loss of population too early in the war as shown in figure 3.c. Furthermore, if more troops die early in the war, the wight population would get an early boost in population due to more dead humans becoming wights. This boost in wight population can be seen in figure 3.a where decrease in wight population is much more gradual over the battles than in figure 2.a.This supports the notion that leaders should pay attention to the optimal retreat threshold when devising their battle strategies.

The other losing outcome is driven by a low cooperation rate. Despite having a higher retreat threshold, the humans fare worse when the cooperation between cities is whats causing the loss. Notice how in figure 4.a and 4.b that the populations of the walkers across the battles are much higher than in any of the other scenarios. Figure 4.c further portrays the bleak outcome of not cooperating: each battle shows a whole city being annihilated. In this specific case, the walkers defeat Winterfell, Casterly Rock and finally King's Landing in battles 1, 2, and 3 respectively. We can infer that the humans have no hope of defeating the walkers and their army of wights if they do not cooperate.



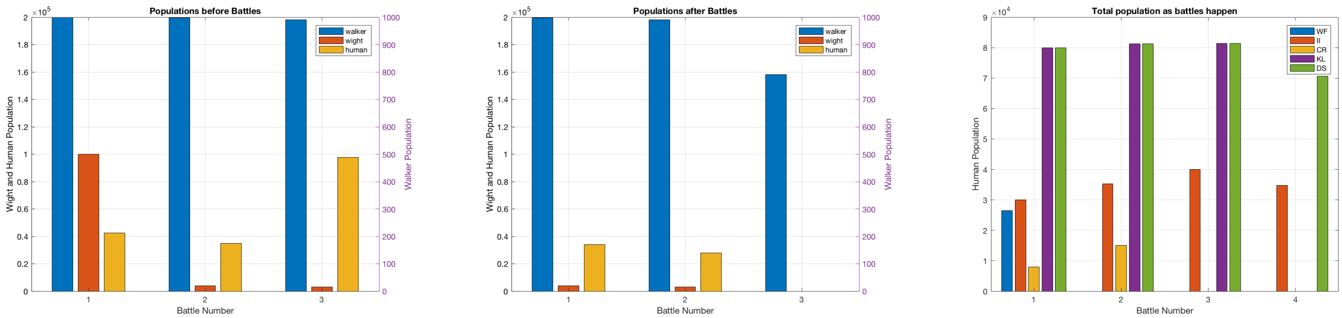
(a) Population Before Battle (b) Population After Battle (c) Population Distribution After Battles

Figure 2: The change in human,white walker, and wight populations before and after each battle, and the total populations by city as battles occur. Note that in this situation $C = [0.9, 0.9, 0.9, 0.9, 0.9]$ and $r = 0.8$ - meaning all cities are willing to send 90% of their troops and human force will retreat from a battle upon losing 20% of the original force. (a) The population before the battles with the left axis being for the wights and humans, and the right axis being for the walkers. Blue, red, and yellow represent walker, wight, and human populations respectively. (b) Same as (a) but for after the battles. (c) The population by city starting with initial populations. Blue, red, yellow, purple, and green represent Winterfell, the Iron Islands, Casterly Rock, King's Landing, and Dragonstone respectively.



(a) Population Before Battle (b) Population After Battle (c) Population Distribution After Battles

Figure 3: Same descriptive graphs as in Figure 2. Note that in this scenario $C = [0.9, 0.9, 0.9, 0.9, 0.9]$ and $r = 0.1$. (a) the population before the battles with the left axis being for the wights and humans, and the right axis being for the walkers. Blue, red, and yellow represent walker, wight, and human populations respectively. (b) Same as (a) but for after the battles. (c) Tracking the population in each cities starting with initial populations. Blue, red, yellow, purple, and green represent Winterfell, Iron Islands, Casterly Rock, King's Landing, and Dragonstone respectively.



(a) Population Before Battle

(b) Population After Battle

(c) Population Distribution After Battles

Figure 4: Same descriptive graphs as in Figures 2 and 3. Note that in this situation $C = [0.1, 0.1, 0.1, 0.1, 0.1]$ and $r = 0.8$. (a) the population before the battles with the left axis being for the wights and humans, and the right axis being for the walkers. Blue, red, and yellow represent walker, wight, and human populations respectively. (b) Same as (a) but for after the battles. (c) Tracking the population in each cities starting with initial populations. Blue, red, yellow, purple, and green represent Winterfell, Iron Islands, Casterly Rock, King's Landing, and Dragonstone respectively.

4.2 Parameter Analysis

Now that we have established the importance of the cooperation coefficients and the retreat thresholds in determining the outcome, we test three different walker strategies: 1) The walkers attack Winterfell, then Casterly Rock, and then King's Landing; 2) The walkers attack Winterfell and then King's Landing; 3) The walkers attack Winterfell and then send half their horde to Casterly Rock and the other half King's Landing. From these we find the optimal level of cooperation and retreat thresholds that result in the humans winning the war.

Case 1:

We first simulate the first walker strategy over a grid of possible parameters to obtain the heat map depicting ending human population shown in Figure 5 - we set all cities to one cooperation coefficient so that we can have a 2-parameter model. From this figure we see that the optimal human strategy would be for the cities to have a cooperative coefficient of 1 with a retreat threshold of about 0.8. Notice that even at the optimal condition, the minimum number of casualties would be about 55% of the total population.

Then, we simulate this scenario with different cities choosing to not cooperate. Figure 6 shows the relative importance of the contributions from the cities other than Winterfell. We do not consider the contribution of Winterfell due to our assumption that the first battle would be at Winterfell (which forces all troops at Winterfell to fight).

From Figure 6, we see that contributions from the Iron Islands and Dragonstone play a crucial role in winning the war. No help from either of these cities would result in no possible winning strategy for humans. On the other hand, King's Landing lack of contribution seems to seems to improve the possible outcome of of the war at high cooperation levels with a minimum mortality rate of 45%. Intuitively, this reflects the fact that after other cities have fallen, King's Landing will have a healthy army that will still defend King's Landing, and other cities will still send troops because of the rallying effect. Perhaps then, it would be in the best interest of those in King's Landing their best interest to not cooperate with the other cities.

At the same time, we also think that these results could be due to our model over-prioritizing protecting King's Landing versus other cities - this is a weakness we discuss later. Nevertheless, we see that overall for this scenario, the best human strategy is a high retreat threshold and a moderate to high cooperation with other cities.

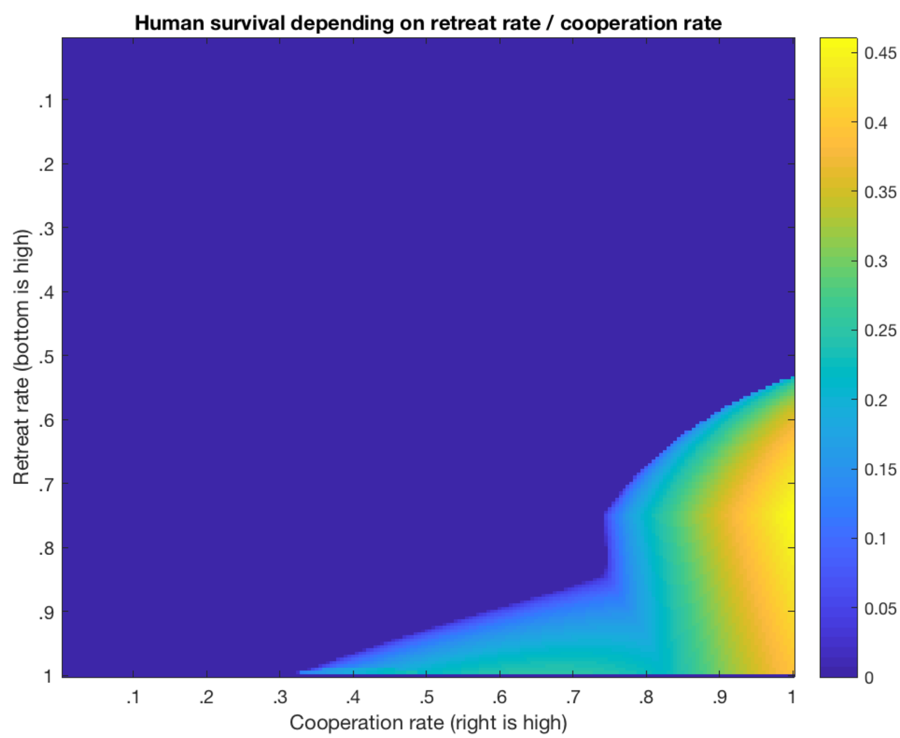
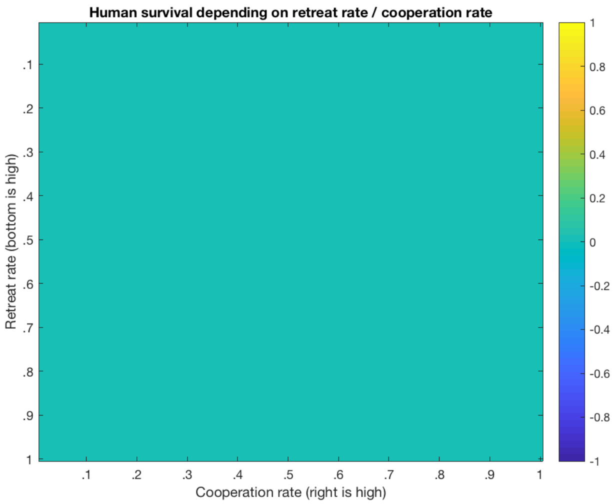
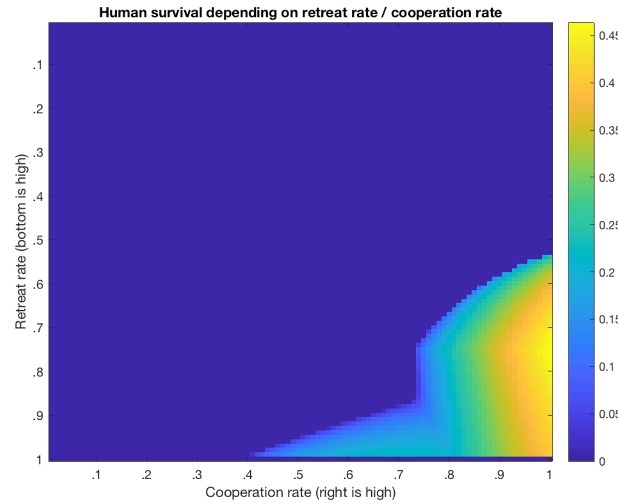


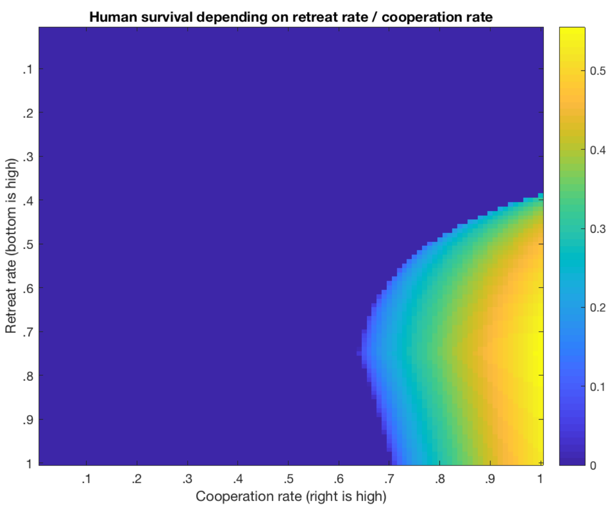
Figure 5: The heat map representing the survival of the total human population after the war depending on the cooperative coefficients and the retreat threshold.



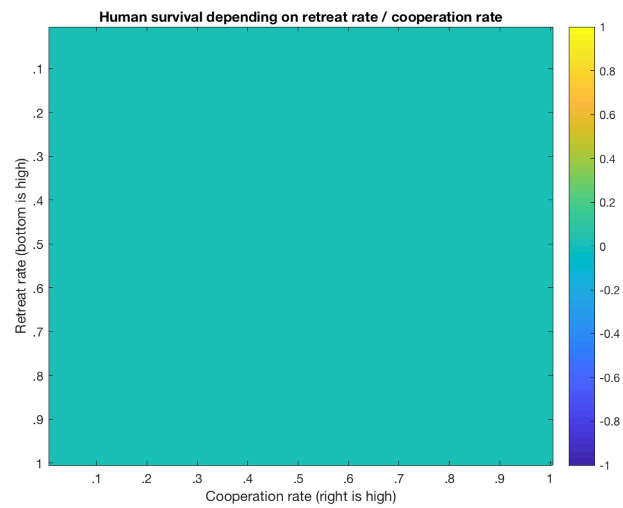
(a) Iron Islands: $c_i = 0$



(b) Casterly Rock: $c_c = 0$



(c) King's Landing: $c_k = 0$



(d) Dragonstone: $c_d = 0$

Figure 6: The heat maps representing the outcomes if the respective cities had a contribution coefficient of 0.

Case 2:

When we simulate the second walker scenario (They go to Winterfell and then King's Landing) we obtain the heat map shown in Figure 7. From this figure we see that the optimal human strategy would be for the cities to have a cooperative coefficient of 1 with a retreat threshold of 1. In essence, this result suggests that the best strategy would be for everyone to wait at King's Landing for the final fight. That this result isn't necessarily false as such a strategy likely does have its merits; however, it does shed light on the design fault we mentioned in the earlier case where we over prioritize protecting King's Landing.Despite this, we can still infer that the human strategy for this case should involve a high level of cooperation and a high retreat threshold.

From Figure 8, we see that this is the worst case scenario for the humans an ally defects and does not cooperate. If any other city other than King's Landing is uncooperative, then there exists no winning strategy. Similar to case 1, if King's Landing is the only one that was not cooperative, the winning is still possible with enough cooperation. With that under consideration, this would also be the best strategy for the walkers and the wights our of our scenarios covered in this analysis.

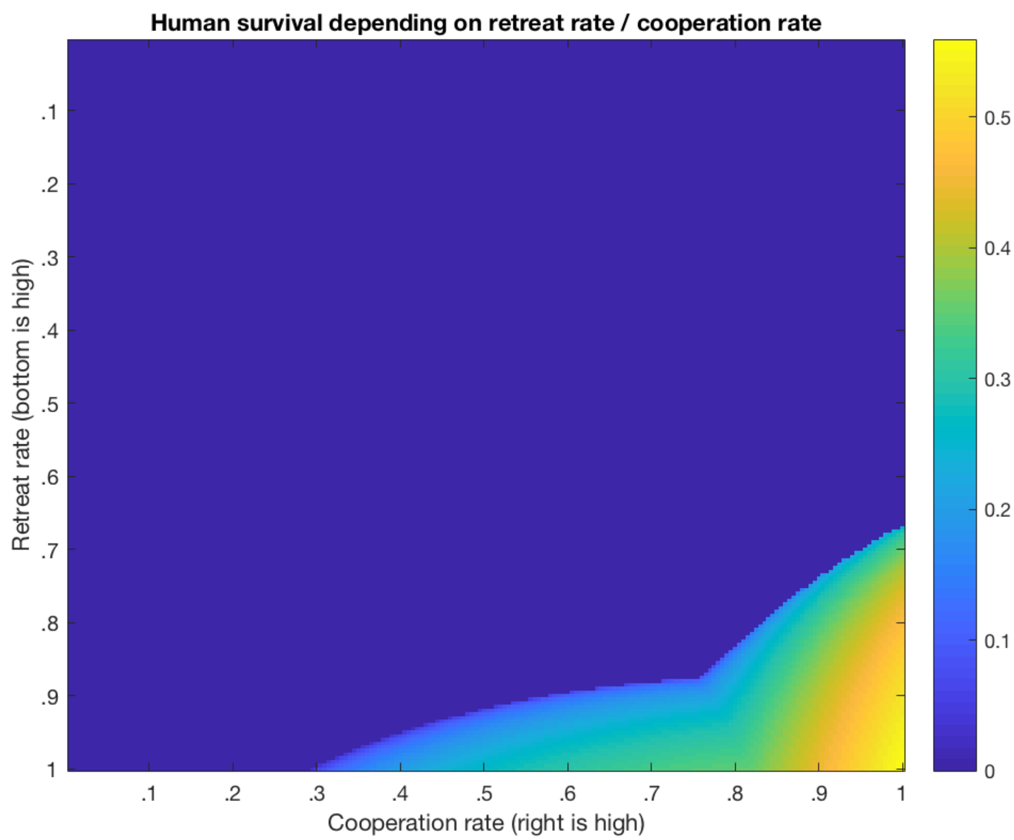
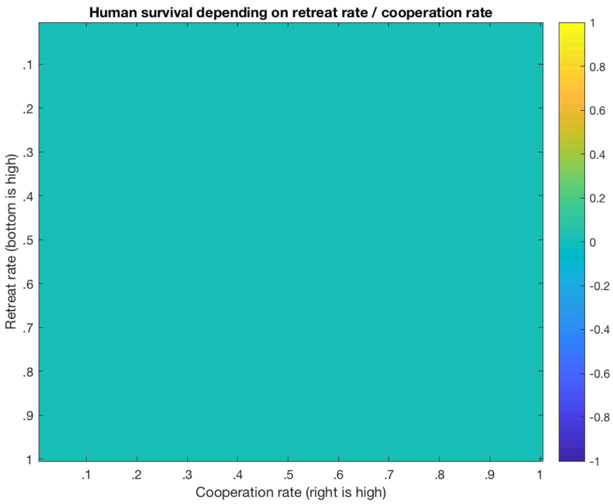
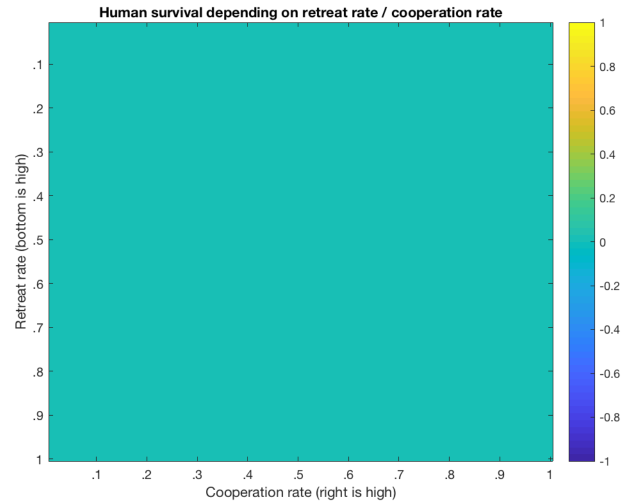


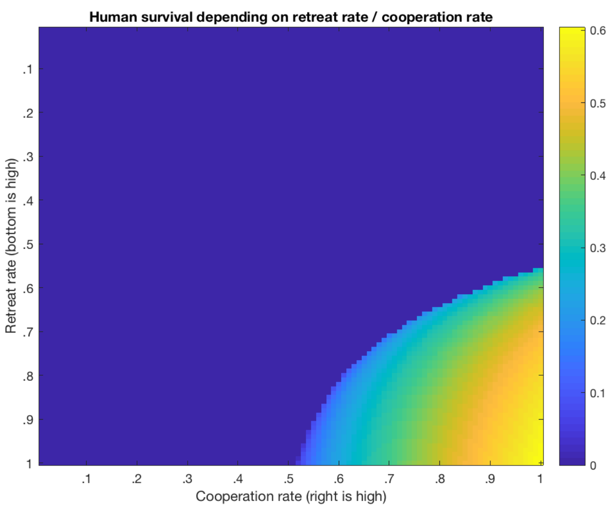
Figure 7: The heat map representing the survival of the total human population after the war depending on the cooperative coefficients and the retreat threshold.



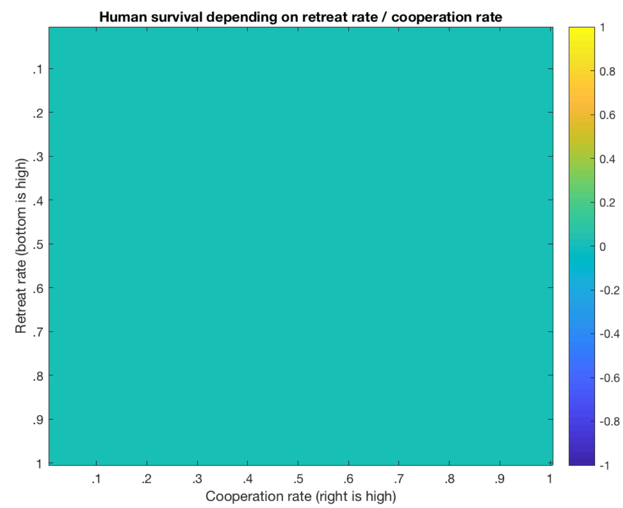
(a) Iron Islands: $c_i = 0$



(b) Casterly Rock: $c_c = 0$



(c) King's Landing: $c_k = 0$



(d) Dragonstone: $c_d = 0$

Figure 8: The heat maps representing the outcomes if the respective cities had a contribution coefficient of 0.

Case 3:

The last walker strategy where they split their forces between Casterly Rock and King's Landing proves to be the easiest one for humans to deal with. Figure 9, compared to the other two heat maps, demonstrates the wider range of winning strategies for humans in this scenario. Furthermore, more humans survive as the best cases here have a final mortality rate of around 20 to 25% or the total population.

Additionally, Figure 10 shows that unlike other white walker strategies, it is impossible for one city to cause the entire war to be lost by defecting. cause the war to be lost. This is an interesting insight that suggests that the more united the white walker forces are, the more important cooperation is for human survival.

We can also infer from Figure 10.d that that Dragonstone seems to have the greatest impact on the outcome. It is interesting to note that with no contribution from Dragonstone, the winning strategy is to have a very low retreat threshold, in contrast to a higher retreat threshold generally being better. Furthermore, figure 10.d also shows that the outcomes are very extreme when Dragonstone defects: either the majority of the population lives or everyone dies.

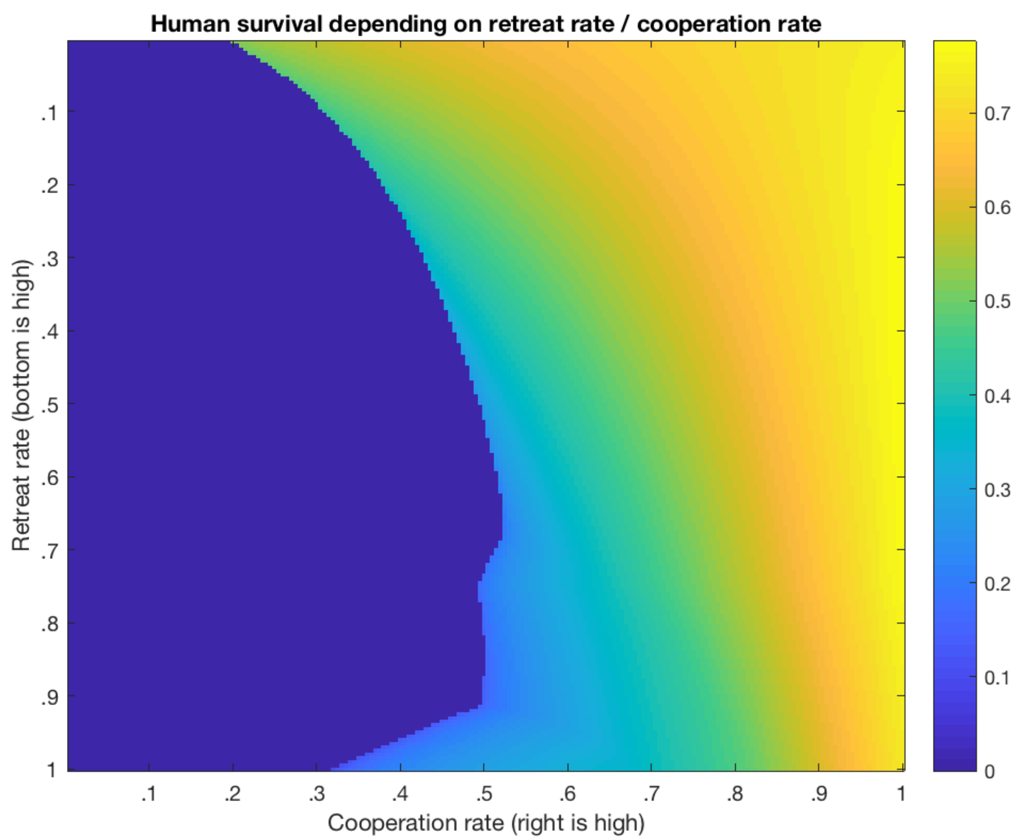
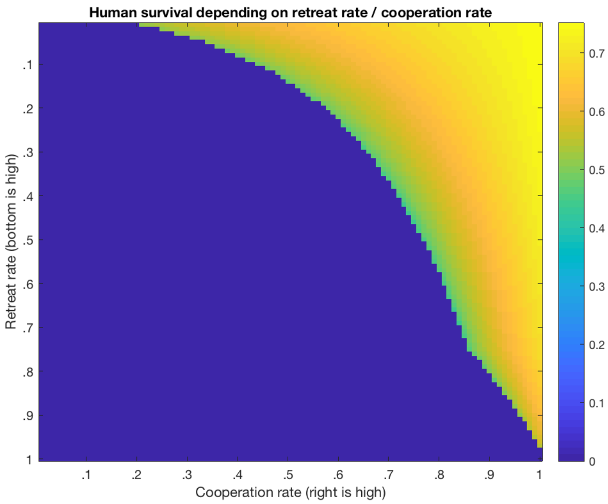
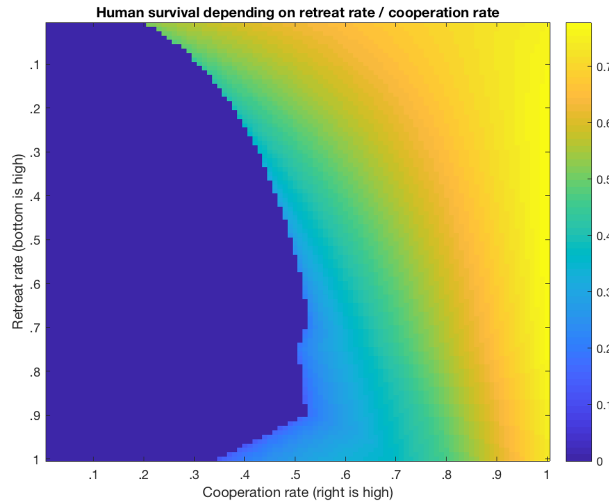


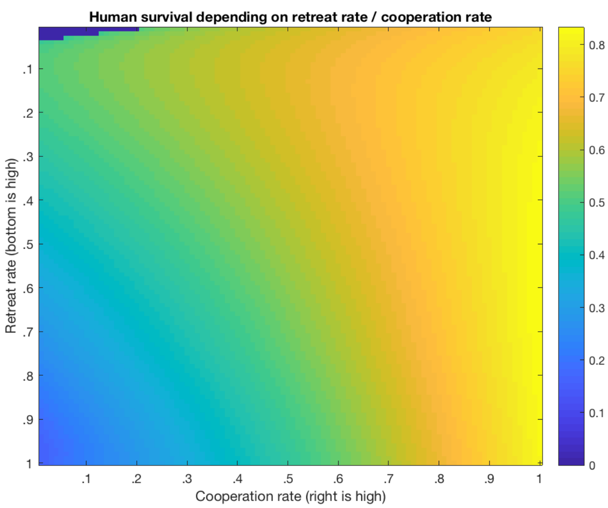
Figure 9: The heat map representing the survival of the total human population after the war depending on the cooperative coefficients and the retreat threshold.



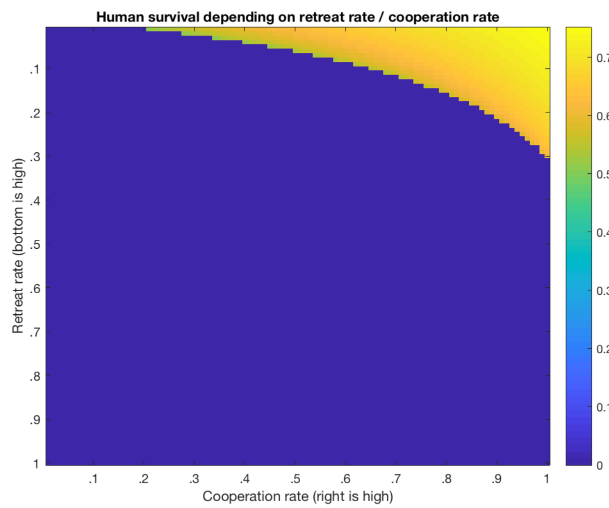
(a) Iron Islands: $c_i = 0$



(b) Casterly Rock: $c_c = 0$



(c) King's Landing: $c_k = 0$



(d) Dragonstone: $c_d = 0$

Figure 10: The heat maps representing the outcomes if the respective cities had a contribution coefficient of 0.

5 Strengths and Weaknesses

5.1 Strengths

1. A major strength of our model is the robustness of results to changes of small details - since we wish to analyze abstract concepts like a city's willingness to cooperate, the results gained from our model matter on a macro, high-level. This is in contrast to a model that zones in more on specific details of the human strategy - such a model should be more likely to be heavily affected by small changes in the model parameters.

2. On the flip side, we believe that actually constructing a model for individual battles makes sense and allows us to somewhat accurately model medieval combat. Our results should be more sophisticated than a similar war simulation model that simplifies combat.

3. Another strength of our model is that it is completely deterministic given input parameters. This allows us to test more efficiently because we don't need sampling to evaluate how a specific set of inputs perform. Thus in the same time, we can evaluate a wider range of situations with our model than with a non-deterministic model that runs in similar time.

5.2 Weaknesses

1. **Modeling War as a Turn-Based Simulation**

   The main effect of our core simplifying assumptions, that humans are much faster than walkers (3.2.1) and that battles happen only at cities(3.2.2), is that entire wars essentially become a turn-based process: human's know what the white walkers move will be and can respond with minimal limitations on movement and resources. There are many reasons why this isn't realistic - Mobilizing armies may be slower than we anticipated, humans may want to have the element of surprise, etc. This discrete system imposes many limitations on our model. The number of battles is fixed at a relatively low number, making it difficult to add in time-dependent factors external to battles, such as dragonglass mining or white walker births from the Night King, etc.

   The solution would be to model the entire simulation with a structure that utilizes time steps and representations of distances. This should allow us to simulate an arbitrary number of battles at arbitrary locations. Thus, we can analyze how travel speed limitations would effect human survival. Moreover, such a model would let us incorporate numerous time-dependent, battle-external factors, as mentioned above.

2. **Assuming That Each City Treats Every Other City Uniformly**

   Through our cooperation coefficient vector, our model currently considers each city as their own faction and then considers all other cities as identical from the perspective of any particular city. This simplification severely restricts the expressiveness in what we can analyze with our model - for example, it's impossible to model factions between the 5 cities which are obviously very real in the fictional universe.

   We considered the idea of having our input be a matrix of cooperation coefficients between particular cities. Using such a matrix would allow us to better model intricate and complex political relationships among cities - however,we encountered some difficulties when thinking about how to evaluate/visualize the date succinctly. This is certainly an area we would have pursued further if time was not a limitation.

3. **No Penalty for Losing Cities**

   A major design flaw of our model is that humans are not penalized for losing cities. In fact, humans benefit from losing cities due to the rallying effect that causes more troops to be sent to the next battle. This is not very realistic, as losing cities should mean a loss in human resources and morale.

   We could tackle this issue in several ways. One way is to model the generation of wights by walkers from their captured regions. So, the more cities walkers capture, the quicker they create wights - this would require that we have some time-step model, as mentioned above. Another way to place significance in cities is by introducing a resource coefficient that functions as a multiplier on the effectiveness of human troops in combat. Essentially, such a coefficient would have some positive correlation with the number of human-occupied cities. The second approach in particular seemed very reasonable to implement and is something we would do if given more time.

4. **No Sophisticated Model for Dragonglass Logistics**

   Our model makes the rather strong assumption that dragonglass will be evenly distributed before the first battle at Winterfell. It is not clear whether or not this assumption is strong as dragonglass seems quite rare in the fictional universe and only recently mined by humans at Dragonstone in the latest season of the show.

   A more sophisticated model accounting for the creation and transportation of dragonglass - or other resources - could provide interesting simulation insights about the supply chain of the human side.

5. **Heavy Reliance on Assumptions about White Walker/Wight/Human Combat Interactions**

   Our model relies heavily on many sweeping assumptions about combat mechanics - how effective different entities are against eachother, how white walkers convert humans to wights, etc. Most of these parameters were based on film snippets or aspects of the storyline. However, we believe that this issue is likely not unique to our model as any model studying a war with fictional enemies would run into the same issue.

   Assumption 1 (3.2.2) stated that due to the durable nature of walkers, all battles would occur outside cities. Although this assumption allowed us to simplify battles, our research on Medieval warfare suggested that defenders in a siege had a significant advantage when attacked upon.

6 Conclusion

6.1 Final Thoughts

Despite the simplifications made in creating our model, we think that our results still deliver meaningful insights in evaluating human strategies against the white walkers. Given the initial conditions, we found that it is possible for the humans to win the war as long as they cooperate with each other. Even if they fail to completely cooperate, depending on the walkers' strategy, the humans can usually still win the war with guerrilla like battle strategies (having a high retreat threshold). However, with little to no cooperation, or if major players defect and do not send troops, the humans have no hope of winning. Thus, we can infer that unity between the cities is the single most important factor for any human strategy to be successful.

7 Appendix

7.1 Code Description

We simulated our constructed models and created our visuals using Matlab. There are four files: run_battle.m, rand_script.m, sim_war.m, and sim_matrix.m. They have the following functions

1. run_battle.m - a function that takes as input the initial populations of the enemy and the humans that are going to participate in the battle and the maximum number of days the battle will last. This function will return an array of values for the populations of the walkers, the wights, the humans, and the converted wights (corpses converted to humans during battle). Note that this is where the battle model is simulated, and this function is used throughout the remaining scripts.

2. rand_script.m - a script that simulates the outcomes of a war and plots the populations of walkers, wights, and people before and after a battle. This script also plots the populations of all the human cities after every battle. This script outputs three bar graphs that represents these populations.

3. sim_war.m - a function that takes as input a vector of cooperation coefficients, the retreat threshold, the strategy of the white walkers, and whether or not that strategy involves the white walkers splitting their forces. This function will return an array of values tracking the populations of every city, the number of people before and after battle, and the proportion of people that survived the war. Note that the returned proportion is used to find the optimal strategy in our heat maps.

4. sim_matrix.m - a script that simulates and plots all the final proportions of people that survived the war for different retreat thresholds and cooperation coefficients. By plotting all the results from $r \in [0, 1]$ and $(\forall c \in C) c \in [0, 1]$, we obtain the heat maps used in our results. Note that when studying the effect of cities defecting, we just fixed a specific city's cooperation coefficient to .01.

7.2 Code

7.2.1 run_battle.m

```
function [W_1, w_1, h_1, lambda_1, index, windex] = run_battle(W0, w0, h0, num_days)

% Adjustible Parameters
t_max = num_days;
dt = 1/1440;
```

```
% Fixed Parameters
W_0 = W0;
w_0 = w0;
h_0 = h0;

% Percentages
focus = 0.25;
nonfocus = 1-focus;
D = 0.05;
F = 0.20;
B = 1- D - F;

% Humans killing enemy
d_dW = 0.03;
d_dw = 1;
d_f = 1.5;
d_b = .8;

% Enemy killing humans
d_W = 10;
d_w = .7;
lambda_hw = 0.5;

% Derivative Definitions
dWdt = @(W, w, h) -d_dW*D*focus*(max(0,h));
dwdt = @(W, w, h) -(d_dw*D*nonfocus + d_f*F + d_b*B)*(max(0,h))...
        -d_dW*D*focus*(max(0,h))*(w/W) + lambda_hw*(d_W*max(0,W) + d_w*max(0,w));
dhdt = @(W, w, h) -d_W*(max(W,0)) - d_w*(max(0,w));
dlambdadt = @(W, w, h) lambda_hw*(d_W*max(0,W) + d_w*max(0,w));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

n_time_steps = t_max/dt;

W = zeros(1, n_time_steps);
w = zeros(1, n_time_steps);
h = zeros(1, n_time_steps);
lambda = zeros(1, n_time_steps);

W(1) = W_0;
w(1) = w_0;
h(1) = h_0;
```

```
lambda(1) = 0;

indicator = 0;
wight_index = n_time_steps;
for i = 2:n_time_steps
   W(i) = W(i-1) + dWdt(W(i-1), w(i-1), h(i-1))*dt;
   w(i) = w(i-1) + dwdt(W(i-1), w(i-1), h(i-1))*dt;
   h(i) = h(i-1) + dhdt(W(i-1), w(i-1), h(i-1))*dt;
   lambda(i) = lambda(i-1) + dlambdadt(W(i-1), w(i-1), h(i-1))*dt;
   if w(i) < 0 && indicator == 0
       wight_index = i;
       indicator = 1;
   end
   if (W(i) < 0) || (h(i) < 0)
       break
   end
end

% Zero all negative terms
W(i) = max(0, W(i));
w(wight_index) = max(0, w(wight_index));
h(i) = max(0, h(i));

windex = wight_index;
index = i;
W_1 = W;
w_1 = w;
h_1 = h;
lambda_1 = lambda;
```

7.2.2 rand_script.m

```
% Parameters to change
split = 0;
path = [1,3,4];
c = [.1,.1,.1,.1,.1];
retreat = .8;

% Run the war sim
[win,b,p] = sim_war(c',path,retreat,split);

% A split causes an extra battle
```

```
num_battle = length(path) + split;

% Graphing logic
y = zeros(num_battle, 3,2);
for k = 1:num_battle
    y(k,:,1) = [b(2,k,1)*200,b(3,k,1), b(1,k,1)];
    y(k,:,2) = [b(2,k,2)*200,b(3,k,2), b(1,k,2)];
end
hold on
figure(1)
clf
H = bar(1:num_battle, y(:,:,2));
grid on
title('Populations after Battles');
ylabel('Wight and Human Population')
xlabel('Battle Number');
legend(H, 'walker', 'wight', 'human')
ylim([0 2*10^5])
yyaxis right
ylim([0 1000])
ylabel('Walker Population');
hold off

hold on
figure(2)
clf
H = bar(1:num_battle, y(:,:,1));
grid on
title('Populations before Battles');
ylabel('Wight and Human Population')
xlabel('Battle Number');
legend(H, 'walker', 'wight', 'human')
ylim([0 2*10^5])
yyaxis right
ylim([0 1000])
ylabel('Walker Population');
hold off

hold on
figure(3)
clf
H = bar(1:(num_battle+1), p');
```

```
grid on
title('Total population as battles happen');
legend(H, 'WF','II','CR','KL','DS');
ylabel('Human Population')
xlabel('Battle Number');
hold off
```

7.2.3 sim_war.m

Note: Locations are as follows:1 - WF, 2-II, 3-CR, 4-KL, 5-DS

```
function [win, battle,pop] = sim_war(coop_vec,ww_path,ret_percent,split)
    num_battles = length(ww_path);
% Simulates a sequence of battles between humans and white walkers
%     Logic with ww's splitting is a bit buggy - mainly hardcoded

    % Parameters not to be changed

    % Penalty for army retreating
    ret_penalty = .1;
    % Percent of dead people after battles converted to wights
    % (we assume white walkers can convert higher% of people to wights
    % after battle)
    wight_conv = 0.9;
    % "Rally factor" causes humans to work together more/send more troops
    % if more cities are taken by white walkers
    rally = [1.25,1,.75];

    % Population Holder
    pop = zeros(5,num_battles+1);

    % 1 - WF, 2-II, 3-CR, 4-KL, 5-DS
    % Initial Populations given in prompt
    pop(:,1) = [26500;30000;8000;80000;80000];

    % 1st dimension is type of entity, 1-Humans, 2-WW, 3-wights
    % 2nd dimension is index of battle
    % 3rd dimension is before or after battle (1 is before,2 is after)
    battle = zeros([3,num_battles,2]);
    % Set initial pre-battle numbers for the first battle for ww and wights
    battle(2,1,1) = 1000;
```

```
battle(3,1,1) = 100000;

% Iterate through number of battles. For our purposes this should
% usually be 2,3,or 4.
for i = 1:num_battles
    % Current city ww are attacking.
    curr_city = ww_path(i);

    % How many troops people send is a function of population,
    % cooperation coefficient, and rallying factor
    allocations = coop_vec.*pop(:,i);
    allocations = min(allocations/rally(i),pop(:,i));
    if split==1 && i>1
        % If a split happens, humans must also split their forces
        allocations = allocations/2;
    end
    % All troops in current city must fight
    allocations(curr_city) = 1*pop(curr_city,i);
    battle(1,i,1) = sum(allocations);
    % Troops leave cities for battle
    pop(:,i+1) = pop(:,i)-allocations;
    if i>1
        if (split ==1)
            % If white walkers split, their forces at 3rd and 2nd
            % battle are forces left after 1st battle
            battle(2,2,1) = battle(2,1,2)/2;
            battle(2,3,1) = battle(2,1,2)/2;
            battle(3,2,1) = battle(3,1,2)/2;
            battle(3,3,1) = battle(3,1,2)/2;
        else
            % usually, ww forces are the survivors from last battle
            battle(2,i,1) = battle(2,i-1,2);
            battle(3,i,1) = battle(3,i-1,2);
        end
    end
    % run a battle simulation. attached in separate file.
    [W_t, w_t, h_t, lambda, index, windex] =...
    run_battle(battle(2,i,1), battle(3,i,1), battle(1,i,1), 100);
    if (curr_city ==4)
        % If current city is king's landing.

        % Grab the outcome of battle in terms of ww,wights,humans
```

```
        battle(2,i,2) = W_t(index);
        battle(3,i,2) = max(w_t(min(index, windex)),0);
        battle(1,i,2) = h_t(index);

        alive_cities = pop(:,i+1)~=0;
        % check if curr city goes down
        if battle(2,i,2)==0 && battle(3,i,2)==0
            % If all ww/wights are dead then city is alive
            alive_cities(curr_city)=1;
        end
        % Allocate survivors evenly among surviving cities
        pop(alive_cities,i+1) = pop(alive_cities,i+1) + ...
            ones([sum(alive_cities),1])*battle(1,i,2)/sum(alive_cities);

    else
        % Generic city
        for j = 1:index-1
            % If humans fall under ret_percent * initial forces
            if (h_t(j+1) <= ret_percent*battle(1,i,1))
                break;
            end
        end
        battle(2,i,2) = W_t(j);
        battle(1,i,2) = h_t(j);

        % Update wights (ww convert more wights after battle)
        add_wight = max((battle(1,i,1)-battle(1,i,2) - lambda(j)),0) *...
                    wight_conv;
        battle(3,i,2) = max(min(w_t(j), w_t(windex)), 0) + add_wight;

        alive_cities = pop(:,i+1)~=0;
        % check if curr city goes down
        if ~(battle(2,i,2)==0 && battle(3,i,2)==0)
            % if ww/wights are not all dead, then this city's pop is
            % split among other cities
            temp = pop(curr_city,i+1);
            pop(curr_city,i+1) = 0;
            alive_cities = pop(:,i+1)~=0;
            % Allocate them evenly
            pop(alive_cities,i+1) = pop(alive_cities,i+1) + ...
              ones([sum(alive_cities),1])*temp/sum(alive_cities);
        end
```

```
            % Allocate retreaters evenly
            pop(alive_cities,i+1) = pop(alive_cities,i+1) + ...
                ones([sum(alive_cities),1])*(1-ret_penalty)*...
                battle(1,i,2)/sum(alive_cities);
        end
    end

    % If ww split and they win @ casterly rock but lose at kings
    % landing...another battle is needed at kings landing. Hard coded..
    if split==1 && pop(4,i)>0
        % assume 4
        battle = [battle, zeros(3,1,2)];
        pop = [pop, zeros(5,1)];

        allocations = coop_vec.*pop(:,4);
        allocations = allocations/rally(3);
        allocations(4) = 1*pop(curr_city,4);
        battle(1,4,1) = sum(allocations);
        pop(:,5) = pop(:,4)-allocations;
        battle(2,4,1) = battle(2,2,2);
        battle(3,4,1) = battle(3,2,2);

        [W_t, w_t, h_t, lambda, index, windex] =...
        run_battle(battle(2,4,1), battle(3,4,1), battle(1,4,1), 100);

        battle(2,4,2) = W_t(index);
        battle(3,4,2) = max(w_t(min(index, windex)),0);
        battle(1,4,2) = h_t(index);

        alive_cities = pop(:,5)~=0;
        % check if curr city goes down
        if battle(2,4,2)==0 && battle(3,4,2)==0
            alive_cities(4)=1;
        end
        % Allocate survivors evenly
        pop(alive_cities,5) = pop(alive_cities,5) + ...
            ones([sum(alive_cities),1])*battle(1,4,2)/sum(alive_cities);
    end

    % Humans win if there are people in kings landing
    if pop(4,num_battles+1) > 0
        win = sum(pop(:,num_battles+1))/224500;
```

```
    else
        win = 0;
    end
end
```

7.2.4 sim_matrix.m

```
% Simulate wars based on a matrix of parameters

split = 0;
path = [1,4];
n = 10;


x = zeros(n);

% Double for loop - test a grid of cooperation vectors and retreat
% thresholds between 0 and 1
for i = 1:n
    for j = 1:n
        c = i/n;
        [win,a,b] = sim_war([c,c,c,c,c]',path,j/n,split);
        x(i,j) = win;
    end
end

% Graph it based on win
clf
figure(1)
imagesc(x)
ylabel('Retreat rate (bottom is high)')
xlabel('Cooperation rate (right is high)');
title('Human survival depending on retreat rate / cooperation rate');


xticks(n/10*(1:n))
xticklabels({'.1','.2','.3','.4','.5','.6','.7','.8','.9','1'});
yticks(n/10*(1:n))
yticklabels({'.1','.2','.3','.4','.5','.6','.7','.8','.9','1'});
```

Notes

[1]Davis, Paul K.. Aggregation, Disaggregation, and the 3:1 Rules in Ground Combat. Santa Monica, CA: RAND Corporation, 1995. https://www.rand.org/pubs/monograph$_r eports/MR638.html.Alsoavailableinprintform.$

[1]The Sieging of Medieval Castles, medievalcastles.stormthecastle.com/sieging$_c astles.htm.$

[2]1Malgus. The Frozen Lake Battle (Jon's Squad Daenerys's Dragons vs White Walkers) - Game of Thrones S7E6. YouTube, YouTube, 22 Aug. 2017, www.youtube.com/watch?v=Dh3UpBKOhE4t=4s.

[3]Wights. Game of Thrones Wiki, gameofthrones.wikia.com/wiki/Wights.

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