CLUTTER RESEARCH 6

Ulf Grenander

April 1999

Anuj has raised the question whether it maybe possible to find an analytical expression for the probability density in "Clutter 5". Here is the answer.

Denote by $f_p\cdot$ the probability density whose Fourier transform is

$$\phi_p(z) \;=\; \int_{-\infty}^{\infty} exp(ixz)f_p(x)dx \;=\; \frac{1}{(1+z^2)^p}$$

so that the Fourier inversion formula gives

$$f_p(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} exp(-ixz)\frac{1}{(1+z^2)^p}dz$$

If $p > 1/2$, so that $f_p\cdot$ is (absolutely) integrable, this formula makes sense. Otherwise it has to be replaced by

$$\frac{1}{2\pi}V.P.[\int_{-\infty}^{\infty} exp(-ixz)\frac{1}{(1+z^2)^p}dz]$$

with the classical definition of the "valeur principale"

$$V.P.[\int_{-\infty}^{\infty} g(t)dt] \;=\; lim_{T\to\infty}\int_{-T}^{T} g(t)dt$$

We should therefore calculate the integral

$$f_p(z) \;=\; \frac{1}{\pi}\int_{0}^{\infty} cos(xz)\frac{1}{(1+z^2)^p}dz$$

but this is a variation of the Bessel functions, namely

$$f_p(z) \;=\; \frac{x^{p-1/2}}{\pi^{3/2}2^{p-1/2}\Gamma(p)}K_{p-1/2}(x)$$

Better chck the algebra! A reference to K_ν functions can be found in S.Bochner:"Vorlesungen ueber Forierschen Integralen", p. 60; I believe there is also an English edition. If you cannot find this reference you could consult the encyclopedic Bateman project for special functions. In MATLAB it is called "besselk" so that it is easy to compute.