CLUTTER RESEARCH 5

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After hearing David's talk in Baltimore last week my interest in clutter was fired up again. Here is a resulting thought.

Let us start from the "transported generator model" for clutter that I have used together with Michael and Paritosh, in which the image intensity (or its log if you prefer) is represented as

$$I(x) = \sum_{\nu} A_\nu g(x - x_\nu);\ x = (x_1, x_2), x_\nu = (x_{1\nu}, x_{2\nu}), \in \mathbf{R}^2$$

where the amplitudes A_ν are iid $N(0, \sigma^2)$ and the generators $g(\cdot)$ describe the shape of the clutter objects. The points x_ν form a Poisson process in the plane with an intensity λ. This is one of many mathematical formalizations of Stuart's dictum that the material world is made up of objects and that clutter analysis should take this into account.

REMARK. The model may be extended in several ways, some of which are:

(i) The Poisson model may be replaced by a clustering model, say of Neyman-Scott type

(ii) The generators $g(\cdot)$ need not be the translates of a single fixed function; they can themselves be made random

(iii) The addition operator in the representation can perhaps be replaced by a non-additive one, expressing obscuration, perspective transformation, etc, but this has not been studied

(iv) The Gaussian assumption for the amplitudes can be relaxed.

Now consider the gradient component

$$G(x) = \frac{\partial I(x)}{\partial x_1} = \sum_{\nu} A_\nu g_1(x - x_\nu)$$

with the notation

$$g_1(x) = \frac{\partial g(x)}{\partial x_1}$$

What is the marginal distribution of $G(x)$?

The conditional distribution of $G(x)$ given the Poisson points is $N(0, \sigma^2 \sum_\nu g_1^2(x - x_\nu))$. Introduce the Fourier transform of this distribution

$$\phi(z) \ = \ E[exp(izG(x))|x_\nu] \ = = exp[-\frac{1}{2}z^2\sigma^2 \sum_\nu g_1^2(x - x_\nu] \ = \ exp[-\frac{1}{2}z^2\sigma^2 u]$$

where we have introduced the random variable

$$u \ = \ \sum_\nu g_1^2(x - x_\nu)$$

But this random variable has the Fourier transform

$$\psi(z) \ = \ Eexp(izu) \ = \ exp[\lambda \int_{\mathbf{R}^2} E[exp(izg_1^2(x-y) - 1)]dy$$

using an argument from renewal theory, see Feller II, Chapter XI. If we know $g(\cdot)$ and hence $g_1(\cdot)$ we can calculate $\psi(\cdot), \phi(\cdot)$ and hence the probability density of G. But this is not what we are going to do.

The random variable u has some distribution on the positive real line and we propose to approximate it by a scaled Γ-density

$$p(u) \ = \ \frac{1}{c\Gamma(p)}(u/c)^{p-1} exp(-u/c); 0 < p < \infty$$

whose parameters p, c we determine by the method of moments. Elementary and boring calculations give the first two cumulants

$$\kappa_1 \ = \ E(u) \ = pc$$

$$\kappa_2 \ = \ Var(u^2) \ = pc^2$$

On the othe hand the first two cumulants k_1, k_2 obtained from the cumulant generating function $log[\psi(z)]$ are

$$k_1 \ = \ \lambda\sigma^2 \int_{\mathbf{R}^2} g_1^2(y)dy$$

$$k_2 \ = \ \lambda\sigma^4 \int_{\mathbf{R}^2} g_1^4(y)dy$$

(better check the algebra!) We therefore estimate the parameters in the scaled Γ-distribution by

$$c = k_2/k_1; \; p = k_1^2/k_2$$

With this approximation we should now integrate over the Poisson variation

$$\phi(z) = \int_0^\infty exp[-\frac{1}{2}z^2\sigma^2 u]p(u)du$$

If we denote the Laplace transform of $p(\cdot)$ by

$$L(w) = \int_0^\infty exp(-wu)p(u)du$$

we get

$$\phi(z) = L(\frac{1}{2}z^2\sigma^2)$$

But

$$L(w) = \frac{1}{(1+cw)^p}$$

so that we get,with the approximation made,

$$\phi(z) = \frac{1}{(1+1/2\sigma^2 z^2)^p}$$

We can now separate cases. If $p = 1$ then $\phi(\cdot)$ is the Fourier transform of the double exponential distribution; let us call this *exponential clutter*. In log scale it appears as a triangle. Otherwise the distribution is the pth convolution power of the double exponential distribution. If $p > 1$, *super-exponential clutter*, we get closer to the Gaussian, especially if $p >> 1$. In log scale it appears like a parabola. On the other hand, if $p < 1$, *sub-exponetial clutter*, the cusp of the density at zero becomes more pronounced.

This seems to agree with David's conjecture about convolution powers! Also with data ("trust no data that have not been confirmed by theory").

Sub-exponential clutter occurs when

$$\lambda < \frac{\int_{\mathbf{R}^2} g_1^4(y)dy}{[\int_{\mathbf{R}^2} g_1^2(y)dy]^2}$$

The right hand side is a criterion expressing how peaked $g_1^2(\cdot)$ is. When the shapes are well defined, with clear delimitations, the criterion is big and we can expect sub-exponential clutter unless the Poisson intensity λ is very large.