Notes on a Spike Count Variability Test

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In this technical report, we collect some proofs and other technical details associated with [1].

1 Proof of Proposition

We say the nonnegative integer-valued random variables $X_1, X_2, ..., X_n$ are multinomially distributed with parameters $\{N; p_1, p_2, ..., p_n\}$ if

$$P(X_1 = m_1, X_2 = m_2, ..., X_n = m_n) = \begin{cases} \binom{N}{m_1\, m_2\, \dots\, m_n} \prod_{i=1}^n p_i^{m_i} & \text{if } \sum_{i=1}^n m_i = N \\ 0 & \text{otherwise,} \end{cases}$$

where $(p_1, p_2, ..., p_n)$ satisfy $p_i \geq 0$ and $\sum_{i=1}^n p_i = 1$. In this case we write $X_1, X_2, ..., X_n \sim \mathcal{M}(N; p_1, p_2, ..., p_n)$.

Our goal is

Proposition. *If $m_1, m_2, ..., m_n$ are independent Poisson random variables with rates $\lambda_1, \lambda_2, ..., \lambda_n$, respectively, then for all r, and for all $\hat{\mu}$, we have*

$$\max_{\lambda_1,\lambda_2,\dots,\lambda_n} P\left(\sum_{i=1}^n m_i^2 \leq r \,\middle|\, \hat{\mu}\right) = P\left(\sum_{i=1}^n X_i^2 \leq r\right)$$

where $X_1, X_2, ..., X_n$ are distributed multinomially with parameters $\{n\hat{\mu}; 1/n, 1/n, ..., 1/n\}$.

We first prove some lemmas about the multinomial distribution which will be useful.

Lemma 1.1. *If*

$$X_1, X_2, ..., X_n \sim \mathcal{M}(N; p_1, p_2, ..., p_n),$$

and

$$Y_1, Y_2, ..., Y_k \sim \mathcal{M}\left(N - \sum_{i=k+1}^n m_i\, ; \frac{p_1}{\sum_{i=1}^k p_i}, \frac{p_2}{\sum_{i=1}^k p_i}, ..., \frac{p_k}{\sum_{i=1}^k p_i}\right),$$

with $k < n$ then

$$P(X_1 = m_1, X_2 = m_2, ..., X_k = m_k | X_{k+1} = m_{k+1}, ..., X_n = m_n) = P(Y_1 = m_1, Y_2 = m_2, ..., Y_k = m_k).$$

Proof of Lemma.

$$P(X_1 = m_1, X_2 = m_2, ..., X_k = m_k | X_{k+1} = m_{k+1}, ..., X_n = m_n)$$

$$= \frac{P(X_1 = m_1, X_2 = m_2, ..., X_n = m_n)}{P(X_{k+1} = m_{k+1}, ..., X_n = m_n)}$$

$$= \frac{\binom{N}{m_1\, m_2 \dots m_n} \prod_{i=1}^{n} p_i^{m_i}}{\sum_{\substack{x_1,\dots,x_k \\ \sum_{i=1}^k x_i = N - \sum_{i=k+1}^n m_i}} \binom{N}{x_1 \dots x_k\, m_{k+1} \dots m_n} \prod_{i=1}^{k} p_i^{x_i} \prod_{i=k+1}^{n} p_i^{m_i}}$$

$$= \frac{\frac{1}{\prod_{i=1}^k m_i!} \prod_{i=1}^{k} p_i^{m_i}}{\frac{1}{(N - \sum_{i=k+1}^n m_i)!} \left(\sum_{i=1}^{k} p_i \right)^{N - \sum_{i=k+1}^n m_i}}$$

by an application of the multinomial formula,

$$= \binom{N - \sum_{i=k+1}^n m_i}{m_1 \dots m_k} \prod_{i=1}^{k} \left(\frac{p_i}{\sum_{i=1}^k p_i} \right)^{m_i}$$

$$= P(Y_1 = m_1, Y_2 = m_2, ..., Y_k = m_k).$$

Corollary 1.2. *If $X_1, X_2, ..., X_n \sim \mathcal{M}(N; p_1, p_2, ..., p_n)$,*

$$P(X_3, ..., X_n | X_1 = j, X_2 = k - j) = P(X_3, ..., X_n | X_1 + X_2 = k) \quad \forall\, k,\, j \leq k.$$

Proof. This follows from Lemma 1.1, since $X_3, ..., X_n$ depends on X_1 and X_2 only through the sum $X_1 + X_2$.

Lemma 1.3. *If X_1, X_2 are multinomially distributed with parameters $\{N; p, 1-p\}$, then for all N and for all r,*

$$\max_p P_{(p,1-p)} \left(X_1^2 + X_2^2 \leq r \right)$$

is achieved by $p = \frac{1}{2}$.

Proof of Lemma. Two observations simplify the problem. The first is that (X_1, X_2) has the same distribution as $(Y, N - Y)$, where Y is binomially distributed with parameters (N, p). (This is essentially the observation that the multinomial distribution generalizes the binomial distribution.) The second is algebraic: for every r, $\exists r''$ such that

$$\left\{Y^2 + (N-Y)^2 \le r\right\} = \left\{\left|Y - \frac{N}{2}\right| \le r''\right\}.$$

To see this, define

$$f(Y) = Y - \frac{N}{2}$$

then

$$\begin{aligned}
\left\{Y^2 + (N-Y)^2 \le r\right\} &= \left\{\left(\frac{N}{2} + f(Y)\right)^2 + \left(\frac{N}{2} - f(Y)\right)^2 \le r\right\} \\
&= \left\{2\left(\frac{N}{2}\right)^2 + 2f(Y)^2 \le r\right\} \\
&= \left\{f(Y)^2 \le r'\right\} \\
&= \left\{\left|Y - \frac{N}{2}\right| \le r''\right\},
\end{aligned}$$

using $r' = \frac{r - 2\left(\frac{N}{2}\right)^2}{2}$, $r'' = \sqrt{r'}$. Thus if we take Y as a binomially-distributed random variable with parameters (N, p), it suffices to prove that for all N, and all r

$$\arg\max_p P\left(\left|Y - \frac{N}{2}\right| \le r\right) = \frac{1}{2}.$$

We seek to maximize the binomial probability $g(p)$ with respect to the parameter p:

$$g(p) := \sum_{j=\lceil \frac{N}{2} - r \rceil}^{\lfloor \frac{N}{2} + r \rfloor} \binom{N}{j} p^j (1-p)^{N-j} = \sum_{j=k}^{j=N-k} \binom{N}{j} p^j (1-p)^{N-j}.$$

Differentiating g with respect to p, we obtain:

$$g'(p) = \sum_{j=k}^{N-k} \left[\binom{N}{j} j p^{j-1} (1-p)^{N-j} - \binom{N}{j} (N-j) p^j (1-p)^{N-j-1}\right]$$

Let us define a_j and b_j as follows:

$$a_j := \binom{N}{j} j p^{j-1} (1-p)^{N-j} \qquad b_j := \binom{N}{j} (N-j) p^j (1-p)^{N-j-1}$$

i.e., so that $g'(p) = \sum_{j=k}^{N-k} a_j - b_j$. Now using the identity $\binom{N}{j}(N-j) = \binom{N}{j+1}(j+1)$, we can observe

$$\begin{aligned}
b_j &= \binom{N}{j} (N-j) p^j (1-p)^{N-j-1} \\
&= \binom{N}{j+1} (j+1) p^j (1-p)^{N-j-1} \\
&= a_{j+1}.
\end{aligned}$$

So $g'(p)$ forms a telescoping sum

$$\begin{aligned} g'(p) &= \sum_{j=k}^{N-k} a_j - b_j \\ &= a_k - b_{N-k} \\ &= \binom{N}{k} k \left[p^{k-1}(1-p)^{N-k} - p^{N-k}(1-p)^{k-1}\right] \\ &= \binom{N}{k} k \left[p(1-p)\right]^{k-1} \left[(1-p)^{N-2k+1} - p^{N-2k+1}\right], \end{aligned}$$

assuming $p > 0$. We can conclude by inspection that $g'(p) = 0$ at $p = \frac{1}{2}$, $g'(p) > 0$ for $0 < p < \frac{1}{2}$, and $g'(p) < 0$ for $\frac{1}{2} < p < 1$, by inspection. Thus $g(p)$ attains its maximum at $p = \frac{1}{2}$, establishing the lemma.

Lemma 1.4. *If $X_1, X_2, ..., X_n$ are multinomial random variables distributed with parameters $\{N; p_1, p_2, ..., p_n\}$, (respecting $\sum_{i=1}^n p_i = 1$), then for all r, and for all N,*

$$\max_{(p_1, p_2, \ldots, p_n)} P\left(\sum_{i=1}^n X_i^2 \le r\right)$$

is achieved by $(p_1, p_2, ..., p_n) = (1/n, 1/n, ..., 1/n)$.

Proof. When $p = (p_1, p_2, ..., p_n)$, we write

$$P_p\left(\sum_{i=1}^n X_i^2 \le r\right)$$

to denote the probability that $X_i^2 \le r$ when $X_1, X_2, ..., X_n$ are distributed multinomially with parameters $\{N; p_1, p_2, ..., p_n\}$. Define

$$p^* := (1/n, 1/n, ..., 1/n),$$

and fix an arbitrary $p^{(0)} \in \{\vec{p} \in \mathbb{R}^n \,|\, 0 \le p_i \le 1 \,\forall i, \sum_{i=1}^n p_i = 1\}$. Then we would like to show that

$$P_{p^{(0)}}\left(\sum_{i=1}^n X_i^2 \le r\right) \le P_{p^*}\left(\sum_{i=1}^n X_i^2 \le r\right)$$

We will construct a sequence of vectors $\vec{p_0}, \vec{p_1}, \vec{p_2}, ...$ such that

$$P_{p^{(j)}}\left(\sum_{i=1}^n X_i^2 \le r\right) \le P_{p^{(j+1)}}\left(\sum_{i=1}^n X_i^2 \le r\right) \quad \forall i \tag{1}$$

and

$$\lim_{j\to\infty} p^{(j)} = p^* \tag{2}$$

The continuity of $P_p\left(\sum_{i=1}^n X_i^2 \le r\right)$ in p will then imply

$$P_{p^{(0)}}\left(\sum_{i=1}^n X_i^2 \le r\right) \le \lim_{j\to\infty} P_{p^{(j)}}\left(\sum_{i=1}^n X_i^2 \le r\right) = P_{p*}\left(\sum_{i=1}^n X_i^2 \le r\right).$$

We construct the sequence $\{p^{(j)}\}$ as follows. Where $p^{(j)} = (p_1^{(j)}, p_2^{(j)}, ..., p_n^{(j)})$, choose

$$\alpha^j := \arg\max_{1\le k\le n} p_k^{(j)} \qquad \beta^j := \arg\min_{1\le k\le n} p_k^{(j)}$$

Then define $p^{(j+1)}$ via

$$p_k^{(j+1)} := \begin{cases} \frac{p_{\alpha^j}^{(j)} + p_{\beta^j}^{(j)}}{2} & \text{if } k \in \{\alpha^j, \beta^j\}, \\ p_k^{(j)} & \text{otherwise.} \end{cases}$$

First, we establish (1). Without loss of generality, we assume $\alpha^j = 1$ and $\beta^j = 2$ (this solely acts to simplify the indexing notation). Observe

$$P_{p^{(j)}}(X_1 + X_2 = k) = P_{p^{(j+1)}}(X_1 + X_2 = k), \quad \forall k, \tag{3}$$

since

$$\begin{aligned} P_p(X_1 + X_2 = k) &= \sum_{\substack{x_1,...,x_n \\ x_1+x_2=k \\ x_3+...+x_n=N-k}} \binom{N}{x_1\, x_2\, ...\, x_n} \prod_{i=1}^n p_i^{x_i} \\ &= \sum_{\substack{x_3,...,x_n \\ x_3+...+x_n=N-k}} \frac{N!}{x_3!...x_n!} \prod_{i=3}^n p_i^{x_i} \sum_{\substack{x_1,x_2 \\ x_1+x_2=k}} \frac{1}{x_1!x_2!} p_1^{x_1} p_2^{x_2} \\ &= \sum_{\substack{x_3,...,x_n \\ x_3+...+x_n=N-k}} \frac{N!}{x_3!...x_n!} \prod_{i=3}^n p_i^{x_i} \cdot \frac{1}{k!} \sum_{j=0}^k \frac{k!}{j!(k-j)!} p_1^j p_2^{k-j} \\ &\overset{(a)}{=} \sum_{\substack{x_3,...,x_n \\ x_3+...+x_n=N-k}} \frac{N!}{x_3!...x_n!} \prod_{i=3}^n p_i^{x_i} \cdot \frac{1}{k!} (p_1 + p_2)^k, \end{aligned}$$

where (a) follows from an application of the binomial formula. Since $p_1^{(j+1)} + p_2^{(j+1)} = p_1^{(j)} + p_2^{(j)}$ and $p_k^{(j+1)} = p_k^{(j)}$ for $3 \le k \le n$, this implies Eq. (3). Furthermore,

$$P_{p^{(j)}}(X_3, X_4, ..., X_n | X_1 + X_2 = k) = P_{p^{(j+1)}}(X_3, X_4, ..., X_n | X_1 + X_2 = k) \tag{4}$$

since

$$\begin{aligned} P_{p^{(j)}}(X_3, X_4, ..., X_n | X_1 + X_2 = k) &\overset{(a)}{=} P_{p^{(j)}}(X_3, X_4, ..., X_n | X_1 = 0, X_2 = k) \\ &\overset{(b)}{=} P_{p^{(j+1)}}(X_3, X_4, ..., X_n | X_1 = 0, X_2 = k) \\ &\overset{(a)}{=} P_{p^{(j+1)}}(X_3, X_4, ..., X_n | X_1 + X_2 = k). \end{aligned}$$

where (a) follows from Corollary (1.2), (b) from Lemma (1.1).

Equations (3) and (4) imply

$$P_{p^{(j)}}(X_3, X_4, ..., X_n) = P_{p^{(j+1)}}(X_3, X_4, ..., X_n) \tag{5}$$

since

$$\begin{aligned} &P_{p^{(j)}}(X_3, X_4, ..., X_n) \\ &= \sum_k P_{p^{(j)}}(X_3, , ..., X_n | X_1 + X_2 = k) P_{p^{(j)}}(X_1 + X_2 = k)) \\ &\overset{(a)}{=} \sum_k P_{p^{(j+1)}}(X_3 =, ..., X_n = | X_1 + X_2 = k) P_{p^{(j+1)}}(X_1 + X_2 = k) \\ &= P_{p^{(j+1)}}(X_3, X_4, ..., X_n) \end{aligned}$$

where (a) follows from (4) and (3).

Now we can combine these arguments to conclude

$$\begin{aligned} &P_{p^{(j)}}\left(\sum_{i=1}^n X_i^2 \le r\right) \\ &= \sum_{\substack{m_3,\dots,m_n \\ \sum_{i=3}^n m_i \le N \\ \sum_{i=3}^n m_i^2 \le r}} P_{p^{(j)}}\left(X_1^2 + X_2^2 \le r - \sum_{i=3}^n m_i^2 \,\middle|\, X_3 = m_3, ..., X_n = m_n\right) P_{p^{(j)}}(X_3 = m_3, ..., X_n = m_n) \\ &\overset{(a)}{=} \sum_{\substack{m_3,\dots,m_n \\ \sum_{i=3}^n m_i \le N \\ \sum_{i=3}^n m_i^2 \le r}} P_{p^{(j)}}\left(X_1^2 + X_2^2 \le r - \sum_{i=3}^n m_i^2 \,\middle|\, X_3 = m_3, ..., X_n = m_n\right) P_{p^{(j+1)}}(X_3 = m_3, ..., X_n = m_n) \\ &\overset{(b)}{\le} \sum_{\substack{m_3,\dots,m_n \\ \sum_{i=3}^n m_i \le N \\ \sum_{i=3}^n m_i^2 \le r}} P_{p^{(j+1)}}\left(X_1^2 + X_2^2 \le r - \sum_{i=3}^n m_i^2 \,\middle|\, X_3 = m_3, ..., X_n = m_n\right) P_{p^{(j+1)}}(X_3 = m_3, ..., X_n = m_n) \\ &= P_{p^{(j+1)}}\left(\sum_{i=1}^n X_i^2 \le r\right) \end{aligned}$$

where (a) follows from (5), and (b) follows from an application of Lemma 1.1 and Lemma 1.3.

Finally we return to Eq (2)

$$\lim_{j \to \infty} p^{(j)} = p^*.$$

Note that

$$p_{\alpha^j}^{(j)} = \max_{1 \le k \le n} p_k^{(j)} \ge \frac{1}{n} \sum_{l=1}^n p_l^{(j)} = \frac{1}{n} \quad \forall j \tag{6}$$

and analogously

$$p^{(j)}_{\beta^j} = \min_{1\le k\le n} p^{(j)}_k \le \frac{1}{n}\sum_{l=1}^{n} p^{(j)}_l = \frac{1}{n} \quad \forall j. \tag{7}$$

Hence we have

$$p^{(j+1)}_{\alpha^j} = \frac{1}{2}\left(p^{(j)}_{\alpha^j} + p^{(j)}_{\beta^j}\right) \le \frac{1}{2}\left(p^{(j)}_{\alpha^j} + \frac{1}{n}\right). \tag{8}$$

So that

$$\begin{aligned} p^{(j+1)}_{\alpha^{j+1}} &\le \max\left\{\frac{1}{2}\left(p^{(j)}_{\alpha^j} + \frac{1}{n}\right), \max_{k\ne\alpha^j} p^{(j)}_k\right\} \\ p^{(j+2)}_{\alpha^{j+2}} &\le \max\left\{\frac{1}{2}\left(p^{(j)}_{\alpha^j} + \frac{1}{n}\right), \frac{1}{2}\left(p^{(j+1)}_{\alpha^{j+1}} + \frac{1}{n}\right), \max_{k\notin\{\alpha^j,\alpha^{j+1}\}} p^{(j)}_k\right\} \\ &\vdots \\ p^{(j+n)}_{\alpha^{j+n}} &\le \max\left\{\frac{1}{2}\left(p^{(j)}_{\alpha^j} + \frac{1}{n}\right), \frac{1}{2}\left(p^{(j+1)}_{\alpha^{j+1}} + \frac{1}{n}\right), ..., \frac{1}{2}\left(p^{(j+n-1)}_{\alpha^{j+n-1}} + \frac{1}{n}\right)\right\} \\ &\le \frac{1}{2}\left(p^{(j)}_{\alpha^j} + \frac{1}{n}\right), \end{aligned} \tag{9}$$

since $p^{(j+1)}_{\alpha^{j+1}} \le p^{(j)}_{\alpha^j}\,\forall j$, by construction. (9) implies $\limsup_{j\to\infty} \max_{1\le k\le n} p^{(j)}_k \le \frac{1}{n}$ (One can see this by considering the first order linear difference equation $y^{(j+1)} = \frac{1}{2}(y^{(j)} + \frac{1}{n})$, and noting $y^{(nj)} \ge p^{(j)}_{\alpha^j}$). Since $\sum_{k=1}^{n} p^{(j)}_k = 1\,\forall j$, we have

$$\lim_{j\to\infty} \vec{p^j} = \vec{p^*},$$

establishing the lemma.

Proposition. *If $m_1, m_2, ..., m_n$ are independent Poisson random variables with rates $\lambda_1, \lambda_2, ..., \lambda_n$, respectively, then for all r, and for all $\hat{\mu}$, we have*

$$\max_{\lambda_1,\lambda_2,...,\lambda_n} P\left(\sum_{i=1}^{n} m_i^2 \le r \,\middle|\, \hat{\mu}\right) = P\left(\sum_{i=1}^{n} X_i^2 \le r\right)$$

where $X_1, X_2, ..., X_n$ are distributed multinomially with parameters $\{n\hat{\mu}; 1/n, 1/n, ..., 1/n\}$.

Proof. *Conditioned* on the event $\{\sum_{i=1}^{n} m_i = n\hat{\mu}\}, X_1, X_2, ...X_n$ is distributed multinomially with parameters $\{n\hat{\mu}; \frac{\lambda_1}{\sum_{i=1}^{n}\lambda_i}, \frac{\lambda_2}{\sum_{i=1}^{n}\lambda_i}, ..., \frac{\lambda_n}{\sum_{i=1}^{n}\lambda_i}\}$:

$$\begin{aligned} \frac{P\left(m_1 = x_1, ..., m_n = x_n\right)}{P\left(\sum_{i=1}^{n} m_i = n\hat{\mu}\right)} &= \frac{\displaystyle\prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}}{\dfrac{e^{-\sum_{i=1}^{n}\lambda_i}\left(\sum_{i=1}^{n}\lambda_i\right)^{n\hat{\mu}}}{(n\hat{\mu})!}} \\ &= \binom{n\hat{\mu}}{m_1\, m_2\, ...\, m_n} \prod_{i=1}^{n}\left(\frac{\lambda_i}{\sum_{i=1}^{n}\lambda_i}\right)^{x_i} \end{aligned} \tag{10}$$

As a consequence, the proposition follows as a corollary of Lemma 1.4.

2 Dynamic Programming Algorithm

Dynamic programming [2] is a technique for computing functionals of functions on high-dimensional spaces which exploits "conditional independence"-like relations (literally in somes cases such as in Markov random fields) among variables by breaking the computational problem down into smaller subproblem which are then recombined, for the purpose of gains in computational efficiency. The technique is quite general (see [4, 5]), but an elementary example suffices for our purposes. Suppose the function $f : X^N \to \mathbb{R}$, where X is some finite state space, can be decomposed as follows:

$$f(x_1, x_2, ..., x_n) = f_1(x_1, x_2) f_2(x_2, x_3) \cdots f_{n-1}(x_{n-1}, x_n), \tag{11}$$

and we would like to compute

$$\sum_{(x_1, x_2, \ldots, x_n) \in X^N} f(x_1, x_2, ..., x_n), \tag{12}$$

then one could utilize the decomposition (11) as follows:

$$\begin{aligned}
&\sum_{x_1, x_2, \ldots, x_n} f(x_1, x_2, ..., x_n) \\
&\quad = \sum_{x_1, x_2, \ldots, x_n} f_1(x_1, x_2) f_2(x_2, x_3) \cdots f_{n-1}(x_{n-1}, x_n) \\
&\quad = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} f_1(x_1, x_2) f_2(x_2, x_3) \cdots f_{n-1}(x_{n-1}, x_n) \\
&\quad = \sum_{x_1} \sum_{x_2} f_1(x_1, x_2) \sum_{x_3} f_2(x_2, x_3) \sum_{x_4} \cdots \sum_{x_n} f_{n-1}(x_{n-1}, x_n)
\end{aligned} \tag{13}$$

As a consequence, to obtain (12) one can work backwards through the sums in (13):

$$\begin{aligned}
g_{n-1}(x_{n-1}) &:= \sum_{x_n} f_{n-1}(x_{n-1}, x_n) \\
g_{n-2}(x_{n-2}) &:= \sum_{x_{n-1}} f_{n-2}(x_{n-2}, x_{n-1}) g_{n-1}(x_{n-1}) \\
g_{n-3}(x_{n-3}) &:= \sum_{x_{n-2}} f_{n-3}(x_{n-3}, x_{n-2}) g_{n-2}(x_{n-2}) \\
&\vdots \\
g_2(x_2) &:= \sum_{x_3} f_2(x_2, x_3) g_3(x_3) \\
g_1(x_1) &:= \sum_{x_2} f_1(x_1, x_2) g_2(x_2)
\end{aligned} \tag{14}$$

and then (through (13))

$$\sum_{(x_1,x_2,...,x_n)\in X^N} f(x_1, x_2, ..., x_n) = \sum_{x_1} g_1(x_1). \tag{15}$$

The computational savings can be considerable: in this problem the brute-force approach (i.e., enumerating all possibilities and summing directly) would require $O(|X|^N)$ operations, whereas dynamic programming computes the sum $O((n-1)|X|^2)$ operations, with $|X|$ the cardinality of the space X.

Returning to our multinomial probability, we seek to compute

$$P\left(\sum_{i=1}^{n} X_i^2 \le k\right) = \sum_{\substack{m_1,m_2,...,m_n \\ \sum_{i=1}^n m_i=N \\ \sum_{i=1}^n m_i^2\le k}} \binom{N}{m_1\, m_2\, \cdots\, m_n} \left(\frac{1}{n}\right)^N, \tag{16}$$

where $X_1, X_2, ..., X_n \sim \mathcal{M}(N; \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$. This can be decomposed into a form amenable to dynamic programming: the basic idea is to transform the apparently global constraints $\sum_{i=1}^n m_i = N$ and $\sum_{i=1}^n m_i^2 \le k$ into local constraints by working directly with the partial sums $\sum_{i=1}^j m_i$ and $\sum_{i=1}^j m_i^2$. If we employ the substitution

$$s_j = \sum_{k=1}^{j} m_k \qquad \tilde{s}_j = \sum_{k=1}^{j} m_k^2, \tag{17}$$

we observe that the constraint sets

$$\left\{ m_1, m_2, ..., m_n : \sum_{i=1}^n m_i = N, \sum_{i=1}^n m_i^2 \le k \right\} \tag{18}$$

and

$$\begin{aligned} B := \{s_1, s_2, ..., s_n, \tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n :& s_n = N, \tilde{s}_n \le k, \\ & s_{j-1} \le s_j \;\forall j, \\ & \tilde{s}_j = \tilde{s}_{j-1} + (s_j - s_{j-1})^2 \;\forall j\}, \end{aligned} \tag{19}$$

are equivalent (with respect to (18)). Therefore,

$$\sum_{\substack{m_1,m_2,\ldots,m_n \\ \sum_{i=1}^n m_i = N \\ \sum_{i=1}^n m_i^2 \le k}} \binom{N}{m_1\, m_2\, \cdots\, m_n} \left(\frac{1}{n}\right)^N$$

$$= \sum_{\substack{s_1,\ldots,s_n \\ \tilde{s}_1,\ldots,\tilde{s}_n \\ s_n = N \\ \tilde{s}_n \le k \\ s_{j-1} \le s_j\ \forall j \\ \tilde{s}_j = \tilde{s}_{j-1} + (s_j - s_{j-1})^2\ \forall k}} \binom{N}{N - s_{n-1}\; s_{n-2} - s_{n-1} \ldots s_2 - s_1\; s_1} \left(\frac{1}{n}\right)^N$$

$$= \sum_{\substack{s_1,\ldots,s_n \\ \tilde{s}_1,\ldots,\tilde{s}_n}} \mathbf{1}_B(s_1,...,s_n,\tilde{s}_1,...,\tilde{s}_n) \binom{N}{N - s_{n-1}\; s_{n-2} - s_{n-1} \ldots s_2 - s_1\; s_1} \left(\frac{1}{n}\right)^N \tag{20}$$

Now, using the pairwise subconstraints

$$\begin{aligned} C := \{(s_j, s_{j-1}, \tilde{s}_j, \tilde{s}_{j-1}) :& s_{j-1} \le s_j \\ & \tilde{s}_j = \tilde{s}_{j-1} + (s_j - s_{j-1})^2 \\ & \tilde{s}_j \le k, s_j \le N\}, \end{aligned} \tag{21}$$

we have

$$\mathbf{1}_B(s_1,...,s_n,\tilde{s}_1,...\tilde{s}_n) = \mathbf{1}_{\{N\}}(s_n) \cdot \mathbf{1}_C(0, s_1, \tilde{s}_1, 0) \cdot \prod_{i=2}^{n} \mathbf{1}_C(s_j, s_{j-1}, \tilde{s}_j, \tilde{s}_{j-1}), \tag{22}$$

(the second indicator function on the righthand side is analogous to implicitly enforcing $s_0 = 0, \tilde{s}_0 = 0$). And returning to (20)

$$\sum_{\substack{s_1,\ldots,s_n\\ \tilde{s}_1,\ldots,\tilde{s}_n}} \mathbf{1}_B(s_1,...,s_n,\tilde{s}_1,...,\tilde{s}_n)\binom{N}{N-s_{n-1}\; s_{n-2}-s_{n-1}\; ...\; s_2-s_1\; s_1}\left(\frac{1}{n}\right)^N$$

$$= \sum_{\substack{s_1,\ldots,s_n\\ \tilde{s}_1,\ldots,\tilde{s}_n}} \mathbf{1}_{\{N\}}(s_n)\cdot\mathbf{1}_C(0,s_1,\tilde{s}_1,0)\cdot\prod_{i=2}^{n}\mathbf{1}_C(s_j,s_{j-1},\tilde{s}_j,\tilde{s}_{j-1})$$

$$\cdot\binom{N}{N-s_{n-1}\; s_{n-2}-s_{n-1}\; ...\; s_2-s_1\; s_1}\left(\frac{1}{n}\right)^N$$

$$= \sum_{\substack{s_1,\ldots,s_n\\ \tilde{s}_1,\ldots,\tilde{s}_n}} \mathbf{1}_{\{N\}}(s_n)\cdot\mathbf{1}_C(0,s_1,\tilde{s}_1,0)\cdot\prod_{i=2}^{n}\mathbf{1}_C(s_j,s_{j-1},\tilde{s}_j,\tilde{s}_{j-1})$$

$$\cdot\binom{N}{s_{n-1}}\binom{s_{n-1}}{s_{n-2}}\cdots\binom{s_2}{s_1}\cdot\left(\frac{1}{n}\right)^N$$

$$= \sum_{\substack{s_n\\ \tilde{s}_n}}\sum_{\substack{s_{n-1}\\ \tilde{s}_{n-1}}}\cdots\sum_{\substack{s_1\\ \tilde{s}_1}} \mathbf{1}_{\{N\}}(s_n)\cdot\mathbf{1}_C(0,s_1,\tilde{s}_1,0)\cdot\prod_{i=2}^{n}\mathbf{1}_C(s_j,s_{j-1},\tilde{s}_j,\tilde{s}_{j-1})$$

$$\cdot\binom{N}{s_{n-1}}\binom{s_{n-1}}{s_{n-2}}\cdots\binom{s_2}{s_1}\cdot\left(\frac{1}{n}\right)^N$$

$$= \left(\frac{1}{n}\right)^N\cdot\sum_{\substack{s_n\\ \tilde{s}_n}}\mathbf{1}_{\{N\}}(s_n)\sum_{\substack{s_{n-1}\\ \tilde{s}_{n-1}}}\binom{N}{s_{n-1}}\mathbf{1}_C(s_n,s_{n-1},\tilde{s}_n,\tilde{s}_{n-1})$$

$$\cdot\sum_{\substack{s_{n-2}\\ \tilde{s}_{n-2}}}\binom{s_{n-1}}{s_{n-2}}\mathbf{1}_C(s_{n-1},s_{n-2},\tilde{s}_{n-1},\tilde{s}_{n-2})\cdots\sum_{\substack{s_1\\ \tilde{s}_1}}\binom{s_2}{s_1}\mathbf{1}_C(s_2,s_1,\tilde{s}_2,\tilde{s}_1)$$

$$= \cdot\sum_{\substack{s_n\\ \tilde{s}_n}}\mathbf{1}_{\{N\}}(s_n)\sum_{\substack{s_{n-1}\\ \tilde{s}_{n-1}}}\left(\frac{1}{N}\right)^{N-s_{n-1}}\binom{N}{s_{n-1}}\mathbf{1}_C(s_n,s_{n-1},\tilde{s}_n,\tilde{s}_{n-1})$$

$$\cdot\sum_{\substack{s_{n-2}\\ \tilde{s}_{n-2}}}\left(\frac{1}{N}\right)^{s_{n-1}-s_{n-2}}\binom{s_{n-1}}{s_{n-2}}\mathbf{1}_C(s_{n-1},s_{n-2},\tilde{s}_{n-1},\tilde{s}_{n-2})$$

$$\cdot\sum\cdots\sum_{\substack{s_1\\ \tilde{s}_1}}\binom{s_2}{s_1}\left(\frac{1}{N}\right)^{s_2}\mathbf{1}_C(s_2,s_1,\tilde{s}_2,\tilde{s}_1), \tag{23}$$

As in (14), one can compute

$$
\begin{aligned}
g_2(s_2, \tilde{s}_2) &:= \sum_{\substack{s_1\\ \tilde{s}_1}} \binom{s_2}{s_1} \left(\frac{1}{N}\right)^{s_2} \mathbf{1}_C(s_2, s_1, \tilde{s}_2, \tilde{s}_1)\\
g_3(s_3, \tilde{s}_3) &:= \sum_{\substack{s_2\\ \tilde{s}_2}} \binom{s_3}{s_2} \left(\frac{1}{N}\right)^{s_3 - s_2} \mathbf{1}_C(s_3, s_2, \tilde{s}_3, \tilde{s}_2) g_2(s_2, \tilde{s}_2)\\
&\vdots\\
g_{n-1}(s_{n-1}, \tilde{s}_{n-1}) &:= \sum_{\substack{s_{n-2}\\ \tilde{s}_{n-2}}} \binom{s_{n-1}}{s_{n-2}} \left(\frac{1}{N}\right)^{s_{n-1} - s_{n-2}} \mathbf{1}_C(s_{n-1}, s_{n-2}, \tilde{s}_{n-1}, \tilde{s}_{n-2}) g_{n-2}(s_{n-2}, \tilde{s}_{n-2})\\
g_n(\tilde{s}_n) &:= \sum_{\substack{s_{n-1}\\ \tilde{s}_{n-1}}} \binom{N}{s_{n-1}} \left(\frac{1}{N}\right)^{N - s_{n-1}} \mathbf{1}_C(N, s_{n-1}, \tilde{s}_n, \tilde{s}_{n-1}) g_{n-1}(s_{n-1}, \tilde{s}_{n-1})
\end{aligned}
\tag{24}
$$

and then

$$
P\left(\sum_{i=1}^n X_i^2 \le k\right) = \sum_{\tilde{s}_n} g_n(\tilde{s}_n). \tag{25}
$$

3 Monte Carlo Estimation

The dynamic programming solution to the calculation of the multinomial probability (16), $p = P(\sum_{i=1}^n X_i^2 \le k)$ can be computationally infeasible if $\sum_{i=1}^n m_i$ and $\sum_{i=1}^n m_i^2$ are too large. An alternative way to compute it is via approximation by Monte Carlo methods [3]. The idea is to produce M i.i.d. vector samples $X^{(1)}, X^{(2)}, ..., X^{(M)}$ where $X^{(j)} = (X_1^{(j)}, X_2^{(j)}, ..., X_n^{(j)})$ and $X_1^{(j)}, X_2^{(j)}, ..., X_n^{(j)} \sim \mathcal{M}(N; \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$ for each j. Then defining

$$
Y^{(j)} := \mathbf{1}_{\{\sum_{i=1}^n (X_i^{(j)})^2 \le k\}}(X^{(j)}), \tag{26}
$$

the law of large of numbers implies

$$
\lim_{M\to\infty} \frac{1}{M} \sum_{j=1}^M Y^{(j)} = E\left[Y^{(1)}\right] = P\left(\sum_{i=1}^n X_i^2 \le k\right), \text{(w.p.1)} \tag{27}
$$

which provides the estimate

$$
\hat{p} := \frac{1}{M} \sum_{j=1}^M Y^{(j)} \approx P\left(\sum_{i=1}^n X_i^2 \le k\right), \tag{28}
$$

with M large. In order to assess the accuracy of the estimate $\hat{p}$, we seek to form α-level *confidence intervals* of tolerance ϵ which satisfy

$$
P(|\hat{p} - p| \le \epsilon) \ge \alpha. \tag{29}
$$

One can get a handle on ϵ by invoking a central limit theorem approximation,

$$\hat{p} = \frac{1}{M}\sum_{j=1}^{M} Y^{(j)} \sim \text{Bin}\left(p, \frac{p(1-p)}{M}\right) \approx \mathcal{N}\left(p, \frac{p(1-p)}{M}\right). \tag{30}$$

Then

$$P\left(|\hat{p}-p| \leq 3\cdot\sqrt{\frac{\hat{p}(1-\hat{p})}{M}}\right) \underset{\sim}{\geq} .99, \tag{31}$$

which provides 99% approximate confidence intervals for the Monte Carlo approximation $\hat{p}$. The central limit theorem approximation works well: as Freiberger and Grenander [3] point out the randomness of the error bound "is more nearly a psychological difficulty than a real one."

4 Computing g by recursion

$$\begin{aligned} g(\vec{n}, \hat{\mu}_1, ..., \hat{\mu}_N) := \min &\left\{t : P\left(\sum_{i=1}^{n} Z_i \geq t\right) \leq \beta\right\} \\ &\text{where } Z_1, Z_2, ..., Z_N \text{ areindependent, and} \\ &Z_i \sim \text{Be}(r^*(n_i, \alpha, \hat{\mu}_i))\,\forall i, \end{aligned} \tag{32}$$

For simplicity we will denote

$$p_i := r^*(n_i, \alpha, \hat{\mu}_i). \tag{33}$$

Then observe that

$$\begin{aligned} P\left(\sum_{i=1}^{j} Z_i = t\right) &= P\left(\sum_{i=1}^{j-1} Z_i = t\right) P\left(Z_j = 0\right) + P\left(\sum_{i=1}^{j-1} Z_i = t-1\right) P\left(Z_j = 1\right) \\ &= P\left(\sum_{i=1}^{j-1} Z_i = t\right)(1-p_j) + P\left(\sum_{i=1}^{j-1} Z_i = t-1\right) p_j \end{aligned} \tag{34}$$

and

$$P(Z_1 = 1) = p_1 \qquad P(Z_1 = 0) = 1 - p_1, \tag{35}$$

which determines a recursion that can be computed in less than $3N^2$ operations.

5 The most reliable outcome

One cause for a *failure* to reject the null hypothesis in a single cell-stimulus pair might simply be a lack of data, either in the form of a paucity of trials or a paucity of spikes. Indeed, for some values of n, the number of trials, and of N, the total number of spikes (i.e., $\sum_{i=1}^{n} m_i$), it is impossible to reject the null hypothesis for *any* configuration of the data. This is the case if $P(\sum_{i=1}^n X_i^2 \leq k_*) > \alpha$ where

$$k_* := \min_{\substack{m_1,...,m_n \in \mathbb{Z}^n \\ \sum_{m_i} = N}} \sum_{i=1}^n m_i^2 = N, \tag{36}$$

and $X_1, X_2, ..., X_n \sim \mathcal{M}(N; 1/n, 1/n, ..., 1/n)$. Obtaining k_*, the value of $\sum_{i=1}^n m_i^2$ corresponding to the *most reliable outcome* (in mean square sense) consistent with a given n and N, is helpful in the optimizing the code for computing f, the threshold for significant rejection. Rather than use dynamic programming again, we can get this directly.

Lemma 5.1. *(The most reliable outcome)*

$$\begin{aligned} k_* &:= \min_{\substack{m_1,...,m_n \in \mathbb{Z}^n \\ \sum_{i=1}^n m_i = N}} \sum_{i=1}^n m_i^2 \\ &= \left[n \left(\left\lfloor \frac{N}{n} \right\rfloor + 1 \right) - N \right] \left\lfloor \frac{N}{n} \right\rfloor^2 + \left(N - n \left\lfloor \frac{N}{n} \right\rfloor \right) \left(\left\lfloor \frac{N}{n} \right\rfloor + 1 \right)^2 \end{aligned} \tag{37}$$

Proof. In the continuum, it is straightforward to see that

$$\min_{\substack{x_1,...,x_n \in \mathbb{R}^n \\ \sum_{i=1}^n x_i = N}} \sum_{i=1}^n x_i^2, \tag{38}$$

is achieved by

$$x_i = N/n \quad \forall i, \tag{39}$$

since any solution which satisfies $\sum_{i=1}^n x_i = N$, can be represented as

$$x_i = N/n + \epsilon_i \tag{40}$$

with $\epsilon_1, \epsilon_2, ..., \epsilon_n$ satisfying $\sum_{i=1}^n \epsilon_i = 0$. But then we can expand

$$\sum_{i=1}^n x_i^2 = \sum_{i=1}^n (N/n + \epsilon_i)^2 = N^2/n + \sum_{i=1}^n \epsilon^2, \tag{41}$$

which is evidently minimized by $\epsilon_i = 0 \, \forall i$. Thus the minimal $x^* = (x_1^*, ..., x_n^*)$ has $x_i^* = N/n \, \forall i$. (This of course immediately reveals the solution to the integer minimization problem in the case where $N \mod n = 0$, and is consistent with (37)).

Now a geometric argument reveals the effect of restricting $m_1, ..., m_n$ to the integer space $\mathbb{Z}^n$. Define the vertices of the hypercube on the integer lattice surrounding the point $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ as $H(x)$:

$$H(x_1, x_2, ..., x_n) = \{(y_1, y_2, ..., y_n) \in \mathbb{R}^n : y_i \in \{\lfloor x_i \rfloor, \lfloor x_i \rfloor + 1\} \; \forall i\}. \tag{42}$$

Then

$$\underset{\substack{m_1,\dots,m_n\in\mathbb{Z}^n \\ \sum_{i=1}^n m_i=N}}{\arg\min} \sum_{i=1}^n m_i^2 \in H(x^*), \tag{43}$$

To show this, suppose not: then there exists a point $z \in \mathbb{Z}^n$ such that $z \notin H(x^*)$, and $\sum_{i=1}^n z_i^2 < \min_{y\in H(x^*)} \sum_{i=1}^n y_i^2$. But since $\sum_{i=1}^n x_i^2$ is radially symmetric, one can then trace an arc A from z to z' such that $\sum_{i=1}^n y_i^2 = \sum_{i=1}^n z_i^2\ \forall y \in A$, and such that the ray from x^* to z' intersects a vertex p (of the hypercube) in $H(x^*)$. Therefore, $\sum_{i=1}^n (z_i')^2 < \min_{y\in H(x^*)} \sum_{i=1}^n y_i^2$. But $\sum_{i=1}^n (x_i^*)^2 \leq \sum_{i=1}^n (z_i')^2$ and $\sum_{i=1}^n (z_i')^2 < \sum_{i=1}^n p_i^2$ contradicts the convexity of

Thus there exists $m_1^*, m_2^*, ..., m_n^* \in H(x^*)$ such that

$$\sum_{i=1}^n (m_i^*)^2 = \min_{\substack{m_1,\dots,m_n \\ \sum_{i=1}^n m_i^2}} m_i^2. \tag{44}$$

This turns out to identify m^* because the condition $\sum_{i=1}^n m_i^* = N$ uniquely characterizes $m \in H(x^*)$. Essentially, $m_i = \left\lfloor \frac{N}{n} \right\rfloor$, or $m_i = \left\lfloor \frac{N}{n} \right\rfloor + 1$. Let j denote the number of variables among $m_1, ..., m_n$ that take the value $\left\lfloor \frac{N}{n} \right\rfloor$ (so $n - j$ variables take the value $\left\lfloor \frac{N}{n} \right\rfloor + 1$.) Then

$$\sum_{i=1}^n m_i = j\left\lfloor \frac{N}{n} \right\rfloor + (n-j)\left(\left\lfloor \frac{N}{n} \right\rfloor + 1\right) = N \tag{45}$$

Solving for j, we get

$$j = n\left(\left\lfloor \frac{N}{n} \right\rfloor + 1\right) - N. \tag{46}$$

This determines the unique point $m^* \in H(x^*)$ on the hyperplane $\sum_{i=1}^n m_i = N$. Plugging m^* into $\sum_{i=1}^2 m_i$ finally produces (37).

6 Table of Significance Thresholds

Table 1: Significance Thresholds

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
2	1	1	0.000000	1	0.000000
2	2	2	0.000000	2	0.000000
2	3	5	0.000000	5	0.000000
2	4	8	0.000000	8	0.000000
2	5	13	0.000000	13	0.000000
2	6	18	0.000000	18	0.000000
2	7	25	0.000000	25	0.000000
2	8	32	0.000000	32	0.000000

continued on next page

Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
2	9	41	0.000000	41	0.000000
2	10	50	0.000000	50	0.000000
2	11	61	0.000000	61	0.000000
2	12	72	0.000000	72	0.000000
2	13	85	0.000000	85	0.000000
2	14	98	0.000000	98	0.000000
2	15	113	0.000000	113	0.000000
2	16	128	0.000000	128	0.000000
2	17	145	0.000000	145	0.000000
2	18	162	0.000000	162	0.000000
2	19	181	0.000000	181	0.000000
2	20	200	0.000000	200	0.000000
3	1	1	0.000000	1	0.000000
3	2	2	0.000000	2	0.000000
3	3	3	0.000000	3	0.000000
3	4	6	0.000000	6	0.000000
3	5	9	0.000000	9	0.000000
3	6	12	0.000000	12	0.000000
3	7	17	0.000000	17	0.000000
3	8	22	0.000000	22	0.000000
3	9	27	0.000000	27	0.000000
3	10	34	0.000000	34	0.000000
3	11	41	0.000000	41	0.000000
3	12	48	0.000000	48	0.000000
3	13	57	0.000000	57	0.000000
3	14	66	0.000000	66	0.000000
3	15	75	0.000000	75	0.000000
3	16	86	0.000000	86	0.000000
3	17	97	0.000000	97	0.000000
3	18	109	0.044275	108	0.044275
3	19	121	0.000000	121	0.000000
3	20	134	0.000000	134	0.000000
3	21	148	0.038151	147	0.038151
3	22	162	0.000000	162	0.000000
3	23	177	0.000000	177	0.000000
3	24	193	0.033515	192	0.033515
3	25	209	0.000000	209	0.000000
3	26	226	0.000000	226	0.000000
3	27	244	0.029883	243	0.029883
3	28	262	0.000000	262	0.000000

continued on next page

Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
3	29	281	0.000000	281	0.000000
3	30	301	0.026961	300	0.026961
4	1	1	0.000000	1	0.000000
4	2	2	0.000000	2	0.000000
4	3	3	0.000000	3	0.000000
4	4	4	0.000000	4	0.000000
4	5	7	0.000000	7	0.000000
4	6	10	0.000000	10	0.000000
4	7	13	0.000000	13	0.000000
4	8	17	0.038452	16	0.038452
4	9	21	0.000000	21	0.000000
4	10	26	0.000000	26	0.000000
4	11	31	0.000000	31	0.000000
4	12	37	0.022030	36	0.022030
4	13	43	0.000000	43	0.000000
4	14	50	0.000000	50	0.000000
4	15	57	0.000000	57	0.000000
4	16	65	0.014683	64	0.014683
4	17	74	0.049922	73	0.049922
4	18	82	0.000000	82	0.000000
4	19	92	0.042683	91	0.042683
4	20	101	0.010671	100	0.010671
4	21	112	0.037348	111	0.037348
4	22	122	0.000000	122	0.000000
4	23	134	0.032809	133	0.032809
4	24	145	0.008202	145	0.008202
4	25	158	0.029294	157	0.029294
4	26	171	0.040802	170	0.040802
4	27	184	0.026230	183	0.026230
4	28	197	0.006558	197	0.006558
4	29	212	0.023771	211	0.023771
4	30	227	0.033428	226	0.033428
4	31	242	0.021589	241	0.021589
4	32	257	0.005397	257	0.005397
4	33	274	0.019790	273	0.019790
4	34	291	0.028036	290	0.028036
4	35	308	0.018171	307	0.018171
4	36	325	0.004543	325	0.004543
4	37	344	0.016808	343	0.016808
4	38	363	0.023952	362	0.023952

continued on next page

Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
4	39	382	0.015569	381	0.015569
4	40	403	0.046352	401	0.003892
5	1	1	0.000000	1	0.000000
5	2	2	0.000000	2	0.000000
5	3	3	0.000000	3	0.000000
5	4	4	0.000000	4	0.000000
5	5	6	0.038400	5	0.038400
5	6	8	0.000000	8	0.000000
5	7	11	0.000000	11	0.000000
5	8	14	0.000000	14	0.000000
5	9	17	0.000000	17	0.000000
5	10	21	0.011612	20	0.011612
5	11	26	0.042578	25	0.042578
5	12	30	0.000000	30	0.000000
5	13	35	0.000000	35	0.000000
5	14	41	0.027553	40	0.027553
5	15	46	0.005511	46	0.005511
5	16	53	0.022042	52	0.022042
5	17	60	0.037472	59	0.037472
5	18	67	0.033724	66	0.033724
5	19	74	0.016019	73	0.016019
5	20	81	0.003204	81	0.003204
5	21	90	0.013456	89	0.013456
5	22	99	0.023683	98	0.023683
5	23	108	0.021788	107	0.021788
5	24	117	0.010458	116	0.010458
5	25	128	0.036953	126	0.002092
5	26	137	0.009064	137	0.009064
5	27	148	0.016315	147	0.016315
5	28	161	0.047676	158	0.015227
5	29	172	0.045210	170	0.007360
5	30	183	0.026706	181	0.001472
5	31	196	0.040043	194	0.006519
5	32	209	0.037569	206	0.011920
5	33	222	0.035723	219	0.011239
5	34	235	0.034118	233	0.005459
5	35	250	0.045274	246	0.001092
5	36	263	0.030706	261	0.004913
5	37	278	0.029034	276	0.009089
5	38	293	0.027762	291	0.008635

continued on next page

Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n,.05,\hat{\mu})$	$r^*(n,.05,\hat{\mu})$	$f(n,.01,\hat{\mu})$	$r^*(n,.01,\hat{\mu})$
5	39	310	0.046406	306	0.004209
5	40	325	0.035764	321	0.000842
5	41	342	0.042622	338	0.003835
5	42	357	0.023108	355	0.007159
5	43	374	0.022195	372	0.006841
5	44	393	0.037524	389	0.003344
5	45	410	0.028962	406	0.000669
5	46	429	0.034746	425	0.003077
5	47	448	0.047222	444	0.005784
5	48	467	0.045409	463	0.005553
5	49	486	0.030965	482	0.002721
5	50	507	0.048443	501	0.000544
6	1	1	0.000000	1	0.000000
6	2	2	0.000000	2	0.000000
6	3	3	0.000000	3	0.000000
6	4	4	0.000000	4	0.000000
6	5	5	0.000000	5	0.000000
6	6	7	0.015432	6	0.015432
6	7	9	0.000000	9	0.000000
6	8	12	0.000000	12	0.000000
6	9	15	0.000000	15	0.000000
6	10	18	0.000000	18	0.000000
6	11	22	0.020630	21	0.020630
6	12	25	0.003438	25	0.003438
6	13	30	0.014899	29	0.014899
6	14	35	0.028971	34	0.028971
6	15	40	0.032190	39	0.032190
6	16	45	0.021460	44	0.021460
6	17	50	0.008107	50	0.008107
6	18	57	0.031753	55	0.001351
6	19	62	0.006418	62	0.006418
6	20	69	0.013371	68	0.013371
6	21	76	0.015599	75	0.015599
6	22	85	0.048261	82	0.010725
6	23	92	0.037000	90	0.004111
6	24	99	0.017130	97	0.000685
6	25	108	0.030833	106	0.003426
6	26	117	0.033650	115	0.007423
6	27	126	0.030731	124	0.008907
6	28	135	0.029014	133	0.006235

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
6	29	146	0.047281	142	0.002411
6	30	155	0.035561	151	0.000402
6	31	166	0.041209	162	0.002076
6	32	175	0.021574	173	0.004614
6	33	186	0.019937	184	0.005639
6	34	199	0.044778	195	0.003994
6	35	210	0.031826	206	0.001553
6	36	223	0.046789	219	0.006916
6	37	234	0.028282	230	0.001368
6	38	247	0.035635	243	0.003095
6	39	260	0.039655	256	0.003832
6	40	273	0.031774	269	0.002737
6	41	286	0.022612	282	0.001069
6	42	301	0.033803	297	0.004854
6	43	316	0.046221	312	0.009335
6	44	331	0.046786	325	0.002194
6	45	346	0.041612	342	0.010000
6	46	361	0.042653	357	0.009670
6	47	376	0.038243	372	0.007635
6	48	393	0.047000	387	0.003560
6	49	408	0.035057	404	0.006928
6	50	425	0.035715	421	0.007972
6	51	442	0.031869	438	0.007522
6	52	459	0.032860	455	0.007309
6	53	478	0.042820	472	0.005791
6	54	495	0.036653	491	0.009738
6	55	514	0.039513	508	0.005308
6	56	533	0.044377	527	0.006148
6	57	552	0.047071	546	0.005826
6	58	571	0.041306	565	0.005683
6	59	590	0.034080	584	0.004514
6	60	611	0.045867	605	0.007654
7	1	1	0.000000	1	0.000000
7	2	2	0.000000	2	0.000000
7	3	3	0.000000	3	0.000000
7	4	4	0.000000	4	0.000000
7	5	5	0.000000	5	0.000000
7	6	7	0.042839	6	0.042839
7	7	8	0.006120	8	0.006120
7	8	11	0.024480	10	0.024480

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n,.05,\hat{\mu})$	$r^*(n,.05,\hat{\mu})$	$f(n,.01,\hat{\mu})$	$r^*(n,.01,\hat{\mu})$
7	9	14	0.047211	13	0.047211
7	10	16	0.000000	16	0.000000
7	11	20	0.044160	19	0.044160
7	12	23	0.022711	22	0.022711
7	13	26	0.007029	26	0.007029
7	14	31	0.029122	29	0.001004
7	15	34	0.005021	34	0.005021
7	16	39	0.011477	38	0.011477
7	17	44	0.015484	43	0.015484
7	18	49	0.013272	48	0.013272
7	19	56	0.044831	54	0.007205
7	20	61	0.028020	59	0.002287
7	21	68	0.049218	64	0.000327
7	22	73	0.022015	71	0.001797
7	23	80	0.027753	78	0.004429
7	24	87	0.026636	85	0.006327
7	25	94	0.024290	92	0.005649
7	26	103	0.049674	99	0.003147
7	27	110	0.036216	106	0.001012
7	28	117	0.024423	115	0.005000
7	29	126	0.030343	122	0.000838
7	30	135	0.035516	131	0.002155
7	31	144	0.040675	140	0.003182
7	32	153	0.037458	149	0.002909
7	33	162	0.027912	158	0.001646
7	34	173	0.046076	169	0.007194
7	35	182	0.029208	178	0.002741
7	36	193	0.040349	189	0.006166
7	37	204	0.045981	200	0.008258
7	38	215	0.042397	211	0.008243
7	39	226	0.039888	222	0.007727
7	40	237	0.038046	233	0.006753
7	41	250	0.049348	244	0.004355
7	42	261	0.039761	257	0.008588
7	43	274	0.043919	268	0.003822
7	44	287	0.046626	281	0.005205
7	45	298	0.027994	294	0.005256
7	46	313	0.048250	307	0.004968
7	47	326	0.039888	320	0.004374
7	48	339	0.033654	335	0.008345

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
7	49	354	0.042552	348	0.005690
7	50	367	0.030367	363	0.007481
7	51	382	0.032552	378	0.009169
7	52	397	0.035808	391	0.003554
7	53	412	0.034195	406	0.003381
7	54	429	0.047378	423	0.007858
7	55	444	0.040718	438	0.005798
7	56	461	0.047332	455	0.008726
7	57	476	0.037250	470	0.005262
7	58	493	0.039867	487	0.006506
7	59	510	0.038834	504	0.007849
7	60	527	0.037226	521	0.007491
7	61	546	0.049196	538	0.005669
7	62	563	0.041284	555	0.004191
7	63	582	0.049260	574	0.006370
7	64	599	0.038289	593	0.009310
7	65	618	0.042252	610	0.004782
7	66	637	0.043675	629	0.005807
7	67	656	0.042138	648	0.005568
7	68	675	0.037804	669	0.009825
7	69	696	0.047521	688	0.007607
7	70	715	0.038158	707	0.004791
8	1	1	0.000000	1	0.000000
8	2	2	0.000000	2	0.000000
8	3	3	0.000000	3	0.000000
8	4	4	0.000000	4	0.000000
8	5	5	0.000000	5	0.000000
8	6	6	0.000000	6	0.000000
8	7	8	0.019226	7	0.019226
8	8	9	0.002403	9	0.002403
8	9	12	0.010815	11	0.010815
8	10	15	0.023657	14	0.023657
8	11	18	0.032528	17	0.032528
8	12	21	0.030495	20	0.030495
8	13	24	0.019822	23	0.019822
8	14	27	0.008672	27	0.008672
8	15	32	0.034843	30	0.002323
8	16	35	0.011131	33	0.000290
8	17	40	0.024681	38	0.001645
8	18	45	0.034039	43	0.004319

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n,.05,\hat{\mu})$	$r^*(n,.05,\hat{\mu})$	$f(n,.01,\hat{\mu})$	$r^*(n,.01,\hat{\mu})$
8	19	50	0.034763	48	0.006839
8	20	55	0.031344	53	0.007124
8	21	60	0.027010	58	0.004987
8	22	65	0.019862	63	0.002285
8	23	72	0.037313	68	0.000626
8	24	77	0.021844	75	0.003364
8	25	84	0.029493	82	0.008189
8	26	91	0.033876	87	0.001390
8	27	98	0.038095	94	0.002346
8	28	105	0.037980	101	0.002566
8	29	112	0.031013	108	0.001860
8	30	119	0.022686	117	0.008035
8	31	128	0.037120	124	0.004296
8	32	135	0.023279	133	0.009492
8	33	144	0.031461	140	0.003545
8	34	153	0.038307	149	0.005473
8	35	162	0.038303	158	0.006048
8	36	171	0.035045	167	0.005738
8	37	180	0.032945	176	0.005150
8	38	191	0.047001	185	0.003920
8	39	200	0.038887	196	0.008197
8	40	211	0.047526	205	0.004844
8	41	220	0.033859	216	0.007034
8	42	231	0.035671	227	0.008534
8	43	242	0.038143	236	0.003224
8	44	253	0.038649	247	0.003102
8	45	264	0.033694	260	0.008757
8	46	277	0.049012	271	0.006480
8	47	288	0.039003	282	0.004651
8	48	301	0.048346	295	0.007099
8	49	312	0.034695	306	0.004083
8	50	325	0.039058	319	0.005032
8	51	338	0.039139	332	0.006024
8	52	351	0.036986	345	0.006330
8	53	364	0.035191	358	0.005338
8	54	379	0.045318	371	0.003966
8	55	392	0.038512	386	0.007098
8	56	407	0.044445	399	0.004433
8	57	420	0.034960	414	0.006384
8	58	435	0.037384	429	0.008207

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
8	59	450	0.039356	444	0.008518
8	60	465	0.039029	459	0.007992
8	61	480	0.035928	474	0.007710
8	62	497	0.046149	489	0.006694
8	63	512	0.039373	506	0.009817
8	64	529	0.046664	521	0.007195
8	65	544	0.036159	538	0.008933
8	66	561	0.038934	555	0.009711
8	67	578	0.038913	570	0.005789
8	68	595	0.037875	587	0.005459
8	69	614	0.049446	606	0.009803
8	70	631	0.044271	623	0.008147
8	71	648	0.039455	640	0.006842
8	72	667	0.044343	659	0.008848
8	73	686	0.048556	676	0.006283
8	74	703	0.038006	695	0.006873
8	75	722	0.039416	714	0.007621
8	76	743	0.049637	733	0.007954
8	77	762	0.048554	752	0.007037
8	78	781	0.045363	771	0.005859
8	79	800	0.039198	792	0.008704
8	80	821	0.044603	811	0.006426
9	1	1	0.000000	1	0.000000
9	2	2	0.000000	2	0.000000
9	3	3	0.000000	3	0.000000
9	4	4	0.000000	4	0.000000
9	5	5	0.000000	5	0.000000
9	6	6	0.000000	6	0.000000
9	7	8	0.037935	7	0.037935
9	8	9	0.008430	9	0.008430
9	9	12	0.034656	10	0.000937
9	10	13	0.004683	13	0.004683
9	11	16	0.011448	15	0.011448
9	12	19	0.017808	18	0.017808
9	13	22	0.019292	21	0.019292
9	14	25	0.015005	24	0.015005
9	15	28	0.008336	28	0.008336
9	16	33	0.033212	31	0.003176
9	17	36	0.014746	34	0.000750
9	18	41	0.032075	39	0.004082

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n,.05,\hat{\mu})$	$r^*(n,.05,\hat{\mu})$	$f(n,.01,\hat{\mu})$	$r^*(n,.01,\hat{\mu})$
9	19	46	0.045318	42	0.000528
9	20	49	0.016448	47	0.001563
9	21	54	0.018847	52	0.002837
9	22	59	0.017917	57	0.003468
9	23	66	0.048993	62	0.002954
9	24	71	0.037214	67	0.001751
9	25	76	0.026643	74	0.008104
9	26	83	0.042144	79	0.003679
9	27	88	0.025956	86	0.008924
9	28	95	0.033975	91	0.002862
9	29	102	0.041704	98	0.004904
9	30	109	0.044285	105	0.005958
9	31	116	0.041287	112	0.005895
9	32	123	0.037634	119	0.005333
9	33	130	0.033224	126	0.004370
9	34	139	0.047846	135	0.009995
9	35	146	0.037275	142	0.006627
9	36	155	0.046295	149	0.003390
9	37	162	0.031502	158	0.005473
9	38	171	0.034161	167	0.006896
9	39	180	0.035561	176	0.008048
9	40	189	0.036628	185	0.008711
9	41	198	0.033965	194	0.007998
9	42	207	0.028388	203	0.006253
9	43	218	0.041898	212	0.004548
9	44	227	0.031990	223	0.007998
9	45	238	0.040700	232	0.004909
9	46	249	0.046432	243	0.006912
9	47	260	0.049031	254	0.009040
9	48	269	0.033550	263	0.003984
9	49	280	0.032078	276	0.009730
9	50	293	0.048089	287	0.009111
9	51	304	0.042308	298	0.008247
9	52	315	0.036143	309	0.006418
9	53	328	0.044714	320	0.004261
9	54	339	0.035015	333	0.006672
9	55	352	0.040082	346	0.008844
9	56	365	0.044135	359	0.009944
9	57	378	0.045037	370	0.005668
9	58	391	0.043870	383	0.005522

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
9	59	404	0.041623	396	0.005214
9	60	417	0.038570	411	0.009005
9	61	432	0.047268	424	0.007486
9	62	445	0.041214	437	0.005907
9	63	460	0.046410	452	0.007845
9	64	473	0.037431	467	0.009620
9	65	488	0.039076	480	0.006030
9	66	503	0.040341	495	0.006532
9	67	518	0.040496	510	0.006977
9	68	533	0.038745	525	0.006634
9	69	550	0.048740	540	0.005619
9	70	565	0.043725	557	0.008897
9	71	580	0.037203	572	0.006799
9	72	597	0.042512	589	0.009124
9	73	614	0.046541	604	0.006181
9	74	631	0.048172	621	0.007375
9	75	648	0.048728	638	0.007983
9	76	665	0.049464	655	0.007789
9	77	682	0.047570	672	0.007454
9	78	699	0.043365	689	0.006995
9	79	716	0.039626	708	0.009507
9	80	735	0.045350	725	0.007989
9	81	752	0.039077	744	0.009572
9	82	771	0.042126	761	0.007352
9	83	790	0.044860	780	0.008066
9	84	809	0.045974	799	0.008634
9	85	828	0.044751	818	0.008959
9	86	847	0.043209	837	0.008617
9	87	866	0.041373	856	0.007671
9	88	887	0.047083	875	0.006612
9	89	906	0.042810	896	0.008606
9	90	927	0.046083	915	0.006714
10	1	1	0.000000	1	0.000000
10	2	2	0.000000	2	0.000000
10	3	3	0.000000	3	0.000000
10	4	4	0.000000	4	0.000000
10	5	5	0.000000	5	0.000000
10	6	6	0.000000	6	0.000000
10	7	7	0.000000	7	0.000000
10	8	9	0.018144	8	0.018144

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
10	9	10	0.003629	10	0.003629
10	10	13	0.016692	11	0.000363
10	11	16	0.037921	14	0.001996
10	12	17	0.005389	17	0.005389
10	13	20	0.009341	20	0.009341
10	14	23	0.011442	22	0.011442
10	15	26	0.010298	25	0.010298
10	16	31	0.040211	29	0.006865
10	17	34	0.027927	32	0.003335
10	18	37	0.015256	35	0.001125
10	19	42	0.033262	40	0.005940
10	20	47	0.049072	43	0.001449
10	21	50	0.023533	48	0.004158
10	22	55	0.028935	53	0.007470
10	23	60	0.031862	58	0.009608
10	24	65	0.033103	63	0.009874
10	25	70	0.030884	68	0.008989
10	26	75	0.025088	73	0.007536
10	27	82	0.046537	78	0.005470
10	28	87	0.033031	83	0.003093
10	29	94	0.048790	90	0.007644
10	30	99	0.030656	95	0.003410
10	31	106	0.039476	102	0.005955
10	32	113	0.044126	109	0.007852
10	33	120	0.045380	116	0.009125
10	34	127	0.045971	123	0.009926
10	35	134	0.044049	130	0.009624
10	36	141	0.038137	137	0.008026
10	37	148	0.031264	144	0.006002
10	38	157	0.045061	151	0.004187
10	39	164	0.033722	160	0.007429
10	40	173	0.042472	167	0.004368
10	41	182	0.049654	176	0.006193
10	42	189	0.032351	185	0.008109
10	43	198	0.034916	194	0.009496
10	44	207	0.034372	203	0.009629
10	45	216	0.032074	212	0.009010
10	46	227	0.047529	221	0.008203
10	47	236	0.040890	230	0.006842
10	48	245	0.034522	239	0.004926

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
10	49	256	0.042180	250	0.008036
10	50	265	0.032841	259	0.005006
10	51	276	0.037063	270	0.006947
10	52	287	0.040911	281	0.008240
10	53	298	0.042828	292	0.008876
10	54	309	0.042240	303	0.009361
10	55	320	0.040183	314	0.009270
10	56	331	0.037428	325	0.008192
10	57	344	0.047823	336	0.006804
10	58	355	0.041691	347	0.005508
10	59	368	0.049045	360	0.007899
10	60	379	0.040259	371	0.005628
10	61	392	0.044019	384	0.006976
10	62	405	0.047333	397	0.008261
10	63	418	0.049009	410	0.009223
10	64	431	0.048278	423	0.009306
10	65	444	0.046233	436	0.008846
10	66	457	0.043729	449	0.008320
10	67	470	0.040167	462	0.007363
10	68	485	0.047557	475	0.005877
10	69	498	0.041125	490	0.008240
10	70	513	0.045838	503	0.005968
10	71	528	0.049258	518	0.007416
10	72	541	0.039487	533	0.008347
10	73	556	0.040092	548	0.008866
10	74	571	0.040417	563	0.009227
10	75	586	0.039726	578	0.009146
10	76	603	0.048757	593	0.008370
10	77	618	0.045266	608	0.007283
10	78	633	0.040099	623	0.006204
10	79	650	0.046157	640	0.007999
10	80	665	0.039096	655	0.006210
10	81	682	0.042558	672	0.007300
10	82	699	0.044335	689	0.008370
10	83	716	0.044963	706	0.009036
10	84	733	0.045174	723	0.009113
10	85	750	0.044334	740	0.008829
10	86	767	0.041834	757	0.008352
10	87	786	0.049317	774	0.007598
10	88	803	0.044749	793	0.009837

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Table 1: *continued*

Number of Trials n	Number of Spikes $n\hat{\mu}$	$f(n, .05, \hat{\mu})$	$r^*(n, .05, \hat{\mu})$	$f(n, .01, \hat{\mu})$	$r^*(n, .01, \hat{\mu})$
10	89	820	0.039298	810	0.008335
10	90	839	0.043501	829	0.009860
10	91	858	0.046751	846	0.007678
10	92	877	0.048193	865	0.008359
10	93	896	0.048867	884	0.008804
10	94	915	0.049032	903	0.009076
10	95	934	0.048047	922	0.008960
10	96	953	0.045816	941	0.008404
10	97	972	0.042650	960	0.007538
10	98	993	0.048420	981	0.009765
10	99	1012	0.043401	1000	0.008166
10	100	1033	0.047134	1021	0.009694

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