GLOBAL EXISTENCE
FOR A CLASS OF TRIANGULAR PARABOLIC SYSTEMS
ON DOMAINS OF ARBITRARY DIMENSION

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ABSTRACT. A class of triangular parabolic systems given on bounded domains of $\mathbb{R}^n$ with arbitrary n is investigated. Sufficient conditions on the structure of the systems are found to assure that weak solutions exist globally.

1. INTRODUCTION

The purpose of this paper is to investigate the global existence problem for a general class of strongly coupled parabolic systems of the type

$$(1.1) \qquad \begin{aligned} u_t &= \nabla(P(u,v)\nabla u + R(u,v)\nabla v) + F(u,v), \\ v_t &= \nabla(Q(u,v)\nabla v) + G(u,v), \end{aligned}$$

which is also supplied with the boundary conditions $\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0$ on the boundary $\partial\Omega$ of a bounded domain Ω in $\mathbb{R}^n$. The initial conditions are described by $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$, $x \in \Omega$. Here $u_0, v_0 \in W^{1,p}(\Omega)$ for some $p > n$.

The fundamental theory of strongly coupled systems such as (1.1) was studied in [1]. The concept of $W^{1,p}$ weak solutions and their local existence was formulated there. One of the important issues, the global existence of solutions, was also discussed. It was pointed out that solutions to (1.1) exist globally in time if one has control on their L^∞ norms. It is not surprising that many classical methods, which were developed successfully for regular reaction-diffusion systems to obtain a priori estimates of the supremum norms of solutions, failed to handle (1.1). Not much is known for the global solvability for (1.1).

Firstly, invariance principles were used in [4, 9] to study the boundedness of weak solutions for certain strongly coupled systems. Of course, this method required severe restrictions on the initial data of the solutions. L^p techniques in [6, 8, 2] and the Lyapunov functional approach in [11, 12, 13] were employed to attack this question. However, not only that these authors must assume certain special structure conditions on their systems but their use of Sobolev imbedding inequalities

forced the domain Ω to be of dimension at most 5 (or even 2). In particular, they considered the following system:

$$(1.2) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta[(d_1 + a_{11}u + a_{12}v)u] + F(u,v), \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + a_{22}v)v] + G(u,v), \end{cases}$$

which was proposed by Shigesada, Kawasaki and Teramoto in [10] to study spatial segregation of interacting species. The functions F, G were taken to be of the form

$$(1.3) \quad F(u,v) = u(b_1 - c_{11}u - c_{12}v), \quad G(u,v) = v(b_2 - c_{21}u - c_{22}v), \text{ with } b_i, c_{ij} > 0.$$

It is mathematically interesting to know whether global solvability still holds under a more general setting where the dimension of the domain Ω can be arbitrary. We answer this question in Section 2. Our results essentially improve the aforementioned works. To illustrate this, we apply our results in Section 2 to (1.2). In fact, we allow a more general system than (1.2) by considering the following forms of P, Q, R in (1.1) (notice the presence of u in $Q(u,v)$ below):

$$(1.4) \qquad \begin{aligned} P(u,v) &= d_1 + a_{11}u + a_{12}v, \quad R(u,v) = b_{11}u, \\ Q(u,v) &= d_2 + a_{21}u + a_{22}v. \end{aligned}$$

Since u, v are population densities, only positive solutions are of interest. We then study these solutions in Section 4, and give the proof of the following.

Theorem 1.1. *Assume* (1.4) *and that* $d_i, a_{ij} > 0, b_{11} > 0$, $i, j = 1, 2$. *In addition, suppose that* $F(0,v) = G(u,0) = 0$ *for all* u, v, *and*

$$(1.5) \qquad F(u,v) \text{ and } G(u,v) \text{ are negative if either } u \text{ or } v \text{ is sufficiently large,}$$

$$(1.6) \qquad a_{11} > a_{21}, \ a_{22} > a_{12}, \ \text{ and } a_{22} \neq a_{12} + b_{11}.$$

Then weak solutions to (1.1) *with nonnegative initial data are classical and exist globally.*

In population dynamics terms, (1.6) means that self-diffusion rates are stronger than cross-diffusion ones. Obviously, the reactions F, G given by (1.3) satisfy our assumption (1.5). Note also that the condition $F(0,v) = G(u,0) = 0$ and maximum principles imply that the solutions stay positive if their initial data are nonnegative.

We would like to remark that the L^p bootstrapping methods in [6, 8, 2] cannot apply to our case here. Indeed, a crucial ingredient in those techniques is an estimate of ∇v that will be used in the bootstrapping argument on the equation for u. Such an estimate, using standard results for scalar regular parabolic equations (see [5]) for the equation of v, is no longer available here. This is because of the presence of u, whose regularity is not yet known, in the diffusion term $Q(u,v)$ of the equation for v.

Finally, for the sake of simplicity, we consider here systems of two equations with homogeneous Neumann boundary conditions, even if our main results here could apply to those of more equations and suitable other boundary conditions. Moreover, our method can be generalized to treat strongly coupled parabolic systems with full diffusion matrices to obtain not only L^∞ bounds for weak solutions but also their Hölder regularity. Results in this direction will be discussed in [7].

2. The general case

In this section, we study a solution (u, v) of the general system (1.1) and give sufficient conditions for global existence of (u, v). Since (1.1) is a triangular system, thanks to the results of [1], we need only to show that the L^∞ norms u, v are bounded. Our first assumption on the coefficients of (1.1) is

(P.1): The functions P, Q, R, F, G are continuous functions in (u, v). Moreover, P, Q are positive for nonnegative u, v, and u_0, v_0 are nonnegative on Ω.

Throughout this section, we consider $W^{1,p}$ weak solutions of (1.1). The reader is referred to [1] for the precise definition of these weak solutions and their local existence. Let (u, v) be a weak solution that exists on $\Omega_T = \Omega \times (0, T)$ for some $T > 0$.

First of all, as a simple consequence of the maximum principles for scalar parabolic equations, one can prove that u, v stay nonnegative.

By multiplying the equation of v in (1.1) with $(v - K_v)_+$ and using the assumption on G we easily prove the following.

Lemma 2.1. *There exists a constant $K_v > 0$, which depends on the initial data v_0, such that $v(x, t) \leq K_v$ for all $(x, t) \in \Omega_T$.*

Consider the following subset of $\mathbb{R}^2$:

$$(2.1) \qquad \Gamma = \{(u, v) : u > 0, \quad 0 < v < K_v\},$$

and the following assumptions.

(H.0): There exist a C^2 function $H(u, v)$ defined on a neighborhood Γ_0 of Γ, and a constant K_0 such that $(H_u F + H_v G)(H - K)_+ \leq 0$ for every $(u, v) \in \Gamma_0$ and $K \geq K_0$.

(H.1): There exists $\lambda_1 > 0$ such that

$$(2.2) \qquad [H_u(P\nabla u + R\nabla v) + H_v Q \nabla v]\nabla H \geq \lambda_1 |\nabla H|^2,$$

$$(2.3) \qquad (P\nabla u + R\nabla v)\nabla H_u + Q\nabla v \nabla H_v \geq 0,$$

for every $(u, v) \in \Gamma_K := \Gamma \cap \{(u, v) : H(u, v) \geq K\}$, $K \geq K_0$, with K_0 being given in (H.0).

(H.2): If $u \to \infty$ in $\mathbb{R}^2$, then $H(u, v) \to \infty$.

Here, we write $H_u = \frac{\partial}{\partial u} H(u, v)$, $H_{uu} = \frac{\partial^2}{\partial u \partial u} H(u, v)$, $\nabla H = \nabla_x H(\vec{u}(x))$, and so on. Furthermore, w_+ will denote the nonnegative part $\sup\{w, 0\}$ of a function w.

Our main result on the boundedness of weak solutions is the following.

Theorem 2.2. *The conditions* (P.1), (H.0), (H.1) *and* (H.2) *imply that u, v are bounded.*

Proof. Firstly, for nonnegative $\eta \in W^{1,2}(\Omega)$, we can test the equations of u, v respectively by $H_u \eta$ and $H_v \eta$, add the results, and use (2.3) to get

$$(2.4)$$
$$\int_\Omega \frac{\partial H}{\partial t} \eta \, dx + \int_\Omega (H_u(P\nabla u + R\nabla v) + H_v Q \nabla v)\nabla \eta \, dx \leq C \int_\Omega (H_u F + H_v G)\eta \, dx.$$

Here, we have used the homogeneous Neumann boundary conditions so that the boundary integrals, which appear in the integration by parts, are all zero.

We set $H_0 = \sup_{x \in \Omega} H(u_0(x), v_0(x))$, which is finite because u_0, v_0 are bounded on Ω. Let $K \geq \max\{K_0, H_0\}$ and η be $(H - K)_+$ in (2.4). Integrate the result in t and use (H.0), (2.2) to obtain

$$\int_\Omega (H - K)_+^2 \, dx \mid_0^t + \lambda_1 \int_0^t \int_{H \geq K} |\nabla H|^2 dx ds \leq 0.$$

Since $(H - K)_+ = 0$ when $t = 0$ (as $K \geq H_0$), the above shows that $(H - K)_+ = 0$ for all t. We conclude that $H \leq K$ on Ω. Condition (H.2) basically says that boundedness of u, v comes from that of $H(u, v)$. Thus, u, v are bounded by some constant depending on K_0 and the initial data u_0, v_0. $\qquad\square$

3. The existence of H

We now see that the assumption on the existence of a function H, satisfying (H.1), is crucial for our main results in the previous section. Obviously, it is not clear whether this function ever exists. In this section we will find sufficient conditions on the structure of (1.1) such that we can find such H.

Clearly, the conditions (2.2), (2.3) are satisfied if the following quadratics (in $U, V \in \mathbb{R}^n$) are positive definite:

$$(3.1) \qquad A_1 := (P - \lambda)H_u^2 U^2 + [RH_u H_v + (Q - \lambda)H_v^2]V^2$$
$$+ [RH_u^2 + (Q + P - 2\lambda)H_u H_v]UV,$$
$$(3.2) \qquad A_2 := PH_{uu}U^2 + (RH_{uv} + QH_{vv})V^2 + [RH_{uu} + (P + Q)H_{uv}]UV.$$

A_1 is positive definite if the coefficients of U^2, V^2 are nonnegative and its discriminant Θ_1 is nonpositive. However, a simple calculation shows that

$$\Theta_1 = (PH_u H_v - RH_u^2 - QH_u H_v)^2 = H_u^2((P - Q)H_v - RH_u)^2.$$

This suggests that we will require H to fulfill $(P - Q)H_v = RH_u$. In other words, we will consider the following equations:

$$(3.3) \qquad\qquad\qquad f(u, v) = (P - Q)/R,$$
$$(3.4) \qquad\qquad\qquad H_u = f(u, v)H_v.$$

Lemma 3.1. *Assume that* (3.4) *holds. There exists $\lambda > 0$ such that A_1 is positive definite.*

Proof. By (3.3) and (3.4), the coefficients of U^2, V^2 in A_1 are respectively $H_u^2(P - \lambda)$ and $H_v^2(Rf + Q - \lambda) = H_v^2(P - \lambda)$. They are nonnegative if we choose $\lambda = \inf_\Gamma P$. $\quad\square$

To verify the positivity of A_2 in (3.2), we consider its discriminant Θ_2. An easy computation shows that

$$\Theta_2 := (RH_{uu} + PH_{uv} + QH_{uv})^2 - 4PH_{uu}(RH_{uv} + QH_{vv}).$$

Differentiating $H_u = fH_v$, we get $H_{uu} = f_u H_v + f H_{uv}$ and $H_{uv} = f_v H_v + f H_{vv}$. Substitute these into Θ_2 and simplify to obtain

$$(3.5) \qquad\qquad\qquad \Theta_2 := \alpha_1 H_{vv}^2 + \alpha_2 H_{vv} H_v + \alpha_3 H_v^2.$$

Using (3.3), we easily see that $\alpha_1 = 0$. Similarly, we have

$$\begin{aligned}
\alpha_2 &= 2\left(R\left(f_u + ff_v\right) + Pf_v + Qf_v\right)\left(Rf^2 + Pf + Qf\right) \\
&\quad - 4P[(f_u + ff_v)\left(Rf + Q\right) + Rf^2 f_v] \\
&= 4(-PRf^2 f_v + P^2 f_v f - PQf_u) \\
&= -4PQ(f_u - f_v f),
\end{aligned}$$

$$\begin{aligned}
\alpha_3 &= (R(f_u + ff_v) + Pf_v + Qf_v)^2 - 4P(f_u + ff_v)Rf_v \\
&= (f_u + ff_v)^2 R^2 + (P + Q)^2 f_v^2 + 2Rf_v[(f_u + ff_v)(Q - P)] \\
&= (f_u + ff_v)^2 R^2 + (P - Q)^2 f_v^2 + 2Rf_v(f_u + ff_v)(Q - P) + 4PQf_v^2 \\
&= [(f_u + ff_v)R + f_v(Q - P)]^2 + 4PQf_v^2 \\
&= R^2 f_u^2 + 4PQf_v^2.
\end{aligned}$$

Let g be a solution to (3.4) and G be any differentiable function on $\mathbb{R}$. We observe that $H(u,v) = G(g(u,v))$ is also a solution to (3.4). We will make the following main assumptions of this section.

(H.3): Assume that there exists a connected neighborhood Γ_K^0 of Γ_K such that g belongs to $C^2(\Gamma_K^0)$. Moreover,

$$(3.6) \qquad g_v \neq 0, \text{ and } \alpha_2 = -4PQ(f_u - f_v f) \neq 0, \quad \forall (u,v) \in \Gamma_K^0.$$

(H.4): The quantities $g_{vv}/g_v^2 + \alpha_3/(\alpha_2 g_v)$, $\delta_{12}/(f\delta_{11})$ and $f\delta_{21}/\delta_{22}$ are bounded on Γ_K. Here, we denote

$$\delta_{12} = P[f^2 g_{vv} + (f_u + ff_v)g_v], \quad \delta_{21} = Pg_{vv} + Rf_v g_v,$$

and $\delta_{11} = \delta_{22} = Pfg_v^2$.

The existence of H is then given by

Theorem 3.2. *Assume* (H.3), (H.4) *and let* $H(u,v) = \exp(\mu g(u,v))$. *There exists* μ *such that* (H.1) *holds.*

Proof. Thanks to Lemma 3.1 and the choice of g, we need only to check the positivity of A_2. We first show that $\Theta_2 < 0$ on Γ_K for a suitable choice of μ. Let $G(x) = \exp(\mu x)$. As $H_v = G'g_v$, $H_{vv} = (G''g_v^2 + G'g_{vv})$, and $G''/G' = \mu$, we have

$$\Theta_2 = H_{vv}H_v\alpha_2 + H_v^2\alpha_3 = (G')^2 g_v^3 \alpha_2 \left[\mu + (\frac{g_{vv}}{g_v^2} + \frac{\alpha_3}{\alpha_2 g_v})\right].$$

Thanks to our assumption (3.6) and because Γ_K^0 is connected, the coefficient of μ never vanishes on Γ_K. That is, either $g_v^3\alpha_2 < 0$ or $g_v^3\alpha_2 > 0$ on Γ_K. Because $g_{vv}/g_v^2 + \alpha_3/(\alpha_2 g_v)$ is bounded on Γ_K and $g_v, G' \neq 0$, the above shows that $\Theta_2 < 0$ on Γ_K for a suitable choice of μ with $|\mu|$ being sufficiently large.

Finally, we show that the coefficients of U^2, V^2 in A_2 are positive. It suffices to show that the following quantities $\delta_1 = PH_{uu}$ and $\delta_2 = (RH_{uv} + QH_{vv})$ are strictly positive on Γ_K. A similar calculation as before yields

$$\delta_1 = P[f^2 H_{vv} + (f_u H_v + ff_v H_v)] = \exp(\mu g)f\delta_{11}\left[\mu\frac{\delta_{12}}{f\delta_{11}} + \mu^2\right],$$

$$\delta_2 = (Rf + Q)H_{vv} + Rf_v H_v = \exp(\mu g)\frac{\delta_{22}}{f}\left[\mu\frac{f\delta_{21}}{\delta_{22}} + \mu^2\right],$$

where δ_{ij} are defined as in (H.4). Since the coefficients of μ, $\delta_{12}/(f\delta_{11})$ and $f\delta_{21}/\delta_{22}$ are bounded on Γ_K, and $f\delta_{11}$, δ_{22}/f are positive, we can choose $|\mu|$ large to have that $\delta_1, \delta_2 > 0$ on Γ_K. $\square$

4. Proof of Theorem 1.1

The boundedness of v was proven in Lemma 2.1 so that we will only concern ourselves with the boundedness of u here. We apply Theorem 2.2 to establish Theorem 1.1.

By Lemma 2.1, we can take Γ_0 to be the strip $\{(u,v)|u > 0, 0 < v < K_v\}$. We also see that f of (3.3) is given by

$$f(u,v) = \frac{d + au - bv}{u},$$

where

$$d = \frac{d_1 - d_2}{b_{11}}, \quad a = \frac{a_{11} - a_{21}}{b_{11}}, \quad b = \frac{a_{22} - a_{12}}{b_{11}}.$$

Our assumption (1.6) simply means $a > 0$, $b \neq 1$ and $b > 0$. Moreover, the equation (3.4) can be solved by methods of characteristics (see [3]). In fact, it is elementary to see that the general solution of (3.4) is given by

$$g(u,v) = L\left(\frac{u^b}{d(b-1) + abu - b(b-1)v}\right),$$

where L can be any C^1 function on $\mathbb{R}$.

Since $a^2 b > 0$ and $F(u,v), G(u,v) \leq 0$ if u is large, we can find $K_1 > 0$ such that if $u \geq K_1$, then $F(u,v), G(u,v) \leq 0$ and $a[d(b-1) + abu - b(b-1)K_v] > 1$. We define

$$\Gamma_1 := \{(u,v) \in \Gamma_0 \mid u \geq K_1\}$$

and

$$(4.1) \qquad \hat{g}(u,v) = (b-1)\log\left(\frac{u^b}{a[d(b-1) + abu - b(b-1)v]}\right), \quad (u,v) \in \Gamma_1.$$

Put $G_0 = \sup\{\hat{g}(u,v) \mid u = K_1, 0 < v \leq K_v\}$. Let $g(u,v)$ be a C^1 extension of $\hat{g}(u,v)$ on Γ_0 that satisfies $\sup_{\Gamma_0 \setminus \Gamma_1} g(u,v) \leq G_0 + 1$. We then set $G_1 := G_0 + 2$. Obviously, we have

$$(4.2) \qquad\qquad g(u,v) \geq G_1 \Rightarrow (u,v) \in \Gamma_1 \Rightarrow u \geq K_1.$$

We study the function g on Γ_1. Firstly, we compute and find

$$(4.3) \qquad\qquad g_v = \frac{b(b-1)^2}{d(b-1) + abu - b(b-1)v}, \quad g_{vv} = \frac{g_v^2}{b-1},$$

$$(4.4) \qquad\qquad f_u = \frac{bv - d}{u^2}, \quad f_v = -\frac{b}{u}.$$

We then prove the following lemmas.

Lemma 4.1. *For $(u,v) \in \Gamma_1$, we have $g_v \alpha_2 < 0$ and $\alpha_3/(\alpha_2 g_v)$ is bounded.*

Proof. By (4.4), we have

$$f_u - ff_v = \frac{d(b-1) + abu - b(b-1)v}{u^2} \neq 0, \quad \forall (u,v) \in \Gamma_1.$$

Thus, by (4.3), $g_v\alpha_2 = -4PQb(b-1)^2/u^2 < 0$ on Γ_1. On the other hand, we write

$$\frac{\alpha_3}{\alpha_2 g_v} = \frac{R^2 f_u^2}{\alpha_2 g_v} - \frac{f_v^2}{(f_u - ff_v)g_v},$$

which can be simplified to

$$-\frac{(bv-d)^2 b_{11}^2}{4b(b-1)^2 PQ} - \frac{b}{(b-1)^2}.$$

The above quantity is bounded on Γ_1 since $P \geq d_1$, $Q \geq d_2$ and v is bounded. The proof of this lemma is complete. $\qquad\square$

Lemma 4.2. $\delta_{12}/(f\delta_{11})$ and $f\delta_{21}/\delta_{22}$ are bounded on Γ_1.

Proof. We have

$$\frac{\delta_{12}}{f\delta_{11}} = \frac{g_{vv}}{g_v^2} + \frac{f_u + ff_v}{f^2 g_v} = \frac{1}{b-1} + \frac{f_u + ff_v}{f^2 g_v}.$$

The last fraction is

$$-\frac{(d(1+b) + abu - b(b+1)v)(d(b-1) + abu - b(b-1)v)}{(d + au - bv)^2 b(b-1)^2},$$

which is bounded because v is bounded on Γ_1 and the powers of u in the numerator and denominator are equal (so that the fraction is bounded when u is large).

Next, we have

$$\frac{f\delta_{21}}{\delta_{22}} = \frac{g_{vv}}{g_v^2} + \frac{Rf_v}{Pg_v} = \frac{1}{b-1} - \frac{b_{11}(d(b-1) + abu - b(b-1)v)}{(d_1 + a_{11}u + a_{12}v)(b-1)^2}.$$

The last fraction is bounded on Γ_1 by the same reason as before. $\qquad\square$

We have shown that the conditions (H.3) and (H.4) are satisfied on the set Γ_1. In particular, because $g_v\alpha_2 < 0$, we see that the factor μ in the proof of Theorem 3.2 can be chosen to be positive and sufficiently large. Fixing such a constant μ, we then define $H(u,v) = \exp(\mu g(u,v))$. Let $K_0 = \exp(\mu G_1)$. We see that $H(u,v) \geq K_0 \Rightarrow g(u,v) \geq G_1$. Therefore, thanks to (4.2), we have

$$(4.5) \qquad \Gamma_{K_0} = \{(u,v) \in \Gamma_0 \,|\, H(u,v) \geq K_0\} \subset \Gamma_1.$$

The definition of Γ_1, Theorem 3.2 and the above lemmas show that (H.0) and (H.1) are verified on Γ_1. By (4.5), they also hold on Γ_{K_0}. It is easy to see that $g(u,v) \sim \log(u^{(b-1)^2})$ when u is large so that $H(u,v) \sim u^{\mu(b-1)^2}$. Since $\mu > 0$ and $H(u,v)$ is bounded on $\Gamma_0 \setminus \Gamma_1$ (by $\exp(\mu G_1)$) we easily see that $H(u,v) \to \infty$ iff $u \to \infty$. Hence (H.2) also holds. Theorem 2.2 asserts that u is bounded. Our proof of Theorem 1.1 is complete.

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