SIAM J. NUMER. ANAL.
Vol. 48, No. 1, pp. 79–94
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MIXED FINITE ELEMENT METHODS FOR INCOMPRESSIBLE FLOW: STATIONARY NAVIER–STOKES EQUATIONS*

ZHIQIANG CAI†, CHUNBO WANG†, AND SHUN ZHANG†

Abstract. In [Z. Cai, C. Tong, P. S. Vassilevski, and C. Wang, *Numer. Methods Partial Differential Equations*, to appear], the authors developed and analyzed a mixed finite element method for the stationary Stokes equations based on the pseudostress-velocity formulation. The pseudostress and the velocity are approximated by a stable pair of finite elements: Raviart–Thomas elements of index $k \ge 0$ and discontinuous piecewise polynomials of degree $k \ge 0$, respectively. This paper extends the method to the stationary, incompressible Navier–Stokes equations. Under appropriate assumptions, we show that the pseudostress-velocity formulation of the Navier–Stokes equation and its discrete counterpart have branches of nonsingular solutions, and error estimates of the mixed finite element approximations are established as well.

Key words. Navier–Stokes equations, mixed finite element, incompressible Newtonian flow

AMS subject classifications. 65M60, 65M15

DOI. 10.1137/080718413

1. Introduction.

Let Ω be a bounded, open, connected subset of $\mathbb{R}^d$ ($d = 2$ or 3) with a Lipschitz continuous boundary $\partial\Omega$. Let $\mathbf{f} = (f_1, \ldots, f_d)$ and $\mathbf{g} = (g_1, \ldots, g_d)$ be the given external body force and the prescribed velocity on the boundary, respectively. Denote $\mathbf{u} = (u_1, \ldots, u_d)$ and p to be the respective velocity vector and pressure. Consider the following stationary, incompressible Navier–Stokes equations:

$$(1.1)\qquad \begin{cases} \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot(2\nu\,\boldsymbol{\epsilon}(\mathbf{u})) + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \end{cases}$$

with the Dirichlet boundary condition

$$(1.2)\qquad \mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega.$$

Here, $\nu > 0$ is the kinematic viscosity, ∇ and $\nabla\cdot$ denote the respective gradient and divergence operators, and $\boldsymbol{\epsilon}(\mathbf{u}) = \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right)/2$ is the deformation rate tensor. Assume that $\mathbf{g} = (g_1, \ldots, g_d)$ satisfies the compatibility condition

$$(1.3)\qquad \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{g}\,ds = 0.$$

Let $\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_{ij})_{d\times d}$ be the stress tensor; then the Navier–Stokes equation in (1.1) is derived from the following stress-velocity-pressure formulation:

$$(1.4)\qquad \begin{cases} \tilde{\boldsymbol{\sigma}} + p\,\boldsymbol{\delta} - 2\,\nu\,\boldsymbol{\epsilon}(\mathbf{u}) = \mathbf{0} & \text{in } \Omega, \\ \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot\tilde{\boldsymbol{\sigma}} = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega. \end{cases}$$

*Received by the editors March 14, 2008; accepted for publication (in revised form) December 29, 2009; published electronically April 2, 2010. This work was sponsored in part by the National Science Foundation under grant DMS-0511430.
http://www.siam.org/journals/sinum/48-1/71841.html
†Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067 (zcai@math.purdue.edu, cwang@math.purdue.edu, zhang@math.purdue.edu).

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These equations are based on the respective physical principles: the constitutive law, the balance of linear momentum, and the conservation of mass.

A tremendous amount of work has been done over many years on the computation of incompressible Navier–Stokes equations based on the velocity-pressure formulation in (1.1) or its variants (see, e.g, the books by Girault and Raviart [17], Pironneau [23], Gunzburger [18], and the references therein). However, the practical need of the stress tensor coupled with a rising interest in non-Newtonian flows have motivated extensive studies of mixed finite element methods in the stress-velocity-pressure formulation (1.4). There are at least two major advantages of this formulation. First, it provides a unified framework for both the Newtonian and the non-Newtonian flows. It has also been pointed out in [20] that an accurate and efficient numerical scheme for Newtonian flows under formulation (1.4) is necessary for the successful computation of non-Newtonian flows. Another advantage is that a physical quantity such as the stress is computed directly instead of by taking derivatives of the velocity. This avoids degrading the accuracy, which is inevitable in the process of numerical differentiation. However, the stress-velocity-pressure formulation has some obvious disadvantages. The most significant ones are the increase in the number of unknowns and the symmetry requirement for the stress tensor [3]. Both of them pose extra difficulty in the numerical computation.

In order to keep the pros and improve the cons of the stress-velocity-pressure formulation, Cai et al. recently analyzed and implemented mixed finite element methods based on the pseudostress-velocity formulation for the stationary Stokes equations in [10]. Raviart–Thomas (RT) elements of index $k \geq 0$ [25] are used for approximating each row of the pseudostress which is a nonsymmetric tensor, and discontinuous piecewise polynomials of degree $k \geq 0$ for approximating each component of the velocity. It is shown that this pair of mixed finite elements is stable and yields optimal accuracy $O(h^{k+1})$ in the $H(\mathrm{div})$ and L^2 norms for the respective pseudostress and velocity. For lower order elements, the total number of degrees of freedom for the discretization is comparable to that for the velocity-pressure formulation using Crouzeix–Raviart elements [15, 24, 8] (nonconforming velocity and discontinuous pressure), and both approaches have the same accuracy for the H^1 seminorm of the velocity and the L^2 norm of the pressure. The indefinite system of linear equations resulting from the discretization is decoupled by the penalty method. The penalized pseudostress system is solved by the $H(\mathrm{div})$ type of multigrid method, and the velocity is then calculated explicitly/locally. It is shown theoretically in [11] that the convergence rate of the $V(1,1)$-cycle multigrid is independent of the mesh size, the number of levels, and the penalty parameter. This is also confirmed numerically in [10, 11] with remarkably fast convergence rates around 0.21 for the RT element of index zero and 0.14 for the BDM element of index one [5].

The purpose of this paper is to extend and analyze the method for the stationary Navier–Stokes equations. Numerical studies of the method for several benchmark test problems, such as the driven cavity problem and flows past cylinder, are reported in [12]. The extension of the method is rather straightforward; the only modification needed is to replace $\nabla \mathbf{u}$ in the nonlinear term by the deviatoric pseudostress since the velocity $\mathbf{u}$ is approximated by *discontinuous* piecewise polynomials. But the analysis is nontrivial. First, it is not known that the general theory on the well-posedness of the saddle point problem (see, e.g., [6]) can be extended for problems with the linear or nonlinear convection term. So we prove the well-posedness of the pseudostress-velocity formulation for the Navier–Stokes equations via establishing its equivalence

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with the velocity-pressure formulation. To analyze the discrete problem (the well-posedness and error estimates), we employ the theory of Brezzi, Rappaz, and Raviart [7, 17] for the approximation of branches of nonsingular solutions. The verification of assumptions required by the theory is usually not a trivial task. This is even true for the velocity-pressure formulation (see [17]). For the pseudostress-velocity formulation, an extra difficulty is that the nonlinear term is not well defined in the trial space of the linear Stokes equations because the momentum equation is required to be valid in $L^2(\Omega)^d$. Hence, we have to use a slightly smoother trial space. This small change causes many complications in the analysis. Under appropriate assumptions, we show that for sufficiently small mesh size, the discrete problem has a branch of nonsingular solutions in a neighborhood of the solution of the continuous problem. Moreover, we establish accuracy $O(h^{k+1-\frac{d}{6}})$ in the $L^3(\Omega)^{d\times d} \times L^3(\Omega)^d$ norms. For a different analysis for nonlinear scalar elliptic equations in the mixed form, see [22] by Milner and Park.

The paper is organized as follows. In the remainder of this section, assumptions on the domain are made and notations and definitions of spaces are introduced. Section 2 begins with the introduction of the pseudostress-velocity formulation for stationary Navier–Stokes equations and follows by the proof of its well-posedness. The mixed finite element approximation based on the pseudostress-velocity formulation is described in section 3. Finally, convergence analysis using a framework of Brezzi, Rappaz, and Raviart [7, 17] is presented in section 4.

1.1. Notation. Assume, throughout the paper, that the boundary of Ω is of class $\mathscr{C}^2$, a two-dimensional convex polygon, or convex polyhedra. We use the standard notations and definitions for the Sobolev spaces $W^{s,p}(\Omega)^d$ and $W^{s,p}(\partial\Omega)^d$ for $s \geq 0$ and $p \in [1, \infty]$. The standard associated inner products are denoted by $(\cdot,\cdot)_{s,p,\Omega}$ and $(\cdot,\cdot)_{s,p,\partial\Omega}$, and their respective norms are denoted by $\|\cdot\|_{s,p,\Omega}$ and $\|\cdot\|_{s,p,\partial\Omega}$. (We suppress the superscript d because the dependence on dimension will be clear by context. We also omit the subscript Ω from the inner product and norm designation when there is no risk of confusion.) For $s = 0$, $W^{s,p}(\Omega)^d$ coincides with $L^p(\Omega)^d$. Moreover, the space $W^{s,2}(\Omega)^d$ will generally be written in the shorthand form $H^s(\Omega)^d$. Set

$$W_0^{1,2}(\Omega) := \left\{ q \in W^{1,2}(\Omega) \,\middle|\, q = 0 \text{ on } \partial\Omega \right\}$$

and

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \,\middle|\, \int_\Omega q\, dx = 0 \right\}.$$

Denote by $\mathscr{D}(\Omega)$ the linear space of infinitely differentiable functions with compact support on Ω.

Let

$$H(\text{div};\, \Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^d \,\middle|\, \nabla\cdot\mathbf{v} \in L^2(\Omega). \right\}, \tag{1.5}$$

which is a Hilbert space equipped with the norm

$$\|\mathbf{v}\|_{H(\text{div})} = \left(\|\mathbf{v}\|_{0,2}^2 + \|\nabla\cdot\mathbf{v}\|_{0,2}^2 \right)^{\frac{1}{2}}.$$

Next, we will introduce a space which is an analogue of $H(\text{div};\, \Omega)$ and is used for the solution space of the pseudostress. For any $s > 2$, let

$$W^{0,s}(\text{div};\, \Omega) = \left\{ \mathbf{v} \in L^s(\Omega)^d \,\middle|\, \nabla\cdot\mathbf{v} \in L^2(\Omega) \right\} \subset H(\text{div};\, \Omega), \tag{1.6}$$

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which is a Banach space equipped with the norm

$$\|\mathbf{v}\|_{W^{0,s}(\mathrm{div})} = \left(\|\mathbf{v}\|_{0,s}^2 + \|\nabla\cdot\mathbf{v}\|_{0,2}^2\right)^{\frac{1}{2}}.$$

Finally, we define their subspaces

$$\widehat{H}(\mathrm{div};\,\Omega)^d = \left\{\boldsymbol{\tau}\in H(\mathrm{div};\,\Omega)^d \,\middle|\, \int_\Omega \mathrm{tr}\,\boldsymbol{\tau}\,d\mathbf{x} = 0\right\}$$

and

$$\widehat{W}^{0,s}(\mathrm{div};\,\Omega)^d = \left\{\boldsymbol{\tau}\in W^{0,s}(\mathrm{div};\,\Omega)^d \,\middle|\, \int_\Omega \mathrm{tr}\,\boldsymbol{\tau}\,d\mathbf{x} = 0\right\}.$$

2. Pseudostress-velocity formulation. This section describes the pseudo-stress-velocity formulation of the stationary incompressible Navier–Stokes equations. The well-posedness of the formulation is proved through establishing its equivalence with the velocity-pressure formulation.

For a vector function $\mathbf{v} = (v_1,\ldots,v_d)$, define its gradient as a $d\times d$ tensor

$$\nabla\mathbf{v} = \begin{pmatrix} \dfrac{\partial v_1}{\partial x_1} & \cdots & \dfrac{\partial v_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial v_d}{\partial x_1} & \cdots & \dfrac{\partial v_d}{\partial x_d} \end{pmatrix} = \left(\frac{\partial v_i}{\partial x_j}\right)_{d\times d}.$$

For a tensor function $\boldsymbol{\tau} = (\tau_{ij})_{d\times d}$, let $\boldsymbol{\tau}_i = (\tau_{i1},\ldots,\tau_{id})$ denote its ith row for $i = 1,\ldots,d$. Define the dot product of a vector $\mathbf{v}$ and a tensor $\boldsymbol{\tau}$ by

$$\mathbf{v}\cdot\boldsymbol{\tau} = (\mathbf{v}\cdot\boldsymbol{\tau}_1,\ldots,\mathbf{v}\cdot\boldsymbol{\tau}_d),$$

and the divergence, normal, and trace of $\boldsymbol{\tau}$ by

$$\nabla\cdot\boldsymbol{\tau} = (\nabla\cdot\boldsymbol{\tau}_1,\ldots,\nabla\cdot\boldsymbol{\tau}_d),\quad \mathbf{n}\cdot\boldsymbol{\tau} = (\mathbf{n}\cdot\boldsymbol{\tau}_1,\ldots,\mathbf{n}\cdot\boldsymbol{\tau}_d),\quad\text{and}\quad \mathrm{tr}\,\boldsymbol{\tau} = \sum_{i=1}^d \tau_{ii},$$

respectively. Let $\mathcal{A}:\mathbb{R}^{d\times d}\to\mathbb{R}^{d\times d}$ be a linear map, which is singular, defined by

$$\mathcal{A}\boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{d}(\mathrm{tr}\,\boldsymbol{\tau})\,\boldsymbol{\delta}. \tag{2.1}$$

It is easy to see that $\mathcal{A}\boldsymbol{\tau}$ is trace free and that $\boldsymbol{\tau}\in\mathbb{R}^{d\times d}$ has the following orthogonal decomposition:

$$\boldsymbol{\tau} = \mathcal{A}\boldsymbol{\tau} + \frac{1}{d}(\mathrm{tr}\,\boldsymbol{\tau})\,\boldsymbol{\delta} \tag{2.2}$$

with respect to the product of tensors

$$\boldsymbol{\sigma}:\boldsymbol{\tau} \equiv \sum_{i,j=1}^d \sigma_{ij}\tau_{ij}$$

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for any $\boldsymbol{\sigma}$, $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$.

Introducing a new independent, nonsymmetric tensor variable, the *pseudostress*, as follows:

$$\boldsymbol{\sigma} = \nu\,\nabla\,\mathbf{u} - p\,\boldsymbol{\delta}, \tag{2.3}$$

taking the trace of (2.3), and using the divergence free condition in the third equation of (1.4) give

$$p = -\frac{1}{d}\,\mathrm{tr}\,\boldsymbol{\sigma}. \tag{2.4}$$

Then (2.3) may be rewritten as

$$\mathcal{A}\left(\frac{\boldsymbol{\sigma}}{\nu}\right) - \nabla\,\mathbf{u} = \mathbf{0}.$$

For incompressible fluids, since the divergence of $(\nabla\,\mathbf{u})^t$ vanishes, the stress and pseudostress have the same divergence; i.e.,

$$\nabla\cdot\boldsymbol{\sigma} = \nabla\cdot\tilde{\boldsymbol{\sigma}}.$$

Hence, we have the following pseudostress-velocity formulation of the stationary Navier–Stokes equation:

$$\left\{\begin{array}{l} \mathcal{A}\left(\dfrac{\boldsymbol{\sigma}}{\nu}\right) - \nabla\,\mathbf{u} = \mathbf{0}, \\ \nabla\cdot\boldsymbol{\sigma} + \mathbf{u}\cdot\mathcal{A}\left(\dfrac{\boldsymbol{\sigma}}{\nu}\right) = -\mathbf{f}. \end{array}\right. \tag{2.5}$$

Here, we replaced $\nabla\mathbf{u}$ in the nonlinear term by $\mathcal{A}\,(\frac{\boldsymbol{\sigma}}{\nu})$ since $\mathbf{u}$ will be approximated by discontinuous piecewise polynomials. Note also that the incompressibility condition is implicitly contained in the "constitutive" equation, the first equation in (2.5).

Rescaling the stress and the right-hand side by $\boldsymbol{\sigma}/\nu \to \boldsymbol{\sigma}$ and $\mathbf{f}/\nu \to \mathbf{f}$, respectively, system (2.5) may be rewritten as

$$\left\{\begin{array}{l} \mathcal{A}\,\boldsymbol{\sigma} - \nabla\,\mathbf{u} = \mathbf{0}, \\ \nabla\cdot\,\boldsymbol{\sigma} - \dfrac{1}{\nu}\,c(\boldsymbol{\sigma},\,\mathbf{u}) = -\mathbf{f}, \end{array}\right. \tag{2.6}$$

where $c(\boldsymbol{\sigma},\,\mathbf{u}) = \mathbf{u}\cdot\mathcal{A}\,\boldsymbol{\sigma} - \nabla\nu\cdot\boldsymbol{\sigma}$. For simplicity of presentation, we assume in the remainder of the paper that the viscosity parameter is a positive constant so that

$$c(\boldsymbol{\sigma},\,\mathbf{u}) = \mathbf{u}\cdot\mathcal{A}\,\boldsymbol{\sigma}.$$

Now, the corresponding variational form of (2.6) and (1.2) is to find $(\boldsymbol{\sigma}, \mathbf{u}) \in \widehat{W}^{0,3}(\mathrm{div};\Omega)^d \times L^2(\Omega)^d$ such that

$$\left\{\begin{array}{ll} (\mathcal{A}\,\boldsymbol{\sigma},\,\boldsymbol{\tau}) + (\nabla\cdot\boldsymbol{\tau},\,\mathbf{u}) = g(\boldsymbol{\tau}) & \forall\,\boldsymbol{\tau} \in \widehat{H}(\mathrm{div};\,\Omega)^d, \\ (\nabla\cdot\boldsymbol{\sigma},\,\mathbf{v}) - \frac{1}{\nu}\,(\mathbf{u}\cdot\mathcal{A}\boldsymbol{\sigma},\,\mathbf{v}) = f(\mathbf{v}) & \forall\,\mathbf{v} \in L^2(\Omega)^d, \end{array}\right. \tag{2.7}$$

where the linear forms are defined by

$$g(\boldsymbol{\tau}) = \int_{\partial\Omega} (\mathbf{n}\cdot\boldsymbol{\tau})\cdot\mathbf{g}\,\mathrm{d}s \quad \text{and} \quad f(\mathbf{v}) = -\int_{\Omega} \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}.$$

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Remark 2.1. The variational form in (2.7) uses a smaller trial space $\widehat{W}^{0,3}(\text{div};\Omega)^d \subset \widehat{H}(\text{div};\Omega)^d$ than the Stokes equations for the pseudostress. By doing so, we can guarantee the nonlinear term, $\mathbf{u}\cdot\mathcal{A}\boldsymbol{\sigma}$, to be in $L^2(\Omega)^d$ and, hence, the second equation in (2.7) is well defined. To see why, note first that the first equation in (2.7) implies that $\mathbf{u}\in H^1(\Omega)$ (see Lemma 2.3 below). Since the imbedding theorem implies that $H^1(\Omega)$ is continuously imbedded in $L^r(\Omega)$ with $r\in[1,\infty)$ for $d=2$ and $r\in[1,6]$ for $d=3$, it then follows from the Hölder inequality with $p=3$ and $q=3/2$ that

$$\|\mathbf{u}\cdot\mathcal{A}\boldsymbol{\sigma}\|_{0,2}\le C\,\|\mathbf{u}\|_{0,6}\|\boldsymbol{\sigma}\|_{0,3}\le C\,\|\mathbf{u}\|_{1,2}\|\boldsymbol{\sigma}\|_{0,3}. \tag{2.8}$$

In the velocity-pressure formulation of the stationary Navier–Stokes equation, rescaling the pressure and the right-hand side by $p/\nu\to p$ and $\mathbf{f}/\nu\to\mathbf{f}$, respectively, we then have

$$\begin{cases}\dfrac{1}{\nu}\,\mathbf{u}\cdot\nabla\mathbf{u}-\Delta\,\mathbf{u}+\nabla\,p=\mathbf{f},\\ \nabla\cdot\mathbf{u}=0.\end{cases} \tag{2.9}$$

Let $H^1_{\mathbf{g}}(\Omega)^d=\{\mathbf{u}\in H^1(\Omega)^d \mid \mathbf{u}|_{\partial\Omega}=\mathbf{g}\}$. The variational form of (2.9) and (1.2) is to find $(\mathbf{u},p)\in H^1_{\mathbf{g}}(\Omega)^d\times L^2_0(\Omega)$ such that

$$\begin{cases}(\nabla\,\mathbf{u},\,\nabla\,\mathbf{v})+\dfrac{1}{\nu}\,(\mathbf{u}\cdot\nabla\cdot\mathbf{u},\,\mathbf{v})-(p,\nabla\cdot\mathbf{v})=-f(\mathbf{v}) & \forall\,\mathbf{v}\in H^1_0(\Omega)^d,\\ (\nabla\cdot\mathbf{u},\,q)=0 & \forall\,q\in L^2_0(\Omega).\end{cases} \tag{2.10}$$

Under the assumption on the boundary in section 1.1, the following theorem is well known (see, e.g., [14, 17]).

THEOREM 2.2. *For* $\mathbf{f}\in H^{l-1}(\Omega)^d$ *and* $\mathbf{g}\in H^{l+1/2}(\partial\Omega)^d$ *with* $l=0,1$, *system* (2.10) *has solutions,* $(\mathbf{u},p)$, *belonging to* $H^{l+1}(\Omega)^d\times H^l(\Omega)^d$. *Moreover, if* $\nu>\nu_0(\Omega;\mathbf{f},\mathbf{g})$,[1] *then the solution* $(\mathbf{u},\,p)$ *is unique.*

In the rest of this section, we will establish the well-posedness and uniqueness of system (2.7).

LEMMA 2.3. *For* $\mathbf{v}\in L^2(\Omega)^d$, *assume that there exists* $\boldsymbol{\Psi}\in L^2(\Omega)^{d\times d}$ *such that*

$$(\mathbf{v},\nabla\cdot\boldsymbol{\Phi})=-(\boldsymbol{\Psi},\boldsymbol{\Phi})$$

for all $\boldsymbol{\Phi}\in\mathscr{D}(\Omega)^{d\times d}$. *Then* $\mathbf{v}$ *is in* $H^1(\Omega)^d$, *and*

$$\nabla\mathbf{v}=\boldsymbol{\Psi}\quad\text{in }\Omega.$$

Proof. The lemma is an immediate consequence of the definition of weak derivatives (see, e.g., [17]). ∎

[1] Here $\nu_0(\Omega;\mathbf{f},\mathbf{g})$ is defined by

$$\nu_0=\inf\{\rho(\mathbf{u}_0)+(\mathcal{N}\,\|l(\mathbf{f};\mathbf{u}_0)\|_{V'})^{1/2}\mid\mathbf{u}_0\in H^1_{\mathbf{g}}(\Omega)^d\text{ and }\nabla\cdot\mathbf{u}_0=0\}\quad\text{with}$$

$$V=\{\mathbf{v}\in H^1_0(\Omega)^d\mid\nabla\cdot\mathbf{u}_0=0\},\rho(\mathbf{u}_0)=\sup_{\mathbf{v}\in V}\frac{(\mathbf{v}\cdot\nabla\mathbf{u}_0,\mathbf{v})}{|\mathbf{v}|^2_{1,\Omega}},$$

$$\mathcal{N}=\sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in V}\frac{(\mathbf{w}\cdot\nabla\mathbf{u},\,\mathbf{v})}{|\mathbf{u}|_{1,\Omega}|\mathbf{v}|_{1,\Omega}|\mathbf{w}|_{1,\Omega}},\|l(\mathbf{f};\mathbf{u}_0)\|_{V'}=\sup_{\mathbf{v}\in V}\frac{\nu(\mathbf{f},\mathbf{v})-\nu\,(\nabla\mathbf{u}_0,\nabla\mathbf{u}_0)-(\mathbf{u}_0\cdot\nabla\mathbf{u}_0,\mathbf{v})}{|\mathbf{v}|_{1,\Omega}}.$$

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LEMMA 2.4. *For* $\mathbf{f} \in L^2(\Omega)^d$ *and* $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)^d$, *let* $(\boldsymbol{\sigma}^*, \mathbf{u}^*) \in \widehat{W}^{0,3}(\mathrm{div};\,\Omega)^d \times L^2(\Omega)^d$ *be a solution of* (2.7)*; then* $(\mathbf{u}^*,\, p^*) \equiv \left(\mathbf{u}^*, -\frac{1}{d}\,\mathrm{tr}\boldsymbol{\sigma}^*\right)$ *satisfies* (2.10).

Proof. Let $(\boldsymbol{\sigma}^*, \mathbf{u}^*) \in \widehat{W}^{0,3}(\mathrm{div};\,\Omega)^d \times L^2(\Omega)^d$ be a solution of the pseudostress-velocity system in (2.7). For any $\boldsymbol{\tau} \in \mathscr{D}(\Omega)^{d\times d} \subset H(\mathrm{div};\,\Omega)^d$, the first equation in (2.7) and the fact that $g(\boldsymbol{\tau}) = 0$ give

$$(\nabla\cdot\boldsymbol{\tau}, \mathbf{u}^*) = (-\mathcal{A}\boldsymbol{\sigma}^*,\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathscr{D}(\Omega)^{d\times d},$$

which, together with Lemma 2.3 and the fact that $\mathcal{A}\boldsymbol{\sigma}^* \in L^3(\Omega)^{d\times d} \subset L^2(\Omega)^{d\times d}$, implies

$$\nabla\mathbf{u}^* = \mathcal{A}\boldsymbol{\sigma}^* \qquad \text{in } \Omega. \tag{2.11}$$

Using (2.11) and integrating by parts in the first equation of (2.7) yield

$$g(\boldsymbol{\tau}) = (\nabla\mathbf{u}^*, \boldsymbol{\tau}) + (\nabla\cdot\boldsymbol{\tau}, \mathbf{u}^*) = \int_{\partial\Omega} (\mathbf{n}\cdot\boldsymbol{\tau})\cdot\mathbf{u}^*\ \mathrm{d}s$$

for all $\boldsymbol{\tau} \in H(\mathrm{div};\,\Omega)^d$. So $\mathbf{u}^* = \mathbf{g}$ on the boundary $\partial\Omega$ and, hence, $\mathbf{u}^* \in H^1_{\mathbf{g}}(\Omega)^d$.

Define $p^* = -\frac{1}{d}\,\mathrm{tr}\boldsymbol{\sigma}^*$; then $p^* \in L^2_0(\Omega)$ and

$$\boldsymbol{\sigma}^* = \mathcal{A}\boldsymbol{\sigma}^* - p^*\,\boldsymbol{\delta} = \nabla\mathbf{u}^* - p^*\,\boldsymbol{\delta}. \tag{2.12}$$

Now, for $\mathbf{f} \in L^2(\Omega)^d$ and $\mathbf{v} \in H^1_0(\Omega)^d \subset L^2(\Omega)^d$, it follows from the second equation in (2.7), integration by parts, (2.11), and (2.12) that

$$\begin{aligned}(\mathbf{f}, \mathbf{v}) &= -(\nabla\cdot\boldsymbol{\sigma}^*, \mathbf{v}) + \frac{1}{\nu}\,(\mathbf{u}^*\cdot\mathcal{A}\boldsymbol{\sigma}^*,\,\mathbf{v}) = (\boldsymbol{\sigma}^*, \nabla\mathbf{v}) + \frac{1}{\nu}\,(\mathbf{u}^*\cdot\nabla\mathbf{u}^*,\,\mathbf{v})\\ &= (\nabla\mathbf{u}^*, \nabla\mathbf{v}) - (p^*, \nabla\cdot\mathbf{v}) + \frac{1}{\nu}\,(\mathbf{u}^*\cdot\nabla\mathbf{u}^*, \mathbf{v}).\end{aligned}$$

The trace of (2.11) leads to $\nabla\cdot\mathbf{u}^* = 0$ which, in turn, implies that $\mathbf{u}^*$ satisfies the second equation in (2.10). Hence, $(\mathbf{u}^*, p^*)$ is a solution of (2.10). This completes the proof of the lemma. ∎

LEMMA 2.5. *For* $\mathbf{f} \in L^2(\Omega)^d$ *and* $\mathbf{g} \in H^{1+\gamma}(\partial\Omega)^d$ *with* $\frac{1}{2} \le \gamma \le 1$, *let* $(\mathbf{u}^\dagger, p^\dagger) \in (H^{1+\gamma}(\Omega)^d \cap H^1_{\mathbf{g}}(\Omega)^d) \times (H^\gamma(\Omega)/\mathbb{R})$ *be a solution of* (2.10)*; then* $(\nabla\mathbf{u}^\dagger - p^\dagger\,\boldsymbol{\delta},\ \mathbf{u}^\dagger)$ *satisfies* (2.7).

Proof. Let $(\mathbf{u}^\dagger, p^\dagger)$ be the solution of (2.10), and let $\boldsymbol{\sigma}^\dagger = \nabla\mathbf{u}^\dagger - p^\dagger\,\boldsymbol{\delta}$. Then the first equation in (2.9) gives

$$\nabla\cdot\boldsymbol{\sigma}^\dagger = \frac{1}{\nu}\,\mathbf{u}^\dagger\cdot\nabla\mathbf{u}^\dagger - \mathbf{f}.$$

The smoothness of $(\mathbf{u}^\dagger, p^\dagger)$ implies that $\boldsymbol{\sigma}^\dagger$ lies in $L^3(\Omega)^{d\times d}$ and that the above equation holds in the L^2 sense. It is then straightforward to see that $(\boldsymbol{\sigma}^\dagger, \mathbf{u}^\dagger) = (\nabla\mathbf{u}^\dagger - p^\dagger\,\boldsymbol{\delta},\ \mathbf{u}^\dagger)$ is a solution of (2.7). ∎

THEOREM 2.6. *Let* $\mathbf{f}$ *be in* $L^2(\Omega)^d$. *For* $\mathbf{g} \in H^{1+\gamma}(\partial\Omega)^d$ *with* $\frac{1}{2} \le \gamma \le 1$, *system* (2.7) *has solutions,* $(\boldsymbol{\sigma},\,\mathbf{u})$*, belonging to* $H^\gamma(\Omega)^{d\times d} \times H^{1+\gamma}(\Omega)^d$. *Moreover, if* $\nu > \nu_0(\Omega; \mathbf{f}, \mathbf{g})$*, then the solution* $(\boldsymbol{\sigma}, \mathbf{u})$ *is unique.*

Proof. The theorem is a direct consequence of Theorem 2.2 and Lemmas 2.4 and 2.5. ∎

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3. Finite element approximation. Assume that Ω is a polygonal/polyhedral domain, and let $\mathcal{T}_h$ be a quasi-regular triangulation of Ω with (triangular/tetrahedral or rectangular/cubic) elements of size $O(h)$. We first denote the space of polynomials on an element $K \subset \mathbb{R}^d$ by the following:

$$P_k(K) \text{ is the space of polynomials of degree } \le k,$$

$$P_{k_1,k_2}(K) = \{p(x_1,\, x_2)\,|\, p(x_1,\, x_2) \textstyle\sum_{i\le k_1,\ j\le k_2} a_{ij} x_1^i x_2^j\}, \qquad d=2,$$

$$P_{k_1,k_2,k_3}(K) = \{p(x_1,\, x_2,\, x_3)\,|\, p(x_1,\, x_2,\, x_3) \textstyle\sum_{i\le k_1,\ j\le k_2,\ k\le k_3} a_{ijk} x_1^i x_2^j x_3^k\}, \quad d=3.$$

We also define

$$Q_k(K) = \begin{cases} P_{k,k}(K) & \text{for } d=2,\\ P_{k,k,k}(K) & \text{for } d=3. \end{cases}$$

Denote the local RT space of index $k \ge 0$ on an element K:

$$RT_k(K) = \begin{cases} P_k(K)^d + (x_1,\ldots,x_d)P_k(K), & K = \text{triangle/tetrahedron},\\[2mm] Q_k(K)^d + (x_1,\ldots,x_d)Q_k(K), & K = \text{rectangle/cube}. \end{cases}$$

In two dimensions, degrees of freedom for $RT_0(K) = (a+bx_1,\, c+bx_2)$ on a triangle or $RT_0(K) = (a+bx_1,\, c+dx_2)$ on a rectangle are normal components of vector fields on all edges. (For the RT_k space of index $k \ge 1$, please see [6] to find the choice of degrees of freedom to ensure the continuity of the normal component of the vector field at the interfaces of elements.) We can now define the $H(\mathrm{div};\Omega)$ conforming RT space of order $k \ge 0$ [25] by

$$RT_k = \big\{\mathbf{v} \in H(\mathrm{div};\Omega)\,\big|\, \mathbf{v}|_K \in RT_k(K) \quad \forall\ K \in \mathcal{T}_h\big\}.$$

Let the space of the divergences of the vectors in $RT_k(K)$ be

$$D_k(K) = \begin{cases} P_k(K), & K = \text{triangle/tetrahedron},\\[2mm] Q_k(K), & K = \text{rectangle/cube}. \end{cases}$$

Denote the space of discontinuous piecewise polynomials by

$$P_k = \big\{q \in L^2(\Omega)\,\big|\, q|_K \in D_k(K) \quad \forall\ K \in \mathcal{T}_h\big\}\,.$$

Denote the product spaces by $RT_k^d = \prod_{i=1}^d RT_k$ and $P_k^d = \prod_{i=1}^d P_k$, and define

$$\widehat{RT}_k^d = \left\{\boldsymbol{\tau} \in RT_k^d \,\middle|\, \int_\Omega \mathrm{tr}\,\boldsymbol{\tau}\,d\mathbf{x} = 0\right\}.$$

Then our mixed finite element approximation is to find a pair $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \widehat{RT}_k^d \times P_k^d$ such that

$$\begin{cases} (\mathcal{A}\,\boldsymbol{\sigma}_h,\, \boldsymbol{\tau}) + (\nabla\cdot\boldsymbol{\tau}, \mathbf{u}_h) = g(\boldsymbol{\tau}) & \forall\ \boldsymbol{\tau} \in \widehat{RT}_k^d,\\[2mm] (\nabla\cdot\, \boldsymbol{\sigma}_h,\, \mathbf{v}) - \dfrac{1}{\nu}\,(\mathbf{u}_h\cdot\mathcal{A}\,\boldsymbol{\sigma}_h,\, \mathbf{v}) = f(\mathbf{v}) & \forall\ \mathbf{v} \in P_k^d. \end{cases} \tag{3.1}$$

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Let $\mathcal{P}_h : L^2(\Omega) \mapsto P_k$ be the standard L^2 projection and let $\Pi_h : H(\text{div}; \Omega) \cap L^t(\Omega)^d \mapsto RT_k$ for fixed $t > 2$ be the well-known RT projection operator which satisfies the commutativity property

$$\nabla \cdot (\mathit{\Pi}_h \, \mathbf{v}) = \mathcal{P}_h \, \nabla \cdot \mathbf{v} \qquad \forall \;\; \mathbf{v} \in H(\text{div}; \Omega) \cap L^t(\Omega)^d.$$

The following approximation properties hold [21] for $2 \le r \le \infty$:

$$\begin{aligned} \|q - \mathcal{P}_h q\|_{0,r} &\le C h^s \|q\|_{s,r}, && 0 \le s \le k+1, \\ \|\mathbf{v} - \mathit{\Pi}_h \mathbf{v}\|_{0,r} &\le C h^s \|\mathbf{v}\|_{s,r}, && \frac{1}{r} \le s \le k+1, \\ \|\nabla \cdot (\mathbf{v} - \mathit{\Pi}_h \mathbf{v})\|_{0,2} &\le C h^s \|\nabla \cdot \mathbf{v}\|_{s,2}, && 0 \le s \le k+1. \end{aligned}$$

We define an interpolation operator $\boldsymbol{\Pi}_h \, : \, \widehat{H}(\text{div}; \Omega)^d \cap L^t(\Omega)^{d\times d} \to \widehat{RT_k^d}$ by

$$\boldsymbol{\Pi}_h \boldsymbol{\tau} = (\mathit{\Pi}_h \boldsymbol{\tau}_1, \ldots, \mathit{\Pi}_h \boldsymbol{\tau}_d) - b\,\mathbf{I} \quad \text{with} \quad b = \frac{1}{d\,|\Omega|} \int_\Omega \text{tr}\,(\mathit{\Pi}_h \boldsymbol{\tau}_1, \ldots, \mathit{\Pi}_h \boldsymbol{\tau}_d)\, d\mathbf{x},$$

where $|\Omega|$ is the area or volume of Ω, and the L^2 projection operator into P_k^d by

$$\mathbf{P}_h \mathbf{v} = (\mathcal{P}_h v_1, \ldots, \mathcal{P}_h v_d).$$

It is then easy to check the validity of the commutativity property

$$\nabla \cdot (\boldsymbol{\Pi}_h \, \boldsymbol{\tau}) = \mathbf{P}_h \, \nabla \cdot \boldsymbol{\tau} \qquad \forall \;\; \boldsymbol{\tau} \in H(\text{div}; \Omega)^d \cap L^t(\Omega)^{d\times d} \tag{3.2}$$

and the approximation properties

$$\|\mathbf{v} - \mathbf{P}_h \mathbf{v}\|_{0,r} \le C h^s \|\mathbf{v}\|_{s,r}, \qquad 0 \le s \le k+1, \tag{3.3}$$

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{0,r} \le C h^s \|\boldsymbol{\tau}\|_{s,r}, \qquad \frac{1}{r} \le s \le k+1, \tag{3.4}$$

$$\|\nabla \cdot (\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau})\|_{0,2} \le C h^s \|\nabla \cdot \boldsymbol{\tau}\|_{s,2}, \qquad 0 \le s \le k+1. \tag{3.5}$$

4. Approximation of branches of nonsingular solutions. To analyze the convergence of the finite element approximation, we make use of the abstract approximation theory of Brezzi, Rappaz, and Raviart [7, 17]. This section cites the version of the abstract theory given in [17].

Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces, and denote by $\mathcal{L}(\mathbf{Y}, \mathbf{X})$ the set of all linear and continuous operators from $\mathbf{Y}$ into $\mathbf{X}$. Let $\Lambda \subset \mathbb{R}$ be a compact interval and assume that $G : \Lambda \times \mathbf{X} \mapsto \mathbf{Y}$ is a $\mathscr{C}^2$ map. Let $T \in \mathcal{L}(\mathbf{Y}, \mathbf{X})$ be a linear operator independent of ν. Consider the following nonlinear problem: find $(\nu, \phi(\nu)) \in \Lambda \times \mathbf{X}$ such that

$$F(\nu, \phi) \equiv \phi + TG(\nu, \phi) = 0. \tag{4.1}$$

The set $\{(\nu, \phi(\nu)) |\, \nu \in \Lambda\}$ is called a branch of solutions of (4.1) if $F(\nu, \phi(\nu)) = 0$ for $\nu \in \Lambda$ and the map $\nu \mapsto \phi(\nu)$ is a continuous function from Λ into $\mathbf{X}$. If, in addition, the Fréchet derivative $D_\phi F(\nu, \phi(\nu))$ of F with respect to ϕ is an isomorphism from $\mathbf{X}$ onto $\mathbf{X}$ for all $\nu \in \Lambda$, then the branch of solutions $\{(\nu, \phi(\nu)) |\, \nu \in \Lambda\}$ is called *nonsingular*.

Let $\mathbf{X}_h \subset \mathbf{X}$ be a finite-dimensional subspace and let $T_h \in \mathcal{L}(\mathbf{Y}, \mathbf{X}_h)$ be a linear operator independent of ν which presumably approximates the linear operator T. Approximations of (4.1) are to find $(\nu, \phi_h(\nu)) \in \Lambda \times \mathbf{X}_h$ such that

$$F_h(\nu, \phi_h) \equiv \phi_h + T_h G(\nu, \phi_h) = 0. \tag{4.2}$$

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The well-posedness and approximation property of (4.2) are stated below (see, e.g., Theorem 3.3 and Remark 3.4 in Chapter IV of [17]).

THEOREM 4.1. *Assume that the following hypotheses hold:*

(1) *G is a $\mathscr{C}^2$-mapping from $\Lambda \times \mathbf{X}$ into $\mathbf{Y}$, and the second-order Fréchet derivatives of G are bounded on all bounded subsets of $\Lambda \times \mathbf{X}$.*

(2) *There exists a Banach space $\mathbf{Z}$ continuously imbedded in $\mathbf{Y}$ such that for all $\nu \in \Lambda$ and all $\phi \in \mathbf{X}$,*

$$D_\phi G(\nu, \phi) \in \mathcal{L}(\mathbf{X}, \mathbf{Z}). \tag{4.3}$$

(3) *For all $\mathbf{g} \in \mathbf{Y}$,*

$$\lim_{h\to 0} \|(T - T_h)\mathbf{g}\|_{\mathbf{X}} = 0 \tag{4.4}$$

and

$$\lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(\mathbf{Z},\mathbf{X})} = 0. \tag{4.5}$$

(4) *$\{(\nu, \phi(\nu)) |\, \nu \in \Lambda\}$ is a branch of nonsingular solutions of* (4.1).

Then for sufficiently small h, there exist a neighborhood $\mathcal{O}$ of the origin in $\mathbf{X}$ and a unique $\mathscr{C}^2$ function $\nu \in \Lambda \to \phi_h \in \mathbf{X}_h$ such that $\{(\nu, \phi_h(\nu)) |\, \nu \in \Lambda\}$ is a branch of nonsingular solutions of (4.2) *and that $\phi_h(\nu) - \phi(\nu) \in \mathcal{O}$ for all $\nu \in \Lambda$.*

Moreover, there exists a constant $C > 0$, independent of h and ν, such that

$$\|\phi(\nu) - \phi_h(\nu)\|_{\mathbf{X}} \le C\, \|TG(\nu, \phi(\nu)) - T_h G(\nu, \phi(\nu))\|_{\mathbf{X}} \qquad \forall\, \nu \in \Lambda. \tag{4.6}$$

5. Convergence analysis. To apply the abstract theory in the previous section to the pseudostress-velocity formulation, similar to the velocity-pressure formulation, it is natural to choose

$$G(\nu, \phi) = \left(-\mathbf{g},\ \frac{1}{\nu}\,\mathbf{u} \cdot \mathcal{A}\,\boldsymbol{\sigma} - \mathbf{f}\right) \tag{5.1}$$

for $\phi = (\boldsymbol{\sigma}, \mathbf{u})$, which contains the nonlinear term and the given data. Since the momentum equation is required to be valid in $L^2(\Omega)^d$ (see (2.7)), we set

$$\mathbf{Y} = H^{\frac{3}{2}}(\partial\Omega)^d \times L^2(\Omega)^d.$$

Here we assumed $\mathbf{g} \in H^{\frac{3}{2}}(\partial\Omega)^d$ to guarantee that the exact solution $(\boldsymbol{\sigma}, \mathbf{u})$ is in $H^2(\Omega) \times H^1(\Omega)$.

A simple calculation gives that the Fréchet derivative of the operator G with respect to ϕ is

$$D_\phi G(\nu, \phi)[\psi] = \left(\mathbf{0},\ \frac{1}{\nu}\,(\mathbf{u} \cdot \mathcal{A}\,\boldsymbol{\tau} + \mathbf{v} \cdot \mathcal{A}\,\boldsymbol{\sigma})\right) \tag{5.2}$$

for $\psi = (\boldsymbol{\tau}, \mathbf{v})$. We consider $\mathbf{Z} = \mathbf{Y}$; then condition (4.5) implies condition (4.4). To guarantee that $D_\phi G(\nu, \phi)[\psi] \in \mathbf{Z}$ for all ψ, we let

$$\mathbf{X} = L^3(\Omega)^{d\times d} \times L^3(\Omega)^d.$$

The Banach space $\mathbf{X}$ is equipped with the norm

$$\|(\boldsymbol{\tau}, \mathbf{v})\|_{\mathbf{X}} = \left(\|\boldsymbol{\tau}\|_{0,3}^2 + \|\mathbf{v}\|_{0,3}^2\right)^{\frac{1}{2}}.$$

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This choice of $\mathbf{X}$ yields $D_\phi G(\nu, \phi)[\psi]$ defined in (5.2) belonging to $\mathbf{Z}$ for any $\psi \in \mathbf{X}$ if $\phi = (\boldsymbol{\sigma}, \mathbf{u})$ is the exact solution of (2.7).

In the remainder of this section, we define the Stokes solution operators and estimate the approximation error in the $\|\cdot\|_{\mathbf{X}}$ norm needed in the abstract theory in subsection 5.1. Subsection 5.2 proves that a unique solution of (2.7) is also a branch of the nonsingular solution of (4.1). Finally, error estimates for the Navier–Stokes equation are established in subsection 5.3.

5.1. The Stokes solution operators. For any $(\mathbf{g}_*, \mathbf{f}_*) \in \mathbf{Y}$, define $T : \mathbf{Y} \to \mathbf{X}$ by

$$T(\mathbf{g}_*, \mathbf{f}_*) = (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X}, \tag{5.3}$$

where $(\boldsymbol{\sigma}, \mathbf{u})$ is the solution of the Stokes equations in the pseudostress-velocity formulation

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma},\, \boldsymbol{\tau}) + (\mathbf{u},\, \nabla\cdot\boldsymbol{\tau}) = g_*(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \widehat{H}(\mathrm{div};\Omega)^d, \\ (\nabla\cdot\boldsymbol{\sigma},\, \mathbf{v}) = f_*(\mathbf{v}) & \forall\, \mathbf{v} \in L^2(\Omega)^d, \end{cases} \tag{5.4}$$

where the linear forms are defined by

$$g_*(\boldsymbol{\tau}) = \int_{\partial\Omega} (\mathbf{n}\cdot\boldsymbol{\tau})\cdot\mathbf{g}_*\,\mathrm{d}s \quad \text{and} \quad f_*(\mathbf{v}) = -\int_\Omega \mathbf{f}_*\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}.$$

LEMMA 5.1. *For any* $(\mathbf{g}_*, \mathbf{f}_*) \in \mathbf{Y}$, *problem* (5.4) *has a unique solution* $(\boldsymbol{\sigma}, \mathbf{u}) = T(\mathbf{g}_*, \mathbf{f}_*)$ *which is in* $H^1(\Omega)^{d\times d} \times H^2(\Omega)^d \subset \mathbf{X}$.

Proof. It is shown in [10] that problem (5.4) has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}) \in \widehat{H}(\mathrm{div};\,\Omega)^d \times L^2(\Omega)^d$. An argument similar to that of Lemma 2.4 yields that $(\mathbf{u}, p) \equiv (\mathbf{u},\, -\frac{1}{d}\mathrm{tr}\boldsymbol{\sigma}) \in H^1_{\mathbf{g}_*}(\Omega)^d \times L^2_0(\Omega)$ satisfies the corresponding Stokes equation and that

$$\boldsymbol{\sigma} = -p\,\boldsymbol{\delta} + \nabla\,\mathbf{u}.$$

Now, the H^2 full regularity, $(\mathbf{u}, p) \in H^2(\Omega)^d \times H^1(\Omega)$ (see, e.g., [17]), of the stationary Stokes equation implies that $(\boldsymbol{\sigma}, \mathbf{u}) \in H^1(\Omega)^{d\times d} \times H^2(\Omega)^d \subset \mathbf{X}$. ∎

Denote by

$$\mathbf{X}_h = \widehat{RT}_k^d \times P_k^d \subset \mathbf{X}$$

the finite element space, and define $T_h : \mathbf{Y} \to \mathbf{X}_h$ by

$$T_h(\mathbf{g}_*, \mathbf{f}_*) = (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{X}_h, \tag{5.5}$$

where $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ is the solution of the discrete counterpart of (5.4):

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma}_h,\, \boldsymbol{\tau}) + (\mathbf{u}_h,\, \nabla\cdot\boldsymbol{\tau}) = g_*(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \widehat{RT}_k^d, \\ (\nabla\cdot\boldsymbol{\sigma}_h,\, \mathbf{v}) = f_*(\mathbf{v}) & \forall\, \mathbf{v} \in P_k^d. \end{cases} \tag{5.6}$$

See [10] for the well-posedness, the a priori estimate, and the standard $H(\mathrm{div}) \times L^2$ error estimates. The next theorem shows that the operators T and T_h satisfy condition (4.5), and estimates the error bound in the $\|\cdot\|_{\mathbf{X}}$ norm.

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THEOREM 5.2. *For any* $(\mathbf{g}_*, \mathbf{f}_*) \in \mathbf{Y}$*, let* $(\boldsymbol{\sigma}, \mathbf{u}) = T(\mathbf{g}_*, \mathbf{f}_*)$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) = T_h(\mathbf{g}_*, \mathbf{f}_*)$ *be the solutions of* (5.4) *and* (5.6)*, respectively. Then*

$$\lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(\mathbf{Z},\mathbf{X})} = 0. \tag{5.7}$$

Moreover, assume that $(\boldsymbol{\sigma}, \mathbf{u}) \in H^r(\Omega)^{d\times d} \times H^r(\Omega)^d$ *for* $1 \le r \le k+1$*; then there exists a positive constant* C *independent of* h *such that*

$$\|T(\mathbf{g}_*, \mathbf{f}_*) - T_h(\mathbf{g}_*, \mathbf{f}_*)\|_{\mathbf{X}} \le C\, h^{r-\alpha} \left(\|\boldsymbol{\sigma}\|_{r,2} + \|\mathbf{u}\|_{r,2} \right), \tag{5.8}$$

where $\alpha = d/6$.

Proof. For any $(\mathbf{g}_*, \mathbf{f}_*) \in \mathbf{Y}$, we have $(\boldsymbol{\sigma}, \mathbf{u}) \in H^1(\Omega)^{d\times d} \times H^2(\Omega)^d$ by Lemma 5.1. Let $E_\sigma = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and $E_u = \mathbf{u} - \mathbf{u}_h$; then

$$E_\sigma = E_{\Pi\sigma} + E_\sigma^h \qquad \text{and} \qquad E_u = E_{Pu} + E_u^h, \tag{5.9}$$

with $E_{\Pi\sigma} = \boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}$, $E_\sigma^h = \boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$, $E_{Pu} = \mathbf{u} - \mathbf{P}_h\mathbf{u}$, and $E_u^h = \mathbf{P}_h\mathbf{u} - \mathbf{u}_h$, where the interpolation operator $\boldsymbol{\Pi}_h$ and the $L^2(\Omega)$ projection operator $\mathbf{P}_h$ are defined in section 3. To estimate the error in the $\|\cdot\|_{\mathbf{X}}$ norm, by (5.9) and the triangle inequality we have

$$\begin{aligned} \|T(\mathbf{g}_*, \mathbf{f}_*) - T_h(\mathbf{g}_*, \mathbf{f}_*)\|_{\mathbf{X}} &\le \|E_\sigma\|_{0,3} + \|E_u\|_{0,3} \\ &\le \left(\left\|E_\sigma^h\right\|_{0,3} + \left\|E_u^h\right\|_{0,3} \right) + \left(\|E_{\Pi\sigma}\|_{0,3} + \|E_{Pu}\|_{0,3} \right). \end{aligned} \tag{5.10}$$

The difference of the first equations of systems (5.4) and (5.6) gives

$$(\mathcal{A}E_\sigma, \boldsymbol{\tau}) + (E_u, \nabla\cdot\boldsymbol{\tau}) = 0 \qquad \forall\, \boldsymbol{\tau} \in \widehat{RT}_k^d.$$

Since $(E_{Pu}, \nabla\cdot\boldsymbol{\tau}) = 0$ for all $\boldsymbol{\tau} \in \widehat{RT}_k^d$, then by the above equation, (5.9), and $\nabla\cdot E_\sigma^h = 0$, (E_σ^h, E_u^h) satisfies the following discrete system:

$$\begin{cases} \left(\mathcal{A}E_\sigma^h, \boldsymbol{\tau}\right) + (E_u^h, \nabla\cdot\boldsymbol{\tau}) = (-\mathcal{A}E_{\Pi\sigma}, \boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \widehat{RT}_k^d, \\ \left(\nabla\cdot E_\sigma^h, \mathbf{v}\right) = 0 & \forall\, \mathbf{v} \in P_k^d. \end{cases} \tag{5.11}$$

Using the a priori estimate of Lemma 3.4 in [10] with $\varepsilon = 0$ and $f' = 0$, we have

$$\left\|E_\sigma^h\right\|_{H(\mathrm{div})} + \left\|E_u^h\right\|_{0,2} \le C\, \|\mathcal{A}E_{\Pi\sigma}\|_{0,2} \le C\, \|E_{\Pi\sigma}\|_{0,2}\,.$$

This, along with the inverse inequality (see Lemma 4.5.3 in [4]), yields

$$\left\|E_\sigma^h\right\|_{0,3} + \left\|E_u^h\right\|_{0,3} \le Ch^{-\alpha} \left(\left\|E_\sigma^h\right\|_{0,2} + \left\|E_u^h\right\|_{0,2} \right) \le Ch^{-\alpha}\, \|E_{\Pi\sigma}\|_{0,2}\,. \tag{5.12}$$

From the Sobolev imbedding theorem (see, e.g., Theorem 7.58 in [1]),

$$H^r(\Omega)^d \hookrightarrow W^{r-\alpha,3}(\Omega)^d \qquad \text{and} \qquad H^r(\Omega)^{d\times d} \hookrightarrow W^{r-\alpha,3}(\Omega)^{d\times d}.$$

It then follows from approximation properties (3.3) and (3.4) and the imbedding relations with $r = 1$ that

$$\|E_{\Pi\sigma}\|_{0,2} \le C\, h\, \|\boldsymbol{\sigma}\|_{1,2} \tag{5.13}$$

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and that

$$\|E_{\Pi\sigma}\|_{0,3} + \|E_{Pu}\|_{0,3} \le C\, h^{1-\alpha} \left(\|\boldsymbol{\sigma}\|_{1-\alpha,3} + \|\mathbf{u}\|_{1-\alpha,3} \right)$$
$$\le C\, h^{1-\alpha} \left(\|\boldsymbol{\sigma}\|_{1,2} + \|\mathbf{u}\|_{1,2} \right). \tag{5.14}$$

Now, (5.10), (5.12), (5.13), and (5.14) yield

$$\|T(\mathbf{g}_*, \mathbf{f}_*) - T_h(\mathbf{g}_*, \mathbf{f}_*)\|_{\mathbf{X}} \le C\, h^{1-\alpha} \left(\|\boldsymbol{\sigma}\|_{1,2} + \|\mathbf{u}\|_{1,2} \right), \tag{5.15}$$

which proves the validity of (5.7).

If $(\boldsymbol{\sigma}, \mathbf{u}) \in H^r(\Omega)^{d\times d} \times H^r(\Omega)^d$, then a proof similar to the above shows

$$\|T(\mathbf{g}_*, \mathbf{f}_*) - T_h(\mathbf{g}_*, \mathbf{f}_*)\|_{\mathbf{X}} \le C\, h^{r-\alpha} \left(\|\boldsymbol{\sigma}\|_{r,2} + \|\mathbf{u}\|_{r,2} \right).$$

This completes the proof of the theorem. ∎

5.2. A branch of nonsingular solutions. Let $\Lambda \subset (0, \infty)$ be a compact interval. Given $(\mathbf{g}, \mathbf{f}) \in \mathbf{Y}$, for any $(\nu, \phi) = (\nu, (\boldsymbol{\sigma}, \mathbf{u})) \in \Lambda \times \mathbf{X}$, define $F : \Lambda \times \mathbf{X} \to \mathbf{X}$ as in (4.1) by

$$F(\nu, \phi) = \phi + TG(\nu, \phi).$$

It is easy to see, by (5.2), that the Fréchet derivative of the operator F with respect to ϕ is

$$D_\phi F(\nu, \phi)[\psi] = \psi + T D_\phi G(\nu, \phi)[\psi] = \psi + T \left(\mathbf{0},\ \frac{1}{\nu} (\mathbf{u} \cdot \mathcal{A}\,\boldsymbol{\tau} + \mathbf{v} \cdot \mathcal{A}\,\boldsymbol{\sigma}) \right) \tag{5.16}$$

for any $\psi = (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}$.

LEMMA 5.3. *Let* $(\mathbf{g}, \mathbf{f}) \in \mathbf{Y}$. *Then* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X}$ *is a solution of* (2.7) *if and only if* $(\nu, \phi) = (\nu, (\boldsymbol{\sigma}, \mathbf{u}))$ *is a solution of* (4.1) *with the operators* T *and* G *defined in the respective equations* (5.3) *and* (5.1).

Proof. Rewrite problem (2.7) as follows:

$$\begin{cases} (\mathcal{A}\,\boldsymbol{\sigma},\, \boldsymbol{\tau}) + (\nabla \cdot \boldsymbol{\tau}, \mathbf{u}) = g(\boldsymbol{\tau}) & \forall\ \boldsymbol{\tau} \in H(\mathrm{div};\, \Omega)^d, \\ (\nabla \cdot \boldsymbol{\sigma},\, \mathbf{v}) = \left(\dfrac{1}{\nu}\, \mathbf{u} \cdot \mathcal{A}\,\boldsymbol{\sigma} - \mathbf{f}, \mathbf{v} \right) & \forall\ \mathbf{v} \in L^2(\Omega)^d. \end{cases}$$

By the definitions of the operators of T in the previous section and G in (5.1), the problem above is equivalent to

$$\phi = (\boldsymbol{\sigma},\, \mathbf{u}) = -T \left(-\mathbf{g},\ \frac{1}{\nu} \mathbf{u} \cdot \mathcal{A}\,\boldsymbol{\sigma} - \mathbf{f} \right) = -TG(\nu, \phi).$$

That is, $F(\nu, \phi) = 0$. ∎

Remark 5.4. Similarly, $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ is a solution of (3.1) if and only if $(\nu, \phi_h) = (\nu, (\boldsymbol{\sigma}_h,\, \mathbf{u}_h))$ is a solution of (4.2) with the operator T_h defined in (5.5).

THEOREM 5.5. *For* $(\mathbf{g}, \mathbf{f}) \in \mathbf{Y}$, *assume that* $\nu > \nu_0(\Omega; \mathbf{f}, \mathbf{g})$. *Then* (4.1), *with the operators* T *and* G *defined in the respective equations* (5.3) *and* (5.1), *has a branch of nonsingular solutions.*

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Proof. Let $(\boldsymbol{\sigma}, \mathbf{u})$ be the unique solution of (2.7). By Lemma 5.3, $(\nu, \phi) = (\nu, (\boldsymbol{\sigma}, \mathbf{u}))$ is a branch of solutions of (4.1). To prove that (ν, ϕ) is nonsingular, i.e., $D_\phi F(\nu, \phi)$ is an isomorphism from $\mathbf{X}$ onto $\mathbf{X}$, it suffices to show that for every $\omega = (\boldsymbol{\zeta}, \mathbf{w}) \in \mathbf{X}$, there exists a unique $\psi' = (\boldsymbol{\tau}', \mathbf{v}') \in \mathbf{X}$ such that $D_\phi F(\nu, \phi)[\psi'] = \omega$. By (5.16), this becomes

$$\omega - \psi' = T\left(\mathbf{0},\, \frac{1}{\nu}\left(\mathbf{u} \cdot \mathcal{A}\,\boldsymbol{\tau}' + \mathbf{v}' \cdot \mathcal{A}\,\boldsymbol{\sigma}\right)\right).$$

By the definition of T, this is equivalent to finding a unique $\psi = (\boldsymbol{\tau}, \mathbf{v}) = (\boldsymbol{\zeta} - \boldsymbol{\tau}', \mathbf{w} - \mathbf{v}') \in \mathbf{X}$ such that

$$\text{(5.17)} \quad \begin{cases} (\mathcal{A}\,\boldsymbol{\tau}, \boldsymbol{\xi}) + (\nabla \cdot \boldsymbol{\xi}, \mathbf{v}) = 0 & \forall\, \boldsymbol{\xi} \in \widehat{H}(\mathrm{div};\, \Omega)^d, \\ (\nabla \cdot\, \boldsymbol{\tau}, \mathbf{z}) - \dfrac{1}{\nu}\,(\mathbf{u} \cdot \mathcal{A}\,\boldsymbol{\tau} + \mathbf{v} \cdot \mathcal{A}\,\boldsymbol{\sigma}, \mathbf{z}) = -\dfrac{1}{\nu}\,(\mathbf{f}_*, \mathbf{z}) & \forall\, \mathbf{z} \in L^2(\Omega)^d, \end{cases}$$

where $\mathbf{f}_* = \mathbf{u} \cdot \mathcal{A}\,\boldsymbol{\zeta} + \mathbf{w} \cdot \mathcal{A}\,\boldsymbol{\sigma}$. It is easy to check that the corresponding velocity-pressure formulation of (5.17) is to find $(\mathbf{v}, q) = \left(\mathbf{v}, -\frac{1}{d}\,\mathrm{tr}\boldsymbol{\tau}\right) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$\text{(5.18)} \quad \begin{cases} (\nabla\mathbf{v}, \nabla\mathbf{z}) - (q, \nabla \cdot \mathbf{z}) + \dfrac{1}{\nu}(\mathbf{u} \cdot \nabla\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{u}, \mathbf{z}) = \dfrac{1}{\nu}(\mathbf{f}_*, \mathbf{z}) & \forall\, \mathbf{z} \in H_0^1(\Omega)^d, \\ (\nabla \cdot \mathbf{v}, r) = 0 & \forall\, r \in L_0^2(\Omega). \end{cases}$$

Under the assumption that $\nu > \nu_0(\Omega; \mathbf{f}, \mathbf{g})$, it is proved in Chapter IV of [17] that (5.18) has a unique solution $(\mathbf{v}, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$. Since $\mathbf{u} \in H^2(\Omega)^d$, $\mathbf{f}_* - \mathbf{u} \cdot \nabla\mathbf{v} - \mathbf{v} \cdot \nabla\mathbf{u}$ belongs to $L^2(\Omega)^d$. By the H^2 regularity of the Stokes equation, we have that $(\mathbf{v}, q)$ is in $H^2(\Omega)^d \times H^1(\Omega)$. Now, an argument similar to that of Lemma 2.5 shows that (5.17) has a unique solution and, hence, (ν, ϕ) is a branch of nonsingular solutions. ∎

5.3. Error estimates.

THEOREM 5.6. *Let* $(\mathbf{f}, \mathbf{g}) \in \mathbf{Y}$. *Assume that* $\nu > \nu_0(\Omega; \mathbf{f}, \mathbf{g})$ *and that* $(\nu, \phi(\nu)) = (\nu, (\boldsymbol{\sigma}(\nu), \mathbf{u}(\nu)))$ *is a branch of nonsingular solutions of* (4.1). *Then for h sufficiently small, there exist a neighborhood* $\mathcal{O}$ *of the origin in* $\mathbf{X}$ *and a unique* $\mathscr{C}^2$ *function* $\nu \to \phi_h \in \mathbf{X}_h$ *such that* $\{(\nu, \phi_h(\nu)) | \nu \in \Lambda\}$ *is a branch of nonsingular solutions of* (4.2) *and that* $\phi(\nu) - \phi_h(\nu) \in \mathcal{O}$ *for all* $\nu \in \Lambda$.

Moreover, assume that $(\boldsymbol{\sigma}(\nu), \mathbf{u}(\nu)) \in H^r(\Omega)^{d\times d} \times H^r(\Omega)^d$ *for* $1 \le r \le k+1$. *Then there exists a constant* $C > 0$, *independent of* h, *such that*

$$\text{(5.19)} \quad \|\boldsymbol{\sigma}(\nu) - \boldsymbol{\sigma}_h(\nu)\|_{0,3} + \|\mathbf{u}(\nu) - \mathbf{u}_h(\nu)\|_{0,3} \le C h^{r-\alpha}\left(\|\boldsymbol{\sigma}(\nu)\|_{r,2} + \|\mathbf{u}(\nu)\|_{r,2}\right)$$

for $\alpha = d/6$ *and any* $\nu \in \Lambda$.

Proof. To show the validity of the first part of the theorem, we simply verify the assumptions in Theorem 4.1. First, it is easy to check that the operator G satisfies the hypothesis (2). The hypotheses (1) and (4) are proved in Theorems 5.5 and 5.2, respectively. By the definition of the space $\mathbf{Z} = \mathbf{Y}$, now it suffices to show

$$\text{(5.20)} \quad D_\phi G(\nu, \phi) \in \mathcal{L}(\mathbf{X}, \mathbf{Z}).$$

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To this end, for $(\mathbf{f}, \mathbf{g}) \in \mathbf{Y}$, by Theorem 2.6, the unique solution of (2.7), $\phi = (\boldsymbol{\sigma}, \mathbf{u})$ is in $H^1(\Omega)^{d\times d} \times H^2(\Omega)^d$, which is continuously imbedded in $L^6(\Omega)^{d\times d} \times L^6(\Omega)^d$. By the triangle and Hölder inequalities, for any $\psi = (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}$, we have

$$\begin{aligned}\|\mathbf{u}\cdot\mathcal{A}\boldsymbol{\tau} + \mathbf{v}\cdot\mathcal{A}\boldsymbol{\sigma}\|_{0,2} &\le \|\mathbf{u}\cdot\mathcal{A}\boldsymbol{\tau}\|_{0,2} + \|\mathbf{v}\cdot\mathcal{A}\boldsymbol{\sigma}\|_{0,2}\\ &\le \|\mathbf{u}\|_{0,6}\|\boldsymbol{\tau}\|_{0,3} + \|\mathbf{v}\|_{0,3}\|\boldsymbol{\sigma}\|_{0,6}\\ &\le \|\mathbf{u}\|_{1,2}\|\boldsymbol{\tau}\|_{0,3} + \|\mathbf{v}\|_{0,3}\|\boldsymbol{\sigma}\|_{1,2}.\end{aligned}$$

This proves (5.20) and, hence, the well-posedness of the discrete problem in (4.2) by Theorem 4.1.

To establish the error bound in (5.19), assume that $(\boldsymbol{\sigma}(\nu), \mathbf{u}(\nu)) \in H^r(\Omega)^{d\times d} \times H^r(\Omega)^d$ with $1 \le r \le k+1$. Thus the error estimate in (5.19) follows Theorem 5.2. This completes the proof of the theorem. ∎

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