APMA0340 Midterm

Name:

Question 1. (10%) Write the equation

$$\frac{d^4x}{dt^4} + 3\frac{d^3x}{dt^3} + 5\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 2x + g(t) = 0$$

as a system of first order equations

$$\begin{cases} u = x' \\ v = u' \\ w = v' \\ w' + 3w + 5v + 6u + 2x + g(x) = 0 \end{cases}$$

$$\begin{bmatrix} x \\ u \\ v \\ w \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -6 & -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ u \\ v \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g(x) \end{bmatrix}$$

Question 2. (30%) Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2p & -q \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2a) (10%) Under which condition on p and q does the system have a unique equilibrium point and what it is.

$$\det\begin{bmatrix} -2p & -q \\ 1 & 0 \end{bmatrix} = \boxed{q \neq 0}.$$ equilibrium point $\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

2b) (10%) Show that the eigenvalues of linear system are given as

$$\lambda_1 = -p + \sqrt{p^2 - q}, \quad \lambda_2 = -p - \sqrt{p^2 - q}$$

$$\det(A - \lambda I) = \det\begin{vmatrix} -2p-\lambda & -q \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda + 2p) + q$$
$$= \lambda^2 + 2p\lambda + q$$

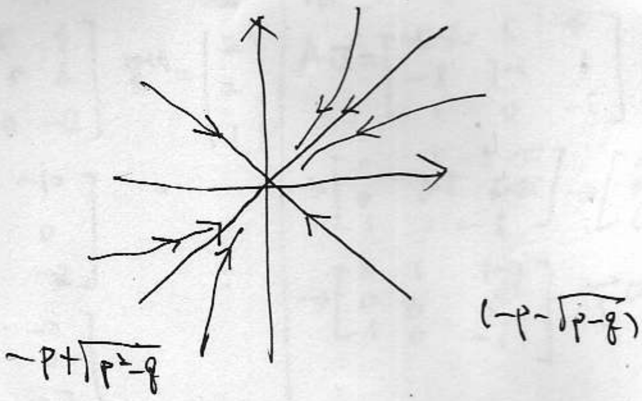
$$\lambda = \frac{-2p \pm \sqrt{4p^2 - 4q}}{2} = -p \pm \sqrt{p^2 - q}.$$

2c) (10%) sketch the phase portrait and state whether the equilibrium point is stable or not in the following cases:
$p^2 - q > 0$, $q > 0$, $p > 0$

when $p^2 - q > 0 \Rightarrow$ real

$p > 0, q > 0 \Rightarrow -p - \sqrt{p^2 - q} < 0$ elliptic point, stable.

$-p + \sqrt{p^2 - q} < 0$

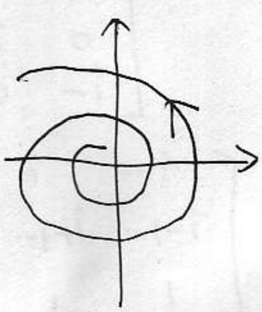


$p^2 - q < 0$, $p < 0$

$p^2 - q < 0 \Rightarrow$ complex

unstable. spiral point

$p < 0 \Rightarrow$ real part > 0



Question 3. (35%) Consider the following homogeneous linear system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

with

$$Det(A - \lambda I) = -(\lambda - 2)(\lambda^2 + 1)$$

3a) (20%) Find 3 linearly independent solutions and write the general solution using these.

$\lambda_1 = 2.$

$$A-2I = \begin{bmatrix} -7 & 5 & 4 \\ -8 & 5 & 6 \\ 1 & 0 & -2 \end{bmatrix} \quad \vec{\xi}^{(1)} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

$$\rightarrow \begin{bmatrix} 0 & 5 & -10 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$\lambda_2 = i$

$$A-iI = \begin{bmatrix} -5-i & 5 & 4 \\ -8 & 7-i & 6 \\ 1 & 0 & -i \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 5 & 5-5i \\ 0 & 7-i & 6-8i \\ 1 & 0 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1-i \\ 0 & 7-i & 6-8i \\ 1 & 0 & -i \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & 1-i \\ 0 & 0 & 0 \\ 1 & 0 & -i \end{bmatrix} \quad \vec{\xi}^{(2)} = \begin{pmatrix} i \\ -1+i \\ 1 \end{pmatrix}$$

$\lambda_3 = -i$

$$\vec{\xi}^{(3)} = \begin{pmatrix} -i \\ -1-i \\ 1 \end{pmatrix}.$$

$$\vec{x} = c_1\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} i \\ -1+i \\ 1 \end{pmatrix}e^{it} + c_3\begin{pmatrix} -i \\ -1-i \\ 1 \end{pmatrix}e^{-it}$$

3b) (5%) Can the matrix A, be diagonalized?

Yes !

3c) (10%) If yes, write down T and D such that $T^{-1}AT = D$. If no, write down T such that $T^{-1}AT$ brings A into Jordan form and write the Jordan form.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

$$T = \begin{bmatrix} 2 & i & -i \\ 2 & -1+i & -1-i \\ 1 & 1 & 1 \end{bmatrix}$$

Question 4 (25%) Find the complete solution to the problem:

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix}1&2\\2&1\end{bmatrix}\mathbf{x} + e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$$

$\det(A-\lambda I) = \det\begin{pmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{pmatrix} = (1-\lambda)^2-4 = \lambda^2-2\lambda-3 = (\lambda-3)(\lambda+1)$

$\lambda_1 = 3. \quad A-3I = \begin{pmatrix}-2&2\\2&-2\end{pmatrix} \quad \xi^{(1)} = \begin{pmatrix}1\\1\end{pmatrix}.$

$\lambda_2=-1 \quad A+I=\begin{pmatrix}2&2\\2&2\end{pmatrix} \quad \xi^{(2)}=\begin{pmatrix}1\\-1\end{pmatrix}.$

$e^{-t}\begin{bmatrix}1\\0\end{bmatrix}$

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method 1. $T=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$, $T^{-1}=\frac12\begin{bmatrix}1&1\\1&-1\end{bmatrix}$, let $\vec y = T^{-1}D$. $\frac{d\vec y}{dt} = \begin{bmatrix}3&0\\0&-1\end{bmatrix}\vec y + T^{-1}e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$

$\frac{d\vec y}{dt} = \begin{bmatrix}3&0\\0&-1\end{bmatrix}\vec y + e^{-t}\begin{bmatrix}1\\0\end{bmatrix}$, $\frac{dy_1}{dt} = 3y_1 + e^{-t}$, $\frac{dy_2}{dt} = -y_2$

$\Rightarrow y_1 = -\frac14 e^{-t}$, $y_2=0$. $\quad x^p = T\vec y = -\frac14 e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$.

$\vec x = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} - \frac14 e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$

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method 2. $x^p = \vec a e^{-t} + \vec b te^{-t}$. $\quad Ax^p + e^{-t}\begin{bmatrix}1\\1\end{bmatrix} = A\vec a e^{-t} + A\vec b te^{-t} + e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$.

$\frac{dx^p}{dt} = -\vec a e^{-t} + \vec b e^{-t} - \vec b te^{-t}$.

$\Rightarrow -\vec a+\vec b = A\vec a + \begin{pmatrix}1\\1\end{pmatrix}$. $\quad \Rightarrow \vec b = k\begin{pmatrix}1\\-1\end{pmatrix}$. $\quad (A+I)\vec a = \vec b - \begin{pmatrix}1\\1\end{pmatrix}$

$-\vec b = A\vec b$

$\Rightarrow \begin{pmatrix}2&2\\2&2\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}k-1\\-k-1\end{pmatrix} \to \begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}\frac12(k-1)\\-k\end{pmatrix}$

has a solution $\Rightarrow k=0 \Rightarrow a_1+a_2 = -\frac12$. pick $\begin{pmatrix}0\\-\frac12\end{pmatrix}$.

so a $x^p = \begin{pmatrix}0\\-\frac12\end{pmatrix}e^{-t}$. $\quad \vec x = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} + \begin{pmatrix}0\\-\frac12\end{pmatrix}e^{-t}$

note that $-\frac14\begin{pmatrix}1\\1\end{pmatrix} = -\frac14\begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}0\\-\frac12\end{pmatrix}$.
so the 2 solutions are actually the same.

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method 3. $\Psi = \begin{pmatrix}e^{3t}&e^{-t}\\e^{3t}&-e^{-t}\end{pmatrix}$. $\quad x^p = \Psi\vec u \Rightarrow \Psi\vec u' = e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$. $\begin{pmatrix}e^{3t}&e^{-t}\\e^{3t}&-e^{-t}\end{pmatrix}\begin{pmatrix}u_1'\\u_2'\end{pmatrix} = \begin{pmatrix}e^{-t}\\e^{-t}\end{pmatrix} \to \left(\begin{array}{cc|c}e^{4t}&1&1\\e^{4t}&-1&1\end{array}\right) \to \left(\begin{array}{cc|c}e^{4t}&1&1\\0&-2&0\end{array}\right)$

$\Rightarrow u_2'=0$, $u_1' = e^{-4t}$, pick $u_2=0$, $u_1 = -\frac14 e^{-4t}$. $\quad x^p = \Psi\vec u = -\frac14 e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$

$\vec x = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-t} - \frac14 e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$.