Recovery-Based a Posteriori Error Estimators for Interface Problems: Mixed and Nonconforming Elements

Zhiqiang Cai Shun Zhang

Department of Mathematics
Purdue University

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Outline

A Priori Error Estimation

- Continuous Problem

$$\mathcal{L}u = f \quad \text{in} \quad \Omega$$

- Discrete Problem

$$\mathcal{L}_h u_h = f_h$$

- a Priori Error Estimation

$$\|u - u_h\| \leq C(u)h^{\alpha} \to 0 \quad \text{as} \quad h \to 0$$

- Error Control ϵ

$$\|u - u_h\| \leq \epsilon$$

A Posteriori Error Estimations

- A Posteriori Error Estimations
  - Indicator $\eta_K(u_h)$ - a computable quantity for each $K \in \mathcal{T}$
  - Estimator $\eta(u_h) = \left(\sum_{K\in\mathcal{T}} \eta_K^2\right)^{1/2}$
- Error Control: Reliability Bound

$$\| u - u_h \| \le C_r \eta + h.o.t$$

- Adaptive Control of Meshing: Efficiency Bound

$$\eta_K \le c_e \| u - u_h \|_K + h.o.t \quad \forall\, K \in \mathcal{T}$$

$$\eta \le C_e \| u - u_h \| + h.o.t$$

A Posteriori Error Estimations

- Robustness
  C_r and C_e are independent of the parameters inherent in the differential equations, such as jumps of the diffusion coefficients, reaction/convetion parameters, e.t.c
- Effectivity Constant

$$\frac{\eta}{|||u - u_h|||}$$

  If the effectivity constant is close to 1, the estimator is accurate.

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Recovery-Based Estimators

- Recovery-based Estimators: $\sigma(u_h)$ is a quantity of mathematical or physical meaning, such as gradient, flux or stress, recover it to get $G(\sigma(u_h))$ in an *appropriate* function space with an *appropriate* norm (most time the energy norm from the differential equation)

$$\eta_G = \|G(\sigma(u_h)) - \sigma(u_h)\|$$

- Possible Good Quality of Recovery-based Estimators: Effectivity constant $\dfrac{\eta}{\|u - u_h\|}$ is close to 1

An Analysis of a Model Problem

- Diffusion Equations

$$-\nabla \cdot (A\nabla u) = f \in \Omega$$

- Smoothness of the Problem
  - Solution:

    $$u \in H^1(\Omega)$$

    Continuous (in the weak sense)
  - Gradient:

    $$\nabla u \in H(\text{curl}; \Omega)$$

    Tangential Component is Continuous
  - Flux:

    $$\sigma = -A\nabla u \in H(\text{div}; \Omega)$$

    Normal Component is Continuous

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Guidelines of Recovery-Based Error Estimators

- Recover a quantity whose continuity is *violated* by the discretization method (Conforming FEM, Mixed Methods, Nonconforming Element Methods...)
- In the corresponding conforming finite element space(without introducing unnecessary continuity),
- Measure the difference in the right norm (Energy Norm) as the indicator.

Interface Problems

$$\begin{cases} -\nabla \cdot (a(x)\nabla u) &= f \quad \text{in } \Omega \subset \mathcal{R}^d \\ u &= 0 \quad \text{on } \Gamma_D \\ a\nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_N \end{cases}$$

$a(x)$ is positive piecewise constant w.r.t $\bar{\Omega} = \cup_{i=1}^{n} \bar{\Omega}_i$,
$a(x) = a_i > 0$ in Ω_i

Low Order Mixed FEM for Interface Problems

- The corresponding mixed variational formulation is to find $(\boldsymbol{\sigma}, u) \in H_N(\mathrm{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} (a^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\nabla\cdot\boldsymbol{\tau}, u) &= 0 & \forall\, \boldsymbol{\tau} \in H_N(\mathrm{div};\Omega), \\ (\nabla\cdot\boldsymbol{\sigma}, v) &= (f, v) & \forall\, v \in L^2(\Omega). \end{cases}$$

- Discrete Problem: The mixed finite element method is to find $(\boldsymbol{\sigma}_m, u_m) \in RT_0 \times P_0$ such that

$$\begin{cases} (a^{-1}\boldsymbol{\sigma}_m, \boldsymbol{\tau}) - (\nabla\cdot\boldsymbol{\tau}, u_m) &= 0 & \forall\, \boldsymbol{\tau} \in RT_0, \\ (\nabla\cdot\boldsymbol{\sigma}_m, v) &= (f, v) & \forall\, v \in P_0. \end{cases}$$

Mixed FEM for Interface Problems

- Comparison of Continuous and Discrete Solutions:

Solution	$u \in H^1_D(\Omega)$	$u_h \in P_0 \not\subset H^1_D(\Omega)$
Gradient	$\nabla u \in H(\mathrm{curl};\Omega)$	$-a^{-1}\sigma_m \not\subset H(\mathrm{curl};\Omega)$
Flux	$\sigma = -a\nabla u \in H(\mathrm{div};\Omega)$	$\sigma_m \in RT_0 \subset H(\mathrm{div};\Omega)$

- Quantity to recover: the Gradient (from $-a^{-1}\boldsymbol{\sigma}_m$).
- In what space? $H(curl;\Omega)$-conforming element spaces $\mathbb{ND}$ (Nedlec edge element spaces of type 1 and 2)
- What if recover in global continuous space S_1^2? Will introduce unnecessary over refinements!

$$S_1 = \{v : v \in C^0(\Omega), v|_K \in P_1(K), \forall\, K \in \mathcal{T}\}$$

Robust Gradient Recovery Error Estimators for Interface Problems: Mixed FEs

- L^2-Projection Gradient Recovery: Find $\boldsymbol{\rho}_m \in \mathbb{ND}_2$ such that

$$(a\,\boldsymbol{\rho}_m, \boldsymbol{\tau}) = -(\boldsymbol{\sigma}_m, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{ND}_2.$$

- Explicit Recovery: See Cai & Zhang 08 for details.
- Error Estimator:

$$\eta_{m,K} = \|a^{1/2}\boldsymbol{\rho}_m + a^{-1/2}\sigma_m\|_{0,K}, \quad \eta_m = \|a^{1/2}\boldsymbol{\rho}_m + a^{-1/2}\sigma_m\|_{0,\Omega},$$

Robust Gradient Recovery Error Estimators for Interface Problems: Mixed FEs

- Robustness: C_e and C_r is independent of the jumps of the coefficients across the interfaces

$$C_e^{-1}\eta_m + h.o.t \le \|a^{1/2}\nabla u + a^{-1/2}\sigma_m\|_{0,\Omega} \le C_r\eta_m + h.o.t$$

- Accurateness: Observed in numerical tests that the effectivity index is close to 1.
- No over refinements along the interface!

A Benchmark Test Ptoblem

- **interface problem**

$$\begin{cases} -\nabla \cdot (a \nabla u) & = & f \;\text{ in } \Omega = (-1,\,1)^2 \\ u & = & g \;\text{ on } \partial\Omega \end{cases}$$

  with $a = R$ in $(0,1)^2 \cup (-1,0)^2$ and 1 in $(-1,0) \times (0,1) \cup (0,1) \times (-1,0)$

- **exact solution**
  $u(r,\theta) = r^{\alpha}\mu(\theta) \in H^{1+\alpha-\epsilon}(\Omega)$ with

$$\mu(\theta) = \begin{cases} \cos((\frac{\pi}{2} - \sigma)\alpha) \cdot \cos((\theta - \frac{\pi}{2} + \rho)\alpha) & \text{if} \quad 0 \le \theta \le \frac{\pi}{2}, \\ \cos(\rho\alpha) \cdot \cos((\theta - \pi + \sigma)\alpha) & \text{if} \quad \frac{\pi}{2} \le \theta \le \pi, \\ \cos(\sigma\alpha) \cdot \cos((\theta - \pi - \rho)\alpha) & \text{if} \quad \pi \le \theta \le \frac{3\pi}{2}, \\ \cos((\frac{\pi}{2} - \rho)\alpha) \cdot \cos((\theta - \frac{3\pi}{2} + \sigma)\alpha) & \text{if} \quad \frac{3\pi}{2} \le \theta \le 2\pi. \end{cases}$$

- **example** $\alpha = 0.1 \Rightarrow u \in H^{1.1-\epsilon}(\Omega)$
  $R \approx 161.448$, $\rho = \pi/4$, and $\sigma \approx -14.923$.

Numerical Results by Robust Gradient Recovery Error Estimators: Mixed FEs

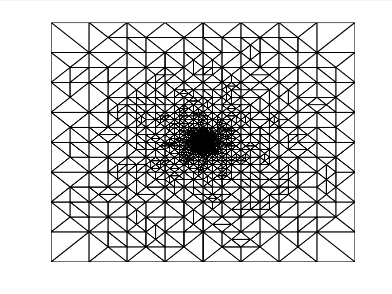

Figure: mesh generated by η_m

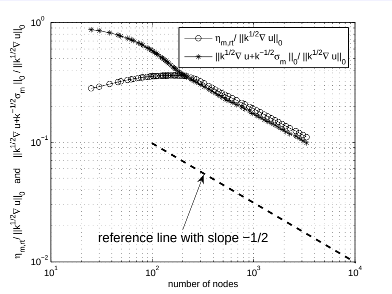

Figure: error and η_m

Gradient/Flux Recovery Error Estimators in Continuous S_1^2

Let $\boldsymbol{\sigma}_m$ be the solution, and let $\boldsymbol{\rho}_{m,f} \in S_1^2$ and $\boldsymbol{\rho}_{m,g} \in S_1^2$ satisfy the following problems

$$\begin{aligned} (a^{-1}\boldsymbol{\rho}_{m,f}, \boldsymbol{\tau}) &= (a^{-1}\boldsymbol{\sigma}_m, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in S_1^2 \\ \text{and} \quad (a\boldsymbol{\rho}_{m,g}, \boldsymbol{\tau}) &= -(\boldsymbol{\sigma}_m, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in S_1^2, \end{aligned}$$

respectively. Then the corresponding error estimators are defined by

$$\eta_{m,CB,f} = \|a^{-1/2}(\boldsymbol{\sigma}_m - \boldsymbol{\rho}_{m,f})\|_{0,\Omega}$$

$$\eta_{m,CB,g} = \|a^{-1/2}\boldsymbol{\sigma}_m + a^{1/2}\boldsymbol{\rho}_{m,g}\|_{0,\Omega}.$$

Gradient/Flux Recovery Error Estimators in Continuous S_1^2

Lots of over refinements along the interface because of recovering in a space asking too much continuity!

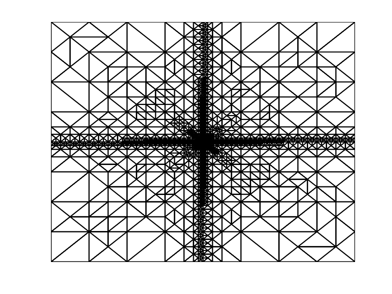

Figure: mesh by $\eta_{m,CB,f}$

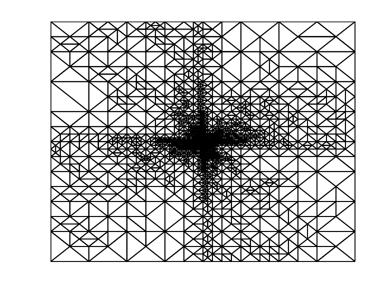

Figure: mesh by $\eta_{m,CB,g}$

Linear Nonconforming FEM for Interface Problems

- Variational Problem: To find $u \in H^1_D(\Omega)$, such that

$$(a\nabla u, \nabla v) = (f, v) \quad \forall \;\; v \in H^1_D(\Omega)$$

- Linear Nonconforming FE Space(Crouzeix-Raviart):
  $V^{nc} = \{v \in L^2(\Omega) \; : \; v|_K \in P_1(K) \forall K \in \mathcal{T}, \text{ and } v \text{ is continuous at } m_e \,\forall\, e \in \mathcal{E}_\Omega, v = 0 \text{ on} \Gamma_D\}$
- Discrete Problem: The nonconforming finite element method is to find $u_{nc} \in V^{nc}$ such that

$$(a\nabla_h u_{nc}, \nabla_h v_{nc}) = (f, v_{nc}) \quad \forall \; v_{nc} \in V^{nc}$$

Nonconforming FEM for Interface Problems

- Comparison of Continuous and Discrete Solutions:

Solution	$u \in H^1_D(\Omega)$	$u_h \in V^{nc} \not\subset H^1_D(\Omega)$
Gradient	$\nabla u \in H(\mathrm{curl};\Omega)$	$\nabla_h u_{nc} \not\subset H(\mathrm{curl};\Omega)$
Flux	$\sigma = -a\nabla u \in H(\mathrm{div};\Omega)$	$-a\nabla_h u_{nc} \not\subset H(\mathrm{div};\Omega)$

- Quantities to recover: Both the Gradient (from $\nabla_h u_{nc}$) and the Flux (from $-a\nabla_h u_{nc}$).
- In what spaces? the Gradient in $H(curl;\Omega)$-conforming element spaces $\mathbb{ND}$ and the Flux in $H(div;\Omega)$-conforming element spaces
- What if in S^2_1? Will introduce unnecessary refinements!

Robust Gradient/Gradient Recovery Error Estimators for Interface Problems: Nonconforming FEs

- L^2-Projection Gradient Recovery: Find $\boldsymbol{\rho}_{nc} \in \mathbb{ND}_1$ such that

$$(a\,\boldsymbol{\rho}_{nc},\, \boldsymbol{\tau}) = -(a\nabla_h u_{nc},\, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{ND}_1.$$

- L^2-Projection Flux Recovery: Find $\boldsymbol{\sigma}_{nc} \in RT_0$ such that

$$(a^{-1}\,\boldsymbol{\sigma}_{nc},\, \boldsymbol{\tau}) = -(\nabla_h u_{nc},\, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in RT_0.$$

- Explicit Recovery: See Cai & Zhang 08 for details.
- Error Estimator:

$$\eta_{nc}^2 = c^2\eta_{nc,1}^2 + (1-c^2)\eta_{nc,2}^2 \quad \text{for} \quad 0 < c < 1.$$

with

$$\begin{aligned}\eta_{nc,1} &= \|a^{-1/2}\boldsymbol{\sigma}_{nc} + a^{1/2}\nabla_h u_{nc}\|_{0,\Omega}\\ \eta_{nc,2} &= \|a^{1/2}(\boldsymbol{\rho}_{nc} + \nabla_h u_{nc})\|_{0,\Omega}\end{aligned}$$

Robust Gradient Recovery Error Estimators for Interface Problems: Mixed FEs

- Robustness: C_e and C_r is independent of the jumps of the coefficients across the interfaces

$$C_e^{-1}\eta_{nc} + h.o.t \le \|a^{1/2}\nabla u - a^{1/2}\nabla_h u_{nc}\|_{0,\Omega} \le C_r\eta_{nc} + h.o.t$$

- Accurateness: Observed in numerical tests that the effectivity constant is close to 1.
- No over refinements along the interface!

Numerical Results by Robust Gradient/Flux Recovery Error Estimators: Nonconforming FEs

The test problem is the same interface problem introduced before.

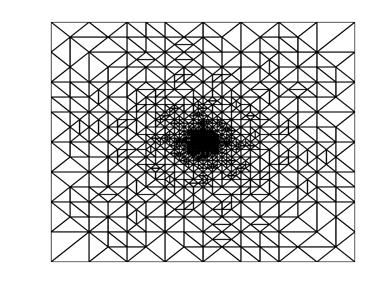

Figure: mesh generated by η_{nc}

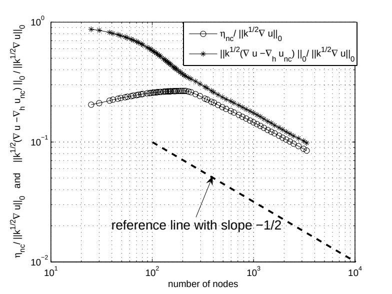

Figure: error and η_{nc}

Gradient/Flux Recovery Error Estimators in Continuous S_1^2 for Nonconforming FEs

let $\boldsymbol{\rho}_{nc,f} \in S_1^2$ and $\boldsymbol{\rho}_{nc,g} \in S_1^2$ satisfy the following problems

$$\begin{aligned} (a^{-1}\boldsymbol{\rho}_{nc,f}, \boldsymbol{\tau}) &= (-\nabla_h u_{nc}, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in S_1^2 \\ \text{and} \quad (a\boldsymbol{\rho}_{nc,g}, \boldsymbol{\tau}) &= (a\nabla_h u_{nc}, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in S_1^2. \end{aligned}$$

Then the corresponding error estimators are defined by

$$\eta_{nc,CB,f} = \|a^{1/2}\nabla_h u_{nc} + a^{-1/2}\boldsymbol{\rho}_{nc,f}\|_{0,\Omega}$$

$$\eta_{nc,CB,g} = \|a^{1/2}(\nabla_h u_{nc} - \boldsymbol{\rho}_{nc,g})\|_{0,\Omega}.$$

Gradient/Flux Recovery Error Estimators in Continuous S_1^2 for Nonconforming FEs

Lots of over refinements along the interface because of recovering in a space aking for too much continuity!

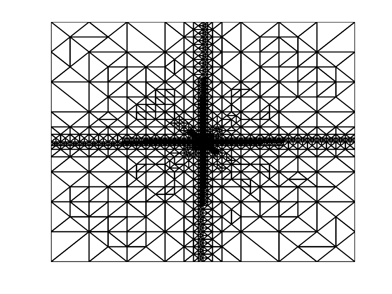

Figure: mesh by $\eta_{m,CB,f}$

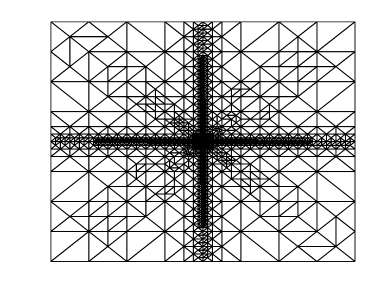

Figure: mesh by $\eta_{m,CB,g}$

Conclusions

- Guidelines for choose quantities, spaces in recovery based a posteriori error estimators
- Robust recovery error estimators for lower order mixed and nonconforming elements