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# ROBUST EQUILIBRATED RESIDUAL ERROR ESTIMATOR FOR DIFFUSION PROBLEMS: CONFORMING ELEMENTS*

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**Abstract.** This paper analyzes an equilibrated residual a posteriori error estimator for the diffusion problem. The estimator, which is a modification of those in [D. Braess and J. Schöberl, *Math. Comput.*, 77 (2008), pp. 651–672; R. Verfürth, *SIAM J. Numer. Anal.*, 47 (2009), pp. 3180–3194], is based on the Prager–Synge identity and on a local recovery of an equilibrated flux. Numerical results for an interface test problem show that the modification is necessary for the robustness of the estimator. When the distribution of diffusion coefficients is local quasi-monotone, it is shown theoretically that the estimator is robust with respect to the size of jumps.

**Key words.** a posteriori error estimator, equilibrated residual error estimator, finite element

**AMS subject classifications.** 65N30, 65N15

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**1. Introduction.** Let $\Omega$ be a bounded polygonal/polyhedronal domain in $\Re^d$, $d = 2$ or 3, with boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and measure$(\Gamma_D) \neq 0$, and let $\mathbf{n}$ be the outward unit vector normal to the boundary. Consider the diffusion equation

$$-\nabla \cdot (A\, \mathrm{grad}\, u) = f \quad \text{in} \quad \Omega \tag{1.1}$$

with boundary conditions

$$-A\, \mathrm{grad}\, u \cdot \mathbf{n} = g_N \quad \text{on} \quad \Gamma_N \quad \text{and} \quad u = g_D \quad \text{on} \quad \Gamma_D. \tag{1.2}$$

We use the standard notation and definitions for the Sobolev spaces. Let

$$H^1_{g,D}(\Omega) = \{v \in H^1(\Omega) \,|\, v = g_D \text{ on } \Gamma_D\} \quad \text{and} \quad H^1_D(\Omega) = H^1_{0,D}(\Omega).$$

Then the corresponding variational problem is to find $u \in H^1_{g,D}(\Omega)$ such that

$$a(u,\, v) \equiv (A\, \mathrm{grad}\, u, \mathrm{grad}\, v) = (f, v) - (g_N, v)_{\Gamma_N} \quad \forall\, v \in H^1_D(\Omega), \tag{1.3}$$

where $(\cdot,\cdot)_\omega$ is the $L^2$ inner product on the set $\omega$. The subscript $\omega$ is omitted when $\omega = \Omega$.

For simplicity of presentation, consider only triangular/tetrahedral elements in two or three dimensions. Let $\mathcal{T} = \{K\}$ be a finite element partition of the domain $\Omega$ that is regular, and denote by $h_K$ the diameter of the element $K$. Let $P_k(K)$ be the space of polynomials of degree less than or equal to $k \geq 0$ on element $K$. Denote

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the continuous Lagrange finite element space of degree $k \geq 1$ associated with the triangulation $\mathcal{T}$ by

$$\mathcal{S}^k = \{v \in H^1(\Omega) \mid v|_K \in P_k(K) \, \forall \, K \in \mathcal{T}\}.$$

Furthermore, assume that $f|_K \in P_{k-1}(K)$ for every $K \in \mathcal{T}$, that $g_{_D}$ and $g_{_N}$ are piecewise polynomials of degrees $k$ and $k-1$, respectively, and that $A$ is a symmetric, positive definite piecewise constant matrix. The finite element approximation of (1.3) is to find $u_{_\mathcal{T}} \in \mathcal{S}^k \cap H^1_{g,D}(\Omega)$ such that

$$(A \operatorname{grad} u_{_\mathcal{T}}, \operatorname{grad} v) = f(v) \quad \forall \, v \in \mathcal{S}^k \cap H^1_D(\Omega). \tag{1.4}$$

Let

$$H(\mathrm{div};\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^d \,|\, \nabla \cdot \boldsymbol{\tau} \in L^2(\Omega)\},$$

which is a Hilbert space equipped with norm

$$\|\boldsymbol{\tau}\|_{H(\mathrm{div})} = \left(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\nabla \cdot \boldsymbol{\tau}\|^2_{0,\Omega}\right)^{1/2}.$$

Let

$$H_{g,N}(\mathrm{div};\Omega) = \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \,|\, \boldsymbol{\tau}\cdot\mathbf{n} = g_{_N} \text{ on } \Gamma_{_N}\} \quad \text{and} \quad H_N(\mathrm{div};\Omega) = H_{0,N}(\mathrm{div};\Omega).$$

Let $u$ be the solution of the diffusion equation in (1.1)–(1.2) and let $\boldsymbol{\tau} \in H_{g,N}(\mathrm{div};\Omega)$ satisfy the equilibrium equation

$$\nabla \cdot \boldsymbol{\tau} = f \quad \text{in} \quad L^2(\Omega). \tag{1.5}$$

Then it is well known [25] (see also [9, 26]) that the Prager–Synge identity

$$\|A^{1/2} \operatorname{grad}(u-v)\|^2_{0,\Omega} + \|A^{1/2} \operatorname{grad} u + A^{-1/2}\boldsymbol{\tau}\|^2_{0,\Omega} = \|A^{1/2} \operatorname{grad} v + A^{-1/2}\boldsymbol{\tau}\|^2_{0,\Omega} \tag{1.6}$$

holds for all $v \in H^1_{g,D}(\Omega)$. (1.6) is a direct consequence of the following orthogonality:

$$\begin{aligned}(A^{1/2} \operatorname{grad}(v-u),\, A^{1/2} \operatorname{grad} u + A^{-1/2}\boldsymbol{\tau}) &= (u - v,\, \nabla \cdot (A \operatorname{grad} u + \boldsymbol{\tau})) \\ &\quad + \int_{\partial\Omega} (v-u)\, \mathbf{n} \cdot (A \operatorname{grad} u + \boldsymbol{\tau})\, ds = 0.\end{aligned}$$

Let $u_{_\mathcal{T}}$ be the solution of (1.4) and $\boldsymbol{\tau} \in H_{g,N}(\mathrm{div};\Omega)$ be any computable vector field satisfying the equilibrium equation in (1.5); then the Prager–Synge identity in (1.6) with $v = u_{_\mathcal{T}}$ implies that the quantity $\|A^{1/2} \operatorname{grad} u_{_\mathcal{T}} + A^{-1/2}\boldsymbol{\tau}\|_{0,\Omega}$ is a reliable a posteriori error estimator, i.e.,

$$\|A^{1/2} \operatorname{grad} u_{_\mathcal{T}} + A^{-1/2}\boldsymbol{\tau}\|_{0,\Omega} \geq \|A^{1/2} \operatorname{grad}(u - u_{_\mathcal{T}})\|_{0,\Omega}. \tag{1.7}$$

Based on this idea (the so-called hypercircle method), various equilibrated residual a posteriori error estimators have been studied recently by many researchers (see [17, 18, 23, 21, 27, 28, 26, 8, 9, 10, 30]). Equilibrated residual estimators can be traced back to [20, 5, 3] (see also the books [29, 4]) based on local error equations with Neumann boundary conditions. Estimators developed in those papers differ mainly in procedures of computing an vector field $\boldsymbol{\tau}$ or $\boldsymbol{\tau} + A\nabla u_{_\mathcal{T}}$, where $\boldsymbol{\tau} \in H_{g,N}(\mathrm{div};\Omega)$ satisfies the equilibrium equation in (1.5). Approximations of these quantities are computed either elementwise [20, 5, 3, 27, 28] or vertex patchwise [17, 18, 8, 9, 10, 30]. The vertex patchwise calculations in [8, 9, 10] for the Poisson equation and in [30] for the reaction-dominant diffusion problem are done through a partition of the unity.

In this paper, we present a systematic study of the equilibrated residual a posteriori error estimator based on the Prager–Synge identity for the diffusion problem (1.1)–(1.2) with an emphasis on the interface problem ($A = \alpha(x)\, I$ and $\alpha(x)$ being piecewise constants). Let $\boldsymbol{\sigma} = -A\nabla u$ be the flux. It is then easy to see that $\boldsymbol{\sigma}$ satisfies a constraint minimization problem (see (2.1)). Therefore, it is natural to recover a flux $\boldsymbol{\sigma}_{\mathcal{T}} \in H_{g,N}(\mathrm{div};\Omega)$ that satisfies the equilibrium equation and that also minimizes the quantity $\|A^{1/2}\,\mathrm{grad}\, u_{\mathcal{T}} + A^{-1/2}\boldsymbol{\tau}\|_{0,\Omega}$. To do so, we localize the problem through a partition of the unity and explicitly recover an equilibrated flux by following the idea in [9] for linear elements and in [30] for higher-order elements. The finite element space for recovering the equilibrated flux in this paper is the Raviart–Thomas elements of index $k-1$ [12], instead of index $k$ as in [10, 30]. The locally recovered flux is further updated through local minimization problems in divergence-free subspaces. For the interface problem, we stress that this correction step is essential for the robustness of the resulting estimator with respect to the size of the jumps. This is shown numerically (see section 6). The idea of using a correction step was explored previously and it was done globally in [18] and locally in [27, 28].

Since the recovered flux in this paper is in $H_{g,N}(\mathrm{div};\Omega)$ and satisfies the equilibrium equation, the reliability of the resulting equilibrated residual estimator is a direct consequence of the Prager–Synge identity. The local efficiency bound is established through a stability estimate of a hybridized mixed saddle point problem and inequalities of Poincaré–Friedrichs type for piecewise $H^1$ functions. This analysis makes use of the fact that the recovered flux satisfies the constraint minimization problem, and it differs from those in [8, 9, 10, 30] where explicit constructions of the error flux play a central role. For the interface problem, under the assumption that the distribution of diffusion coefficients is local quasi-monotone, both the stability of the hybridized mixed problem and the Poincaré–Friedrichs inequalities are shown to be independent of the size of jumps. This, in turn, implies the robustness of the efficiency bound.

For the conforming finite element approximation to the interface problem, robust error estimators have been studied by Bernardi and Verfürth [7] and Petzoldt [24] for the residual-based estimator, Luce and Wohlmuth [21] for an equilibrated estimator on a dual mesh, and by us [14] for the recovery-based error estimator. Ainsworth in [1, 2] studied robust error estimators for nonconforming and mixed methods, respectively. Robust error estimators for locally conserved methods were studied by Kim [19]. Recently, we studied robust recovery-based estimators for lowest-order nonconforming, mixed, and discontinuous Galerkin methods (see [15, 13]) via the $L^2$ recovery and for higher-order conforming elements in [16] via a weighted $H(\mathrm{div})$ recovery.

The paper is organized as follows. Section 2 describes a global constraint minimization problem for the flux and its localization via a partition of the unity. An algorithm to compute local approximations of the flux and the resulting a posteriori error estimator are presented in sections 3 and 4, respectively. Section 5 establishes the local efficiency bound and section 6 provides numerical results for a benchmark test problem.

**2. Flux and localization.** Let $u$ be the solution of the diffusion equation in (1.1)–(1.2), and denote by $\boldsymbol{\sigma} = -A\,\mathrm{grad}\, u$ the flux which satisfies the equilibrium equation: $\nabla\cdot\boldsymbol{\sigma} = f$ in $\Omega$. Let

$$H_{g,N}(\mathrm{div};\Omega,f) := \{\boldsymbol{\tau} \in H_{g,N}(\mathrm{div};\Omega) | \nabla\cdot\boldsymbol{\tau} = f \ \text{ in } \ \Omega\},$$

it is then easy to see that the flux $\boldsymbol{\sigma}$ is the solution of the following minimization problem:

$$\|A^{1/2}\operatorname{grad} u + A^{-1/2}\boldsymbol{\sigma}\|_{0,\Omega} = \min_{\boldsymbol{\tau}\in H_{g,N}(\operatorname{div};\Omega,f)} \|A^{1/2}\operatorname{grad} u + A^{-1/2}\boldsymbol{\tau}\|_{0,\Omega}. \tag{2.1}$$

With the current finite element approximation $u_{\mathcal{T}}$, (2.1) suggests that one should recover a flux in a conforming finite element space of $H_{g,N}(\operatorname{div};\Omega)$ that satisfies the equilibrium equation and that also minimizes the quantity $\|A^{1/2}\operatorname{grad} u_{\mathcal{T}} + A^{-1/2}\boldsymbol{\tau}\|_{0,\Omega}$. However, this procedure requires us to solve a global constraint minimization problem. In this paper, we describe a local recovery procedure.

To this end, let

$$e = u - u_{\mathcal{T}} \quad \text{and} \quad \boldsymbol{\sigma}^{\Delta} = \boldsymbol{\sigma} + A\operatorname{grad} u_{\mathcal{T}} = -A\operatorname{grad} e.$$

Denote by $\mathcal{N}_K$ the set of vertices of $K\in\mathcal{T}$ and the set of vertices of the triangulation $\mathcal{T}$ by

$$\mathcal{N} := \mathcal{N}_I \cup \mathcal{N}_D \cup \mathcal{N}_N,$$

where $\mathcal{N}_I$ is the set of interior vertices and $\mathcal{N}_D$ and $\mathcal{N}_N$ are the sets of boundary vertices belonging to the respective $\overline{\Gamma}_D$ and $\Gamma_N$. Denote by $\phi_z(\mathbf{x}) \in \mathcal{S}^1$ the nodal basis function associated with the vertex $z\in\mathcal{N}$ and its support by $\omega_z = \operatorname{supp}(\phi_z(\mathbf{x}))$. Then $\{\phi_z(\mathbf{x})\}_{z\in\mathcal{N}}$ forms a partition of the unity in $\Omega$:

$$\sum_{z\in\mathcal{N}} \phi_z(\mathbf{x}) = 1 \quad \forall\, \mathbf{x}\in\Omega.$$

Hence, the error in the flux has the following decomposition:

$$\boldsymbol{\sigma}^{\Delta} = \sum_{z\in\mathcal{N}} (\phi_z \boldsymbol{\sigma}^{\Delta}) = \sum_{z\in\mathcal{N}} \boldsymbol{\sigma}_z^{\Delta} \quad \text{with} \quad \boldsymbol{\sigma}_z^{\Delta} = \phi_z \boldsymbol{\sigma}^{\Delta}. \tag{2.2}$$

Next, we derive equations satisfied by the local error flux $\boldsymbol{\sigma}_z^{\Delta}$. To do so, we first introduce some notation. Denote by $\mathcal{E}_K$ the set of edges/faces of element $K\in\mathcal{T}$ and the set of edges/faces of the triangulation $\mathcal{T}$ by

$$\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N,$$

where $\mathcal{E}_I$ is the set of interior element edges and $\mathcal{E}_D$ and $\mathcal{E}_N$ are the sets of boundary edges belonging to the respective $\Gamma_D$ and $\Gamma_N$. For each $F\in\mathcal{E}$, denote by $h_F$ the length/diameter of the edge/face $F$ and by $\mathbf{n}_F$ a unit vector normal to $F$. Let $K_F^-$ and $K_F^+$ be the two elements sharing the common edge/face $F$ such that the unit outward normal vector of $K_F^-$ coincides with $\mathbf{n}_F$. When $F\in\mathcal{E}_D\cap\mathcal{E}_N$, $\mathbf{n}_F$ is the unit outward vector normal to $\partial\Omega$ and denote by $K_F^-$ the element having the edge/face $F$. For a function $v$ defined on $K_F^-\cup K_F^+$, denote its traces on $F$ by $v|_F^-$ and $v|_F^+$, respectively. The jump over the edge/face $F$ is denoted by

$$[\![v]\!]_F := \begin{cases} v|_F^- - v|_F^+, & F\in\mathcal{E}_I,\\ v|_F^-, & F\in\mathcal{E}_D\cup\mathcal{E}_N. \end{cases}$$

(When there is no ambiguity, the subscript or superscript $F$ in the designation of the jump will be dropped.)

For any $K \in \mathcal{T}_z = \{K \in \mathcal{T} : \omega_z \cap K \neq \emptyset\}$, it is easy to see that

$$
\begin{cases} A^{-1}\boldsymbol{\sigma}_z^{\Delta} = -\phi_z \operatorname{grad} e, \\ \nabla \cdot \boldsymbol{\sigma}_z^{\Delta} = \phi_z\, r_K - \operatorname{grad} \phi_z \cdot A \operatorname{grad} e, \end{cases} \tag{2.3}
$$

where $r_K$ is the element residual defined by

$$
r_K = (f + \nabla \cdot (A \operatorname{grad} u_{\mathcal{T}}))\,|_K.
$$

Denote by $\tilde{\mathcal{E}}_z$ the union of all edges/faces having $z \in \mathcal{N}$ as a vertex. Let

$$
\mathcal{E}_z = \{F \in \mathcal{E} :\, F \cap \bar{\omega}_z \neq \emptyset\} = \mathcal{E}_{I,z} \cup \mathcal{E}_{b,z}
$$

with $\mathcal{E}_{I,z}$ and $\mathcal{E}_{b,z}$ being the sets of the respective interior and boundary edges/faces of $\mathcal{T}_z$, and let

$$
\mathcal{E}_{D,z} = \{F \in \mathcal{E}_{b,z} \cap \tilde{\mathcal{E}}_z :\, F \subset \Gamma_D\} \quad \text{and} \quad \mathcal{E}_{N,z} = \{F \in \mathcal{E}_{b,z} \cap \tilde{\mathcal{E}}_z :\, F \subset \Gamma_N\}.
$$

Define

$$
\begin{aligned}
\mathcal{E}_{j,z} &= \begin{cases} \mathcal{E}_{I,z} & \text{if} \quad z \in \mathcal{N}_I, \\ \mathcal{E}_{I,z} \cup \mathcal{E}_{N,z} & \text{if} \quad z \in \mathcal{N}_D \cup \mathcal{N}_N, \end{cases} \\
\mathcal{E}_{0,z} &= \begin{cases} \mathcal{E}_{b,z} & \text{if} \quad z \in \mathcal{N}_I, \\ \{F \in \mathcal{E}_{b,z} : F \not\subset \partial\Omega\} & \text{if} \quad z \in \mathcal{N}_D \cup \mathcal{N}_N, \end{cases} \\
\text{and} \quad \mathcal{E}_{c,z} &= \begin{cases} \mathcal{E}_{b,z} & \text{if} \quad z \in \mathcal{N}_I, \\ \{F \in \mathcal{E}_{b,z} : F \notin \mathcal{E}_{D,z}\} & \text{if} \quad z \in \mathcal{N}_D \cup \mathcal{N}_N. \end{cases}
\end{aligned}
$$

Then the local error flux satisfies the edge/face conditions

$$
\begin{cases} [\![\boldsymbol{\sigma}_z^{\Delta} \cdot \mathbf{n}_F]\!]_F = \phi_z\, j_F & \text{on } F \in \mathcal{E}_{j,z}, \\ \quad \boldsymbol{\sigma}_z^{\Delta} \cdot \mathbf{n}_F = 0 & \text{on } F \in \mathcal{E}_{0,z}, \end{cases} \tag{2.4}
$$

where $j_F$ is the edge/face jump defined by

$$
j_F = \begin{cases} [\![A \operatorname{grad} u_{\mathcal{T}} \cdot \mathbf{n}_F]\!]_F & \forall\, F \in \mathcal{E}_I, \\ g_N + \mathbf{n}_F \cdot (A \operatorname{grad} u_{\mathcal{T}})|_F & \forall\, F \in \mathcal{E}_N. \end{cases}
$$

Due to the assumptions on $f$ and $A$, the residual $r_K$ and the jump $j_F$ are piecewise polynomials of degree less than or equal to $k-1$.

**3. Approximation to the local error flux.** The error estimator to be studied in this paper is based on a finite element approximation of the local error flux $\boldsymbol{\sigma}_z^{\Delta}$. To this end, introduce the following finite element spaces [12, 22] defined on element

$K \in \mathcal{T}$ and on the local triangulation $\mathcal{T}_z$:

$$\begin{aligned}
RT^k(K) &= \{\boldsymbol{\tau} \in L^2(K)^d : \boldsymbol{\tau} = \mathbf{a} + b\,\mathbf{x},\ \mathbf{a} \in P_k(K)^d,\ b \in P_k(K)\},\\
RT^k_{-1,z} &= \{\boldsymbol{\tau} \in L^2(\omega_z)^d : \boldsymbol{\tau}|_{K} \in RT^k(K)\ \forall\, K \in \mathcal{T}_z \text{ and } \boldsymbol{\tau}\cdot\mathbf{n}_{F}|_{F} = 0\ \forall\, F \in \mathcal{E}_{0,z}\},\\
RT^k_z &= RT^k_{-1,z} \cap H(\mathrm{div};\omega_z),\\
P^k_z &= \{v \in L^2(\omega_z) : v|_{K} \in P_k(K)\ \forall\, K \in \mathcal{T}_z\},\\
M^k_z &= \{\mu \in L^2(\mathcal{E}_z) : v|_{F} \in P_k(F)\ \forall\, F \in \mathcal{E}_{j,z}\},\\
S^k_{z,0} &= \{v \in H^1(\omega_z) : v|_{K} \in P_k(K)\ \forall\, K \in \mathcal{T}_z \text{ and } v|_{F} = 0 \text{ on } F \in \mathcal{E}_{c,z}\},\\
Nd^k(K) &= \{\boldsymbol{\tau} \in L^2(K)^d : \boldsymbol{\tau} = \mathbf{a} + \mathbf{b},\ \mathbf{a} \in P_k(K)^d,\ \mathbf{b} \in \tilde{P}_{k+1}(K)^d, \text{ and } \mathbf{b}\cdot\mathbf{x} = 0\},\\
Nd^k_z &= \{\boldsymbol{\tau} \in H(\mathbf{curl};\omega_z) : \boldsymbol{\tau}|_{K} \in Nd^k(K)\ \forall\, K \in \mathcal{T}_z\},\\
Nd^k_{z,0} &= \{\boldsymbol{\tau} \in Nd^k_z : \boldsymbol{\tau}\times\mathbf{n}|_{F} = \mathbf{0} \text{ on } F \in \mathcal{E}_{c,z}\},
\end{aligned}$$

where $\tilde{P}_k(K)$ denotes the space of homogeneous polynomials of order $k$ on element $K$. Let $\Pi_{K}$ and $\Pi_{F}$ be the $L^2$-projections onto $P_{k-1}(K)$ and $P_{k-1}(F)$, respectively, and denote by

$$\bar{r}_{K,z} = \Pi_{K}(\phi_z r_{K}) \quad \text{and} \quad \bar{j}_{F,z} = \Pi_{F}(\phi_z j_{F}).$$

Set

$$\mathcal{H}_z = \{\boldsymbol{\tau} \in RT^{k-1}_{-1,z} : \nabla\cdot\boldsymbol{\tau} = \bar{r}_{K,z}\ \forall\, K \in \mathcal{T}_z \text{ and } [\![\boldsymbol{\tau}\cdot\mathbf{n}_{F}]\!]_{F} = \bar{j}_{F,z}\ \forall\ F \in \mathcal{E}_{j,z}\},$$
$$\text{and} \quad \mathcal{H}_{0,z} = \{\boldsymbol{\tau} \in \mathcal{H}_z : \boldsymbol{\tau}\cdot\mathbf{n}_{F}|_{F} = 0\ \forall\ F \in \mathcal{E}_{0,z}\},$$

then the finite element approximation to the local error flux is to find $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z} \in \mathcal{H}_{0,z}$ such that

$$\|A^{-1/2}\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z}\|_{0,\omega_z} = \min_{\boldsymbol{\tau}\in\mathcal{H}_{0,z}} \|A^{-1/2}\boldsymbol{\tau}\|_{0,\omega_z}. \tag{3.1}$$

It is well known [12] that this constraint minimization problem is equivalent to the following hybridized mixed finite element method. Find $(\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z}, w_z, \lambda_z) \in RT^{k-1}_{-1,z} \times P^{k-1}_z \times M^{k-1}_z$ such that

$$\begin{cases} a_z(\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z},\, \boldsymbol{\tau}) + b_z(\boldsymbol{\tau}, (w_z, \lambda_z)) = 0 & \forall\, \boldsymbol{\tau} \in RT^{k-1}_{-1,z},\\ \qquad b_z(\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z},\, (v,\mu)) = R_z(v) + J_z(\mu) & \forall\, (v,\mu) \in P^{k-1}_z \times M^{k-1}_z, \end{cases} \tag{3.2}$$

where the bilinear forms $a_z(\cdot,\,\cdot)$ and $b_z(\cdot,\,\cdot)$ are defined by

$$a_z(\boldsymbol{\sigma},\boldsymbol{\tau}) = (A^{-1}\boldsymbol{\sigma},\boldsymbol{\tau})_{\omega_z} \quad \text{and} \quad b_z(\boldsymbol{\tau},(v,\mu)) = \sum_{K\in\mathcal{T}_z} (\nabla\cdot\boldsymbol{\tau}, v)_K + \sum_{F\in\mathcal{E}_{j,z}} ([\![\boldsymbol{\tau}\cdot\mathbf{n}_{F}]\!],\mu)_F$$

for $\boldsymbol{\sigma}$, $\boldsymbol{\tau} \in RT^{k-1}_{-1,z}$ and for $(v, \mu) \in P^{k-1}_z \times M^{k-1}_z$, and the linear forms $R_z(\cdot)$ and $J_z(\cdot)$ are defined by

$$R_z(v) = \sum_{K \in \mathcal{T}_z} (\bar{r}_{K,z}, v)_K \quad \text{and} \quad J_z(\mu) = \sum_{F \in \mathcal{E}_{j,z}} (\bar{j}_{F,z}, \mu)_F$$

for $v \in P^{k-1}_z$ and $\mu \in M^{k-1}_z$. The approach of recovering the local error flux by solving (3.2) was briefly mentioned in a remark in [10] but not discussed in detail.

Even though it is not very expensive to solve the indefinite patch problem in (3.2), below we describe a semi-explicit procedure to compute $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z}$ that satisfies (3.1). This procedure consists of explicitly constructing a vector field $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e} \in \mathcal{H}_{0,z}$ and computing a correction $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d}$ by solving a local minimization problem in the divergence-free subspace of $H(\mathrm{div}; \omega_z)$. The recovered local error flux is then

$$\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z} = \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e} + \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d}. \tag{3.3}$$

Before constructing $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}$, let us state the following well-known result on the degrees of freedom of the Raviart–Thomas (RT) elements (see, e.g., Chapter 3 of [12] or Lemma 3.1 of [30]).

LEMMA 3.1. *The vector function $\boldsymbol{\tau}_K \in RT^k(K)$ is uniquely determined by the conditions*

$$\begin{cases} \nabla \cdot \boldsymbol{\tau}_K = f & \text{in} \quad K, \\ \boldsymbol{\tau}_K \cdot \mathbf{n}_F = g_F & \text{on} \quad F \in \mathcal{E}_K, \end{cases}$$

*where the functions $f \in P_k(K)$ and $g_F \in P_k(F)$ satisfy the following compatibility condition:*

$$\sum_{F \in \mathcal{E}_K} \int_F g_F \, ds = \int_K f \, dx.$$

To construct $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e} \in \mathcal{H}_{0,z}$, we follow the idea in [30] (see p. 183 of [8] for linear elements). Denote by $n_z$ the number of elements in $\mathcal{T}_z$ and let

$$\mathcal{T}_z = \{K_1, K_2, \ldots, K_{n_z}\}$$

such that

- $K_i$ and $K_{i+1}$ share an edge/face $F_i = \partial K_i \cap \partial K_{i+1} \in \mathcal{E}_{I,z}$ for $i = 1, \ldots, n_z - 1$;
- For $z \in \mathcal{N}_I$, $K_{n_z}$ and $K_1$ share an edge/face $F_{n_z} = \partial K_{n_z} \cap \partial K_1 \in \mathcal{E}_{I,z}$; and
- For $z \in \mathcal{N}_N$ or $\mathcal{N}_D$, $K_{n_z}$ has an edge/face $F_{n_z} \in \mathcal{E}_{N,z}$ or $\mathcal{E}_{D,z}$, respectively.

ALGORITHM TO CONSTRUCT $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}$.

(1) Compute

$$a_1 = \frac{1}{|F_1|} \left( \sum_{F \in \mathcal{E}_{K_1} \cap \tilde{\mathcal{E}}_z} \int_F \bar{j}_{F,z} \, ds - \int_{K_1} \bar{r}_{K_1,z} \, dx \right)$$

$$\text{and} \quad a_i = \frac{1}{|F_i|} \left( a_{i-1} |F_{i-1}| + \sum_{(F \in \mathcal{E}_K \cap \tilde{\mathcal{E}}_z) \setminus F_{i-1}} \int_F \bar{j}_{F,z} \, ds - \int_{K_i} \bar{r}_{K_i,z} \, dx \right)$$

for $i = 2, \ldots, n_z$.

(2) Set the degrees of freedom of $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}$ in the elements $K_1, \ldots, K_n$ by

$$\nabla \cdot \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}|_{K_1} = \bar{r}_{K_1,z} \quad \text{and}$$

$$\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e} \cdot \mathbf{n}_{K_1} = \begin{cases} \bar{j}_{F,z} & \text{on} \quad (\mathcal{E}_{K_i} \cap \tilde{\mathcal{E}}_z) \backslash F_1, \\ \bar{j}_{F_1,z} - a_1 & \text{on} \quad F_1, \\ 0 & \text{on} \quad \mathcal{E}_K \backslash \tilde{\mathcal{E}}_z, \end{cases}$$

$$\text{and} \quad \nabla \cdot \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}|_{K_i} = \bar{r}_{K_i,z} \quad \text{and}$$

$$\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e} \cdot \mathbf{n}_{K_i} = \begin{cases} \bar{j}_{F,z} & \text{on} \quad (\mathcal{E}_K \cap \tilde{\mathcal{E}}_z) \backslash (F_{i-1} \cup F_i), \\ -a_{i-1} & \text{on} \quad F_{i-1}, \\ \bar{j}_{F_i,z} - a_i & \text{on} \quad F_i, \\ 0 & \text{on} \quad \mathcal{E}_K \backslash \tilde{\mathcal{E}}_z \end{cases}$$

for $i = 2, \ldots, n_z$.

With $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}$ defined above, let

$$\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d} = \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z} - \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}.$$

Then it is easy to see that $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d}$ belongs to

$$\begin{aligned} N_z &= \{\boldsymbol{\tau} \in RT^{k-1}_z : \nabla \cdot \boldsymbol{\tau} = 0 \ \forall K \in \mathcal{T}_z \text{ and } [\![\boldsymbol{\tau} \cdot \mathbf{n}_F]\!]_F = 0 \ \forall F \in \mathcal{E}_{j,z}\} \\ &= \begin{cases} \nabla^{\perp} S^k_{z,0}, & d = 2, \\ \nabla \times Nd^k_{z,0}, & d = 3, \end{cases} \end{aligned}$$

where $\nabla^{\perp} v = (-\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x})$. Hence,

$$\|A^{-1/2}\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z}\|_{0,\omega_z} = \left\|A^{-1/2}\left(\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d} + \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}\right)\right\|_{0,\omega_z} = \min_{\boldsymbol{\tau} \in N_z} \left\|A^{-1/2}\left(\boldsymbol{\tau} + \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}\right)\right\|_{0,\omega_z},$$

which is equivalent to finding $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d} \in N_z$ such that

$$(A^{-1}\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d}, \boldsymbol{\tau})_{\omega_z} = -(A^{-1}\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}, \boldsymbol{\tau})_{\omega_z} \quad \forall \boldsymbol{\tau} \in N_z. \tag{3.4}$$

*Remark* 3.2. When $d = 2$, $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d} = \nabla^{\perp} p_z$ where $p_z \in S^k_{z,0}$ satisfies the following local problem:

$$(A^{-1}\nabla^{\perp} p_z, \nabla^{\perp} q)_{\omega_z} = -(A^{-1}\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}, \nabla^{\perp} q)_{\omega_z} \quad \forall q \in S^k_{z,0}. \tag{3.5}$$

For $k = 1$, problem (3.5) has at most one unknown and, hence, computation of $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z}$ is fully explicit.

When $d = 3$, $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d} = \nabla \times \boldsymbol{\rho}_z$ where $\boldsymbol{\rho}_z \in Nd^k_{z,0}$ satisfies the following local problem:

$$(A^{-1}\nabla \times \boldsymbol{\rho}_z, \nabla \times \boldsymbol{\tau})_{\omega_z} = -(A^{-1}\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}, \nabla \times \boldsymbol{\tau})_{\omega_z} \quad \forall \boldsymbol{\tau} \in Nd^k_{z,0}. \tag{3.6}$$

**4. A posteriori error estimator.** With the local error flux, $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z}$ for all $z \in \mathcal{N}$, computed in the previous section, let

$$\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_K} = \sum_{z \in \mathcal{N}_K} \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z} \quad \text{and} \quad \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}} = \sum_{z \in \mathcal{N}} \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z},$$

then the recovered flux is

$$\boldsymbol{\sigma}_{\mathcal{T}} = \boldsymbol{\sigma}_{\mathcal{T}}^{\Delta} - A \operatorname{grad} u_{\mathcal{T}}. \tag{4.1}$$

LEMMA 4.1. *The recovered flux $\boldsymbol{\sigma}_{\mathcal{T}}$ is in $H_{g,N}(\mathrm{div};\Omega, f)$.*

*Proof.* On each $K \in \mathcal{T}$, by the facts that $\sum_{z\in\mathcal{N}_K} \phi_z(\mathbf{x}) = 1$ and $r_{K,z} \in P_{k-1}(K)$, we have

$$\nabla \cdot \boldsymbol{\sigma}_{\mathcal{T}}^{\Delta}|_K = \sum_{z\in\mathcal{N}_K} \left(\nabla \cdot \boldsymbol{\sigma}_{\mathcal{T}_z}^{\Delta}\right) = \sum_{z\in\mathcal{N}_K} \bar{r}_{K,z} = \Pi_K\left(\sum_{z\in\mathcal{N}_K} \phi_z r_K\right) = \Pi_K(r_K) = r_K,$$

which implies $\nabla \cdot \boldsymbol{\sigma}_{\mathcal{T}}|_K = f$. Similarly, it can be shown that

$$[\![\boldsymbol{\sigma}_{\mathcal{T}} \cdot \mathbf{n}_F]\!]_F = 0 \text{ on } F \in \mathcal{E}_I \quad \text{and} \quad \boldsymbol{\sigma}_{\mathcal{T}} \cdot \mathbf{n}_F|_F = g \text{ on } F \in \mathcal{E}_N.$$

This completes the proof of the lemma. ∎

Define the local indicators and the error estimator by

$$\eta_z = \|A^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}_z}^{\Delta}\|_{0,\omega_z}, \quad \eta_K = \|A^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}_K}^{\Delta}\|_{0,K}, \quad \text{and} \quad \eta = \|A^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}}^{\Delta}\|_{0,\Omega}, \tag{4.2}$$

respectively. It is a direct consequence of Lemma 4.1 and the Prager–Sygne identity in (1.6) with $v = u_{\mathcal{T}}$ and $\boldsymbol{\tau} = \boldsymbol{\sigma}_{\mathcal{T}}$ that the estimator $\eta$ is reliable.

THEOREM 4.2 (reliablility). *The error estimator $\eta$ is reliable with the reliability constant being one; i.e.,*

$$\|A^{1/2} \operatorname{grad}(u - u_{\mathcal{T}})\|_{0,\Omega} \le \eta. \tag{4.3}$$

**5. Efficiency.** In this section, we establish efficiency bounds for the local indicators $\eta_K$ and $\eta_z$ for the interface problem. This is analyzed though a stability estimate of the hybridized mixed finite element problem in (3.2) and inequalities of Poincaré–Friedrichs type for piecewise $H^1$ functions. (Our analysis is an alternative to that in [10].) In order to prove the robustness of the indicators, we show that the stability estimate of (3.2) is independent of jumps. This is done by employing the abstract framework of the saddle point problem (see, e.g., [12]) and by choosing proper mesh- and $\alpha$-dependent norms.

Let $A = \alpha(x)I$ with $\alpha$ being a given scalar, piecewise positive constant function with respect to the triangulation $\mathcal{T}$. For $F = \partial K_F^+ \cap \partial K_F^- \in \mathcal{E}_I$, denote by $\alpha_F^+$ and $\alpha_F^-$ the restriction of $\alpha$ on the respective $K_F^+$ and $K_F^-$. Define the following weighted average:

$$\{v(x)\}_w^F = \begin{cases} w_F^- v_F^- + w_F^+ v_F^+, & F \in \mathcal{E}_I, \\ 0, & F \in \mathcal{E}_D \cup \mathcal{E}_N, \end{cases}$$

$$\text{and} \quad \{v(x)\}_F^w = \begin{cases} w_F^+ v_F^- + w_F^- v_F^+, & F \in \mathcal{E}_I, \\ v_F^-, & F \in \mathcal{E}_D \cup \mathcal{E}_N, \end{cases}$$

where $w_F^- = 1 - w_F^+$ and $w_F^+$ is defined by

$$w_F^+ = \begin{cases} \dfrac{\alpha_F^-}{\alpha_F^- + \alpha_F^+}, & F \in \mathcal{E}_I, \\ 1, & F \in \mathcal{E}_D \cup \mathcal{E}_N. \end{cases}$$

(When there is no ambiguity, the subscript or superscript $F$ in the designation of the weighted average will be dropped.) A simple calculation leads to the following identity:

$$[\![uv]\!]_F = \{v\}^w_F [\![u]\!]_F + \{u\}^F_w [\![v]\!]_F. \tag{5.1}$$

For $F \in \mathcal{E}$ and for $0 \le c \le 1$, denote a weighted average of $\alpha$ by

$$\alpha_F = \begin{cases} c\alpha_{K^-} + (1-c)\alpha_{K^+}, & F \in \mathcal{E}_I, \\ \alpha_{K^-}, & F \in \mathcal{E}_D \cup \mathcal{E}_N. \end{cases}$$

Obviously, $\min\{\alpha_{K^-}, \alpha_{K^+}\} \le \alpha_F \le \max\{\alpha_{K^-}, \alpha_{K^+}\}$ for $F \in \mathcal{E}_I$. Denote the arithmetic and the harmonic averages of $\alpha$ on $F \in \mathcal{E}$ by

$$\alpha_{F,a} = \begin{cases} \dfrac{\alpha_F^+ + \alpha_F^-}{2}, & F \in \mathcal{E}_I, \\ \alpha_F^-, & F \in \mathcal{E}_D \cup \mathcal{E}_N, \end{cases} \quad \text{and} \quad \alpha_{F,h} = \begin{cases} \dfrac{\alpha_F^+ \alpha_F^-}{\alpha_F^+ + \alpha_F^-}, & F \in \mathcal{E}_I, \\ \alpha_F^-, & F \in \mathcal{E}_D \cup \mathcal{E}_N, \end{cases}$$

respectively, which are equivalent to the maximum and the minimum of $\alpha$:

$$\begin{aligned} &\frac{1}{2}\max\{\alpha_F^+, \alpha_F^-\} \le \alpha_{F,a} \le \max\{\alpha_F^+, \alpha_F^-\} \quad \text{and} \\ &\frac{1}{2}\min\{\alpha_F^+, \alpha_F^-\} \le \alpha_{F,h} \le \min\{\alpha_F^+, \alpha_F^-\}. \end{aligned} \tag{5.2}$$

**5.1. Stability estimate of (3.2).** In this section, the hybridized mixed saddle point problem (3.2) is analyzed by using mesh- and $\alpha$-dependent norms. Earlier analysis on the mixed methods using mesh-dependent norms can be found in Babuška, Osborn, and Pitkäranta [6] and Braess and Verfürth [11].

LEMMA 5.1. *The bilinear form $b_z(\cdot,\cdot)$ defined in section 3 has the following representation:*

$$\begin{aligned} b_z(\boldsymbol{\tau}, (v,\mu)) = &- \sum_{K\in\mathcal{T}_z} (\operatorname{grad} v, \boldsymbol{\tau})_K \\ &+ \sum_{F\in\mathcal{E}_{j,z}} ([\![\boldsymbol{\tau}\cdot\mathbf{n}_F]\!], \mu + \{v\}^w)_F + \sum_{F\in\mathcal{E}_{I,z}} (\{\boldsymbol{\tau}\cdot\mathbf{n}_F\}_w, [\![v]\!])_F. \end{aligned} \tag{5.3}$$

*Proof.* Equation (5.3) is a consequence of integration by parts, identity (5.1), and the fact that $\{\boldsymbol{\tau}\cdot\mathbf{n}\}_w = 0$ on $F \in \partial\Omega$:

$$\begin{aligned} b_z(\boldsymbol{\tau}, (v,\mu)) &= - \sum_{K\in\mathcal{T}_z} (\operatorname{grad} v, \boldsymbol{\tau})_K \\ &\quad + \sum_{F\in\mathcal{E}_{j,z}} (([\![\boldsymbol{\tau}\cdot\mathbf{n}_F]\!], \mu + \{v\}^w)_F + (\{\boldsymbol{\tau}\cdot\mathbf{n}_F\}_w, [\![v]\!])_F) \\ &= - \sum_{K\in\mathcal{T}_z} (\operatorname{grad} v, \boldsymbol{\tau})_K \\ &\quad + \sum_{F\in\mathcal{E}_{j,z}} ([\![\boldsymbol{\tau}\cdot\mathbf{n}_F]\!], \mu + \{v\}^w)_F + \sum_{F\in\mathcal{E}_{I,z}} (\{\boldsymbol{\tau}\cdot\mathbf{n}_F\}_w, [\![v]\!])_F. \qquad \square \end{aligned}$$

For all $K \in \mathcal{T}_z$ and for $\boldsymbol{\tau} \in RT^{k-1}_{-1,z}$, $v \in P^{k-1}_z$, and $\mu \in M^{k-1}_z$, let

$$\|\boldsymbol{\tau}\|^2_{\alpha^{-1},h,K} = \|\alpha^{-1/2}\boldsymbol{\tau}\|^2_{0,K} + \sum_{F\in\mathcal{E}_K\cap\mathcal{E}_{j,z}} \frac{h_F}{\alpha_K}\|\boldsymbol{\tau}\cdot\mathbf{n}\|^2_{0,F},$$

$$\|v\|^2_{\alpha,h,K} = \frac{\alpha_K}{h_K^2}\|v\|^2_{0,K} + \sum_{F\in\mathcal{E}_K\cap\mathcal{E}_{j,z}} \frac{\alpha_K}{2\,h_F}\|v\|^2_{0,F},$$

$$\begin{aligned}|\!|\!|(v,\mu)|\!|\!|^2_{\alpha,h,K} &= \|\alpha^{1/2}\,\mathrm{grad}\, v\|^2_{0,K}\\ &\quad + \frac{1}{2}\sum_{F\in\mathcal{E}_K\cap\mathcal{E}_{I,z}}\left(\frac{\alpha_{F,h}}{h_F}\|[\![v]\!]\|^2_{0,F} + \frac{\alpha_K}{h_F}\|\mu+\{v\}^w\|^2_{0,F}\right) + \Lambda(v,\mu,N),\end{aligned}$$

where

$$\Lambda(v,\mu,N) = \begin{cases} 0 & \text{if} \quad z\in\mathcal{N}_I, \\ \displaystyle\sum_{F\in\mathcal{E}_K\cap\mathcal{E}_{N,z}} \frac{\alpha_K}{h_F}\|\mu+v\|^2_{0,F} & \text{if} \quad z\in\mathcal{N}_D\cup\mathcal{N}_N. \end{cases}$$

Define $(\alpha,\, h)$-dependent norms on $\omega_z$ by

$$\|\boldsymbol{\tau}\|_{\alpha^{-1},h,z} = \left(\sum_{K\in\mathcal{T}_z}\|\boldsymbol{\tau}\|^2_{\alpha^{-1},h,K}\right)^{1/2}, \quad \|v\|_{\alpha,h,z} = \left(\sum_{K\in\mathcal{T}_z}\|v\|^2_{\alpha,h,K}\right)^{1/2},$$

$$\text{and} \quad |\!|\!|(v,\mu)|\!|\!|_{\alpha,h,z} = \left(\sum_{K\in\mathcal{T}_z}|\!|\!|(v,\mu)|\!|\!|^2_{\alpha,h,K}\right)^{1/2}.$$

A simple calculation gives

$$|\!|\!|(v,\mu)|\!|\!|^2_{\alpha,h,z} = \|\alpha^{1/2}\,\mathrm{grad}_h\, v\|^2_{0,\omega_z} + \sum_{F\in\mathcal{E}_{j,z}}\left(\frac{\alpha_{F,a}}{h_F}\|\mu+\{v\}^w\|^2_{0,F}\right) + \sum_{F\in\mathcal{E}_{I,z}}\frac{\alpha_{F,h}}{h_F}\|[\![v]\!]\|^2_{0,F}.$$

LEMMA 5.2. *For all $\boldsymbol{\tau}\in RT^k(K)$ and all $v\in P_k(K)$, there exists a positive constant $C$ such that*

$$\sum_{F\in\mathcal{E}_K}\frac{h_F}{\alpha_K}\|\boldsymbol{\tau}\cdot\mathbf{n}\|^2_{0,F} \le C\|\alpha^{-1/2}\boldsymbol{\tau}\|^2_{0,K} \quad \textit{and} \quad \sum_{F\in\mathcal{E}_K}\frac{\alpha_K}{2\,h_F}\|v\|^2_{0,F} \le C\,h_K^{-1}\|\alpha^{1/2}v\|^2_{0,K},$$

*where the constant $C$ depends only on the polynomial degree $k$ and shape parameters of $\mathcal{T}_z$.*

*Proof.* The lemma is a simple consequence of the standard scaling argument and the fact that both $RT^k(K)$ and $P_k(K)$ are finite dimensional spaces. ▯

Lemma 5.2 implies that

$$\|\alpha^{-1/2}\boldsymbol{\tau}\|_{0,K} \le \|\boldsymbol{\tau}\|_{\alpha^{-1},h,K} \le C\|\alpha^{-1/2}\boldsymbol{\tau}\|_{0,K} \quad \forall\ \boldsymbol{\tau}\in RT^k(K), \tag{5.4}$$

$$h_K^{-1/2}\|\alpha^{1/2}v\|_{0,K} \le \|v\|_{\alpha,h,K} \le C\,h_K^{-1}\|\alpha^{1/2}v\|^2_{0,K} \quad \forall\ v\in P_k(K), \tag{5.5}$$

$$\text{and } \|\alpha^{-1/2}\boldsymbol{\tau}\|_{0,\omega_z} \le \|\boldsymbol{\tau}\|_{\alpha^{-1},h,z} \le C\|\alpha^{-1/2}\boldsymbol{\tau}\|_{0,\omega_z} \quad \forall\ \boldsymbol{\tau}\in RT^k_{-1,z}. \tag{5.6}$$

LEMMA 5.3. *For any $v \in P_z^k$, there exists a constant $C$ depending only on the polynomial degree $k$ and shape parameters of $\mathcal{T}_z$ such that*

$$\sum_{K\in\mathcal{T}_z} \frac{\alpha_K}{h_K^2}\|v\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{j,z}} \frac{\alpha_{F,a}}{h_F}\|\{v\}^w\|_{0,F}^2 \le C \sum_{K\in\mathcal{T}_z} \frac{\alpha_K}{h_K^2}\|v\|_{0,K}^2. \tag{5.7}$$

*Proof.* It follows from the definition of $\{v\}^w$, the fact that $w_F^{\pm} \le 1$, the triangle inequality, and Lemma 5.2 that

$$\begin{aligned}
\frac{\alpha_{F,a}}{h_F}\|\{v\}^w\|_{0,F}^2 &= \frac{\alpha_F^+ + \alpha_F^-}{2\,h_F}\Big\|\frac{\alpha_F^+}{\alpha_F^+ + \alpha_F^-}v|_{K_F^+} + \frac{\alpha_F^-}{\alpha_F^+ + \alpha_F^-}v|_{K_F^-}\Big\|_{0,F}^2 \\
&\le \frac{1}{h_F}\left(\frac{(\alpha_F^+)^2}{\alpha_F^+ + \alpha_F^-}\|v_F^+\|_{0,F}^2 + \frac{(\alpha_F^-)^2}{\alpha_F^+ + \alpha_F^-}\|v_F^-\|_{0,F}^2\right) \\
&\le \frac{\alpha_K^+}{h_F}\|v_F^+\|_{0,F}^2 + \frac{\alpha_K^-}{h_F}\|v_F^-\|_{0,F}^2 \le C \sum_{K\in\{K_F^+,\,K_F^-\}} h_K^{-2}\|\alpha^{1/2}v\|_{0,K}^2,
\end{aligned}$$

which, in turn, implies (5.7). This completes the proof of the lemma. ∎

LEMMA 5.4. *The bilinear form $a_z(\cdot,\cdot)$ is continuous and coercive with respect to the norm $\|\cdot\|_{\alpha^{-1},h,z}$ in $\boldsymbol{\tau} \in RT_{-1,z}^{k-1}$; i.e., there exists a positive constant $a_c$ independent of $\alpha$ and the mesh size such that for all $\boldsymbol{\sigma},\ \boldsymbol{\tau} \in RT_{-1,z}^{k-1}$*

$$a_c\|\boldsymbol{\tau}\|_{\alpha^{-1},h,z}^2 \le a_z(\boldsymbol{\tau},\boldsymbol{\tau}) \quad \textit{and} \quad a_z(\boldsymbol{\sigma},\boldsymbol{\tau}) \le \|\boldsymbol{\sigma}\|_{\alpha^{-1},h,z}\|\boldsymbol{\tau}\|_{\alpha^{-1},h,z}.$$

*Proof.* The lemma is a direct consequence of (5.6) and the Cauchy–Schwarz inequality. ∎

LEMMA 5.5. *The bilinear form $b_z(\cdot,\cdot)$ is continuous in $RT_{-1,z}^{k-1}\times(P_z^{k-1}\times M_z^{k-1})$; i.e., there exists a positive constant $C$ independent of the mesh size such that*

$$b_z(\boldsymbol{\tau},(v,\mu)) \le C\|\boldsymbol{\tau}\|_{\alpha^{-1},h,z}|\!|\!|(v,\mu)|\!|\!|_{\alpha,h,z}. \tag{5.8}$$

*Proof.* It follows from (5.3) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
b_z(\boldsymbol{\tau},(v,\mu)) \le& \sum_{K\in\mathcal{T}_z}\|\alpha^{1/2}\operatorname{grad} v\|_{0,K}\|\alpha^{-1/2}\boldsymbol{\tau}\|_{0,K} \\
&+ \sum_{F\in\mathcal{E}_{j,z}} \frac{h_F^{1/2}}{\alpha_{F,a}^{1/2}}\|[\![\boldsymbol{\tau}\cdot\mathbf{n}_F]\!]\|_{0,F}\frac{\alpha_{F,a}^{1/2}}{h_F^{1/2}}\|\mu + \{v\}^w\|_{0,F} \\
&+ \sum_{F\in\mathcal{E}_{I,z}} \frac{h_F^{1/2}}{\alpha_{F,h}^{1/2}}\|\{\boldsymbol{\tau}\cdot\mathbf{n}_F\}_w\|_{0,F}\frac{\alpha_{F,h}^{1/2}}{h_F^{1/2}}\|[\![v]\!]\|_{0,F}.
\end{aligned}$$

Hence,

$$\begin{aligned}
b_z(\boldsymbol{\tau},(v,\mu)) \le& \|\alpha^{1/2}\operatorname{grad} v\|_{0,\omega_z}\|\alpha^{-1/2}\boldsymbol{\tau}\|_{0,\omega_z} \\
&+ \left(\sum_{F\in\mathcal{E}_{j,z}} \frac{h_F\|[\![\boldsymbol{\tau}\cdot\mathbf{n}_F]\!]\|_{0,F}^2}{\alpha_{F,a}}\right)^{1/2}\left(\sum_{F\in\mathcal{E}_{j,z}} \frac{\alpha_{F,a}\|\mu+\{v\}^w\|_{0,F}^2}{h_F}\right)^{1/2} \\
&+ \left(\sum_{F\in\mathcal{E}_{I,z}} \frac{h_F\|\{\boldsymbol{\tau}\cdot\mathbf{n}_F\}_w\|_{0,F}^2}{\alpha_{F,h}}\right)^{1/2}\left(\sum_{F\in\mathcal{E}_{I,z}} \frac{\alpha_{F,h}\|[\![v]\!]\|_{0,F}^2}{h_F}\right)^{1/2}.
\end{aligned}$$

Denote by $\tau^{\pm}_{n_F} = \boldsymbol{\tau}|_{K_F^{\pm}} \cdot \mathbf{n}_F$. The triangle inequality and Lemma 5.2 give

$$\sum_{F\in\mathcal{E}_{j,z}} \frac{h_F \|[\![\boldsymbol{\tau}\cdot\mathbf{n}_F]\!]\|^2_{0,F}}{\alpha_{F,a}} \le 2 \sum_{F\in\mathcal{E}_{j,z}} \frac{h_F\left(\|\tau^-_{n_F}\|^2_{0,F} + \|\tau^+_{n_F}\|^2_{0,F}\right)}{\alpha_{F,a}}$$
$$\le 4 \sum_{F\in\mathcal{E}_{j,z}} \left(\frac{h_F}{\alpha_F^-}\|\tau^-_{n_F}\|^2_{0,F} + \frac{h_F}{\alpha_F^+}\|\tau^+_{n_F}\|^2_{0,F}\right) \le C\,\|\alpha^{-1/2}\boldsymbol{\tau}\|^2_{0,\omega_z}.$$

Similarly, we have

$$\sum_{F\in\mathcal{E}_{j,z}} \frac{h_F\|\{\boldsymbol{\tau}\cdot\mathbf{n}_F\}_w\|^2_{0,F}}{\alpha_{F,h}} \le C\,\|\alpha^{-1/2}\boldsymbol{\tau}\|^2_{0,\omega_z} \quad \text{and}$$
$$\sum_{F\in\mathcal{E}_{I,z}} \frac{\alpha_{F,a}\|\mu+\{v\}^w\|^2_{0,F}}{h_F} \le C\,|\!|\!|(v,\mu)|\!|\!|^2_{\alpha,h,z}.$$

Combining the above inequalities implies (5.8). This completes the proof of the lemma. $\square$

LEMMA 5.6 (inf-sup condition). *The following inf-sup condition holds with constant $\beta > 0$ independent of $\alpha$ and $h$:*

$$\sup_{\boldsymbol{\tau}\in RT^{k-1}_{-1,z}} \frac{b_z(\boldsymbol{\tau},(v,\mu))}{\|\boldsymbol{\tau}\|_{\alpha^{-1},h,z}} \ge \beta|\!|\!|(v,\mu)|\!|\!|_{\alpha,h,z} \quad \forall\,(v,\mu)\in P^{k-1}_z \times M^{k-1}_z. \tag{5.9}$$

*Proof.* Choose a $\tilde{\boldsymbol{\tau}} \in RT^{k-1}_{-1,z}$ such that

$$\int_K (\tilde{\boldsymbol{\tau}} + \alpha\,\mathrm{grad}\,v)\cdot\mathrm{grad}\,q dx = 0 \quad \forall\, q\in P_{k-2}(K) \quad \forall\; K\in\mathcal{T}_z$$

and that

$$\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}_K|_F = \begin{cases} \mathrm{sgn}(K,F)\dfrac{\alpha_{F,h}}{h_F}[\![v]\!] + \dfrac{\alpha_K}{2\,h_F}(\mu+\{v\}^w), & F\in\mathcal{E}_K\cap\mathcal{E}_{I,z},\\ \dfrac{\alpha_K}{2\,h_F}(\mu+v), & F\in\mathcal{E}_K\cap\mathcal{E}_{N,z} \quad \text{if } z\in\mathcal{N}_D\cup\mathcal{N}_N,\\ 0, & F\in\mathcal{E}_K\cap\mathcal{E}_{0,z}, \end{cases} \tag{5.10}$$

where $\mathrm{sgn}(K,F) = \mathbf{n}_K\cdot\mathbf{n}_F$. (See [12] for existence of $\tilde{\boldsymbol{\tau}} \in RT^{k-1}_{-1,z}$ satisfying the above conditions.) Obviously, (5.10) implies

$$[\![\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}_F]\!] = \frac{\alpha_{F,a}}{h_F}(\mu+\{v\}^w) \quad \forall\,F\in\mathcal{E}_{j,z} \quad \text{and} \quad \{\tilde{\boldsymbol{\tau}}\cdot\mathbf{n}_F\}_w = \frac{\alpha_{F,h}}{h_F}[\![v]\!] \quad \forall\,F\in\mathcal{E}_{I,z},$$

which, together with (5.3), gives

$$b(\tilde{\boldsymbol{\tau}},(v,\mu)) = |\!|\!|(v,\mu)|\!|\!|^2_{\alpha,h,z}. \tag{5.11}$$

For every $K\in\mathcal{T}_z$, by the standard scaling argument, there exists a constant $C>0$ independent of $\alpha$ and the mesh size such that

$$\|\tilde{\boldsymbol{\tau}}\|^2_{0,K} \le C\Bigg(\|\alpha_K\,\mathrm{grad}\,v\|^2_{0,K} + h_K \sum_{F\in\mathcal{E}_K\cap\mathcal{E}_{I,z}} \|\mathrm{sgn}(K,F)\frac{\alpha_{F,h}}{h_F}[\![v]\!]$$
$$+ \frac{\alpha_K}{2\,h_F}(\mu+\{v\}^w)\|^2_{0,F} + \alpha_K\Lambda(v,\mu,N)\Bigg),$$

which, together with (5.2), gives

$$\|\alpha_{K}^{-1/2}\tilde{\boldsymbol{\tau}}\|_{0,K}^2 \le C\Bigg(\|\alpha^{1/2}\operatorname{grad} v\|_{0,K}^2 + \sum_{F\in\mathcal{E}_K\cap\mathcal{E}_{I,z}}\left(\frac{\alpha_{F,h}}{h_F}\|[\![v]\!]\|_{0,F}^2 + \frac{\alpha_K}{h_F}\|\mu+\{v\}^w\|_{0,F}^2\right) + \Lambda(v,\mu,N)\Bigg).$$

Hence, there exists a constant $\tilde{C}>0$ independent of $\alpha$ and $h$ such that

$$\|\boldsymbol{\tau}\|_{\alpha^{-1},h,z} \le \tilde{C}|\!|\!|(v,\mu)|\!|\!|_{\alpha,h,z},$$

which, together with (5.11), leads to (5.9) with $\beta = 1/\tilde{C}$. This completes the proof of the lemma. □

THEOREM 5.7. *Problem* (3.2) *has a unique solution* $(\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z}, w_z, \lambda_z) \in RT^{k-1}_{-1,z} \times P_z^{k-1}\times M_z^{k-1}$ *that satisfies the following a priori estimate:*

$$\|\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z}\|_{\alpha^{-1},h,z} + |\!|\!|(w_z,\lambda_z)|\!|\!|_{\alpha,h,z} \le C(a_c,\beta)\sup_{(v,\mu)\in P_z^{k-1}\times M_z^{k-1}}\frac{R_z(v)+J_z(\mu)}{|\!|\!|(v,\mu)|\!|\!|_{\alpha,h,z}}, \tag{5.12}$$

*where the constant* $C(a_c,\beta)>0$ *is independent of the mesh size and jumps.*

*Proof.* The theorem follows from the abstract theory of the saddle point problem (see, e.g., [12]) and Lemmas 5.4, 5.5, and 5.6. □

**5.2. Local efficiency bound.** For any $z\in\mathcal{N}$, let

$$\hat{\omega}_z = \left\{K\in\omega_z \,:\, \alpha_K = \max_{K'\in\omega_z}\alpha_{K'}\right\}.$$

Assume that the distribution of the coefficients $\alpha_K$ for all $K\in\mathcal{T}$ is locally quasi-monotone [24], which is slightly weaker than Hypothesis 2.7 in [7]. For convenience, we restate it here.

DEFINITION 5.8. *Given a vertex* $z\in\mathcal{N}$*, the distribution of the coefficients* $\alpha_K$*,* $K\in\omega_z$*, is said to be* quasi-monotone *with respect to the vertex* $z$ *if there exists a subset* $\tilde{\omega}_{K,z,qm}$ *of* $\omega_z$ *such that the union of elements in* $\tilde{\omega}_{K,z,qm}$ *is a Lipschitz domain and that*

- *If* $z\in\mathcal{N}\backslash\mathcal{N}_D$*, then* $\{K\}\cup\hat{\omega}_z\subset\tilde{\omega}_{K,z,qm}$ *and* $\alpha_K\le\alpha_{K'}$ $\forall K'\in\tilde{\omega}_{K,z,qm}$*;*
- *If* $z\in\mathcal{N}_D$*, then* $K\in\tilde{\omega}_{K,z,qm}$*,* $\partial\tilde{\omega}_{K,z,qm}\cap\Gamma_D\ne\emptyset$*, and* $\alpha_K\le\alpha_{K'}$ $\forall K'\in\tilde{\omega}_{K,z,qm}$*.*

*The distribution of the coefficients* $\alpha_K$*,* $K\in\mathcal{T}$*, is said to be locally* quasi-monotone *if it is quasi-monotone with respect to every vertex* $z\in\mathcal{N}$*.*

For an element $K\in\mathcal{T}_z$ and any $v\in H^1(K)$, let

$$\bar{v}_K = \frac{1}{|K|}\int_K v\,dx \quad\text{and}\quad \bar{v}_{K,F} = \frac{1}{|F|}\int_F v|_K\,dx \quad \forall\, F\in\mathcal{E}_K$$

be the averages of $v|_K$ over $K$ and face $F\in\mathcal{E}_K$, respectively. It is well known (see, e.g., (4.7) of [31]) that the following Poincaré–Friedrichs inequalities hold on element $K$ with diameter $h_K$; i.e., there exists a positive constant $C$ independent of $h_K$ such that

$$\begin{aligned} \|v-\bar{v}_K\|_{0,K} &\le Ch_K\|\operatorname{grad} v\|_{0,K} \quad\text{and}\\ \|v-\bar{v}_{K,F}\|_{0,K} &\le Ch_K\|\operatorname{grad} v\|_{0,K} \quad \forall\, v\in H^1(K). \end{aligned} \tag{5.13}$$

Define the following piecewise $H^1$ function spaces:

$$V_z = \begin{cases} \{v \in L^2(\omega_z) : v|_K \in H^1(K)\}, & z \in \mathcal{N}\backslash\mathcal{N}_D, \\ \{v \in L^2(\omega_z) : v|_K \in H^1(K) \text{ and } v|_{\Gamma_D} = 0\}, & z \in \mathcal{N}_D. \end{cases}$$

Obviously, $P_z^k \subset V_z$. Let $K' \in \hat{\omega}_z$, define $\bar{v}_z = \bar{v}_{K'}$.

Lemma 5.9. *Suppose the distribution of $\alpha$ is quasi-monotone with respect to the vertex $z \in \mathcal{N}$. For any $v \in V_z$, there exists a constant $C$ independent of the mesh size and $\alpha$ such that*

(5.14)
$$\sum_{K\in\mathcal{T}_z} h_K^{-2}\|\alpha^{1/2}(v-\bar{v}_z)\|_{0,K}^2 \le C\left(\sum_{K\in\mathcal{T}_z}\|\alpha^{1/2}\operatorname{grad} v\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{I,z}}\frac{\alpha_F^{1/2}}{h_F^d}\left|\int_F [\![v]\!]ds\right|^2\right)$$

*when $z \in \mathcal{N}\backslash\mathcal{N}_D$ and that*

$$\sum_{K\in\mathcal{T}_z} h_K^{-2}\|\alpha^{1/2}v\|_{0,K}^2 \le C\left(\sum_{K\in\mathcal{T}_z}\|\alpha^{1/2}\operatorname{grad} v\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{I,z}}\frac{\alpha_F^{1/2}}{h_F^d}\left|\int_F [\![v]\!]ds\right|^2\right) \tag{5.15}$$

*when $z \in \mathcal{N}_D$.*

*Proof.* We will only show the validity of (5.14) since (5.15) may be proved in a similar fashion. By the definition of $\bar{v}_z$ and (5.13), there exists a $K' \in \hat{\omega}_z$ such that $\bar{v}_z = \bar{v}_{K'}$ and that

$$\|\alpha^{1/2}(v-\bar{v}_z)\|_{0,K'} = \|\alpha^{1/2}(v-\bar{v}_{K'})\|_{0,K'} \le C h_{K'}\|\alpha^{1/2}\operatorname{grad} v\|_{0,K'}.$$

For any $K \in \mathcal{T}_z$, the triangle inequality gives

$$\|\alpha_K^{1/2}(v-\bar{v}_z)\|_{0,K} \le \|\alpha_K^{1/2}(v-\bar{v}_K)\|_{0,K} + \|\alpha_K^{1/2}(\bar{v}_{K'}-\bar{v}_K)\|_{0,K}.$$

Since $\alpha$ is quasi-monotone with respect to $z$, there is a connected path, $\{K = K_0, K_1, \ldots, K_l = K'\}$ with $K_i \in \mathcal{T}_z$, from $K$ to $K'$ such that

$$\alpha_K \le \alpha_{K_1} \le \cdots \le \alpha_{K'}.$$

Denote by $F_i$ the common face between $K_{i-1}$ and $K_i$; then

$$\alpha_K \le \alpha_{F_1} \le \cdots \le \alpha_{F_l}.$$

Using the triangle inequality, we have

$$\begin{aligned}\|\alpha_K^{1/2}(\bar{v}_{K'}-\bar{v}_K)\|_{0,K} &= \alpha_K^{1/2}|K|^{1/2}|\bar{v}_{K_0}-\bar{v}_{K_l}| \le \alpha_K^{1/2}|K|^{1/2}\sum_{i=0}^{l-1}|\bar{v}_{K_i}-\bar{v}_{K_{i+1}}| \\ &\le \alpha_K^{1/2}|K|^{1/2}\sum_{i=0}^{l-1}\Big(|\bar{v}_{K_i}-\bar{v}_{K_i,F_{i+1}}| + |\bar{v}_{K_i,F_{i+1}}-\bar{v}_{K_{i+1},F_{i+1}}| \\ &\qquad + |\bar{v}_{K_{i+1},F_{i+1}}-\bar{v}_{K_{i+1}}|\Big).\end{aligned}$$

Since $\mathcal{T}_z$ is quasi-uniform, it then follows from the fact that $\alpha_K \le \alpha_{K_i}$, (5.13), and the triangle inequality that

$$\begin{aligned}\alpha_K^{1/2}|K|^{1/2}|\bar{v}_{K_i} - \bar{v}_{K_i,F_{i+1}}| &\le C\|\alpha_{K_i}^{1/2}(\bar{v}_{K_i} - \bar{v}_{K_i,F_{i+1}})\|_{0,K_i} \\ &\le C\left(\|\alpha_{K_i}^{1/2}(v - \bar{v}_{K_i})\|_{0,K_i} + \|\alpha_{K_i}^{1/2}(v - \bar{v}_{K_i,F_{i+1}})\|_{0,K_i}\right) \\ &\le Ch_{K_i}\|\alpha_{K_i}^{1/2}\operatorname{grad} v\|_{0,K_i}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\alpha_K^{1/2}|K|^{1/2}|\bar{v}_{K_{i+1},F_{i+1}} - \bar{v}_{K_{i+1}}| &\le Ch_{K_{i+1}}\|\alpha_{K_{i+1}}^{1/2}\operatorname{grad} v\|_{0,K_{i+1}}, \\ \alpha_K^{1/2}|K|^{1/2}|\bar{v}_{K_i,F_{i+1}} - \bar{v}_{K_{i+1},F_{i+1}}| &\le \alpha_{F_{i+1}}^{1/2} h_{F_{i+1}}^{1-d/2}\left|\int_{F_{i+1}} [\![v]\!] ds\right|.\end{aligned}$$

Combining the above inequalities gives

$$h_K^{-1}\|\alpha_K^{1/2}(v - \bar{v}_z)\|_{0,K} \le C\left(\sum_{i=0}^{l}\|\alpha_{K_i}^{1/2}\operatorname{grad} v\|_{0,K_i} + \sum_{i=1}^{l}\alpha_{F_{i+1}}^{1/2} h_{F_{i+1}}^{-d/2}\left|\int_{F_{i+1}}[\![v]\!]ds\right|\right).$$

Squaring both sides, using the Cauchy–Schwarz inequality, and summing up over all $K \in \mathcal{T}_z$ imply (5.14). This completes the proof of the lemma. ∎

COROLLARY 5.10. *Under the assumptions of Lemma* 5.9, *for any $v \in V_z$, there exists a constant $C$ independent of the mesh size and $\alpha$ such that*

(5.16)
$$\sum_{K\in\mathcal{T}_z} h_K^{-2}\|\alpha^{1/2}(v - \bar{v}_z)\|_{0,K}^2 \le C\left(\sum_{K\in\mathcal{T}_z}\|\alpha^{1/2}\operatorname{grad} v\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{I,z}}\frac{\alpha_F^{1/2}}{h_F}\|[\![v]\!]\|_{0,F}^2\right)$$

*when $z \in \mathcal{N}\backslash\mathcal{N}_D$ and*

$$\sum_{K\in\mathcal{T}_z} h_K^{-2}\|\alpha^{1/2}v\|_{0,K}^2 \le C\left(\sum_{K\in\mathcal{T}_z}\|\alpha^{1/2}\operatorname{grad} v\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{I,z}}\frac{\alpha_F^{1/2}}{h_F}\|[\![v]\!]\|_{0,F}^2\right) \tag{5.17}$$

*when $z \in \mathcal{N}_D$.*

*Proof.* The corollary is an immediate consequence of Lemma 5.9 and the fact that

$$\int_F [\![v]\!]ds \le h_F^{\frac{d-1}{2}}\|[\![v]\!]\|_{0,F}.$$

Denote the vertex-based local residual error indicator by

$$\hat{\eta}_z = \left(\sum_{K\in\mathcal{T}_z}\frac{h_K^2}{\alpha_K}\|r_K\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{j,z}}\frac{h_F}{\alpha_{F,a}}\|j_F\|_{0,F}^2\right)^{1/2}. \tag{5.18}$$

Its local efficiency was proved in [7, 24]; i.e., there exists a constant $C > 0$ is independent of $\alpha$ and the mesh size such that

$$\hat{\eta}_z \le C\|\alpha^{1/2}\operatorname{grad}(u - u_{\mathcal{T}})\|_{0,\tilde{\omega}_z}, \tag{5.19}$$

where $\tilde{\omega}_z = \omega_z \cup \{K : K \text{ and } \partial\omega_z \text{ shares an edge/face }\}$. Note that there are no oscillation terms in (5.19) due to the assumption that $f$ is a piecewise polynomial of degree $k-1$.

It is well known (see, e.g., [29]) that the residual functional has the following $L^2$-representation:

$$\mathcal{R}(v) = f(v) - (A\,\mathrm{grad}\,u_{\mathcal{T}}, \mathrm{grad}\,v) = \sum_{K\in\mathcal{T}} (r_{K}, v)_K - \sum_{F\in\mathcal{E}_I\cup\mathcal{E}_N} (j_{F}, v)_F \quad \forall\, v \in H^1_D(\Omega).$$

Define the local residual on patch $\omega_z$ by

$$\mathcal{R}_z(v) := \mathcal{R}(v\phi_z) = \sum_{K\in\mathcal{T}_z} (r_{K,z}, v)_K - \sum_{F\in\mathcal{E}_{j,z}} (j_{F,z}, v)_F \quad \forall\, v \in H^1_D(\Omega)$$

with $r_{K,z} = \phi_z r_K$ and $j_{F,z} = \phi_z j_F$. By the Galerkin orthorgnality, i.e., $\mathcal{R}_z(1) = \mathcal{R}(\phi_z) = 0$, we then have the following local orthogonality property:

$$\sum_{K\in\mathcal{T}_z} (r_{K,z}, 1)_K - \sum_{F\in\mathcal{E}_{j,z}} (j_{F,z}, 1)_F = 0. \tag{5.20}$$

THEOREM 5.11 (efficiency). *Under the assumptions of Lemma 5.9, the local indicators $\eta_z$ and $\eta_K$ are efficient; i.e., there exists a constant $C > 0$ independent of $\alpha$ and the mesh size such that*

$$\eta_z \le C\|\alpha^{1/2}\,\mathrm{grad}\,e\|_{0,\tilde{\omega}_z} \quad \textit{and} \quad \eta_K \le C \sum_{z\in\mathcal{N}_K} \|\alpha^{1/2}\,\mathrm{grad}\,e\|_{0,\tilde{\omega}_z}. \tag{5.21}$$

*Proof.* Squaring both sides of the first inequality in (5.21) and summing up over all $z \in \mathcal{N}_K$ imply the second inequality in (5.21). To prove the validity of the first inequality in (5.21), by Theorem 5.1 and (5.19), it suffices to show that

$$\sup_{(v,\mu)\in P_z^{k-1}\times M_z^{k-1}} \frac{R_z(v) + J_z(\mu)}{|\!|\!|(v,\mu)|\!|\!|_{\alpha,h,z}} \le C\,\hat{\eta}_z$$

or, equivalently,

$$I \equiv R_z(v) + J_z(\mu) \le C\,\hat{\eta}_z\,|\!|\!|(v,\mu)|\!|\!|_{\alpha,h,z} \quad \forall\ (v,\,\mu) \in P_z^{k-1} \times M_z^{k-1}. \tag{5.22}$$

To do so, from the local orthogonality property in (5.20) and the facts

$$(r_{K,z}, 1)_K = (\bar{r}_{K,z}, 1)_K \quad \forall\, K \in \mathcal{T}_z \quad \text{and} \quad (j_{F,z}, 1)_F = (\bar{j}_{F,z}, 1)_F \quad \forall\, F \in \mathcal{E}_{j,z},$$

we have that for an arbitrary constant $c$

$$R_z(c) - J_z(c) = \sum_{K\in\mathcal{T}_z} (\bar{r}_{K,z}, c)_K - \sum_{F\in\mathcal{E}_{I,z}\cup\mathcal{E}_{N,z}} (\bar{j}_{F,z}, c)_F = 0,$$

which implies

$$I = R_z(v-c) + J_z(\mu+c) = R_z(v-c) - J_z(\{v-c\}^w) + J_z(\mu + \{v\}^w) \tag{5.23}$$

for any $(v,\,\mu) \in P_z^{k-1} \times M_z^{k-1}$. Since $\|\bar{r}_{K,z}\|_{0,K} \le \|r_K\|_{0,K}$ and $\|\bar{j}_{F,z}\|_{0,F} \le \|j_F\|_{0,F}$, it follows from the triangle and the Cauchy–Schwarz inequalities that

$$
\begin{aligned}
I &\le \sum_{K\in\mathcal{T}_z} \|\bar{r}_{K,z}\|_{0,K}\|v-\bar{v}_z\|_{0,K} + \sum_{F\in\mathcal{E}_{j,z}} \|\bar{j}_{F,z}\|_{0,F}\left(\|\{v-\bar{v}_z\}^w\|_{0,F} + \|\mu+\{v\}^w\|_{0,F}\right)\\
&\le \hat{\eta}_z \left(\sum_{K\in\mathcal{T}_z} \frac{\alpha_K}{h_K^2}\|v-\bar{v}_z\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{j,z}} \frac{\alpha_{F,a}}{h_F}\left(\|\{v-\bar{v}_z\}^w\|_{0,F}^2 + \|\mu+\{v\}^w\|_{0,F}^2\right)\right)^{1/2}\\
&\le C\hat{\eta}_z \left(\sum_{K\in\mathcal{T}_z} \frac{\alpha_K}{h_K^2}\|v-\bar{v}_z\|_{0,K}^2 + \sum_{F\in\mathcal{E}_{j,z}} \frac{\alpha_{F,a}}{h_F}\|\mu+\{v\}^w\|_{0,F}^2\right)^{1/2}\\
&\le C\hat{\eta}_z \left(\|\alpha^{1/2}\operatorname{grad}_h v\|_{0,\omega_z}^2 + \sum_{F\in\mathcal{E}_{I,z}} \frac{\alpha_{F,h}}{h_F}\|[\![v]\!]\|_{0,F}^2 + \sum_{F\in\mathcal{E}_{j,z}} \frac{\alpha_{F,a}}{h_F}\|\mu+\{v\}^w\|_{0,F}^2\right)^{1/2}\\
&= C\hat{\eta}_z |\!|\!|(v,\mu)|\!|\!|_{\alpha,h,z}.
\end{aligned}
$$

This proves the validity of (5.22) and, hence, the theorem. ∎

**6. Numerical experiments.** In this section, we report some numerical results for an interface problem with intersecting interfaces used by many authors, e.g., [19, 14, 15, 16], which is considered a benchmark test problem. For this test problem, we show numerically that the local minimization procedure is essential. That is, estimators based on the $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}$ as in [8, 9] will produce a nonoptimal mesh and, hence, are not robust.

To this end, let $\Omega = (-1,1)^2$ and

$$u(r,\theta) = r^{\gamma}\mu(\theta)$$

in the polar coordinates at the origin with $\mu(\theta)$ being a smooth function of $\theta$ (see, e.g., [14]). The function $u(r,\theta)$ satisfies the interface equation with $A = \alpha I$, $\Gamma_N = \emptyset$, $f = 0$, and

$$
\alpha(x) = \begin{cases} R & \text{in } (0,1)^2 \cup (-1,0)^2,\\ 1 & \text{in } \Omega \setminus ([0,1]^2 \cup [-1,0]^2). \end{cases}
$$

The $\gamma$ depends on the size of the jump. In our test problem, $\gamma = 0.1$ is chosen and corresponds to $R \approx 161.4476387975881$. Note that the solution $u(r,\theta)$ is only in $H^{1+\gamma-\epsilon}(\Omega)$ for any $\epsilon > 0$ and, hence, it is very singular for small $\gamma$ at the origin. This suggests that refinement is centered around the origin.

Let $u_{\mathcal{T}} \in S^1$ be the linear finite element approximation. We start with the coarsest triangulation $\mathcal{T}_0$ obtained from halving 16 congruent squares by connecting the bottom left and upper right corners. A sequence of meshes is generated by using standard adaptive meshing algorithm that adopts the maximum marking strategy: mark those elements such that $\eta_K \ge 0.5\max_{K'\in\mathcal{T}}\eta_{K'}$. Marked triangles are refined by bisection. Define the effectivity index, eff-index $:= \frac{\eta}{\|\alpha^{1/2}\operatorname{grad}(u-u_{\mathcal{T}})\|_{0,\Omega}}$, and use the following stopping criteria: rel-err $:= \frac{\|\alpha^{1/2}\operatorname{grad}(u-u_{\mathcal{T}})\|_{0,\Omega}}{\|\alpha^{1/2}\operatorname{grad} u\|_{0,\Omega}} \le$ tol. We report numerical results with tol $= 0.05$.

Broken $RT_0$ elements are used to approximate the error flux. Denote by $\xi$ the error estimator based on the $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e}$ similar to that in [8, 9] and by $\eta = \eta_K$ the error estimator defined in (4.2), i.e., based on $\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z} = \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e} + \boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d}$. Here, the correction

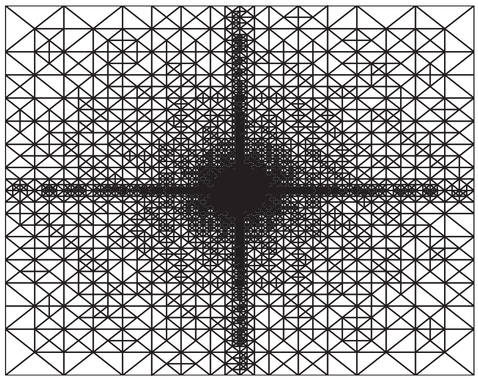

FIG. 1. *Mesh generated by $\xi$.*

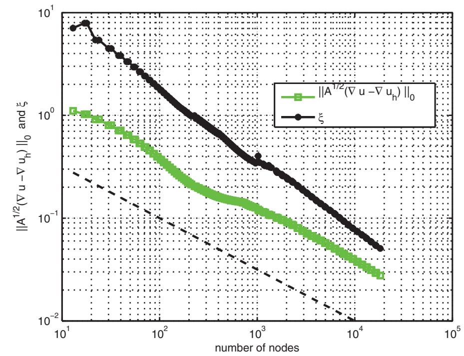


FIG. 2. *Error and extimator $\xi$.*

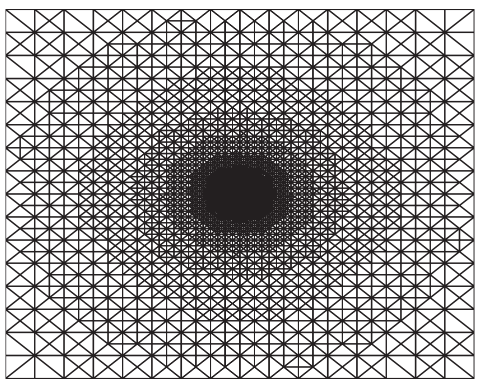

FIG. 3. *Mesh generated by $\eta$.*

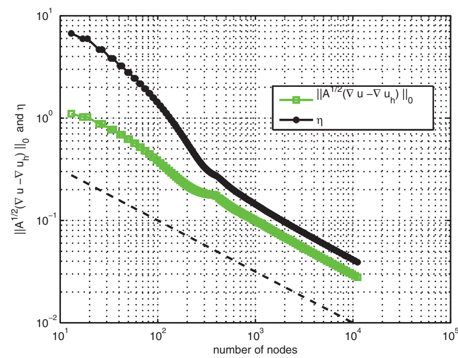


FIG. 4. *Error and extimator $\eta$.*

$\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d}$ is given by (see Remark 3.2)

$$\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,d} = -\frac{\left(A^{-1}\boldsymbol{\sigma}^{\Delta}_{\mathcal{T}_z,e},\, \nabla^{\perp}\phi_z\right)}{(A^{-1}\nabla^{\perp}\phi_z,\, \nabla^{\perp}\phi_z)}\,\nabla^{\perp}\phi_z,$$

where $\phi_z$ is the nodal basis function associated with the vertex $z \in \mathcal{N}$.

Figures 1 and 2 clearly show that adaptive mesh refinement using $\xi$ as the indicator introduces unnecessary refinements along the interfaces and that the convergence rate is not optimal.

Mesh generated by $\eta$ is shown in Figure 3. The refinement is centered at origin, The mesh is optimal with no overrefinement along the interface. Similar meshes for this test problem generated by other error estimators can be found in [14, 15]. The comparison of the error and the $\eta$ is shown in Figure 4. The effectivity index is close to 1.4. Moreover, the slope of the log(dof)-log(relative error) for $\eta$ is $-1/2$, which indicates the optimal decay of the error with respect to the number of unknowns.

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