eigenvalue/eigenfunction

$u'' + \lambda u = 0$ on $(0, L)$

$u_x(0) = u_x(L) = 0$.

① $\lambda < 0$, $\lambda = -\mu^2$

$u = c_1 e^{-\mu x} + c_2 e^{\mu x}$

$u_x = -\mu c_1 e^{-\mu x} + c_2 \mu e^{\mu x}$

$u_x(0) = \mu(-c_1 + c_2) = 0$

$u_x(L) = \mu(-c_1 e^{-\mu L} + c_2 e^{\mu L}) = 0$

$$\begin{bmatrix} -1 & 1 \\ -e^{-\mu L} & e^{\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\det \downarrow = -e^{-\mu L} + e^{\mu L} \neq 0$

$\Rightarrow c_1 = c_2 = 0$. so trivial!

or $u = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$

$u_x = c_1 \mu \sinh(\mu x) + c_2 \mu \cosh(\mu x)$

$0 = u_x(0) = c_1 \mu \cdot 0 + c_2 \mu \cdot 1 = 0$

$\Rightarrow c_2 = 0$.

$0 = u_x(L) = c_1 \mu \sinh(\mu L)$

$\sinh(\mu L) \neq 0 \Rightarrow c_1 = 0$ $\Big\} \Rightarrow$ trivial

② $\lambda = 0$. $u = c_1 + c_2 x$

$u_x = c_2$, $u_x(0) = u_x(L) = 0 \Rightarrow c_2 = 0$

$\Rightarrow$ $\boxed{u = c \text{ or } u = 1}$ is eigenfunction for $\lambda = 0$

③ $\lambda > 0$. $\lambda = \mu^2 > 0$

$u = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

$u_x = -c_1 \mu \sin \mu x + c_2 \mu \cos(\mu x)$

$u_x(0) = 0 \Rightarrow c_2 = 0$, $u_x(L) = 0 \Rightarrow \sin(\mu L) = 0$

$\mu L = n\pi \Rightarrow \mu = \frac{n\pi}{L} \Rightarrow$ $\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad u_n = \cos\left(\frac{n\pi}{L} x\right)}$

Don't worry about

expansion in $\sin\left(\frac{2n-1}{L} \pi\right)$!

no such problem in the exam.

$u'' + \lambda u = 0$ on $(0, L)$.

$u(0) = 0$, $u_x(L) = 0$.

① $\lambda < 0$. $\lambda = -\mu^2$

$u = c_1 e^{-\mu x} + c_2 e^{\mu x}$

$u_x = -\mu c_1 e^{-\mu x} + c_2 \mu e^{\mu x}$

$u(0) = 0 \Rightarrow c_1 + c_2 = 0$

$u_x(L) = 0 \Rightarrow \mu(-c_1 e^{-\mu L} + c_2 e^{\mu L}) = 0$

$$\begin{bmatrix} 1 & 1 \\ -e^{-\mu L} & e^{\mu L} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$\det(\downarrow) = e^{\mu L} + e^{-\mu L} \neq 0$

$\Rightarrow c_1 = c_2 = 0$ trival sol.

or $u = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$

$u_x = c_1 \mu \sinh(\mu x) + c_2 \mu \cosh(\mu x)$

~~$0 = u_x(0)$~~ $0 = u(0) = c_1 \Rightarrow c_1 = 0$

$u_x = c_2 \mu \cosh(\mu x)$. $u_x(L) = 0$

$\Rightarrow c_2 \mu \cosh(\mu L) = 0 \Rightarrow c_2 = 0$

so only trivial solution!

② $\lambda = 0$. $u = c_1 + c_2 x$

$u_x = c_2$. $u(0) = 0 \Rightarrow c_1 = 0$, $u_x(L) = 0 \Rightarrow c_2 = 0$

$\Rightarrow$ trival.

③ $\lambda > 0$. $\lambda = \mu^2 > 0$.

$u = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

$u_x = -c_1 \mu \sin \mu x + c_2 \mu \cos(\mu x)$

$u(0) = 0 \Rightarrow c_1 = 0$

$u_x(L) = 0 \Rightarrow \cos(\mu L) = 0$

$\Rightarrow \mu L = (n - \tfrac{1}{2})\pi$. $\mu = \frac{(n - \frac{1}{2})\pi}{L}$

$$\boxed{\lambda_n = \left(\frac{n - \frac{1}{2}}{L} \pi\right)^2, \quad u_n = \sin\left(\frac{n - \frac{1}{2}}{L} \pi x\right)}$$