Recovery-Based A Posteriori Error Estimators for Elliptic Equations

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Recent Advances in A Posteriori Error Estimation and Adaptive Methods, SIAM Annual Meeting, 2008

Outline

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Scalar Elliptic Equations

- Scalar Elliptic Equations

$$-\nabla \cdot (A(x)\nabla u) + \mathbf{b} \cdot \nabla u + cu = f \quad \text{in } \Omega \subset \mathcal{R}^d$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot A\nabla u = 0 \quad \text{on } \Gamma_N$$

- Let $X\,v = \mathbf{b} \cdot \nabla v + c\,v$, rewrite the equation as

$$-\nabla \cdot (A(x)\nabla u) + X\,u = f \quad \text{in } \Omega \subset \mathcal{R}^d$$

- Diffusion Dominant Case only

L^2 Projection Recovery

- The finite element solution $u_h \in \mathcal{U}_k$, $\mathcal{U}_k$ $(k \geq 1)$ is the piecewise continuous $k-th$ degree polynomial finite element space.

$$(A\nabla u_h, \nabla v) + (Xu_h, v) = (f, v) \quad \forall v \in \mathcal{U}_k$$

- Quantity to recover: the flux $\sigma = -A\nabla u \in H(div)$
- Recovered flux σ_h lies in $H(div)$ conforming finite element space RT_{k-1} or BDM_k.
- Find $\sigma_h \in RT_{k-1}/BDM_k$, s.t.

$$(A^{-1}\sigma_h, \tau) = -(\nabla u_h, \tau) \quad \forall \tau \in RT_{k-1}/BDM_k$$

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L^2 Projection Error Estimators

- L^2 projection error estimator

$$\xi_{L2} = \|A^{1/2}\nabla u_h + A^{-1/2}\sigma_h\|_{0,\Omega}$$

- When Linear Elements and Diffusion Dominated, even for discontinuous A

$$\xi_{L2} \sim \|A^{1/2}\nabla(u - u_h)\|_{0,\Omega}$$

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Existing recovery based error estimators for higher order finite elements

- (Bank, Xu, and B. Zhang 2007)
  Superconvergence, gradient recovery
- (Naga and Z. Zhang 2005)
  Polynomial Preserving Recovery of the gradient on mildly structured mesh
- (Bartels and Carstensen 2002)
  Averaging Scheme for the gradient, Poisson equations

Why L^2 recovery works for linear elements?

- Residual based error estimator:

$$\eta_{Res}^2 := \sum_{e\in\mathcal{E}} h_e \|[A\nabla u_h \cdot \mathbf{n}]\|_{0,e}^2 + \sum_{K\in\mathcal{T}} h_K^2 \|f + \nabla\cdot(A\nabla u_h) - Xu_h\|_{0,K}^2$$

- (Carstensen and Verfürth 1999), for the linear element case, edge jump terms are dominant, and element residual terms are higher order terms.
- $\eta_{edge}^2 = \sum_{e\in\mathcal{E}} h_e \|[A\nabla u_h \cdot \mathbf{n}]\|_{0,e}^2$
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Why L^2 recovery may fail for higher order elements?

- (D.Yu 91), for rectangular grids, edge jump terms are dominant for the odd-order element case, while element residual terms are dominant for the even-order element case.
- Simple L^2 projection recovery of the flux may fail for higher order finite elements.

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How to fix it?

Recover a σ, such that element residual terms are higher order terms.

An H(div) Problem Recovery

- Find $\boldsymbol{\sigma}_h \in RT_{k-1}/BDM_k$, s.t.,

$$(A^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\nabla\cdot\boldsymbol{\sigma}_h, \nabla\cdot\boldsymbol{\tau}) = \\ (-\nabla u_h, \boldsymbol{\tau}) + (f - Xu_h, \nabla\cdot\boldsymbol{\tau}) \quad \forall\boldsymbol{\tau} \in RT_{k-1}/BDM_k$$

- Too costly to solve?
Fast Full-Multigrid H(div) Solvers or Direct Solvers

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New Error Estimator

$$\xi_{hdiv,K} = \|A^{-1/2}\boldsymbol{\sigma}_h + A^{1/2}\nabla u_h\|_{0,K}.$$

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Analysis

- $e = u - u_h$.
- Notation: $\|h\, g(x)\|_0 = (\sum_{K\in\mathcal{T}} h_K^2 \|g\|_{0,K}^2)^{1/2}$
- Reliability bound.

$$\|A^{1/2}\nabla e\|_0 \le C(\xi_{hdiv} + \|h(f - Xu_h - \nabla\cdot\sigma_h)\|_0)$$

- Is $\|h(f - Xu_h - \nabla\cdot\sigma_h)\|_0$ is of higher order compared to ξ_{hdiv}?

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- Since $(f - Xu_h - \nabla \cdot \boldsymbol{\sigma}_h, \nabla \cdot \boldsymbol{\tau}) = (\nabla u_h + A^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau})$, we can prove

$$\|f - Xu_h - \nabla \cdot \boldsymbol{\sigma}_h\|_0 \leq C\xi_{hdiv} + \|R - \mathcal{P}_{k-1}R\|_0$$

  Where $R = f - Xu_h - \nabla \cdot \boldsymbol{\sigma}_h$, and $\mathcal{P}_{k-1}$ is the L^2 projection operator onto the discontinuous piecewise polynomial space of degree $k-1$ with respect to the triangulation $\mathcal{T}$.
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Two Criteria for Recovery Based Error Estimators

- Optimality of the mesh
  - Smooth solution and uniform mesh
    - Poisson equation $-\Delta u = f$ and $u \in H^{1+k}$,
    - $u_h \in \mathcal{U}_k$ is the finite element solution in the piecewise continuous k-th degree finite element space.
    - $\mathcal{T}_h$ is the mesh with uniform mesh size h
    - N: Number of the unknowns $\approx h^{-d}$, $d = 1, 2$, or 3.
    - $\|\nabla e\|_0 \le Ch^k \|D^{1+k} u\|_0 = CN^{-k/d}$.
    - The slope of $\log(N)$-$\log(\|\nabla e\|_0)$ line is $-k/d$.
  - Adaptive mesh generated by error indicators: should have similar error decay.
- Effectivity index $\dfrac{\text{error estimator}}{\text{error}}$ is close to 1

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A 1-D P_2 element example

$-u'' = f$ on $(0, 1)$, $u(0) = u(1) = 0$, with the right-hand side function $f = 30x^4 - 20x^3$ and the exact solution $u = x^5(1 - x)$.

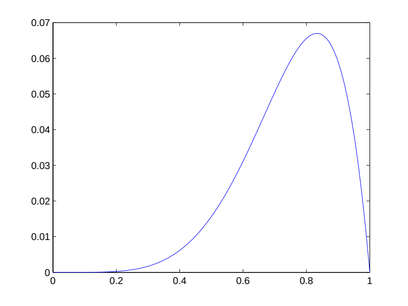


Figure: True solution u

L^2 projection error estimator fails

$u_h \in \mathcal{U}_2$ is the quadratic finite element solution.
L^2 projection recovery:
Find $\sigma_h \in \mathcal{U}_2$,s.t., $(\sigma_h, \tau) = -(u_h', \tau)\ \forall \tau \in \mathcal{U}_2$
$\xi_{L2} = \|\sigma + u_h'\|_0$

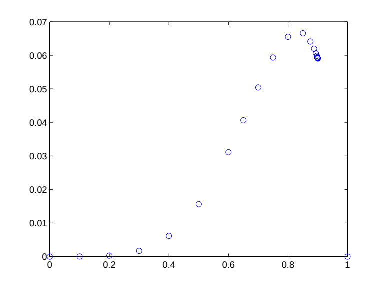


Figure: solution

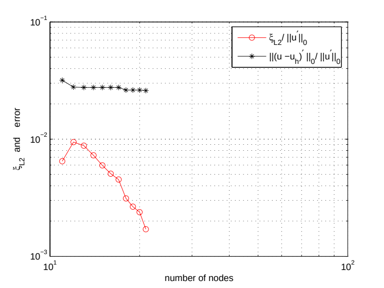


Figure: error and ξ_{L2}

Error estimator ξ_{hdiv}

$u_h \in \mathcal{U}_2$ is the quadratic finite element solution.
Recover $\sigma_h \in \mathcal{U}_2$

$$(\sigma_h, \tau) + (\sigma_h', \tau') = -(u_h', \tau) + (f, \tau') \quad \forall \tau \in \mathcal{U}_2$$

Error Estimator

$$\xi_{hdiv} = \|\sigma_h + u_h'\|_0$$

For P_2 element, error estimator ξ_{hdiv} works

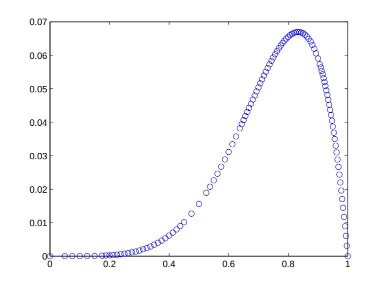


Figure: solution

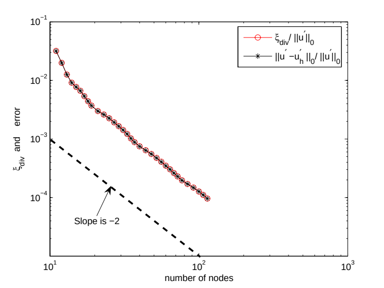


Figure: error and ξ_{hdiv}

2-D P_2 Element Example

- Interface problem

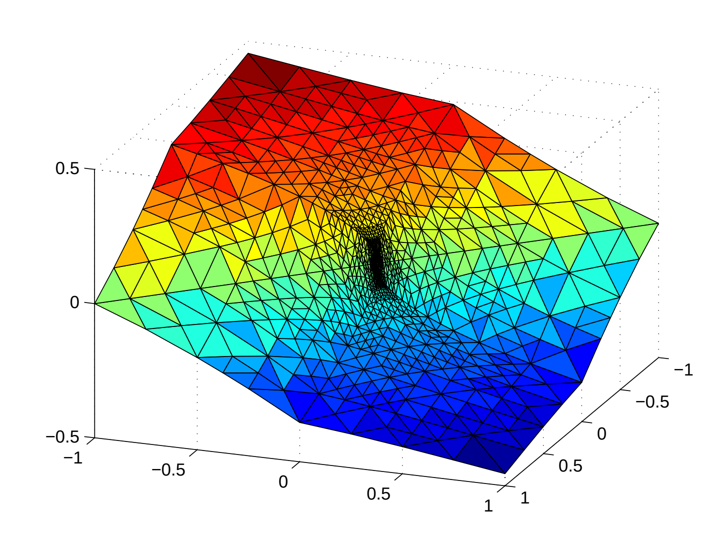
$$\left\{ \begin{array}{rcl} -\nabla \cdot (a\nabla u) & = & f \text{ in } \Omega = (-1,\, 1)^2 \\ u & = & g \text{ on } \partial\Omega \end{array} \right.$$

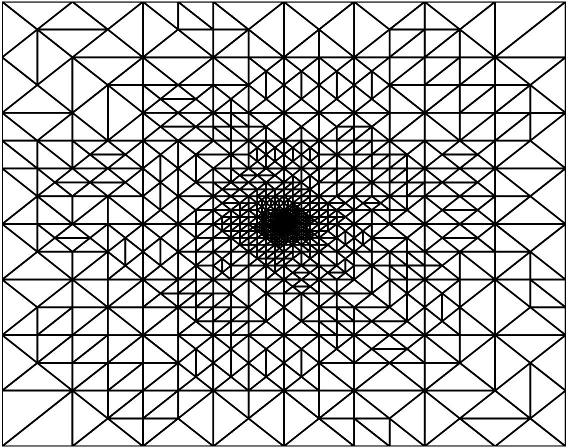
with $a = R$ in $(0,1)^2 \cup (-1,0)^2$ and 1 in $(-1,0) \times (0,1) \cup (0,1) \times (-1,0)$

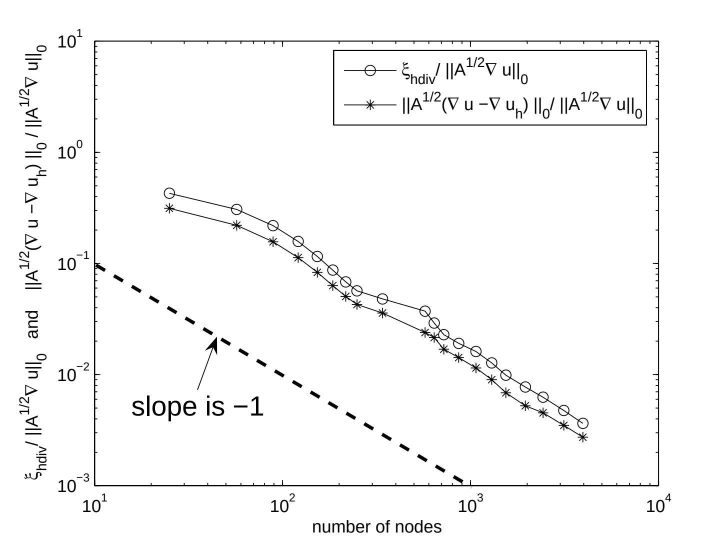
- Exact solution $u(r,\theta) = r^\alpha \mu(\theta) \in H^{1+\alpha-\epsilon}(\Omega)$ with

$$\mu(\theta) = \begin{cases} \cos((\frac{\pi}{2} - \sigma)\alpha) \cdot \cos((\theta - \frac{\pi}{2} + \rho)\alpha) & \text{if } 0 \le \theta \le \frac{\pi}{2}, \\ \cos(\rho\alpha) \cdot \cos((\theta - \pi + \sigma)\alpha) & \text{if } \frac{\pi}{2} \le \theta \le \pi, \\ \cos(\sigma\alpha) \cdot \cos((\theta - \pi - \rho)\alpha) & \text{if } \pi \le \theta \le \frac{3\pi}{2}, \\ \cos((\frac{\pi}{2} - \rho)\alpha) \cdot \cos((\theta - \frac{3\pi}{2} - \sigma)\alpha) & \text{if } \frac{3\pi}{2} \le \theta \le 2\pi. \end{cases}$$

- Example when $\alpha = 0.5$, then $R \approx 5.8284271247461907$, $\rho = \pi/4$, and $\sigma \approx -2.3561944901923448$.


0.5
0
−0.5
−1
−0.5
0
0.5
1
−1
−0.5
0
0.5
1




$\xi_{hdiv}/\ \|A^{1/2}\nabla u\|_0$
$\|A^{1/2}(\nabla u - \nabla u_h)\|_0/\ \|A^{1/2}\nabla u\|_0$
slope is −1
number of nodes
$\xi_{hdiv}/\ \|A^{1/2}\nabla u\|_0$ and $\|A^{1/2}(\nabla u - \nabla u_h)\|_0\ /\ \|A^{1/2}\nabla u\|_0$
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{1}
10^{2}
10^{3}
10^{4}

Concluding Remarks

- An Extension of L^2 Recovery
- Flux Recovery for Higher Order Finite Elements
- No Regularity Assumptions are Required