A Note on Discontinuous Galerkin Methods

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Abstract. In [4], we presented a comprehensive derivation of discontinuous Galerkin methods for elliptic interface problems and established optimal *a priori* error estimates when the solution is only in $H^{1+\alpha}$ with $\alpha \in (0, 1/2]$. In this note, we extend those work to advection-diffusion-reaction problems.

1 Introduction

Recently, Aysuo and Marini in [1] and Ern, Stephansen, and Zunino in [5] studied discontinuous Galerkin (DG) methods for advection-diffusion-reaction problems. Optimal *a priori* error estimates were established for the exact solution being at least in $H^{3/2+\epsilon}$, $\epsilon > 0$, in suitable norms. (The standard notations and definitions for Sobolev spaces will be used in this note (see, e.g., [6]).) Moreover, existing and new discontinuous Galerkin methods were derived in [1] through the so-called weighted residual approach developed in [2]. Since this is a short note, we refer readers to [1, 5] and references therein for comments and remarks on various DG methods studied by various researchers. The purposes of this note are to present a comprehensive derivation of a class of DG methods and to establish optimal *a priori* error estimates of these methods when the underlying problem is not piecewise $H^{3/2+\epsilon}$ regular.

Let Ω be a bounded polygonal domain in $\Re^2$ with boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$ and let $\mathbf{n} = (n_1, n_2)$ be the outward unit vector normal to the boundary. Let $\boldsymbol{\beta} = (\beta_1, \beta_2)^T \in W^{1,\infty}(\Omega)^2$ be the velocity vector field defined on $\overline{\Omega}$. Define inflow and outflow boundaries of $\partial\Omega$ by

$$\Gamma^- = \{x \in \partial\Omega : \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) < 0\} \quad \text{and} \quad \Gamma^+ = \{x \in \partial\Omega : \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) > 0\},$$

respectively, and let

$$\Gamma_D^\pm = \Gamma_D \cap \Gamma^\pm \quad \text{and} \quad \Gamma_N^\pm = \Gamma_N \cap \Gamma^\pm.$$

Consider the following advection-diffusion-reaction problem with discontinuous coefficients:

$$-\nabla \cdot (k(x)\nabla u) + \nabla \cdot (\boldsymbol{\beta} u) + \gamma u = f \quad \text{in } \Omega \tag{1.1}$$

with boundary conditions

$$u = g_{_D} \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \left(\boldsymbol{\beta} u \chi_{_{\Gamma_N^-}} - k\nabla u\right) = g_{_N} \quad \text{on } \Gamma_N, \tag{1.2}$$

where $f \in L^2(\Omega)$, $g_{_D} \in H^{1/2}(\Gamma_D)$, and $g_{_N} \in H^{-1/2}(\Gamma_N)$ are given functions; $\chi_{_{\Gamma_N^-}}$ is the characteristic function of the set Γ_N^-; and diffusion coefficient $k(x)$ is non-negative and piecewise

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constant on polygonal subdomains of Ω with possible large jumps across subdomain boundaries (interfaces):

$$k(x) = k_i \geq 0 \quad \text{in } \Omega_i \quad \text{for } i = 1, ..., n.$$

Here, $\{\Omega_i\}_{i=1}^n$ is a partition of the domain Ω with Ω_i being an open polygonal domain. Assume that

$$\rho(x) = \frac{1}{2}\,\mathrm{div}\boldsymbol{\beta} + \gamma \geq 0$$

and that $k(x)$ and $\rho(x)$ do not vanish in the same subdomain.

2 Jumps and Averages

For simplicity of presentation, consider only triangular elements. Let $\mathcal{T} = \{K\}$ be a finite element partition of the domain Ω. Denote by h_K the diameter of the element K. Assume that the triangulation $\mathcal{T}$ is regular. Furthermore, assume that interfaces $F = \{\partial\Omega_i \cap \partial\Omega_j : i, j = 1, ..., n\}$ do not cut through any element $K \in \mathcal{T}$.

Denote by $\mathcal{E}_K$ the set of three edges of element $K \in \mathcal{T}$. Denote the set of all edges of the triangulation $\mathcal{T}$ by

$$\mathcal{E} := \mathcal{E}_{_I} \cup \mathcal{E}_{_D} \cup \mathcal{E}_{_N},$$

where $\mathcal{E}_{_I}$ is the set of all interior element edges, and $\mathcal{E}_{_D}$ and $\mathcal{E}_{_N}$ are the sets of all boundary edges belonging to the respective Γ_D and Γ_N. Denote by $\mathcal{E}_{\Gamma^+}$ and $\mathcal{E}_{\Gamma^-}$ the sets of all boundary edges belonging to the respective Γ^+ and Γ^-. For each $e \in \mathcal{E}$, denote by h_e the length of the edge e; denote by $\mathbf{n}_e$ a unit vector normal to e. For each interior edge $e \in \mathcal{E}_{_I}$, choose $\mathbf{n}_e$ such that $\boldsymbol{\beta} \cdot \mathbf{n}_e > 0$. Let K_e^- and K_e^+ be the two elements sharing the common edge e such that the unit outward normal vector of K_e^- coincides with $\mathbf{n}_e$. When $e \in \mathcal{E}_{\Gamma^\pm}$, $\mathbf{n}_e$ is the unit outward normal vector and denote the element by $K_e^\pm$. For any $e \in \mathcal{E}$, denote by $v|_e^-$ and $v|_e^+$, respectively, the traces of a function v over e.

Define jumps over edges by

$$[\![v]\!]_e := \begin{cases} v|_e^- - v|_e^+ & e \in \mathcal{E}_I, \\ v|_e^- & e \in \mathcal{E}_{\Gamma^-}, \\ v|_e^+ & e \in \mathcal{E}_{\Gamma^+}. \end{cases}$$

Let w_e^+ and w_e^- be weights defined on e satisfying

$$w_e^+(x) + w_e^-(x) = 1, \tag{2.1}$$

and define the following weighted averages

$$\{v(x)\}_w^e = \begin{cases} w_e^- v_e^- + w_e^+ v_e^+ & e \in \mathcal{E}_I, \\ v|_e^- & e \in \mathcal{E}_{\Gamma^-}, \\ v|_e^+ & e \in \mathcal{E}_{\Gamma^+} \end{cases} \quad \text{and} \quad \{v(x)\}_e^w = \begin{cases} w_e^+ v_e^- + w_e^- v_e^+ & e \in \mathcal{E}_I, \\ v|_e^+ & e \in \mathcal{E}_{\Gamma^-}, \\ v|_e^- & e \in \mathcal{E}_{\Gamma^+} \end{cases}$$

for all $e \in \mathcal{E}$. When there is no ambiguity, the subscript or superscript e in the designation of the jump and the weighted averages will be dropped. A simple calculation leads to the following identity:

$$[\![uv]\!]_e = \{v\}_e^w\, [\![u]\!]_e + \{u\}_w^e\, [\![v]\!]_e. \tag{2.2}$$

Let e be the interface of elements K_e^+ and K_e^-:

$$e = \partial K_e^+ \cap \partial K_e^-,$$

and denote by k_e^+ and k_e^- the diffusion coefficients on K_e^+ and K_e^-, respectively. Denote by

$$W_e = \{k\}_w^e$$

the weighted average of k on edge e. For boundary edges, set

$$w_e^- = 1, \quad W_e = k_e^- \quad \text{if} \quad e \in \Gamma^- \quad \text{and} \quad w_e^+ = 1, \quad W_e = k_e^+ \quad \text{if} \quad e \in \Gamma^+.$$

3 Discontinuous Variational Formulation

Following our previous work in [4], we present a comprehensive derivation of discontinuous Galerkin methods. The key of this derivation is the introduction of a proper solution space in which integrals over inter-edges are well-defined. Moreover, the proper solution space is crucial for *a priori* error estimates of the underlying problem with low regularity.

Let u be the solution of problem (1.1-1.2), then it is well known from the regularity estimate [6] that u is in $H^{1+\alpha}(\Omega)$ for some positive α which could be very small. Since $f \in L^2(\Omega)$, it is then easy to see that divergences of the diffusion and advection fluxes, $-k\nabla u$ and $\boldsymbol{\beta} u$, are square integrable, i.e.,

$$-k\nabla u, \;\; \boldsymbol{\beta} u \in H(\text{div};\Omega) \equiv \{\boldsymbol{\tau} \in L^2(\Omega)^2 \;:\; \nabla \cdot \boldsymbol{\tau} \in L^2(\Omega)\}. \tag{3.1}$$

Moreover, by the imbedding theorem (see, e.g., [6]), we have

$$\boldsymbol{\sigma} \equiv -k\nabla u \in L^r(\Omega) \tag{3.2}$$

for all $2 < r$ if $\alpha \geq 1$ or for all $2 < r < 2/(1-\alpha)$ if $\alpha < 1$. Hence, we consider the following solution space

$$V^{1+\epsilon}(\mathcal{T}) = \{v \in H^{1+\epsilon}(\mathcal{T}) \;:\; \nabla \cdot (k\nabla v) \in L^2(K) \quad \forall\, K \in \mathcal{T}\} \tag{3.3}$$

for $0 < \epsilon \ll 1$, where $H^s(\mathcal{T})$ is the broken Sobolev space of degree $s > 0$ with respect to $\mathcal{T}$:

$$H^s(\mathcal{T}) = \{v \in L^2(\Omega) : v|_K \in H^s(K) \quad \forall\, K \in \mathcal{T}\}.$$

For any w and any v in $V^{1+\epsilon}(\mathcal{T})$ with $\epsilon > 0$ and for any $K \in \mathcal{T}$, one has the following Green's formula:

$$\int_{\partial K} (k\nabla w \cdot \mathbf{n})\, v\, ds := \langle k\nabla w \cdot \mathbf{n}, v\rangle_{\partial K} = (\nabla \cdot (k\nabla w),\, v)_K + (k\nabla w,\, \nabla v)_K. \tag{3.4}$$

By the trace theorem [6], $v|_{\partial K}$ is in $H^{1/2+\epsilon}(\partial K) \subset H^{1/2-\epsilon}(\partial K)$. Hence, the formal boundary integral in the left-hand side of (3.4) may be regarded as the duality pairing between $H^{\epsilon-1/2}(\partial K)$ and $H^{1/2-\epsilon}(\partial K)$, which is defined by the right-hand side of (3.4). Since for each edge $e \subset \partial K$, the trivial extension of functions in $H^{1/2-\epsilon}(e)$ by zero to all of ∂K belongs to $H^{1/2-\epsilon}(\partial K)$ (see, e.g., Theorem 1.5.2.3 in [6]), this interpretation enables us to define the duality pairing on each edge e of ∂K,

$$\int_e (k\nabla w \cdot \mathbf{n})\, v\, ds := \langle k\nabla w \cdot \mathbf{n}, v\rangle_e,$$

where $(k\nabla w\cdot\mathbf{n})|_e \in H^{\epsilon-1/2}(e)$ and $v|_e\in H^{1/2-\epsilon}(e)$. Moreover, by the definition of the dual norm, we have

$$|\int_e (k\nabla w\cdot\mathbf{n})\,v\,ds| \le \|k\nabla w\cdot\mathbf{n}\|_{\epsilon-1/2,e}\|v\|_{1/2-\epsilon,e}. \tag{3.5}$$

Denote the discrete gradient and divergence operators by

$$(\nabla_h v)|_K = \nabla(v|_K) \quad\text{and}\quad (\nabla_h\cdot\boldsymbol{\tau})|_K = \nabla\cdot(\boldsymbol{\tau}|_K),$$

for all $K\in\mathcal{T}$, respectively. Multiplying equation (1.1) by a test function $v\in V^{1+\epsilon}(\mathcal{T})$, integrating by parts, and using boundary conditions (1.2), we have

$$\begin{aligned}
(f,\,v) &= \sum_{K\in\mathcal{T}}(k\nabla u,\,\nabla v)_K - \sum_{K\in\mathcal{T}}\int_{\partial K}(k\nabla u\cdot\mathbf{n})\,v\,ds\\
&\quad+\sum_{K\in\mathcal{T}}(u,\,-\boldsymbol{\beta}\cdot\nabla v+\gamma v)_K+\sum_{K\in\mathcal{T}}\int_{\partial K}(\boldsymbol{\beta}u\cdot\mathbf{n})\,v\,ds\\
&= (k\nabla_h u,\nabla_h v)-\sum_{e\in\mathcal{E}_I}\int_e[\![(k\nabla u\cdot\mathbf{n}_e)\,v]\!]\,ds-\sum_{e\in\mathcal{E}_D}\int_e(k\nabla u\cdot\mathbf{n}_e)\,v\,ds\\
&\quad+(u,\,-\boldsymbol{\beta}\cdot\nabla_h v+\gamma v)+\sum_{e\in\mathcal{E}_I}\int_e[\![\boldsymbol{\beta}u\cdot\mathbf{n}_e v]\!]\,ds+\sum_{e\in\mathcal{E}_{D^-}}\int_e(\boldsymbol{\beta}u\cdot\mathbf{n}_e)\,v\,ds\\
&\quad+\sum_{e\in\mathcal{E}_N}\int_e(-k\nabla u\cdot\mathbf{n}_e)\,vds+\sum_{e\in\mathcal{E}_{N^-}}\int_e(\boldsymbol{\beta}u\cdot\mathbf{n}_e)vds+\sum_{e\in\mathcal{E}_{\Gamma+}}\int_e(\boldsymbol{\beta}u\cdot\mathbf{n}_e)vds\\
&= (k\nabla_h u,\nabla_h v)-\sum_{e\in\mathcal{E}_I}\int_e[\![(k\nabla u\cdot\mathbf{n}_e)\,v]\!]\,ds-\sum_{e\in\mathcal{E}_D}\int_e(k\nabla u\cdot\mathbf{n}_e)\,v\,ds\\
&\quad+(u,\,-\boldsymbol{\beta}\cdot\nabla_h v+\gamma v)+\sum_{e\in\mathcal{E}_I}\int_e[\![\boldsymbol{\beta}u\cdot\mathbf{n}_e v]\!]\,ds+\sum_{e\in\mathcal{E}_{D^-}}\int_e(\boldsymbol{\beta}\cdot\mathbf{n}_e)g_D\,v\,ds\\
&\quad+\sum_{e\in\mathcal{E}_N}\int_e g_N vds+\sum_{e\in\mathcal{E}_{\Gamma+}}\int_e(\boldsymbol{\beta}u\cdot\mathbf{n}_e)vds,
\end{aligned}$$

where $\mathcal{E}_{D^-}=\mathcal{E}_D\cap\Gamma^-$ and $\mathcal{E}_{N^-}=\mathcal{E}_N\cap\Gamma^-$. Note that the Dirichlet boundary condition is used on the inflow boundary. Since (3.1) implies continuities of the diffusion and advection fluxes:

$$\int_e[\![k\nabla u\cdot\mathbf{n}_e]\!]\,\{v\}^w\,ds=0 \quad\text{and}\quad \int_e[\![\boldsymbol{\beta}u\cdot\mathbf{n}_e]\!]\,\{v\}^w\,ds=0 \quad \forall\, e\in\mathcal{E}_I,\quad \forall\, v\in V^{1+\epsilon}(\mathcal{T}), \tag{3.6}$$

by identity (2.2) and the Dirichlet boundary condition in (1.2), we then have

$$\begin{aligned}
&(k\nabla_h u,\nabla_h v)-\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{k\nabla u\cdot\mathbf{n}_e\}_w[\![v]\!]\,ds+(u,\,-\boldsymbol{\beta}\cdot\nabla_h v+\gamma v)\\
&\quad+\sum_{e\in\mathcal{E}_I}\int_e\{\boldsymbol{\beta}u\cdot\mathbf{n}_e\}_w[\![v]\!]\,ds+\sum_{e\in\mathcal{E}_{\Gamma+}}\int_e(\boldsymbol{\beta}u\cdot\mathbf{n}_e)\,v\,ds\\
&= (f,\,v)-\int_{\Gamma_N}g_N\,v\,ds-\sum_{e\in\mathcal{E}_{D^-}}\int_e(\boldsymbol{\beta}\cdot\mathbf{n}_e)g_D\,v\,ds.
\end{aligned} \tag{3.7}$$

for all $v \in V^{1+\epsilon}(\mathcal{T})$.

Since the derivation of (3.7) does not make use of the continuity of the solution, one needs to impose such a continuity in order to achieve stability. To do so, it is natural and well-known to stabilize the diffusion and the advection operators by adding proper jump terms of the solution. Following the idea of [5] (see also [4]), we stabilize the diffusion operator by adding the following equation:

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_\theta h_e^{-1} W_e [\![u]\!][\![v]\!]\, ds = \sum_{\epsilon\in\mathcal{E}_D} \gamma_\theta h_e^{-1} W_e \int_e g_D v ds \quad \forall\, v \in V^{1+\epsilon}(\mathcal{T}). \tag{3.8}$$

Since the diffusion operator is self-adjoint, it is then natural to symmetrize the diffusion part of (3.7) by adding the following equation:

$$\theta \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla v\cdot \mathbf{n}_e\}_w [\![u]\!]\, ds = \theta \sum_{e\in\mathcal{E}_D} \int_e g_D (k\nabla v\cdot \mathbf{n}_e) ds \quad \forall\, v \in V^{1+\epsilon}(\mathcal{T}) \tag{3.9}$$

with $\theta = -1$. Both (3.8) and (3.9) follow from the continuity of $u \in H^{1+\alpha}(\Omega)$ and the Dirichlet boundary condition. When $\theta = 1$, (3.9) plays a role of stabilization and, hence, (3.8) is not needed.

For the advection-reaction term, introduce the following general upwind average:

$$\{\boldsymbol{\beta} u\cdot \mathbf{n}_e\}_{up}^e = \xi_e^-(\boldsymbol{\beta} u^-\cdot \mathbf{n}_e) + \xi_e^+(\boldsymbol{\beta} u^+\cdot \mathbf{n}_e), \quad \xi_e^- + \xi_e^+ = 1, \quad \xi^- > 1/2, \tag{3.10}$$

which is more general than that in [1] since ξ_e^+ could be negative. When $\xi^- = 1$, (3.10) is the classic upwind. As pointed out in [3], the jump-stabilization is more general than the classic upwind. But it is easy to see that the jump-stabilization is equivalent to (3.10). In this note, we employ (3.10) in (3.7).

Now, define bilinear forms for $u, v \in V^{1+\epsilon}(\mathcal{T})$ by

$$\begin{aligned} a_{d,\theta}(u,v) &= (k\nabla_h u, \nabla_h v) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_\theta h_e^{-1} W_e [\![u]\!][\![v]\!]\, ds \\ &\quad - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla u\cdot \mathbf{n}_e\}_w [\![v]\!] ds + \theta \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla v\cdot \mathbf{n}_e\}_w [\![u]\!] ds \end{aligned}$$

for $\theta \in \{-1, 0, 1\}$ and

$$a_c(u,v) = (u, -\boldsymbol{\beta}\cdot\nabla_h v + \gamma v) + \sum_{e\in\mathcal{E}_I} \int_e \{\boldsymbol{\beta} u\cdot \mathbf{n}_e\}_{up} [\![v]\!]\, ds + \sum_{e\in\mathcal{E}_{\Gamma+}} \int_e (\boldsymbol{\beta} u\cdot \mathbf{n}_e)\, v\, ds$$

and a linear form for $v \in V^{1+\epsilon}(\mathcal{T})$ by

$$\begin{aligned} f_\theta(v) &= (f, v) + \sum_{\epsilon\in\mathcal{E}_D} \gamma_\theta h_e^{-1} W_e \int_e g_D v\, ds + \sum_{\epsilon\in\mathcal{E}_N} \int_e g_N v\, ds \\ &\quad + \theta \sum_{e\in\mathcal{E}_D} \int_e g_D (k\nabla v\cdot \mathbf{n}_e)\, ds - \sum_{e\in\mathcal{E}_{D-}} \int_e (\boldsymbol{\beta}\cdot \mathbf{n}_e) g_D\, v\, ds. \end{aligned}$$

Using (3.7), (3.9), and (3.8) and choosing the general upwind average for the advection flux, then the weak solution of (1.1–1.2) satisfies the following variational problem: find $u \in V^{1+\epsilon}(\mathcal{T})$ such that

$$a_\theta(u,\, v) \equiv a_{d,\theta}(u,\, v) + a_c(u,\, v) = f_\theta(v) \quad \forall\, v \in V^{1+\epsilon}(\mathcal{T}). \tag{3.11}$$

4 Discontinuous Finite Element Approximation

Let $P_k(K)$ be the space of polynomials of degree k on element $K \in \mathcal{T}$. Denote the discontinuous Galerkin linear finite element space associated with the triangulation $\mathcal{T}$ by

$$\mathcal{U}^{DG} = \{v \in L^2(\Omega) \,:\, v|_K \in P_1(K) \;\forall\, K \in \mathcal{T}\}.$$

Discontinuous Galerkin (DG) finite element method is to find $u_{\mathcal{T}} \in \mathcal{U}^{DG} \subset V^{1+\epsilon}(\mathcal{T})$ such that

$$a_\theta(u_{\mathcal{T}}, v) = f_\theta(v) \quad \forall\, v \in \mathcal{U}^{DG}. \tag{4.1}$$

The method corresponding to $\theta = -1$ and the classic upwind was introduced and analyzed recently in [5] for different boundary conditions. When $k(x) = \varepsilon$, the methods corresponding to $\theta = 0$, 1 and the classic upwind reproduce the first two methods in [1]; the third (introduced in [7]) and fourth methods in [1] are corresponding to (4.1) with the respective classic and general upwind averages for both the diffusion and advection terms. It is not clear to us if it is necessary to use an upwind average for the diffusion term. *A priori* error bounds for DG methods had been established by various researchers (see [1, 5] and references therein) provided that the solution is at least piecewise $H^{3/2+\epsilon}$ smooth and that γ_θ is large enough.

In the remainder of this section, we describe the coercivity of the bilinear form $a_\theta(\cdot,\,\cdot)$ in $\mathcal{U}^{DG}$ that implies the well-posedness of (4.1). To this end, for any $v \in \mathcal{U}^{DG}$, define the DG norms for the diffusion and advection-reaction parts by

$$|\!|\!| v |\!|\!|_d^2 = \|k^{1/2}\nabla_h v\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e^{-1} W_e \|[\![v]\!]\|_{0,e}^2 \quad \text{and} \quad |\!|\!| v |\!|\!|_c^2 = \|\rho^{1/2} v\|_0^2 + \sum_{e\in\mathcal{E}} \|c_e^{1/2}[\![v]\!]\|_{0,e}^2,$$

respectively, where $\rho(x) = \dfrac{1}{2}\mathrm{div}\boldsymbol{\beta} + \gamma$ and

$$c_e = \begin{cases} \left(\xi^- - \dfrac{1}{2}\right)\boldsymbol{\beta}\cdot\mathbf{n}_e & \text{on} \quad e \in \mathcal{E}_I, \\ \dfrac{1}{2}\boldsymbol{\beta}\cdot\mathbf{n}_e & \text{on} \quad e \in \mathcal{E}_{\Gamma+}, \\ -\dfrac{1}{2}\boldsymbol{\beta}\cdot\mathbf{n}_e & \text{on} \quad e \in \mathcal{E}_{\Gamma-}. \end{cases}$$

The DG norm is defined by

$$|\!|\!| v |\!|\!|_{DG} = \left(|\!|\!| v |\!|\!|_d^2 + |\!|\!| v |\!|\!|_c^2\right)^{1/2}.$$

It was proved in [4] that the bilinear form $a_{d,\theta}(\cdot,\,\cdot)$ is coercive in $\mathcal{U}^{DG}$ with respect to the norm $|\!|\!|\cdot|\!|\!|_d$. For the convenience of readers, we state below.

Lemma 4.1. (i) *The bilinear form $a_{d,1}(\cdot,\cdot)$ is coercive in $\mathcal{U}^{DG}$ with the coercivity constant* $\min\{1,\gamma_1\}$ *provided* $\gamma_1 > 0$, *i.e,*

$$a_{d,1}(v,v) \ge \min\{1,\gamma_1\} |\!|\!| v |\!|\!|_d^2 \quad \forall\, v \in \mathcal{U}^{DG}. \tag{4.2}$$

(ii) *Let w_e^+ and w_e^- be weights satisfying* (2.1)*, then there exists a positive constant α_0 such that*

$$a_{d,\theta}(v,v) \ge \alpha_0 |\!|\!| v |\!|\!|_d^2 \quad \forall\, v \in \mathcal{U}^{DG} \tag{4.3}$$

for $\theta = -1$ and 0, provided that γ_θ is great than a computable constant independent of the jump of the diffusion coefficients and the mesh size.

By noting identity (4.5) below, the coercivity of the bilinear form $a_c(\cdot,\,\cdot)$ may be proved in a similar fashion as that in [3]. The proof is provided here for the convenience of readers.

Lemma 4.2. *For any $v \in \mathcal{U}^{DG}$, we have*

$$a_c(v,v) = \|v\|_c^2. \tag{4.4}$$

Proof. Let $v \in \mathcal{U}^{DG}$. It is easy to check the following algebraic identity

$$\{v\}_{up}[\![v]\!] = \frac{1}{2}[\![v^2]\!] + (\xi^- - 1/2)[\![v]\!]^2,$$

which, combining with the continuity of $\boldsymbol{\beta}\cdot\mathbf{n}_e$, yields

$$\{\boldsymbol{\beta} v\cdot\mathbf{n}_e\}_{up}[\![v]\!]_e = \frac{1}{2}(\boldsymbol{\beta}\cdot\mathbf{n}_e)[\![v^2]\!]_e + c_e[\![v]\!]_e^2 \tag{4.5}$$

on edge $e \in \mathcal{E}_I$ with $c_e = (\xi_e^- - 1/2)(\boldsymbol{\beta}_e\cdot\mathbf{n}_e)$. Hence,

$$\sum_{e\in\mathcal{E}_I}\int_e \{\boldsymbol{\beta} v\cdot\mathbf{n}_e\}_{up}[\![v]\!]_e\,ds = \frac{1}{2}\sum_{e\in\mathcal{E}_I}\int_e(\boldsymbol{\beta}_e\cdot\mathbf{n}_e)[\![v^2]\!]_e\,ds + \sum_{e\in\mathcal{E}_I}\int_e c_e[\![v]\!]_e^2\,ds. \tag{4.6}$$

For any $K\in\mathcal{T}$, integration by parts gives the following identity:

$$-\int_K(\boldsymbol{\beta} v)\cdot\nabla v dx = \int_K(\frac{1}{2}\mathrm{div}\boldsymbol{\beta})v^2dx - \frac{1}{2}\int_{\partial K}(\boldsymbol{\beta}\cdot\mathbf{n})v^2ds.$$

Summing over $K\in\mathcal{T}$ and using the continuity of $\boldsymbol{\beta}\cdot\mathbf{n}_e$ give

$$-(\boldsymbol{\beta}\cdot\nabla_h v,\,v) = \left((\frac{1}{2}\mathrm{div}\boldsymbol{\beta})v,\,v\right) - \frac{1}{2}\sum_{e\in\mathcal{E}_I}\int_e(\boldsymbol{\beta}\cdot\mathbf{n}_e)[\![v^2]\!]ds - \frac{1}{2}\sum_{e\in\mathcal{E}_{\Gamma-}\cup\mathcal{E}_{\Gamma+}}\int_e(\boldsymbol{\beta}\cdot\mathbf{n}_e)v^2ds.$$

Now, it follows from (4.6) that

$$\begin{aligned}
a_c(v,v) &= -(v,\,\boldsymbol{\beta}\cdot\nabla_h v) + (\gamma v,\,v) + \sum_{e\in\mathcal{E}_I}\int_e\{\boldsymbol{\beta} v\cdot\mathbf{n}_e\}_{up}[\![v]\!]\,ds + \sum_{e\in\mathcal{E}_{\Gamma+}}\int_e(\boldsymbol{\beta}\cdot\mathbf{n}_e)\,v^2\,ds \\
&= (\rho v,\,v) - \frac{1}{2}\sum_{e\in\mathcal{E}_{\Gamma-}\cup\mathcal{E}_{\Gamma+}}\int_e(\boldsymbol{\beta}\cdot\mathbf{n}_e)v^2ds + \sum_{e\in\mathcal{E}_I}\int_e c_e[\![v]\!]_e^2\,ds + \sum_{e\in\mathcal{E}_{\Gamma+}}\int_e(\boldsymbol{\beta}\cdot\mathbf{n}_e)v^2\,ds \\
&= (\rho v,\,v) + \sum_{e\in\mathcal{E}}\int_e c_e[\![v]\!]_e^2\,ds = \|v\|_c^2.
\end{aligned}$$

This completes the proof of the lemma. □

Theorem 4.3. *Under the assumptions in* Lemma 4.1*, there exists a positive constant α such that*

$$a_\theta(v,v) \ge \alpha\|v\|_{DG}^2 \quad \forall\ v\in\mathcal{U}^{DG}. \tag{4.7}$$

5 A Priori Error Estimation

Difference of (3.11) and (4.1) yields the following error equation:

$$a_\theta(u - u_{\mathcal{T}},\, v) = 0 \quad \forall\ v \in \mathcal{U}^{DG}. \tag{5.1}$$

Let $P_{\mathcal{T}}$ be the L^2 projection onto $\mathcal{U}^{DG}$. Note that $P_{\mathcal{T}}$ is the local L^2 projection onto $P_1(K)$ for all $K \in \mathcal{T}$. Hence, for any $v \in H^{1+s}(\mathcal{T})$, $s \in [0,1]$, and any $K \in \mathcal{T}$, the following estimate holds

$$\|v - P_{\mathcal{T}} v\|_{0,K} + h\,\|\nabla(v - P_{\mathcal{T}} v)\|_{0,K} + h^{1+\epsilon}\|\nabla(v - P_{\mathcal{T}} v)\|_{\epsilon,K} \le C h^{1+s}\|\nabla v\|_{s,K}. \tag{5.2}$$

For any $v \in H^{1+s}(\mathcal{T})$, $0 < s \le 1$, denote by

$$B_s(h,\, v) = \left(\sum_{K\in\mathcal{T}} h_K^{2(s-\epsilon)}\|k^{1/2}\nabla v\|_{s,K}^2\right)^{1/2} + \left(\sum_{K\in\mathcal{T}} h_K^2\|k^{1/2}\Delta v\|_{0,K}^2\right)^{1/2}.$$

Set

$$z = u - P_{\mathcal{T}} u \quad \text{and} \quad z_{\mathcal{T}} = u_{\mathcal{T}} - P_{\mathcal{T}} u.$$

Lemma 5.1. *Assume that the solution $u \in V^{1+\epsilon}(\mathcal{T})$ of problem* (3.11) *belongs to $H^{1+s}(\mathcal{T})$ with $0 < \epsilon \le s \le 1$. Then*

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla z \cdot \mathbf{n}_e\}_w [\![z_{\mathcal{T}}]\!] ds \le C B_s(h,u) |\!|\!| z_{\mathcal{T}} |\!|\!|_d. \tag{5.3}$$

Proof. (5.3) may be proved in a similar fashion as Lemma 3.1 in [4]. □

Lemma 5.2. *The following inequality holds*

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla z_{\mathcal{T}} \cdot \mathbf{n}_e\}_w [\![z]\!] ds \le |\!|\!| z_{\mathcal{T}} |\!|\!|_d |\!|\!| z |\!|\!|_d. \tag{5.4}$$

Proof. For any $e \in \mathcal{E}_I$, since (see (2.5) in [4])

$$w_e^-(k_e^-)^{1/2},\ w_e^+(k_e^+)^{1/2} \le W_e^{1/2},$$

it follows from the Cauchy-Schwarz, the triangle, and the trace inequalities that

$$\begin{aligned}
& \int_e \{k\nabla z_{\mathcal{T}} \cdot \mathbf{n}_e\}_w [\![z]\!] ds \le \|\{k\nabla z_{\mathcal{T}} \cdot \mathbf{n}_e\}_w\|_{0,e} \|[\![z]\!]\|_{0,e} \\
\le\ & \left(w_e^- \|k^-\nabla z_{\mathcal{T}}|_{K^-} \cdot \mathbf{n}_e\|_{0,e} + w_e^+ \|k^+\nabla z_{\mathcal{T}}|_{K^+} \cdot \mathbf{n}_e\|_{0,e} \right) \|[\![z]\!]\|_{0,e} \\
\le\ & \left(\|(k^-)^{1/2}\nabla z_{\mathcal{T}}|_{K^-} \cdot \mathbf{n}_e\|_{0,e} + \|(k^+)^{1/2}\nabla z_{\mathcal{T}}|_{K^+} \cdot \mathbf{n}_e\|_{0,e} \right) W_e^{1/2} \|[\![z]\!]\|_{0,e} \\
\le\ & \left(\|(k^-)^{1/2}\nabla z_{\mathcal{T}}\|_{0,K^-} + \|(k^+)^{1/2}\nabla z_{\mathcal{T}}\|_{0,K^+} \right) h_e^{-1/2} W_e^{1/2} \|[\![z]\!]\|_{0,e}.
\end{aligned}$$

Similar results hold for $e \in \mathcal{E}_D$. Now, the lemma is a consequence of summing up those inequalities. □

Lemma 5.3. *Assume that the solution u is in $H^{1+s}(\mathcal{T})$ and that $W_e \le C \min\{k_e^+, k_e^-\}$ for all $e \in \mathcal{E}_I$. Then the following estimate holds*

$$\|z\|_{DG} \le C \left(\sum_{K\in\mathcal{T}} h_K^{2s}(k_K + h_K^2\|\rho\|_{0,\infty,K} + h_K\|\boldsymbol{\beta}\|_{0,\infty,K})\|\nabla u\|_{s,K}^2 \right)^{1/2}. \tag{5.5}$$

Proof. It follows from the assumption, the trace inequality, and the approximation property in (5.2) that

$$h_e^{-1}W_e\|[\![z]\!]\|_{0,e}^2 \le C h_e^{-1}\left(\|k_-^{1/2}z\|_{0,e}^2 + \|k_+^{1/2}z\|_{0,e}^2\right) \le C \sum_{K\in\omega_e} h_K^{2s}\|k^{1/2}\nabla u\|_{s,K}^2,$$

which implies

$$\|z\|_d^2 \le C \sum_{K\in\mathcal{T}} h_K^{2s}\|k^{1/2}\nabla u\|_{s,K}^2.$$

It is easy to see from the trace inequality and the approximation property in (5.2) that

$$\|z\|_c^2 \le C \sum_{K\in\mathcal{T}} h_K^{2s}\left(h_K^2\|\rho\|_{0,\infty,K} + h_K\|\boldsymbol{\beta}\|_{0,\infty,K}\right)\|\nabla u\|_{s,K}^2.$$

Now, the lemma is a direct consequence of these two inequalities. □

Note that the harmonic average $W_e = \dfrac{2k_e^+k_e^-}{k_e^+ + k_e^-}$ satisfies the assumption in Lemma 5.3.

Theorem 5.4. *Under the assumptions in* Lemma 5.3*, the following a priori error estimate holds*

$$\|u - u_{\mathcal{T}}\|_{DG} \le C \left(\sum_{K\in\mathcal{T}} h_K^{2s}(k_K + C(h,\boldsymbol{\beta},\rho,K))\|\nabla u\|_{s,K}^2 \right)^{1/2} + B_s(h,u), \tag{5.6}$$

where

$$C(h,\boldsymbol{\beta},\rho,K) = h_K^2\|\rho\|_{0,K,\infty} + h_K^2\|\nabla\boldsymbol{\beta}\|_{0,\infty,K}^2\|\rho\|_{0,\infty,K}^{-1} + h_K\|\boldsymbol{\beta}\|_{0,\infty,K}.$$

Proof. To show the validity of (5.6), by the triangle inequality and Lemma 5.3, it suffices to prove that

$$\|z_{\mathcal{T}}\|_{DG} \le C \left(\sum_{K\in\mathcal{T}} h_K^{2s}(k_K + C(h,\boldsymbol{\beta},\rho,K))\|\nabla u\|_{s,K}^2 \right)^{1/2} + B_s(h,u). \tag{5.7}$$

To do so, by the coercivity of $a_{d,\theta}$ and the error equation in (5.1), we have

$$C\,\|z_{\mathcal{T}}\|_{DG}^2 \le a_\theta(z_{\mathcal{T}}, z_{\mathcal{T}}) = a_\theta(z, z_{\mathcal{T}}) = a_{d,\theta}(z, z_{\mathcal{T}}) + a_c(z, z_{\mathcal{T}}).$$

First, the Cauchy-Schwarz inequality and Lemmas 5.1 and 5.2 yield

$$a_{d,\theta}(z, z_{\mathcal{T}}) \le C\left(\|z\|_d + B_s(h,u)\right)\|z_{\mathcal{T}}\|_d. \tag{5.8}$$

Second, a similar proof as that for Theorem 5.1 in [1] gives

$$a_c(z, z_{\mathcal{T}}) \le C \left(\sum_{K\in\mathcal{T}} C(h,\boldsymbol{\beta},\rho,K)h_K^{2s}\|\nabla u\|_{s,K}^2 \right)^{1/2} \|z_{\mathcal{T}}\|_c. \tag{5.9}$$

Now, (5.8) and (5.9) imply (5.7) and, hence, the theorem. □

References

[1] B. Ayuso and L. D. Marini, *Discontinuous Galerkin methods for advection-diffusion-reaction problems*, SIAM J. Numer. Anal., 47:2 (2009), 1391-1420. 1, 3, 4, 5

[2] F. Brezzi, B. Cockburn, L. D. Marini, and E. Süli, *Stabilization mechanisms in discontinuous Galerkin finite element methods*, Comut. Methods Appl. Mech. Engrg., 195 (2006), 3293-3310. 1

[3] F. Brezzi, L. D. Marini, E. Süli, *Discontinuous Galerkin methods for first-order hyperbolic problems*, Math. Models Methods Appl. Sci., 14 (2004), 1893-1903. 3, 4

[4] Z. Cai, X. Ye, and S. Zhang, *Discontinuous Galerkin finite element methods for interface problems: a priori and a posteriori error estimations*, submitted. (document), 3, 3, 4, 5, 5

[5] A. Ern, A. Stephansen, and P. Zunino, *A discontinuous Galerkin method with weighted averages for advection-diffusion equations with locally small and anisotropic diffusivity*, IMA J. of Numer. Anal., 29:2 (2009), 235-256. 1, 3, 4

[6] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monographs and Studies in Mathematics, Vol. 24, Pitman, Boston, 1985. 1, 3, 3, 3

[7] T.J.R. Hughes, G. Scovazzi, P.B. Bochev, and A. Buffa, *A multiscale discontinuous Galerkin method with the computational structure of a continuous Galerkin method*, Comput. Methods Appl. Mech. Engrg., 195 (2006), 2761-2787. 4