SIAM J. NUMER. ANAL.
Vol. 49, No. 5, pp. 1761–1787

DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS FOR INTERFACE PROBLEMS: A PRIORI AND A POSTERIORI ERROR ESTIMATIONS*

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Abstract. Discontinuous Galerkin (DG) finite element methods were studied by many researchers for second-order elliptic partial differential equations, and a priori error estimates were established when the solution of the underlying problem is piecewise $H^{3/2+\epsilon}$ smooth with $\epsilon > 0$. However, elliptic interface problems with intersecting interfaces do not possess such a smoothness. In this paper, we establish a quasi-optimal a priori error estimate for interface problems whose solutions are only $H^{1+\alpha}$ smooth with $\alpha \in (0, 1)$ and, hence, fill a theoretical gap of the DG method for elliptic problems with low regularity. The second part of the paper deals with the design and analysis of robust residual- and recovery-based a posteriori error estimators. Theoretically, we show that the residual and recovery estimators studied in this paper are robust with respect to the DG norm, i.e., their reliability and efficiency bounds do not depend on the jump, provided that the distribution of coefficients is locally quasi-monotone.

Key words. a priori error estimation, a posteriori error estimator, discontinuous Galerkin methods, interface problems

AMS subject classifications. 65N30, 65N15

DOI. 10.1137/100805133

1. Introduction.

Consider the following interface problem:

$$-\nabla \cdot (k(x)\nabla u) = f \quad \text{in } \Omega \tag{1.1}$$

with boundary conditions

$$u = g_D \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot (k\nabla u) = g_N \quad \text{on } \Gamma_N, \tag{1.2}$$

where f, g_D, and g_N are given scalar-valued functions; Ω is a bounded polygonal domain in $\Re^2$ with boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$; $\mathbf{n} = (n_1, n_2)$ is the outward unit vector normal to the boundary; and diffusion coefficient $k(x)$ is positive and piecewise constant on polygonal subdomains of Ω with possible large jumps across subdomain boundaries (interfaces):

$$k(x) = k_i > 0 \quad \text{in } \Omega_i$$

for $i = 1, \ldots, n$. Here, $\{\Omega_i\}_{i=1}^n$ is a partition of the domain Ω with Ω_i being an open polygonal domain. Define

*Received by the editors August 12, 2010; accepted for publication (in revised form) June 2, 2011; published electronically September 13, 2011. This work was supported in part by the National Science Foundation under grant DMS-0810855.
http://www.siam.org/journals/sinum/49-5/80513.html
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$$k_{\min} = \min_{1\le i\le n} k_i \quad \text{and} \quad k_{\max} = \max_{1\le i\le n} k_i.$$

Assume that f is square integrable over Ω. For simplicity, we assume that g_D and g_N are piecewise linear and constant, respectively, and that Γ_D is not empty (i.e., $\text{mes}\,(\Gamma_D) \neq 0$).

Discontinuous Galerkin (DG) finite element methods for elliptic boundary value problems have been studied since the late 1970s and now constitute an active research area (see, e.g., [6, 7] and recent books [25, 33]). For problems with discontinuous coefficients such as interface problems, the stabilization (edge jump) term in the DG finite element method needs special treatments in order to be robust. Robustness in this paper means that constants in a priori error estimates or in the reliability and efficiency bounds of a posteriori error estimators are independent of the jump of the coefficients. Recently, Ern, Stephansen, and Zunino [23] developed a DG method using general weighted averages and proper weights for the stabilization term for advection-diffusion equations and established a robust a priori error estimate, provided that the solution is piecewise H^2 smooth.

A posteriori error estimation for continuous Galerkin finite element methods has been extensively studied for the past three decades (see, e.g., books by Verfürth [37], Ainsworth and Oden [4], and Babuška and Strouboulis [8] and the references therein). Recently, there has been increasing interest in a posteriori error estimation for the DG finite element method (see, e.g., [27, 9, 26, 2, 20, 34, 35, 36, 18]).

For elliptic interface problems considered in this paper, robust a posteriori error estimators have been investigated. For the continuous Galerkin method, Bernardi and Verfürth [11] and Petzoldt [31] studied a residual-based estimator, and we in [14] studied recovery-based estimators that were further extended to mixed and non-conforming finite element methods in [15]. For a DG finite element method without proper weights, Ainsworth [3] developed an a posteriori error estimator based on the so-called numerical flux. This result is further extended to meshes with hanging nodes in [5]. However, the error estimator and its analysis in [3, 5] depend on the jumps of coefficients. Recently, Ern, Nicaise, and Vohralik [19] and Ern, Stephansen, and Vohralik [21] investigated an equilibrated error estimator using the numerical flux for the DG method with the harmonic average weight; and Ern and Stephansen [22] studied a residual-based error estimator for DG approximations to diffusion and to advection-diffusion-reaction equations. Both the equilibrated and the residual-based estimators studied in [21, 22] contain a so-called nonconforming error term which is the energy norm of difference between the DG approximation and its recovery through the Oswald interpolation. Theoretically, they showed that their reliability constants are independent of the jump of the diffusion coefficients, but their efficiency constants do depend on the jump. This dependency could be removed for the interface problem by using a modified Oswald interpolate (see, e.g., section 4) as remarked in [22], provided that the diffusion coefficient is locally quasi-monotone. Such a hypothesis is assumed in order to prove the robustness of estimators in [11, 31, 14, 15] and in this paper.

For elliptic interface problems with intersecting interfaces, it is well known [28, 24] that their solutions may have only $H^{1+\epsilon}$ regularity with small $\epsilon > 0$ and are not piecewise H^s, $s > 3/2$, smooth. Since the standard a priori error estimate of the DG method requires that the exact solution be piecewise $H^{3/2+\epsilon}$ smooth, there are no known results for elliptic interface problems. One of the purposes of this paper is to fill this theoretical gap. To this end, we introduce a nonstandard variational formulation (see (2.14)) that uses general weighted averages. The formulation is defined in an appropriate solution space V (see (2.9)) that permits discontinuity across interior edges of a triangulation $\mathcal{T}$ and that does not require piecewise H^s, $s > 3/2$, smoothness. The corresponding DG (or *standard Galerkin*) finite element approximation is then the solution of this variational problem in a discontinuous (or *continuous*) finite dimensional subspace. In this setting, the error equation is then obtained in a straightforward manner and, most importantly, unnecessary smoothness of the solution is not assumed.

As usual, this formulation involves a parameter $\theta \in \{-1, 0, 1\}$ and a stabilization parameter γ_θ. For $\theta = 1$, the formulation is stable for all $\gamma_\theta \geq 0$. For $\theta = -1$ or 0, we show that there exists a positive constant γ_0 such that the variational problem in the discontinuous piecewise polynomial space is stable, provided that $\gamma_\theta \geq \gamma_0$. The γ_0 is computable and depends only on the shape of elements but not on the mesh size and the jump of the diffusion coefficient. With this discrete stability and the error equation, we are then able to obtain a robust a priori error estimate of the DG method for the interface problem with low regularity. Note that the DG method corresponding to $\theta = -1$ was first introduced and analyzed in [23] for smooth solution.

The second part of the paper deals with the development and analysis of various robust a posteriori error estimators including residual- and recovery-based a posteriori error estimators. The residual-based estimator studied in the paper is standard and may be regarded as an extension of that by Bernardi and Verfürth [11] and Petzoldt [31] to the DG method. The recovery-based a posteriori error estimators follow ideas of our previous work in [14, 15] for conforming, nonconforming, and mixed finite element methods. Theoretically, we show that the residual and recovery estimators are robust with respect to the DG norm; i.e., their reliability and efficiency bounds do not depend on the jump, provided that the distribution of coefficients is locally quasi-monotone. Finally, we remark that there is numerical evidence showing that both the residual and the recovery estimators are not subject to the locally quasi-monotone assumption (see, e.g., [14, 15]).

This paper is organized as follows. DG finite element methods are introduced and their well-posedness is established in section 2. In section 3, we obtain a robust a priori error estimate in a norm which is stronger than the broken energy norm. Modified Oswald and Clément types of interpolations are described in section 4. A residual-based a posteriori error estimator is introduced and analyzed in section 5. We introduce and analyze flux recoveries and the resulting recovery-based estimators in sections 6 and 7, respectively. Numerical results for a test problem are reported in section 8.

1.1. Notation. For a subdomain $G \subset \Omega$, we use the standard notation and definitions for Sobolev spaces (see, e.g., Lions and Magenes [29], Adams [1], or Grisvard [24]). For G being an open region, denote the Sobolev space by $W^{s,r}(B)$ on $B = G$ or ∂G equipped with the standard Sobolev norm $\| \cdot \|_{s,r,B}$, where s is a real number and $1 \leq r \leq \infty$. When $r = 2$, $W^{s,2}(B)$ is a Hilbert space and is denoted by $H^s(B)$ with the norm $\| \cdot \|_{s,B}$. When $s = 0$, $W^{0,r}(B)$ is the standard $L^r(B)$ space. By the trace theorem [24], the trace of any function in $W^{s,r}(G)$ lies in $W^{s-1/r,r}(\partial G)$, provided that $s - 1/r > 0$ is not an integer.

In two dimensions, for a vector-valued function $\boldsymbol{\tau} = (\tau_1, \tau_2)^t$, define the divergence by

$$\nabla \cdot \boldsymbol{\tau} = \frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2}.$$

For a scalar-valued function v, define the operator $\nabla^{\perp}$ by

$$\nabla^{\perp} v = Q\,\nabla v = \left(-\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1}\right)^t \quad \text{with} \quad Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We shall use the Hilbert space

$$H(\text{div}; G) = \{\boldsymbol{\tau} \in L^2(G)^2 \,:\, \nabla\cdot\boldsymbol{\tau} \in L^2(G)\}$$

equipped with the norm

$$\|\boldsymbol{\tau}\|_{H(\text{div};\,G)} = \left(\|\boldsymbol{\tau}\|_{0,G}^2 + \|\nabla\cdot\boldsymbol{\tau}\|_{0,G}^2\right)^{1/2}.$$

Finally, let

$$H^1_{g,D}(\Omega) = \{v \in H^1(\Omega) \,:\, v = g \text{ on } \Gamma_D\}$$

and

$$H_{g,N}(\text{div};\Omega) = \{\boldsymbol{\tau} \in H(\text{div};\Omega) \,:\, \boldsymbol{\tau}\cdot\mathbf{n} = g_{_N} \text{ on } \Gamma_N\}.$$

2. Discontinuous finite element approximation.

2.1. Finite element spaces. For simplicity of presentation, consider only triangular elements. Let $\mathcal{T} = \{K\}$ be a finite element partition of the domain Ω. Assume that the triangulation $\mathcal{T}$ is regular (see [17]); i.e., for all $K \in \mathcal{T}$, there exists a positive constant κ such that

$$h_K \le \kappa\,\rho_K,$$

where h_K denotes the diameter of the element K and ρ_K the diameter of the largest circle that may be inscribed in K. Note that the assumption of the regularity does not exclude highly locally refined meshes. Furthermore, assume that interfaces

$$F = \{\partial\Omega_i \cap \partial\Omega_j \,:\, i, j = 1, \ldots, n\}$$

do not cut through any element $K \in \mathcal{T}$.

Let $P_k(K)$ be the space of polynomials of degree k on element K. Denote the conforming continuous piecewise linear finite element space and the discontinuous Galerkin linear finite element space associated with the triangulation $\mathcal{T}$ by

$$\mathcal{U} = \{v \in H^1(\Omega) : v|_K \in P_1(K)\ \forall\, K \in \mathcal{T}\}$$

and

$$\mathcal{U}^{DG} = \{v \in L^2(\Omega) \,:\, v|_K \in P_1(K)\ \forall\, K \in \mathcal{T}\},$$

and denote the subspaces of $\mathcal{U}$ by

$$\mathcal{U}_g = \{v \in \mathcal{U} : v = g_{_D} \text{ on } \Gamma_D\} \quad \text{and} \quad \mathcal{U}_0 = \{v \in \mathcal{U} : v = 0 \text{ on } \Gamma_D\}.$$

Denote the local lowest-order Raviart–Thomas (RT) [32, 13] and Brezzi–Douglas–Marini (BDM) spaces [12, 13] on element $K \in \mathcal{T}$ by

$$RT_0(K) = P_0(K)^2 + \mathbf{x}\,P_0(K) \quad \text{and} \quad BDM_1(K) = P_1(K)^2,$$

respectively, where $\mathbf{x} = (x_1, x_2)$. Then the standard $H(\text{div};\Omega)$ conforming RT and BDM spaces are defined by

$$RT_0 = \{\boldsymbol{\tau} \in H(\text{div};\Omega) \,:\, \boldsymbol{\tau}|_K \in RT_0(K) \;\; \forall\, K \in \mathcal{T}\}$$

and

$$BDM_1 = \{\boldsymbol{\tau} \in H(\text{div};\Omega) \,:\, \boldsymbol{\tau}|_K \in BDM_1(K) \;\; \forall\, K \in \mathcal{T}\},$$

respectively. For convenience, denote RT_0 or BDM_1 by $\mathcal{V}$.

2.2. Jumps and averages. Denote by $\mathcal{E}_K$ the set of three edges of element $K \in \mathcal{T}$. Denote the set of all edges of the triangulation $\mathcal{T}$ by

$$\mathcal{E} := \mathcal{E}_{I} \cup \mathcal{E}_{D} \cup \mathcal{E}_{N},$$

where $\mathcal{E}_{I}$ is the set of all interior element edges, and $\mathcal{E}_{D}$ and $\mathcal{E}_{N}$ are the sets of all boundary edges belonging to the respective Γ_D and Γ_N. For each $e \in \mathcal{E}$, denote by h_e the length of the edge e; denote by $\mathbf{n}_e$ a unit vector normal to e. When $e \in \mathcal{E}_{D} \cup \mathcal{E}_{N}$, denote by K_+^e the element with the edge e, and assume that $\mathbf{n}_e$ is the unit outward normal vector. For each interior edge $e \in \mathcal{E}_{I}$, let K_+^e and K_-^e be the two elements sharing the common edge e such that the unit outward normal vector of K_+^e coincides with $\mathbf{n}_e$. For any $e \in \mathcal{E}$, denote by $v|_+^e$ and $v|_-^e$, respectively, the traces of a function v over e.

Define jumps over edges by

$$[\![v]\!]_e := \begin{cases} v|_+^e - v|_-^e, & e \in \mathcal{E}_I, \\ v|_+^e, & e \in \mathcal{E}_D \cup \mathcal{E}_N. \end{cases}$$

Let $w_+^e \in [0,1]$ and $w_-^e \in [0,1]$ be weights defined on e satisfying

$$w_+^e(x) + w_-^e(x) = 1, \tag{2.1}$$

and define the weighted averages

$$\{v(x)\}_w^e = \begin{cases} w_-^e v_-^e + w_+^e v_+^e, & e \in \mathcal{E}_I, \\ v|_+^e, & e \in \mathcal{E}_D \cup \mathcal{E}_N, \end{cases}$$

and

$$\{v(x)\}_e^w = \begin{cases} w_+^e v_-^e + w_-^e v_+^e, & e \in \mathcal{E}_I, \\ 0, & e \in \mathcal{E}_D \cup \mathcal{E}_N, \end{cases}$$

for all $e \in \mathcal{E}$. When there is no ambiguity, the subscript or superscript e in the designation of the jump and the weighted averages will be dropped. A simple calculation leads to the following identity:

$$[\![uv]\!]_e = \{v\}_e^w \,[\![u]\!]_e + \{u\}_w^e \,[\![v]\!]_e. \tag{2.2}$$

Let e be the interface of elements K_+^e and K_-^e,

$$e = \partial K_+^e \cap \partial K_-^e,$$

and denote by k_+^e and k_-^e the diffusion coefficients on K_+^e and K_-^e, respectively. There are several possible choices of the weights:

$$w_{\pm,1}^e = \frac{1}{2}, \quad w_{\pm,2}^e = \frac{k_\mp^e}{k_+^e + k_-^e}, \quad \text{and} \quad w_{\pm,3}^e = \frac{\sqrt{k_\mp^e}}{\sqrt{k_+^e} + \sqrt{k_-^e}}. \tag{2.3}$$

Denote by

$$W_e = \{k\}_w^e$$

the weighted average of k on edge e. For the above choices of the weights, W_e is then the arithmetic, the harmonic, and the geometric averages:

$$W_{e,1} = \frac{k_+^e + k_-^e}{2}, \quad W_{e,2} = \frac{2k_+^e k_-^e}{k_+^e + k_-^e}, \quad \text{and} \quad W_{e,3} = \sqrt{k_-^e k_+^e}, \tag{2.4}$$

respectively. It is well known that these averages have the following relations:

$$W_{e,2} \le W_{e,3} \le W_{e,1}.$$

Since

$$\frac{(w_+^e)^2 k_+^e}{W_e} = \frac{(w_+^e)^2 k_+^e}{w_+^e k_+^e + w_-^e k_-^e} \le \frac{(w_+^e)^2 k_+^e}{w_+^e k_+^e} \le w_+^e \le 1,$$

the same argument shows that

$$\frac{(w_+^e)^2 k_+^e}{W_e} \le 1 \quad \text{and} \quad \frac{(w_-^e)^2 k_-^e}{W_e} \le 1 \tag{2.5}$$

for any weights satisfying (2.1). Letting

$$k_{\min}^e = \min\{k_+^e, k_-^e\} \quad \text{and} \quad k_{\max}^e = \max\{k_+^e, k_-^e\},$$

it is easy to check that

$$\frac{1}{2} k_{\max}^e \le W_{e,1} \le k_{\max}^e, \quad k_{\min}^e \le W_{e,2} \le 2k_{\min}^e, \quad \text{and} \quad k_{\min}^e \le W_{e,3} \le k_{\max}^e, \tag{2.6}$$

which implies that the arithmetic and harmonic averages are equivalent to the maximum and the minimum, respectively. For boundary edge $e \in \partial K_+^e \cap \partial\Omega$, set

$$w_+^e = 1 \quad \text{and} \quad W_e = k_+^e.$$

2.3. DG finite element approximation. To describe the DG finite element method, we introduce a nonstandard variational formulation for (1.1)–(1.2) defined in a proper solution space which is a subspace of piecewise $H^{1+\epsilon}$ functions for $0 < \epsilon \ll 1$. The corresponding DG finite element approximation is the solution of this variational problem in a *discontinuous* finite dimensional subspace. Also, the standard Galerkin method is the variational problem restricted in a *continuous* finite dimensional subspace.

To define a proper solution space, let us start with the following Green's formula:

$$\int_{\partial K} (\nabla w \cdot \mathbf{n})\, v\, ds := \langle \nabla w \cdot \mathbf{n}, v\rangle_{\partial K} = (\Delta w, v)_K + (\nabla w, \nabla v)_K \quad \forall\ K \in \mathcal{T} \tag{2.7}$$

holds for all $w \in H^{1+\epsilon}(K)$ with $\Delta w \in L^2(K)$ and for all $v \in H^{1-\epsilon}(K)$ with $0 < \epsilon < 1/2$. Let $H^{-\epsilon}(K)$ be the dual of $H_0^\epsilon(K)$ which is the closure of $C_0^\infty(K)$ in the $H^\epsilon(K)$ norm. Since $H^\epsilon(K)$ is the same space as $H_0^\epsilon(K)$ for $\epsilon \in (0, 1/2)$ (see, e.g., Theorem 1.4.2.4 in [24]), ∇v is then in $H^{-\epsilon}(K)^2$. That is, the term $(\nabla w, \nabla v)_K$ in (2.7) can be viewed as a duality pairing between $H^\epsilon(K)^2$ and $H^{-\epsilon}(K)^2$. The validity of (2.7) follows from the standard density argument and the fact that (2.7) holds for $C^\infty(\bar{K})$ functions.

By the trace theorem [24], $v|_{\partial K}$ is in $H^{1/2-\epsilon}(\partial K)$. Hence, the formal boundary integral in the left-hand side of (2.7) may be regarded as the duality pairing between $H^{\epsilon-1/2}(\partial K)$ and $H^{1/2-\epsilon}(\partial K)$, which is defined by the right-hand side of (2.7). Since, for each edge $e \subset \partial K$, the trivial extension of functions in $H^{1/2-\epsilon}(e)$ by zero to all of ∂K belongs to $H^{1/2-\epsilon}(\partial K)$ (see, e.g., Theorem 1.5.2.3 in [24]), this interpretation enables us to define the duality pairing on each edge e of ∂K,

$$\int_e (\nabla w \cdot \mathbf{n})\, v\, ds := \langle \nabla w \cdot \mathbf{n}, v \rangle_e,$$

where $(\nabla w \cdot \mathbf{n})|_e \in H^{\epsilon-1/2}(e)$ and $v|_e \in H^{1/2-\epsilon}(e)$.

LEMMA 2.1. *Letting $K \in \mathcal{T}$, $e \in \partial K$, and $0 < \epsilon < 1/2$, for any $\phi \in H^{1+\epsilon}(K)$ with $\Delta\phi \in L^2(K)$, there exists a positive constant C independent of ϕ such that*

$$\|\nabla\phi \cdot \mathbf{n}\|_{\epsilon-1/2,e} \le C \left(\|\nabla\phi\|_{\epsilon,K} + h_K^{1-\epsilon} \|\Delta\phi\|_{0,K} \right). \tag{2.8}$$

Proof. Inequality (2.8) is contained in the proof of Corollary 3.3 on page 1384 of [10]. For the convenience of readers, we provide a proof here. For any $g \in H^{1/2-\epsilon}(e)$, there exists a lifting v_g of g such that $v_g \in H^{1-\epsilon}(K)$, $v_g|_e = g$, $v_g|_{\partial K \setminus e} = 0$, and

$$\|\nabla v_g\|_{-\epsilon,K} + h_K^{\epsilon-1} \|v_g\|_{0,K} \le c \|g\|_{1/2-\epsilon,e}.$$

It then follows from the Green's formula in (2.7), the Cauchy–Schwarz inequality, and the definition of the dual norm that

$$\begin{aligned}
\langle \nabla\phi \cdot \mathbf{n}, g \rangle_e &= \langle \nabla\phi \cdot \mathbf{n}, v_g \rangle_{\partial K} = (\Delta\phi,\, v_g)_K + (\nabla\phi,\, \nabla v_g)_K \\
&\le \|\Delta\phi\|_{0,K} \|v_g\|_{0,K} + \|\nabla\phi\|_{\epsilon,K} \|\nabla v_g\|_{-\epsilon,K} \\
&\le C \left(\|\nabla\phi\|_{\epsilon,K} + h_K^{1-\epsilon} \|\Delta\phi\|_{0,K} \right) \|g\|_{1/2-\epsilon,e},
\end{aligned}$$

which, combining with the definition of the dual norm

$$\|\nabla\phi\|_{\epsilon-1/2,e} = \sup_{g \in H^{1/2-\epsilon}(e)} \frac{\langle \nabla\phi \cdot \mathbf{n}, g \rangle_e}{\|g\|_{1/2-\epsilon,e}},$$

implies (2.8). This completes the proof of the lemma. ∎

Denote by $H^s(\mathcal{T})$ the broken Sobolev space of degree $s > 0$ with respect to $\mathcal{T}$,

$$H^s(\mathcal{T}) = \{ v \in L^2(\Omega) : v|_K \in H^s(K) \quad \forall\, K \in \mathcal{T} \},$$

and denote its subspace by

$$V^s(\mathcal{T}) = \{ v \in H^s(\mathcal{T}) \,:\, \nabla \cdot (k \nabla v) \in L^2(K) \quad \forall\, K \in \mathcal{T} \}. \tag{2.9}$$

Let u be the solution of problem (1.1)–(1.2); then it is well known from the regularity estimate [28, 24] that $u \in H^{1+\alpha}(\Omega)$ for some positive α which could be very small.

Since $f \in L^2(\Omega)$, it is then easy to see that $u \in V^{1+\epsilon}(\mathcal{T})$ for any $0 < \epsilon < \alpha$, and the flux $\boldsymbol{\sigma} = -k\nabla u$ is in $H(\mathrm{div};\Omega) \cap H^{\epsilon}(\mathcal{T})^2$.

Denote the discrete gradient and divergence operators by

$$(\nabla_h v)|_K = \nabla(v|_K) \quad \text{and} \quad (\nabla_h \cdot \boldsymbol{\tau})|_K = \nabla \cdot (\boldsymbol{\tau}|_K)$$

for all $K \in \mathcal{T}$, respectively. Multiplying (1.1) by a test function $v \in V^{1+\epsilon}(\mathcal{T})$, integrating by parts, and using boundary conditions (1.2), we have

$$\begin{aligned}(f,\, v) &= \sum_{K\in\mathcal{T}} (k\nabla u,\, \nabla v)_K - \sum_{K\in\mathcal{T}} \int_{\partial K} (k\nabla u \cdot \mathbf{n}_e)\, v\, ds \\ &= (k\nabla_h u, \nabla_h v) - \sum_{e\in\mathcal{E}_I} \int_e [\![(k\nabla u \cdot \mathbf{n}_e)\, v]\!]\, ds - \sum_{e\in\mathcal{E}_D} \int_e (k\nabla u \cdot \mathbf{n}_e)\, v\, ds - \int_{\Gamma_N} g_{_N}\, v\, ds.\end{aligned}$$

Together with (2.2) and the continuity of the flux, for all $v \in V^{1+\epsilon}(\mathcal{T})$,

$$\int_e [\![k\nabla u \cdot \mathbf{n}_e]\!]\, \{v\}^w\, ds = 0 \quad \forall\, e \in \mathcal{E}_I \tag{2.10}$$

implies that

$$(k\nabla_h u, \nabla_h v) - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla u \cdot \mathbf{n}_e\}_w [\![v]\!]\, ds = (f,\, v) + \int_{\Gamma_N} g_{_N}\, v\, ds \tag{2.11}$$

for all $v \in V^{1+\epsilon}(\mathcal{T})$. By the continuity of $u \in H^{1+\alpha}(\Omega)$ and the Dirichlet boundary condition, we have that

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla v \cdot \mathbf{n}_e\}_w [\![u]\!]\, ds = \sum_{e\in\mathcal{E}_D} \int_e g_{_D} (k\nabla v \cdot \mathbf{n}_e) ds \tag{2.12}$$

and

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_{_\theta} h_e^{-1} W_e [\![u]\!][\![v]\!]\, ds = \sum_{\epsilon\in\mathcal{E}_D} \gamma_{_\theta} h_e^{-1} W_e \int_e g_{_D} v ds \tag{2.13}$$

for all $v \in V^{1+\epsilon}(\mathcal{T})$, where $\gamma_{_\theta} > 0$ is a parameter to be determined.

For $\theta \in \{-1, 0, 1\}$, define a bilinear form for $u, v \in V^{1+\epsilon}(\mathcal{T})$ by

$$\begin{aligned}a_{_\theta}(u, v) = (k\nabla_h u, \nabla_h v) &+ \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_{_\theta} h_e^{-1} W_e [\![u]\!][\![v]\!]\, ds \\ &- \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla u \cdot \mathbf{n}_e\}_w [\![v]\!] ds + \theta \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla v \cdot \mathbf{n}_e\}_w [\![u]\!] ds\end{aligned}$$

and a linear form for $v \in V^{1+\epsilon}(\mathcal{T})$ by

$$\begin{aligned}f_{_\theta}(v) = \sum_{K\in\mathcal{T}} (f, v)_K &+ \sum_{\epsilon\in\mathcal{E}_D} \gamma_{_\theta} h_e^{-1} W_e \int_e g_{_D} v\, ds \\ &+ \sum_{\epsilon\in\mathcal{E}_N} \int_e g_{_N} v\, ds + \theta \sum_{e\in\mathcal{E}_D} \int_e g_{_D} (k\nabla v \cdot \mathbf{n}_e)\, ds.\end{aligned}$$

Note that the bilinear form is symmetric if $\theta = -1$ and nonsymmetric otherwise. Using (2.11), (2.12), and (2.13), the weak solution of (1.1)–(1.2) $u \in H^{1+\alpha}(\Omega) \cap V^{1+\epsilon}(\mathcal{T})$ satisfies the following variational equation:

$$a_\theta(u, v) = f_\theta(v) \quad \forall\, v \in V^{1+\epsilon}(\mathcal{T}). \tag{2.14}$$

Note that (2.13) is used in (2.14) for stabilizing the formulation or, equivalently, enforcing weakly the continuity of the solution. For $\theta = 1$, (2.12) is used for symmetrizing and stabilizing the bilinear form. Note also that the Dirichlet boundary condition is enforced weakly in (2.14). One could enforce it strongly in the solution space by removing all boundary integrals over edge $e \in \mathcal{E}_D$.

REMARK 2.2. *It is obvious that the bilinear form $a_1(\cdot,\cdot)$ is coercive in $V^{1+\epsilon}(\mathcal{T})$ with respect to the energy/DG norm if $\gamma_1 > 0$. But it is difficult, if not impossible, to show that the bilinear form $a_\theta(\cdot,\cdot)$ is coercive in $V^{1+\epsilon}(\mathcal{T})$ for $\theta = -1, 0$. Nonetheless, the fact that the weak solution of* (1.1)–(1.2) *satisfies problem* (2.14) *implies existence of solutions of problem* (2.14)*. Uniqueness of problem* (2.14) *follows from that of* (1.1)–(1.2) *and the fact that solution of problem* (2.14) *satisfies* (1.1)–(1.2) *in the weak sense.*

Now, the corresponding DG finite element method is to find $u_{\mathcal{T}} \in \mathcal{U}^{DG} \subset V^{1+\epsilon}(\mathcal{T})$ such that

$$a_\theta(u_{\mathcal{T}}, v) = f_\theta(v) \quad \forall\, v \in \mathcal{U}^{DG}. \tag{2.15}$$

Methods corresponding to $\theta = -1$, 0, or 1 are the so-called symmetric interior penalty Galerkin (SIPG), incomplete interior penalty Galerkin (IIPG), or nonsymmetric interior penalty Galerkin (NIPG) methods, respectively. A special and interesting case of the NIPG method is $\gamma_1 = 0$, which was studied in [30] by Oden, Babŭska, and Baumann. With the special choices of weights in (2.3), the corresponding DG methods in (2.15) are called the arithmetic, the harmonic, and the geometric weighted DG methods, respectively. The SIPG method defined in (2.15) with general weights but a slightly different stabilization term was introduced and analyzed recently in [23] and a robust a priori error bound was obtained, provided that the solution is piecewise H^2 smooth and that γ_θ is large enough.

2.4. Well-posedness of the DG finite element formulation. For any element $K \in \mathcal{T}$, let $\{\lambda_i(x)\}_{i=1}^3$ be the standard barycentric coordinates of K. Following the idea of Lemma 1 in [3], we introduce an element stiffness matrix of the Laplace operator (instead of the diffusion operator in [3]):

$$\mathbf{S}_K = \left(S_{ij}^K\right)_{3\times 3} \quad \text{and} \quad S_{ij}^K = (\nabla\lambda_i, \nabla\lambda_j)_K.$$

The matrix $\mathbf{S}_K$ is positive semidefinite, and its largest eigenvalue, $\rho(\mathbf{S}_K)$, depends only on the shape of the element K but not on its size h_K. Obviously, $\rho(\mathbf{S}_K)$ is independent of the coefficient k. If shapes of the elements in $\mathcal{T}$ are reasonably regular, $\rho(\mathbf{S}_K)$ is of similar size for all $K \in \mathcal{T}$. Denote the maximum of $\rho(\mathbf{S}_K)$ over $\mathcal{T}$ by

$$\rho_{\mathcal{T}} = \max_{K\in\mathcal{T}} \rho(\mathbf{S}_K).$$

For any $v \in \mathcal{U}^{DG}$, let

$$|\!|\!| v |\!|\!|_{DG,K} = \left(\|k^{\frac{1}{2}}\nabla v\|_{0,K}^2 + \sum_{e\in\mathcal{E}_K\setminus\mathcal{E}_N} h_e^{-1} W_e \|[\![v]\!]\|_{0,e}^2 \right)^{1/2}$$

for any $K \in \mathcal{T}$. The so-called DG norm is defined as follows:

$$\|v\|_{DG} = \left(\|k^{\frac{1}{2}} \nabla_h v\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} h_e^{-1} W_e \|[\![v]\!]\|_{0,e}^2 \right)^{1/2}.$$

LEMMA 2.3 (uniqueness and coercivity).

(i) *The bilinear form $a_1(\cdot,\cdot)$ is coercive in $\mathcal{U}^{DG}$ with the coercivity constant $\min\{1,\gamma_1\}$, provided that $\gamma_1 > 0$, i.e,*

$$a_1(v,v) \ge \min\{1,\gamma_1\} \|v\|_{DG}^2 \quad \forall\, v \in \mathcal{U}^{DG}. \tag{2.16}$$

Thus the NIPG problem in (2.15) *has a unique solution, provided that $\gamma_1 > 0$.*

(ii) *Let w_+^e and w_-^e be weights satisfying* (2.1). *Then SIPG and IIPG problems* (2.15) *have a unique solution, provided that $\gamma_\theta > 2(1+\theta)^2 \rho_{\mathcal{T}}$. Moreover, the symmetric/incomplete bilinear form $a_\theta(\cdot,\cdot)$ for $\theta = -1$ or 0 is coercive in $\mathcal{U}^{DG}$ with a coercivity constant $\alpha_0 \in (0,1)$ independent of the mesh size and the ratio $k_{\max}/k_{\min}$, i.e.,*

$$a_\theta(v,v) \ge \alpha_0 \|v\|_{DG}^2 \quad \forall\, v \in \mathcal{U}^{DG}, \tag{2.17}$$

for $\theta = -1$ and 0, provided that $\gamma_\theta > \frac{2(1+\theta)^2 \rho_{\mathcal{T}}}{1-\alpha_0} + \alpha_0$.

Proof. Let δ be a positive constant to be determined. For any $v \in \mathcal{U}^{DG}$ and for $e \in \mathcal{E}_I \bigcup \mathcal{E}_D$, the Cauchy–Schwarz inequality and the inequality of arithmetic and geometric means give

$$2 \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \{k \nabla v \cdot \mathbf{n}_e\}_w [\![v]\!]\, ds \le \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \frac{\delta\, h_e}{W_e} \|\{k \nabla v \cdot \mathbf{n}_e\}_w\|_{0,e}^2 + \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \frac{W_e}{\delta\, h_e} \|[\![v]\!]\|_{0,e}^2. \tag{2.18}$$

For $e \in \mathcal{E}_I$, let $e = \partial K_+^e \cap \partial K_-^e$. Since $k \nabla v$ is constant on each element,

$$\begin{aligned}
&\frac{h_e}{W_e} \|\{k \nabla v \cdot \mathbf{n}_e\}_w\|_{0,e}^2 \\
&\quad \le 2 h_e^2 \left\{ \frac{(w_+^e)^2 k_+}{W_e} k_+ (\nabla v_+ \cdot \mathbf{n}_e)^2 + \frac{(w_-^e)^2 k_-}{W_e} k_- (\nabla v_- \cdot \mathbf{n}_e)^2 \right\} \\
&\quad \le 2 h_e^2 \max\left\{ \frac{(w_+^e)^2 k_+^e}{W_e}, \frac{(w_-^e)^2 k_-^e}{W_e} \right\} \left(k_+ (\nabla v_+ \cdot \mathbf{n}_e)^2 + k_- (\nabla v_- \cdot \mathbf{n}_e)^2 \right).
\end{aligned}$$

Similarly, for $e \in \mathcal{E}_D$ and $e \subset \partial K$, we have

$$\frac{h_e}{W_e} \|\{k \nabla v \cdot \mathbf{n}_e\}_w\|_{0,e}^2 = k_K^{-1} h_e^2 (k_K \nabla v_K \cdot \mathbf{n}_e)^2 = h_e^2 k_K (\nabla v_K \cdot \mathbf{n}_e)^2.$$

Summing up over all edges in $\mathcal{E}_I \cup \mathcal{E}_D$ and using (2.5) imply that

$$\sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \frac{h_e}{W_e} \|\{k \nabla v \cdot \mathbf{n}\}_w\|_e^2 \le 2 \sum_{K \in \mathcal{T}} \sum_{e \in \mathcal{E}_K} h_e^2 k_K (\nabla v_K \cdot \mathbf{n}_e)^2. \tag{2.19}$$

It is proved in [3] that

$$\sum_{e \in \mathcal{E}_K} h_e^2 (\nabla v \cdot \mathbf{n}_K)^2 = 4 \vec{v}_K^T \mathbf{S}_K^2 \vec{v}_K,$$

where $\vec{v}_K$ is the vector of values of v at vertices of K. Since $k_K \vec{v}_K^T \mathbf{S}_K \vec{v}_K = (k\nabla v, \nabla v)_K$, thus

$$\sum_{e\in\mathcal{E}_K} h_e^2 k_K (\nabla v_K \cdot \mathbf{n}_e)^2 \le 4k_K \vec{v}_K \mathbf{S}_K^2 \vec{v}_K \le 4\rho(\mathbf{S}_K) k_K \vec{v}_K^T \mathbf{S}_K \vec{v}_K = 4\rho(\mathbf{S}_K)\,(k\nabla v, \nabla v)_K,$$

which, together with (2.19), leads to

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \frac{h_e}{W_e} \|\{k\nabla v\cdot\mathbf{n}\}_w\|_e^2 \le 8\rho_{\mathcal{T}}(k\nabla_h v,\, \nabla_h v).$$

Using (2.18), we now have

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla v\cdot\mathbf{n}_e\}_w [\![v]\!]\, ds \le 4\delta\rho_{\mathcal{T}}(k\nabla_h v,\, \nabla_h v) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \frac{W_e}{2\delta\, h_e} \|[\![v]\!]\|_{0,e}^2.$$

Hence,

$$a_\theta(v,\, v) \ge (1 - 4(1+\theta)\delta\rho_{\mathcal{T}})(k\nabla_h v,\, \nabla_h v) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \left(\gamma_\theta - \frac{1+\theta}{2\delta}\right) \frac{W_e}{h_e} \|[\![v]\!]\|_{0,e}^2.$$

For any constant $\alpha_0 \in [0,1)$, assume that $\gamma_\theta > \frac{2(1+\theta)^2\rho_{\mathcal{T}}}{1-\alpha_0} + \alpha_0$; then there exists $\delta > 0$ such that $\frac{2(1-\alpha_0)}{1+\theta} > \delta^{-1} > \frac{4(1+\theta)\rho_{\mathcal{T}}}{1-\alpha_0}$, which is equivalent to

$$1 - 4(1+\theta)\delta\rho_{\mathcal{T}} > \alpha_0 \quad \text{and} \quad \gamma_\theta - \frac{1+\theta}{2\delta} > \alpha_0.$$

This implies the coercivity of $a_\theta(v,v)$ in (2.17) for any $\alpha_0 \in (0,1)$. When $\alpha_0 = 0$, it yields that $a_\theta(v,v)$ is positive and definite in $\mathcal{U}^{DG}$ and, hence, problem (2.15) has a unique solution. This completes the proof of the lemma. ∎

REMARK 2.4. *The constant* γ_θ *that appears in* [5] *is chosen to be greater than* $(1+\theta)^2 \max_{K\in\mathcal{T}} k_K \rho(S_K)$, *which depends on* k *for* $\theta \neq -1$, *and, hence, it is not optimal.*

3. A priori error estimate. Let $e = u - u_{\mathcal{T}}$, where u and $u_{\mathcal{T}}$ are the solutions of (2.14) and (2.15), respectively. The difference of (2.14) and (2.15) yields the following error equation:

$$a_\theta(e,\, v) = 0 \quad \forall\ v \in \mathcal{U}^{DG}. \tag{3.1}$$

Let $\epsilon > 0$ be a very small constant, and define

$$\|v\|_{k,\epsilon,\Omega} = \|k^{1/2}\nabla v\|_{\epsilon,\Omega}.$$

Let $P_{\mathcal{T}} : H_{g,D}^{1+\epsilon}(\Omega) \to \mathcal{U}_g$ be the orthogonal projection operator from $H_{g,D}^{1+\epsilon}(\Omega)$ onto $\mathcal{U}_g$ with respect to the inner product associated with the norm $\|\cdot\|_{k,\epsilon,\Omega}$. Then the standard interpolation argument and an analysis similar to that for Proposition 2.4 in [11] give that for $\phi \in H_{g,D}^{1+\epsilon}(\Omega) \cap H^{1+s}(\mathcal{T})$ with $\epsilon \le s \le 1$,

$$\|k^{1/2}\nabla(\phi - P_{\mathcal{T}}\phi)\|_{\epsilon,\Omega} \le C \left(\sum_{K\in\mathcal{T}} h_K^{2(s-\epsilon)}\, \|k^{1/2}\nabla\phi\|_{s,K}^2\right)^{1/2}, \tag{3.2}$$

where C is a positive constant independent of the mesh size and the ratio $k_{\max}/k_{\min}$. For any $v \in H^{1+s}(\mathcal{T})$, $0 < s \le 1$, denote

$$B_s(h,\, v) = \left(\sum_{K\in\mathcal{T}} h_K^{2(s-\epsilon)} \|k^{1/2}\nabla v\|_{s,K}^2 \right)^{1/2} + \left(\sum_{K\in\mathcal{T}} h_K^2 k_K^{-1} \|f\|_{0,K}^2 \right)^{1/2}.$$

LEMMA 3.1. *Assume that the solution $u \in V^{1+\epsilon}(\mathcal{T})$ of problem (2.14) belongs to $H^{1+s}(\mathcal{T})$ with $0 < \epsilon \le s \le 1$. Then*

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla(P_{\mathcal{T}} u - u)\cdot \mathbf{n}_e\}_w [\![P_{\mathcal{T}} u - u_{\mathcal{T}}]\!]\, ds \le C\, B_s(h,\, u) |\!|\!| P_{\mathcal{T}} u - u_{\mathcal{T}} |\!|\!|_{DG}, \tag{3.3}$$

where C is a positive constant independent of the mesh size and the ratio $k_{\max}/k_{\min}$.

Proof. Let $z = P_{\mathcal{T}} u - u$ and $z_{\mathcal{T}} = P_{\mathcal{T}} u - u_{\mathcal{T}}$. By using the definition of the dual norm, the triangle inequality, the inverse inequality, (2.5), Lemma 2.1, and (3.2), we have

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla(P_{\mathcal{T}} u - u)\cdot \mathbf{n}_e\}_w [\![P_{\mathcal{T}} u - u_{\mathcal{T}}]\!]\, ds = \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla z\cdot \mathbf{n}_e\}_w [\![z_{\mathcal{T}}]\!]\, ds \\
&\le \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \|\{k\nabla z\cdot \mathbf{n}_e\}_w\|_{\epsilon-1/2,e}\, \|[\![z_{\mathcal{T}}]\!]\|_{1/2-\epsilon,e} \\
&\le \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \left(w_-^e \|k_-\nabla z|_{K_-}\cdot \mathbf{n}_e\|_{\epsilon-\frac12,e} + w_+^e \|k_+\nabla z|_{K_+}\cdot \mathbf{n}_e\|_{\epsilon-\frac12,e} \right) h_e^{\epsilon-\frac12} \|[\![z_{\mathcal{T}}]\!]\|_{0,e} \\
&\le \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \left(\|k_-^{1/2}\nabla z|_{K_-}\cdot \mathbf{n}_e\|_{\epsilon-\frac12,e} + \|k_+^{1/2}\nabla z|_{K_+}\cdot \mathbf{n}_e\|_{\epsilon-\frac12,e} \right) h_e^{\epsilon-\frac12} W_e^{1/2} \|[\![z_{\mathcal{T}}]\!]\|_{0,e} \\
&\le C \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \left(\sum_{K\in\omega_e} h^{\epsilon} \|k^{\frac12}\nabla z\|_{\epsilon,K} + h_K \|k^{\frac12}\Delta z\|_{0,K} \right) W_e^{\frac12} h_e^{-\frac12} \|[\![z_{\mathcal{T}}]\!]\|_{0,e} \\
&\le C B_s(h,\, u) |\!|\!| P_{\mathcal{T}} u - u_{\mathcal{T}} |\!|\!|_{DG}.
\end{aligned}$$

This completes the proof of the lemma. ∎

THEOREM 3.2. *Assume that the solution $u \in V^{1+\epsilon}(\mathcal{T})$ of problem (2.14) belongs to $H^{1+s}(\mathcal{T}) \cap H^{1+\epsilon}(\Omega)$ with $0 < \epsilon \le s \le 1$ and that $\gamma_\theta > \frac{2(1+\theta)^2 \rho_{\mathcal{T}}}{1-\alpha_0} + \alpha_0$ for $\theta = +1$, 0, and -1. Then we have the following a priori error bound:*

$$|\!|\!| u - u_{\mathcal{T}} |\!|\!|_{DG} \le C\, B_s(h,\, u), \tag{3.4}$$

where C is a positive constant independent of the mesh size and the ratio $k_{\max}/k_{\min}$.

Proof. The triangle inequality gives

$$|\!|\!| e |\!|\!|_{DG} \le |\!|\!| u - P_{\mathcal{T}} u |\!|\!|_{DG} + |\!|\!| P_{\mathcal{T}} u - u_{\mathcal{T}} |\!|\!|_{DG}.$$

Since $u - P_{\mathcal{T}} u$ is continuous and vanishes on Γ_D, thus

$$|\!|\!| u - P_{\mathcal{T}} u |\!|\!|_{DG} = \|k^{1/2}\nabla(u - P_{\mathcal{T}} u)\|_{0,\Omega} \le \|k^{1/2}\nabla(u - P_{\mathcal{T}} u)\|_{\epsilon,\Omega}.$$

Now, by (3.2) with $\phi = u$ it suffices to show that

$$|\!|\!| P_{\mathcal{T}} u - u_{\mathcal{T}} |\!|\!|_{DG} \le C \left(|\!|\!| u - P_{\mathcal{T}} u |\!|\!|_{DG} + B_s(h,\, u) \right). \tag{3.5}$$

To this end, using the coercivity of $a_\theta(\cdot,\cdot)$ in (2.17), the error equation in (3.1), the Cauchy–Schwarz inequality, and the fact that $[\![P_\mathcal{T} u - u]\!]_{\mathcal{E}_I\cup\mathcal{E}_D} = 0$, we have

$$\begin{aligned}
&\alpha_0 |\!|\!| P_\mathcal{T} u - u_\mathcal{T} |\!|\!|_{DG}^2 \le a_\theta(P_\mathcal{T} u - u_\mathcal{T},\, P_\mathcal{T} u - u_\mathcal{T}) = a_\theta(P_\mathcal{T} u - u,\, P_\mathcal{T} u - u_\mathcal{T})\\
&= (k\nabla_h(P_\mathcal{T} u - u),\, \nabla_h(P_\mathcal{T} u - u_\mathcal{T})) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \gamma_\theta h_e^{-1} W_e [\![P_\mathcal{T} u - u]\!][\![P_\mathcal{T} u - u_\mathcal{T}]\!]\,ds\\
&\quad - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e (\{k\nabla(P_\mathcal{T} u - u)\cdot\mathbf{n}_e\}_w [\![P_\mathcal{T} u - u_\mathcal{T}]\!]\\
&\qquad\qquad - \theta\{k\nabla(P_\mathcal{T} u - u_\mathcal{T})\cdot\mathbf{n}_e\}_w [\![P_\mathcal{T} u - u]\!])\,ds\\
&\le C\,|\!|\!| P_\mathcal{T} u - u |\!|\!|_{DG} |\!|\!| P_\mathcal{T} u - u_\mathcal{T} |\!|\!|_{DG} + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{k\nabla(P_\mathcal{T} u - u)\cdot\mathbf{n}_e\}_w [\![P_\mathcal{T} u - u_\mathcal{T}]\!]\,ds,
\end{aligned}$$

which, together with Lemma 3.1, implies (3.5) and, hence, (3.4). This completes the proof of the theorem. ◻

4. Oswald- and Clément-type interpolations. Denote by $\mathcal{N}$, $\mathcal{N}_D$, and $\mathcal{N}_K$ the sets of all vertices of the triangulation $\mathcal{T}$, on the Γ_D, and of element $K \in \mathcal{T}$, respectively. For any $z \in \mathcal{N}$, denote by ϕ_z the nodal basis function of $\mathcal{U}$, and let

$$\omega_z = \{K \in \mathcal{T} \,:\, K \subset \operatorname{supp}(\phi_z)\} \quad \text{and} \quad \hat{\omega}_z = \left\{K \in \omega_z \,:\, k_K = \max_{K'\in\omega_z} k_{K'}\right\}.$$

The number of elements in $\hat{\omega}_z$ is denoted by $cd(z)$. Also, denote by $\tilde{\mathcal{E}}_K$ the set of edges that share at least a vertex with K.

In this section and sections 5 and 7, assume that the distribution of the coefficients k_K for all $K \in \mathcal{T}$ is locally quasi-monotone [31], which is slightly weaker than Hypothesis 2.7 in [11]. For the convenience of the readers, we restate it here.

Definition 4.1. *Given a vertex $z \in \mathcal{N}$, the distribution of coefficients k_K, $K \in \omega_z$, is said to be* quasi-monotone *with respect to the vertex z if there exists a subset $\tilde{\omega}_{K,z,qm}$ of ω_z such that the union of elements in $\tilde{\omega}_{K,z,qm}$ is a Lipschitz domain and that the following hold:*

- *if $z \in \mathcal{N}\backslash\mathcal{N}_D$, then $\{K\} \cup \hat{\omega}_z \subset \tilde{\omega}_{K,z,qm}$ and $k_K \le \max_{K'\in\omega_z} k_{K'}$;*
- *if $z \in \mathcal{N}_D$, then $K \in \tilde{\omega}_{K,z,qm}$, $\partial\tilde{\omega}_{K,z,qm} \cap \Gamma_D \ne \emptyset$, and $k_K \le \max_{K'\in\omega_z} k_{K'}$.*

The distribution of coefficients k_K, $K \in \mathcal{T}$, is said to be locally quasi-monotone *if it is quasi-monotone with respect to every vertex $z \in \mathcal{N}$.*

For a given function $v \in \mathcal{U}^{DG}$, define the Oswald interpolation operator $\mathcal{I} : \mathcal{U}^{DG} \to \mathcal{U}_g$ by

$$\mathcal{I}v = \sum_{z\in\mathcal{N}} \mathcal{I}v(z)\phi_z(x),$$

where the nodal value of the interpolant $\mathcal{I}v$ at z is defined by

$$\mathcal{I}v(z) = \begin{cases} g_D(z) & \text{if } z \in \mathcal{N}_D,\\[1ex] \dfrac{1}{cd(z)}\displaystyle\sum_{K\in\hat{\omega}_z} v_K(z) & \text{if } z \in \mathcal{N}\backslash\mathcal{N}_D \end{cases}$$

with $v_K = v|_K$ the restriction of v on K.

LEMMA 4.2. *Assume that the triangulation $\mathcal{T}$ is quasi-uniform. Then for any $v \in \mathcal{U}^{DG}$, there exists a positive constant C independent of the ratio $k_{\max}/k_{\min}$ such that*

$$k_K \|v - \mathcal{I}v\|_{0,K}^2 \le C \left(\sum_{e \in \tilde{\mathcal{E}}_K \setminus \mathcal{E}_D} h_e W_e \|[\![v]\!]\|_{0,e}^2 + \sum_{e \in \tilde{\mathcal{E}}_K \cap \mathcal{E}_D} h_e W_e \|v - g_D\|_{0,e}^2 \right) \tag{4.1}$$

and

$$\|k_K^{1/2} \nabla (v - \mathcal{I}v)\|_{0,K}^2 \le C \left(\sum_{e \in \tilde{\mathcal{E}}_K \setminus \mathcal{E}_D} \frac{W_e}{h_e} \|[\![v]\!]\|_{0,e}^2 + \sum_{e \in \tilde{\mathcal{E}}_K \cap \mathcal{E}_D} \frac{W_e}{h_e} \|v - g_D\|_{0,e}^2 \right) \tag{4.2}$$

for all $K \in \mathcal{T}$.

Proof. For any $v \in \mathcal{U}^{DG}$ and any $K \in \mathcal{T}$, the inverse inequality implies that

$$h_K^2 \|k_K^{1/2} \nabla (v - \mathcal{I}v)\|_{0,K}^2 \le C\, k_K \|v - \mathcal{I}v\|_{0,K}^2.$$

Hence, it suffices to establish the validity of (4.1).

To this end, for any $K \in \mathcal{T}$ and any $z \in \mathcal{N}_K$, denote by $\phi_{z,K}(x) = \phi_z(x)|_K$ the restriction of ϕ_z in K. Then

$$v_K(x) = v|_K(x) = \sum_{z \in \mathcal{N}_K} v_K(z) \phi_{z,K}(x) \quad \forall\, v \in \mathcal{U}^{DG}.$$

Since $\|\phi_{z,K}\|_{0,K} \le C\, h_K$ and

$$v_K(z) - \mathcal{I}v(z) = v_K(z) - \frac{1}{cd(z)} \sum_{K' \in \hat{\omega}_z} v_{K'}(z) = \frac{1}{cd(z)} \sum_{K' \in \hat{\omega}_z} (v_K(z) - v_{K'}(z))$$

for all $z \in \mathcal{N} \backslash \mathcal{N}_D$, we then have

$$\begin{aligned}
\|v - \mathcal{I}v\|_{0,K} &= \|\sum_{z \in \mathcal{N}_K} (v - \mathcal{I}v)(z) \phi_{z,K}\|_{0,K} \\
&\le \sum_{z \in \mathcal{N}_K} \|(v - \mathcal{I}v)(z) \phi_{z,K}\|_{0,K} \le C \sum_{z \in \mathcal{N}_K} h_K |v_K(z) - \mathcal{I}v(z)| \\
&\le C \left(\sum_{z \in \mathcal{N}_K \backslash \mathcal{N}_D} \sum_{K' \in \hat{\omega}_z} h_K\, |v_K(z) - v_{K'}(z)| + \sum_{z \in \mathcal{N}_K \cap \mathcal{N}_D} h_K\, |g_D(z) - v_K(z)| \right). \qquad (4.3)
\end{aligned}$$

For $z \in \mathcal{N}_K \backslash \mathcal{N}_D$ and any $K' \in \hat{\omega}_z$, by the fact that the distribution of the coefficients of K is quasi-monotone with respect to z, there is a connected path,

$$\{K = K_0,\, K_1, \ldots, K_l = K'\} \quad \text{with} \quad K_i \in \omega_z,$$

from K to K' such that the diffusion coefficient is monotonically increasing. Denote by e_i the common edge of K_{i-1} and K_i; then we have

$$k_K \le W_{e_1} \le \cdots \le W_{e_l}.$$

Now, it follows from the triangle inequality and the inverse inequality that for any $K' \in \hat{\omega}_z$,

$$k_K^{1/2}|v_K(z) - v_{K'}(z)| \le k_K^{1/2} \sum_{i=0}^{l} |v_{K_i}(z) - v_{K_{i+1}}(z)|$$

$$\le \sum_{i=0}^{l} \sqrt{W_{e_i}} \|[\![v]\!]\|_{\infty,e_i} \le C \sum_{i=0}^{l} \sqrt{\frac{W_{e_i}}{h_{e_i}}} \|[\![v]\!]\|_{0,e_i}. \tag{4.4}$$

For $z \in \mathcal{N}_K \cap \mathcal{N}_D$, by Definition 4.1, there exists a $K' \in \hat{\omega}_z$ such that $\mathcal{E}_{K'} \cap \mathcal{E}_D \neq \emptyset$. Let $e_D = \mathcal{E}_{K'} \cap \mathcal{E}_D \neq \emptyset$; then the triangle inequality and an argument similar to that above yield

$$k_K^{1/2}|g_D - v_K(z)| \le k_K^{1/2}|g_D - v_{K'}(z)| + k_K^{1/2}|v_{K'}(z) - v_K(z)|$$

$$\le \sqrt{W_{e_D}} \|g_D - v\|_{\infty,e_i} + \sum_{i=0}^{l'} \sqrt{W_{e_i}} \|[\![v]\!]\|_{\infty,e_i}$$

$$\le C \sqrt{\frac{W_{e_D}}{h_{e_D}}} \|g_D - v\|_{0,e_i} + C \sum_{i=0}^{l'} \sqrt{\frac{W_{e_i}}{h_{e_i}}} \|[\![v]\!]\|_{0,e_i}. \tag{4.5}$$

Now, (4.1) is a direct consequence of (4.3), (4.4), (4.5), and the Cauchy–Schwarz inequality. This completes the proof of the lemma. ∎

Clément-type interpolation operators (see, e.g., [11, 31]) are often used for establishing the reliability bound of a posteriori error estimators. We define a weighted Clément-type interpolation operator and state its approximation and stability properties. For more details, see [14].

For a given function v, define its weighted average over $\hat{\omega}_z$ by

$$\fint_{\hat{\omega}_z} v\,dx = \frac{\int_{\hat{\omega}_z} v\,\phi_z\,dx}{\int_{\hat{\omega}_z} \phi_z\,dx}. \tag{4.6}$$

Following [11, 31], define the Clément-type interpolation operator $\mathcal{J} : L^2(\Omega) \to \mathcal{U}_0$ by

$$\mathcal{J}v = \sum_{z \in \mathcal{N}} (\pi_z v)\phi_z(x),$$

where the nodal value at z is defined by

$$(\mathcal{J}v)(z) = \pi_z v = \begin{cases} \fint_{\hat{\omega}_z} v\,dx, & z \in \mathcal{N} \setminus \Gamma_D, \\ 0, & z \in \mathcal{N} \cap \Gamma_D. \end{cases}$$

LEMMA 4.3 (see [14]). *For any $K \in \mathcal{T}$ and $v \in H^1_{0,D}(\Omega)$, the estimates*

$$\|v - \mathcal{J}v\|_{0,K} \le C\,h_K k_K^{-1/2} \|k^{1/2}\nabla v\|_{0,\Delta_K} \tag{4.7}$$

and

$$\|\nabla(v - \mathcal{J}v)\|_{0,K} \le C\,k_K^{-1/2} \|k^{1/2}\nabla v\|_{0,\Delta_K} \tag{4.8}$$

hold, where Δ_K is the union of all elements that share at least one vertex with K. For any $e \in \mathcal{E}_I \cup \mathcal{E}_N$ and $v \in H^1_{0,D}(\Omega)$, the estimate

$$\|v - \mathcal{J}v\|_{0,e} \le C\, h_e^{1/2} (W_{e,1})^{-1/2} \|k^{1/2}\nabla v\|_{0,\Delta_e} \tag{4.9}$$

holds, where Δ_e is the union of all elements that share at least one vertex with edge e.

LEMMA 4.4 (see [14]). *For any $v \in H^1_{0,D}(\Omega)$, there exists a positive constant C independent of the mesh size and the ratio $k_{\max}/k_{\min}$ such that*

$$|(f,\, v - \mathcal{J}v)| \le C\, H_f\, \|k^{1/2}\nabla v\|_{0,\Omega} \tag{4.10}$$

with

$$H_f = \left(\sum_{z\in\mathcal{N}\cap(F\cup\partial\Omega)} \sum_{K\subset\omega_z} k_K^{-1} h_K^2\, \|f\|^2_{0,K} + \sum_{z\in\mathcal{N}\setminus(F\cup\partial\Omega)} \sum_{K\subset\omega_z} k_K^{-1} h_K^2\, \|f - \fint_{\omega_z} f\, dx\|^2_{0,K} \right)^{1/2}.$$

REMARK 4.5 (see [16]). *If $f \in L^2(\Omega)$, the second term in H_f is $o(\max_{K\in\mathcal{T}} h_K)$. If $f \in L^p(\Omega)$ with $p > 2$, the same holds for the first term.*

5. Residual-based a posteriori error estimators. In this section, we study the following residual-based a posteriori error estimator:

$$\eta_R = \left(\sum_{K\in\mathcal{T}} \eta^2_{R,K} \right)^{1/2},$$

where the local indicator on $K \in \mathcal{T}$ is defined by

$$\eta_{R,K} = \left(\eta^2_{R_f,K} + \eta^2_{J_\sigma,K} + \eta^2_{J_u,K} + \eta^2_{R_D,K} + \eta^2_{R_N,K} \right)^{1/2}$$

with

$$\eta^2_{R_f,K} = \frac{h_K^2 \|f\|^2_{0,K}}{k_K}, \quad \eta^2_{R_N,K} = \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_N} \frac{h_e}{k_e} \|g_N - k\nabla u_{\mathcal{T}} \cdot \mathbf{n}\|^2_{0,e},$$

$$\eta^2_{R_D,K} = \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_D} \frac{W_e}{h_e} \|g_D - u_{\mathcal{T}}\|^2_{0,e},$$

$$\eta^2_{J_\sigma,K} = \frac{1}{2} \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_I} \frac{h_e \|[\![k\nabla u_{\mathcal{T}} \cdot \mathbf{n}]\!]\|^2_{0,e}}{W_{e,1}}, \quad \text{and} \quad \eta^2_{J_u,K} = \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_I} \frac{W_e}{h_e} \|[\![u_{\mathcal{T}}]\!]\|^2_{0,e}.$$

The $\eta_{R_f,K}$ is the element residual, the $\eta_{R_D,K}$ and $\eta_{R_N,K}$ are the boundary residuals, and the $\eta_{J_\sigma,K}$ and $\eta_{J_u,K}$ are associated with edge jumps of the flux and the solution, respectively. For $k = 1$, i.e., the Poisson equation, this residual-based estimator is identical to that of [27]. For k being a tensor, Ern and Stephansen in [22] recently studied a residual-based error estimator, which can be made robust for scalar k and under the assumption of local monotonicity. Their estimator differs from the estimator $\eta_{R,K}$ in both the flux jump term and the solution jump term. Instead of $\eta_{J_u,K}$, they use the so-called nonconforming error which is the energy norm of difference between the DG approximation and its continuous recovery through the Oswald interpolation.

To analyze the estimator η_R and recovery-based estimators to be introduced in section 7, we employ a standard technique that uses the Helmholtz decomposition. To this end, we cite the following decomposition (see, e.g., [3]): for any given vector-valued function $\boldsymbol{\tau} \in L^2(\Omega)^2$, there exist $p \in H^1_{0,D}(\Omega)$ and $q \in \mathcal{H}$ such that

$$\boldsymbol{\tau} = k(x)\nabla p + \nabla^\perp q, \tag{5.1}$$

where $\mathcal{H}$ is a subspace of $H^1(\Omega)$ having zero mean value and homogeneous tangential derivatives on Γ_N:

$$\mathcal{H} = \left\{ v \in H^1(\Omega) \,:\, \int_\Omega v\,dx = 0 \text{ and } \frac{\partial v}{\partial \mathbf{t}} = 0 \quad \text{on } \Gamma_N \right\}.$$

Integrating by parts gives

$$(\nabla p,\, \nabla^\perp q) = 0 \tag{5.2}$$

for all $p \in H^1_{0,D}(\Omega)$ and all $q \in \mathcal{H}$, which, in turn, implies that the decomposition is orthogonal with respect to the weighted L^2 inner product $(k^{-1}\cdot, \cdot)$:

$$(k^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (k\nabla p, \nabla p) + (k^{-1}\nabla^\perp q, \nabla^\perp q). \tag{5.3}$$

Let $e = u - u_\mathcal{T}$; then there exist $p \in H^1_{0,D}(\Omega)$ and $q \in \mathcal{H}$ such that

$$k\nabla_h e = k\nabla p + \nabla^\perp q \tag{5.4}$$

and

$$\|k^{1/2}\nabla_h e\|^2_{0,\Omega} = \|k^{1/2}\nabla p\|^2_{0,\Omega} + \|k^{-1/2}\nabla^\perp q\|^2_{0,\Omega}. \tag{5.5}$$

Denote the weighted oscillations of the data f over the collection $\mathcal{T}'$ of elements by

$$\operatorname{osc}(f, \mathcal{T}')^2 = \sum_{K\in\mathcal{T}'} \frac{h_K^2}{k_K}\|f - \bar{f}_K\|^2_{0,K},$$

where $\bar{f}_K$ is the average of f over K.

THEOREM 5.1 (reliability). *Assume that $u \in V^{1+\epsilon}(\mathcal{T})$ is the solution of problem* (2.14) *and that the triangulation is quasi-uniform. Then the residual-based a posteriori error estimator η_R satisfies the following global reliability bound:*

$$|\!|\!| u - u_\mathcal{T} |\!|\!|_{DG} \le C\,\eta_R, \tag{5.6}$$

where C is a positive constant independent of the mesh size and the ratio $k_{\max}/k_{\min}$.

Proof. It follows from (5.2), the error equation in (3.1) with $v = \mathcal{J}p$, and the continuity of $\mathcal{J}p \in \mathcal{U}_0$ that

$$\begin{aligned}
\|k^{1/2}\nabla p\|^2_{0,\Omega} &= (k\nabla_h e,\, \nabla p) \\
&= (k\nabla_h e, \nabla(p - \mathcal{J}p)) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \bigg(\{k\nabla e\cdot\mathbf{n}\}_w [\![\mathcal{J}p]\!] - \theta\{k\nabla\mathcal{J}p\cdot\mathbf{n}\}_w[\![e]\!] \\
&\qquad - \gamma_\theta \frac{W_e}{h_e}[\![\mathcal{J}p]\!][\![e]\!]\bigg)\,ds \\
&= (k\nabla_h e, \nabla(p - \mathcal{J}p)) - \theta \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla\mathcal{J}p\cdot\mathbf{n}\}_w[\![e]\!]\,ds. \qquad (5.7)
\end{aligned}$$

For any $e \in \mathcal{E}_I$, using the Cauchy–Schwarz inequality, (2.5), the trace theorem, and the inverse inequality, we have

$$\int_e \{k\nabla \mathcal{J}p\cdot \mathbf{n}\}_w \llbracket e \rrbracket\, ds = \int_e w_+ (k_+ \nabla \mathcal{J}p \cdot \mathbf{n}) \llbracket e \rrbracket\, ds + \int_e w_- (k_- \nabla \mathcal{J}p \cdot \mathbf{n}) \llbracket e \rrbracket\, ds$$

$$\le \left(\frac{h_e^{1/2} w_+ k_+^{1/2}}{W_e^{1/2}} \|k_+^{1/2} \nabla \mathcal{J}p_+ \cdot \mathbf{n}\|_{0,e} + \frac{h_e^{1/2} w_- k_-^{1/2}}{W_e^{1/2}} \|k_-^{1/2} \nabla \mathcal{J}p_- \cdot \mathbf{n}\|_{0,e} \right) \frac{W_e^{1/2}}{h_e^{1/2}} \|\llbracket u_{\mathcal{T}} \rrbracket\|_{0,e}$$

$$\le \left(\|k_+^{1/2} \nabla \mathcal{J}p\|_{0,K_+} + \|k_-^{1/2} \nabla \mathcal{J}p\|_{0,K_-} \right) \frac{W_e^{1/2}}{h_e^{1/2}} \|\llbracket u_{\mathcal{T}} \rrbracket\|_{0,e}.$$

Thus, summing over all $e \in \mathcal{E}_I$ and using the Cauchy–Schwarz inequality lead to

$$\sum_{e\in\mathcal{E}_I} \int_e \{k\nabla \mathcal{J}p \cdot \mathbf{n}\}_w \llbracket e \rrbracket\, ds \le C \left(\sum_{e\in\mathcal{E}_I} \frac{W_e}{h_e} \|\llbracket u_{\mathcal{T}} \rrbracket\|_{0,e}^2 \right)^{1/2} \|k^{1/2} \nabla p\|_{0,\Omega}. \tag{5.8}$$

Similarly, we have

$$\sum_{e\in\mathcal{E}_D} \int_e \{k\nabla \mathcal{J}p \cdot \mathbf{n}\}_w \llbracket e \rrbracket\, ds \le C \left(\sum_{e\in\mathcal{E}_D} \frac{W_e}{h_e} \|g_D - u_{\mathcal{T}}\|_{0,e}^2 \right)^{1/2} \|k^{1/2} \nabla p\|_{0,\Omega}. \tag{5.9}$$

Denote $e_{\mathcal{J}} = p - \mathcal{J}p$. Using integration by parts, (1.1), (2.10), the facts that

$$\nabla \cdot (k\nabla u_{\mathcal{T}}) = 0 \ \text{ on } K \in \mathcal{T} \quad \text{and} \quad e_{\mathcal{J}} \in H_{0,D}^1(\Omega),$$

(1.2), the Cauchy–Schwarz inequality, and Lemma 4.3, we have

$$(k\nabla_h e, \nabla(p - \mathcal{J}p)) = -\sum_{K\in\mathcal{T}} (\nabla\cdot(k\nabla_h e),\, e_{\mathcal{J}}) + \sum_{K\in\mathcal{T}} \int_{\partial K} (k\nabla e \cdot \mathbf{n})\, e_{\mathcal{J}}\, ds \tag{5.10}$$

$$= \sum_{K\in\mathcal{T}} (f,\, e_{\mathcal{J}}) - \sum_{e\in\mathcal{E}_I} \int_e \llbracket k\nabla u_{\mathcal{T}} \cdot \mathbf{n} \rrbracket\, e_{\mathcal{J}}\, ds + \sum_{e\in\mathcal{E}_N} \int_e (g_N - k\nabla u_{\mathcal{T}} \cdot \mathbf{n})\, e_{\mathcal{J}}\, ds$$

$$\le \sum_{K\in\mathcal{T}} \|f\|_{0,K}\, \|e_{\mathcal{J}}\|_{0,K} + \sum_{e\in\mathcal{E}_I} \|\llbracket k\nabla u_{\mathcal{T}} \cdot \mathbf{n} \rrbracket\|_{0,e} \|e_{\mathcal{J}}\|_{0,e}$$

$$+ \sum_{e\in\mathcal{E}_N} \|g_N - k\nabla u_{\mathcal{T}} \cdot \mathbf{n}\|_{0,e}\, \|e_{\mathcal{J}}\|_{0,e}$$

$$\le C \left(\sum_{K\in\mathcal{T}} \frac{h_K \|f\|_{0,K}}{k_K^{1/2}} \|k^{1/2}\nabla p\|_{0,\Delta_K} + \sum_{e\in\mathcal{E}_I} \sqrt{\frac{h_e}{W_{e,1}}}\, \|\llbracket k\nabla u_{\mathcal{T}} \cdot \mathbf{n} \rrbracket\|_{0,e} \|k^{1/2}\nabla p\|_{0,\Delta_e} \right.$$

$$\left. + \sum_{e\in\mathcal{E}_N} \sqrt{\frac{h_e}{k_e}}\, \|g_N - k\nabla u_{\mathcal{T}} \cdot \mathbf{n}\|_{0,e} \|k^{1/2}\nabla p\|_{0,\Delta_e} \right)$$

$$\le C \left(\sum_{K\in\mathcal{T}} (\eta_{R_f,K}^2 + \eta_{J_\sigma,K}^2 + \eta_{R_N,K}^2) \right)^{1/2} \|k^{1/2}\nabla p\|_{0,\Omega}.$$

Combining (5.7), (5.8), (5.9), and (5.11) yields

$$\|k^{1/2}\nabla p\|_{0,\Omega} \le C\,\eta_{_R}.$$

Together with (5.5), then it suffices to show

$$\|k^{-1/2}\nabla^{\perp} q\|_{0,\Omega} \le C\,\eta_{_R}. \tag{5.11}$$

To this end, since $u \in H^1(\Omega)$, $\mathcal{I}u_{_\mathcal{T}} \in H^1(\Omega)$, $u - \mathcal{I}u_{_\mathcal{T}} = 0$ on Γ_D, and $q \in \mathcal{H}$, integrating by parts gives

$$(\nabla(u - \mathcal{I}u_{\mathcal{T}}), \nabla^{\perp} q) = 0.$$

This, together with the Cauchy–Schwarz inequality and (4.2), implies that

$$\begin{aligned}
\|k^{-1/2}\nabla^{\perp} q\|^2_{0,\Omega} &= (\nabla_h e,\, \nabla^{\perp} q) = \sum_{K\in\mathcal{T}} (\nabla(u - u_{_\mathcal{T}}),\, \nabla^{\perp} q)_K \\
&= \sum_{K\in\mathcal{T}} (\nabla(\mathcal{I}u_{_\mathcal{T}} - u_{_\mathcal{T}}),\, \nabla^{\perp} q)_K \\
&\le \sum_{K\in\mathcal{T}} \|k^{1/2}\nabla(\mathcal{I}u_{_\mathcal{T}} - u_{_\mathcal{T}})\|_{0,K} \|k^{-1/2}\nabla^{\perp} q\|_{0,K} \\
&\le \left(\sum_{K\in\mathcal{T}} \|k^{1/2}\nabla(\mathcal{I}u_{_\mathcal{T}} - u_{_\mathcal{T}})\|^2_{0,K}\right)^{\frac{1}{2}} \|k^{-1/2}\nabla^{\perp} q\|_{0,\Omega} \\
&\le C \sum_{K\in\mathcal{T}} \left(\eta^2_{_{J_u},K} + \eta^2_{_{R_D},K}\right)^{1/2} \|k^{-1/2}\nabla^{\perp} q\|_{0,\Omega}.
\end{aligned}$$

Thus

$$\|k^{-1/2}\nabla^{\perp} q\|_{0,\Omega} \le C \sum_{K\in\mathcal{T}} \left(\eta^2_{_{J_u},K} + \eta^2_{_{R_D},K}\right)^{1/2}, \tag{5.12}$$

which proves the validity of (5.11) and, hence, the theorem. ◻

Note that (5.4) indicates that the true error $e = u - u_{_\mathcal{T}}$ comes from two kinds of sources: discontinuous approximations of the normal component of the flux and the tangential component of the gradient. As shown in the following lemma, the element residual, the Neumann boundary residual, and the edge jump of the flux may be bounded by the energy norm of p plus higher-order terms.

LEMMA 5.2. *There exists a positive constant C independent of the mesh size and the ratio $k_{\max}/k_{\min}$ such that*

$$\eta_{_{R_f},K} \le C\left(\|k^{1/2}\nabla p\|_{0,K} + osc(f,K)\right) \quad \forall\ K\in\mathcal{T}, \tag{5.13}$$

$$\sqrt{\frac{h_e}{W_{e,1}}}\,\|[\![k\nabla u_{_\mathcal{T}}\cdot\mathbf{n}]\!]\|_{0,e} \le C\left(\|k^{1/2}\nabla p\|_{0,\omega_e} + osc(f,\omega_e)\right) \quad \forall\ e\in\mathcal{E}_{_I}, \tag{5.14}$$

$$\sqrt{\frac{h_e}{k}}\,\|g_{_N} - k\nabla u_{\mathcal{T}}\|_{0,e} \le C\left(\|k^{1/2}\nabla p\|_{0,\omega_e} + osc(f,\omega_e)\right) \quad \forall\ e\in\mathcal{E}_{_N}, \tag{5.15}$$

where ω_e is the collection of all elements that share the common edge e.

Proof. For any element $K \in \mathcal{T}$, let b_K be the standard cubic bubble function whose support is K. Then (see, e.g., [37])

$$C\, h_K^{1/2} \le \|b_K^{1/2}\|_{0,K}, \quad \|b_K\|_{\infty,K} \le C, \quad \text{and} \quad \|\nabla b_K\|_{\infty,K} \le C\, h_K^{-1}. \tag{5.16}$$

Using (5.4) and (5.2), we have

$$(k\nabla p,\, \nabla v) = (k\nabla_h e,\, \nabla v) \quad \forall\, v \in H^1_{0,D}(\Omega). \tag{5.17}$$

Choosing $v = \bar{f}_K b_K$ in (5.17) and integrating by parts lead to

$$(k\nabla p,\, \nabla(\bar{f}_K b_K))_K = (k\nabla_h e,\, \nabla(\bar{f}_K b_K))_K = (f - \bar{f}_K,\, \bar{f}_K b_K)_K + (\bar{f}_K,\, \bar{f}_K b_K)_K.$$

It follows from the Cauchy–Schwarz inequality and (5.16) that

$$\begin{aligned} C\|\bar{f}_K\|_{0,K}^2 &\le (\bar{f}_K,\, \bar{f}_K b_K)_K = (k\nabla p,\, \nabla(\bar{f}_K b_K))_K - (f - \bar{f}_K,\, \bar{f}_K b_K)_K \\ &\le \|k^{1/2}\nabla p\|_{0,K}\|k^{1/2}\bar{f}_K \nabla b_K\|_{0,K} + \|f - \bar{f}_K\|_{0,K}\|\bar{f}_K b_K\|_{0,K} \\ &\le C\left(\frac{k_K^{1/2}}{h_K}\|k^{1/2}\nabla p\|_{0,K}\|\bar{f}_K\|_{0,K} + \|f - \bar{f}_K\|_{0,K}\ \|\bar{f}_K\|_{0,K}\right). \end{aligned}$$

Hence,

$$\frac{h_K}{k_K^{1/2}}\|\bar{f}_K\|_{0,K} \le C\left(\|k^{1/2}\nabla p\|_{0,K} + \frac{h_K}{k_K^{1/2}}\|f - \bar{f}_K\|_{0,K}\right).$$

Now, (5.13) is a direct consequence of the triangle inequality.

For any edge $e \in \mathcal{E}_I$, let b_e be the standard piecewise quadratic edge bubble function corresponding to the edge e whose support is ω_e. Then (see, e.g., [37])

$$C\, h_e^{1/2} \le \|b_e^{1/2}\|_{0,e}, \quad \|b_e\|_{\infty,\omega_e} \le C, \quad \text{and} \quad \|\nabla b_e\|_{\infty,\omega_e} \le C\, h_e^{-1}. \tag{5.18}$$

Choosing $v = [\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]_e\, b_e$ in (5.17) and integrating by parts lead to

$$\begin{aligned} (k\nabla p,\, \nabla([\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]_e\, b_e))_{\omega_e} &= (k\nabla_h e,\, \nabla([\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]_e\, b_e))_{\omega_e} \\ &= (f,\, [\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]_e\, b_e)_{\omega_e} + \|[\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]\, b_e^{1/2}\|_{0,e}^2. \end{aligned}$$

It follows from (5.18), the Cauchy–Schwarz inequality, and (2.6) that

$$\begin{aligned} &C\|[\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]\|_{0,e}^2 \\ &\quad \le \|[\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]\, b_e^{1/2}\|_{0,e}^2 \\ &\quad \le \|k\nabla p\|_{0,\omega_e}\|[\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]_e \nabla b_e\|_{0,\omega_e} + \|f\|_{0,\omega_e}\|[\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]\, b_e\|_{0,\omega_e} \\ &\quad \le C\, h_e^{-1/2}\|[\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]\|_{0,e} \sum_{K\in\omega_e} k_K^{1/2}\left(\|k^{1/2}\nabla p\|_{0,K} + h_K k_K^{-1/2}\|f\|_{0,K}\right), \end{aligned}$$

which, together with (2.6) and the triangle inequality, implies that

$$\sqrt{\frac{h_e}{W_{e,1}}}\|[\![k\nabla u_\mathcal{T} \cdot \mathbf{n}]\!]\|_{0,e} \le C\left(\|k^{1/2}\nabla p\|_{0,\omega_e} + \sum_{K\in\omega_e}\frac{h_K(\|\bar{f}_K\|_{0,K} + \|f - \bar{f}_K\|_{0,K})}{k_K^{1/2}}\right).$$

Now, (5.14) follows from (5.13). For $e \in \mathcal{E}_N$, (5.15) may be proved in a similar fashion by choosing $v = (g_{_N} - k\nabla u_{\mathcal{T}} \cdot \mathbf{n})\, b_e$. This completes the proof of the lemma. ∎

THEOREM 5.3 (efficiency). *Assume that the diffusion coefficient is locally quasi-monotone. Then there exists a positive constant C independent of the mesh size and the ratio $k_{\max}/k_{\min}$ such that*

$$\eta_{_{R,K}} \le C\,\left(\|e\|_{DG,K} + osc(f,\omega_K)\right) \quad \forall\; K \in \mathcal{T},$$

where ω_K is the collection of elements in $\mathcal{T}$ that share a common edge with K.

Proof. The local efficiency bound is a straightforward consequence of Lemma 5.2 and the definition of the DG norm. ∎

6. Flux recovery.

The flux defined by

$$\boldsymbol{\sigma} = -k(x)\nabla u \quad \text{in } \Omega \tag{6.1}$$

is an important physical quantity which is often the primary concern in practice. In this section, we describe both implicit and explicit recoveries of the flux that are used to design robust a posteriori error estimators in the subsequent section. In addition, we show that the implicitly recovered flux is a good approximation to the flux.

6.1. Implicit approximation.

The implicitly recovered flux is defined as follows: find $\boldsymbol{\sigma}_{\mathcal{T}} \in \mathcal{V}_N$ such that

$$(k^{-1}\boldsymbol{\sigma}_{\mathcal{T}},\, \boldsymbol{\tau}) = -(\nabla_h u_{\mathcal{T}},\, \boldsymbol{\tau}) \quad \forall\; \boldsymbol{\tau} \in \mathcal{V}_N, \tag{6.2}$$

where $\mathcal{V}_N$ is the subspace of $\mathcal{V}$ (RT_0 or BDM_1) satisfying the Neumann boundary conditions

$$\mathcal{V}_N = \{\boldsymbol{\tau} \in \mathcal{V} \,:\, \boldsymbol{\tau}\cdot\mathbf{n} = g_{_N} \;\text{ on } \Gamma_N\}.$$

THEOREM 6.1. *There exists a positive constant C independent of the ratio $k_{\max}/k_{\min}$ such that the a priori error bound*

$$\|k^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{\mathcal{T}})\|_{0,\Omega} \le C\left(\inf_{\boldsymbol{\tau}\in\mathcal{V}_N}\|k^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\tau})\|_{0,\Omega} + \|k^{1/2}(\nabla u - \nabla_h u_{\mathcal{T}})\|_{0,\Omega}\right) \tag{6.3}$$

holds.

Proof. By using the error equation

$$(k^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{\mathcal{T}}),\, \boldsymbol{\tau}) = (\nabla_h u_{\mathcal{T}} - \nabla u,\, \boldsymbol{\tau}) \quad \forall\; \boldsymbol{\tau} \in \mathcal{V}_N,$$

(6.3) may be proved in a fashion similar to that of Theorem 3.1 in [14]. ∎

6.2. Explicit approximations.

Let $\delta_{e\,e'}$ denote the Kronecker delta:

$$\delta_{e\,e'} = \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{if } e \ne e'. \end{cases}$$

For RT_0, its nodal basis function, $\boldsymbol{\phi}_e$, corresponding to the edge $e \in \mathcal{E}$ is uniquely determined by

$$\boldsymbol{\phi}_e \cdot \mathbf{n}_{e'} = \delta_{e\,e'} \quad \forall\; e' \in \mathcal{E}.$$

Define the explicit approximation $\hat{\boldsymbol{\sigma}}_{\mathcal{T}}$ in $RT_0 = \text{span}\{\boldsymbol{\phi}_e : e \in \mathcal{E}\}$ by

$$\hat{\boldsymbol{\sigma}}_{\mathcal{T}} = \sum_{e\in\mathcal{E}} \hat{\sigma}_e\, \boldsymbol{\phi}_e, \tag{6.4}$$

where $\hat{\sigma}_e$ is the normal component of $\hat{\boldsymbol{\sigma}}_{\mathcal{T}}$ on the edge $e \in \mathcal{E}$ defined as a weighted average of the normal components of the approximated flux $-k\nabla_h u_{\mathcal{T}}$; i.e.,

$$\hat{\sigma}_e := \{-k\nabla_h u_{\mathcal{T}} \cdot \mathbf{n}_e\}^e_{w_i} \tag{6.5}$$

with weights w_i defined in (2.3). To ensure the efficiency bound independent of the size of jumps, we choose $i = 2$ or $i = 3$, i.e., the harmonic or the geometric weights. Note that these weights satisfy the following inequality:

$$\max\left\{\frac{(w^e_{+,i})^2}{k^e_-}, \frac{(w^e_{-,i})^2}{k^e_+}\right\} \le \frac{1}{k^e_+ + k^e_-}, \quad i = 2, 3. \tag{6.6}$$

7. Recovery-based a posteriori error estimators. For any element $K \in \mathcal{T}$, based on the implicitly and explicitly recovered fluxes, define

$$\eta_{\sigma,K} = \|k^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}} + k^{1/2}\nabla u_{\mathcal{T}}\|_{0,K} \quad \text{and} \quad \hat{\eta}_{\sigma,K} = \|k^{-1/2}\hat{\boldsymbol{\sigma}}_{\mathcal{T}} + k^{1/2}\nabla u_{\mathcal{T}}\|_{0,K}.$$

Obviously,

$$\eta_{\sigma,K} \le \hat{\eta}_{\sigma,K}. \tag{7.1}$$

Let

$$\eta^2_\sigma = \sum_{K\in\mathcal{T}} \eta^2_{\sigma,K}, \quad \eta^2_{J_u} = \sum_{K\in\mathcal{T}} \eta^2_{J_u,K}, \quad \text{and} \quad \eta^2_{R_D} = \sum_{K\in\mathcal{T}} \eta^2_{R_D,K}.$$

It is easy to see that

$$\eta_\sigma = \|k^{-1/2}\boldsymbol{\sigma}_{\mathcal{T}} + k^{1/2}\nabla_h u_{\mathcal{T}}\|_{0,\Omega} = \min_{\boldsymbol{\tau}\in\mathcal{V}_N} \|k^{-1/2}\boldsymbol{\tau} + k^{1/2}\nabla_h u_{\mathcal{T}}\|_{0,\Omega}.$$

Now, we define the recovery-based a posteriori error estimators as follows:

$$\xi = \left(\eta^2_\sigma + \eta^2_{J_u} + \eta^2_{R_D}\right)^{1/2} \quad \text{and} \quad \hat{\xi} = \left(\hat{\eta}^2_\sigma + \eta^2_{J_u} + \eta^2_{R_D}\right)^{1/2}.$$

THEOREM 7.1 (reliability). *There exists a positive constant C independent of the mesh size and the ratio $k_{\max}/k_{\min}$ such that*

$$|\!|\!| e |\!|\!|_{DG} \le C\,(\xi + H_f) \le C\left(\hat{\xi} + H_f\right). \tag{7.2}$$

Proof. The second inequality in (7.2) follows directly from (7.1). To establish the validity of the first inequality in (7.2), by (5.5), (5.12), and the definition of the DG norm, it suffices to show that

$$\|k^{1/2}\nabla p\|_{0,\Omega} \le C\,(\xi + H_f)\,. \tag{7.3}$$

Letting $e_{\mathcal{J}} = p - \mathcal{J}\,p$, where $\mathcal{J}$ is the Clément-type interpolation operator defined in section 4, it then follows from integration by parts, homogeneous boundary conditions of $e_{\mathcal{J}}$ on Γ_D and $(k\nabla u + \boldsymbol{\sigma}_{\mathcal{T}})\cdot \mathbf{n}$ on Γ_N, the Cauchy–Schwarz inequality, (4.8), the fact that $\nabla_h \cdot (k\nabla_h u_{\mathcal{T}}) = 0$, the inverse inequality, (4.7), and (4.10) that

$$\begin{aligned}(k\nabla_h e, \nabla e_{\mathcal{J}}) &= (k\nabla u + \boldsymbol{\sigma}_{\mathcal{T}}, \nabla e_{\mathcal{J}}) - (\boldsymbol{\sigma}_{\mathcal{T}} + k\nabla_h u_{\mathcal{T}}, \nabla e_{\mathcal{J}})\\ &\le (f - \nabla\cdot\boldsymbol{\sigma}_{\mathcal{T}}, e_{\mathcal{J}}) + C\,\eta_\sigma\,\|k^{1/2}\nabla p\|_{0,\Omega}\\ &= (f, e_{\mathcal{J}}) - (\nabla_h\cdot(\boldsymbol{\sigma}_{\mathcal{T}} + k\nabla_h u_{\mathcal{T}}), e_{\mathcal{J}}) + C\,\eta_\sigma\,\|k^{1/2}\nabla p\|_{0,\Omega}\\ &\le C\,(\eta_\sigma + H_f)\,\|k^{1/2}\nabla p\|_{0,\Omega}.\end{aligned}$$

Combining with (5.7), (5.8), and (5.9) yields the first inequality in (7.3). This completes the proof of the theorem. ∎

REMARK 7.2. *A different explicit recovery-based estimator is derived in* [21]. *Its flux recovery achieves a tighter local connection between* $\nabla\cdot\boldsymbol{\sigma}$ *and* f. *Its reliability bound similar to* (7.2) *is established with* $C = 1$ *and* H_f *being replaced by a superconvergent term if* $f \in H^1(\Omega)$.

LEMMA 7.3. *For any element* $K \in \mathcal{T}$, *let* ω_K *be the union of elements sharing a common edge with* K. *There exists a positive constant* C *independent of the mesh size and the ratio* $k_{\max}/k_{\min}$ *such that*

$$\hat{\eta}_{\sigma,K} \le C\left(\|k^{1/2}\nabla p\|^2_{0,\omega_K} + \operatorname{osc}(f,\omega_K)^2\right)^{1/2} \quad \forall\, K \in \mathcal{T}. \tag{7.4}$$

Proof. To show the validity of (7.4), by (5.14) and (5.15), it suffices to prove that for any element $K \in \mathcal{T}$,

$$\hat{\eta}^2_{\sigma,K} \le C\left(\sum_{e\in\mathcal{E}_K\cap\mathcal{E}_I}\frac{h_e}{W_{e,1}}\|[\![k\nabla u_{\mathcal{T}}\cdot\mathbf{n}_e]\!]\|^2_{0,e} + \sum_{e\in\mathcal{E}_K\cap\mathcal{E}_N}\frac{h_e}{k}\|g_N - k\nabla u_{\mathcal{T}}\cdot\mathbf{n}_e\|^2_{0,e}\right). \tag{7.5}$$

We provide the proof of (7.5) only in the case that $\mathcal{E}_K \cap \mathcal{E}_N = \emptyset$ because it can be proved in a similar fashion in the case that $\mathcal{E}_K \cap \mathcal{E}_N \ne \emptyset$. To this end, for any edge $e \in \mathcal{E}_K$, without loss of generality, let $\mathbf{n}_e$ be the outward unit vector normal to ∂K, and denote by K_e the adjacent element with the common edge e. Since $\boldsymbol{\tau} = -k\nabla_h u_{\mathcal{T}}$ is piecewise constant, $\boldsymbol{\tau}|_K$ may be represented in terms of the nodal basis function of RT_0, $\{\boldsymbol{\phi}_e\}_{e\in\partial K}$, as follows:

$$\boldsymbol{\tau}|_K = \sum_{e\in\mathcal{E}_K}\tau_{e,K}\,\boldsymbol{\phi}_e.$$

For any $\mathbf{x} \in K$, (6.4) and (6.5) give

$$\begin{aligned}\hat{\boldsymbol{\sigma}}_{\mathcal{T}} - \boldsymbol{\tau} &= \sum_{e\in\mathcal{E}_K}(\hat{\sigma}_e - \tau_{e,K})\,\boldsymbol{\phi}_e = \sum_{e\in\mathcal{E}_K}(w^e_{+,i} - 1)\,(\tau_{e,K} - \tau_{e,K_e})\,\boldsymbol{\phi}_e\\ &= -\sum_{e\in\mathcal{E}_K} w_{e,-}[\![\boldsymbol{\tau}\cdot\mathbf{n}_e]\!]\,\boldsymbol{\phi}_e.\end{aligned}$$

Now, it follows from the triangle inequality, the fact that $\int_K |\boldsymbol{\phi}_e|^2\,dx \le C$, and (6.6) that

$$\begin{aligned}\hat{\eta}^2_{\sigma,K} &= \|k^{-1/2}\,(\hat{\boldsymbol{\sigma}}_{\mathcal{T}} - \boldsymbol{\tau})\,\|^2_{0,K} \le C\,k_K^{-1}\|\hat{\boldsymbol{\sigma}}_{\mathcal{T}} - \boldsymbol{\tau}\|^2_{0,K}\\ &\le C\sum_{e\in\mathcal{E}_K} k_K^{-1} w^2_{e,-} h_e \int_e [\![\boldsymbol{\tau}\cdot\mathbf{n}_e]\!]^2\,ds \le C\left(\sum_{e\in\mathcal{E}_K}\frac{h_e}{W_{e,1}}\|[\![k\nabla u_{\mathcal{T}}\cdot\mathbf{n}_e]\!]\|^2_{0,e}\right),\end{aligned}$$

which proves (7.5) and, hence, (7.4). This completes the proof of the lemma. □

THEOREM 7.4 (efficiency). *Assume that the diffusion coefficient is locally quasi-monotone. Then there exists a positive constant C independent of the mesh size and the ratio $k_{\max}/k_{\min}$ such that*

$$\xi \le \hat{\xi} \le C\,\left(|\!|\!| e |\!|\!|_{DG} + \mathrm{osc}\,(f, \mathcal{T})\right). \tag{7.6}$$

Proof. Inequality (7.6) is a direct consequence of (7.1), (7.4), and the definition of the DG norm. □

8. Numerical experiment. In this section, we report some numerical results for an interface problem with intersecting interfaces used by many authors, which is considered as a benchmark test problem. For this test problem, we numerically illustrate the discretization error of the DG method and demonstrate the robustness of our error estimators.

To this end, let $\Omega = (-1, 1)^2$ and

$$u(r, \theta) = r^{\beta}\mu(\theta)$$

in the polar coordinates at the origin with $\mu(\theta)$ being a smooth function of θ (see, e.g., [14]). The function $u(r, \theta)$ satisfies the interface equation with $A = kI$, $\Gamma_N = \emptyset$, $f = 0$, and

$$k(x) = \begin{cases} R & \text{in } (0,1)^2 \cup (-1,0)^2, \\ 1 & \text{in } \Omega \setminus ([0,1]^2 \cup [-1,0]^2). \end{cases}$$

Note that the solution $u(r, \theta)$ is only in $H^{1+\beta-\epsilon}(\Omega)$ for any $\epsilon > 0$ and, hence, it is very singular for small β at the origin. This suggests that refinement is centered around the origin. The β depends on the size of the jump. For the test problem, we choose $\beta = 0.1$ which is corresponding to $R \approx 161$.

For simplicity of presentation, the harmonic weighted incomplete interior penalty Galerkin (IIPG) method is used. Let $u_{\mathcal{T}} \in \mathcal{U}^{DG}$ be the discontinuous finite element approximation of the solution. Denote by N the number of unknowns. We start with the coarsest triangulation $\mathcal{T}_0$ obtained from halving 16 congruent squares by connecting the bottom left and upper right corners.

Numerical results on uniform meshes are reported in Figure 1. The a priori error estimate in (3.4) is of order $h^{0.1}$. This indicates that the slope of the log(dof)-log(error) should be -0.05. Figure 1 shows that the asymptotic convergence rate for this test problem is slightly better than the theoretical prediction.

Starting with the coarse triangulation $\mathcal{T}_0$, a sequence of meshes is generated by using a standard adaptive meshing algorithm that adopts the maximum marking strategy: (1) mark those elements such that $\eta_K \ge 0.5 \max_{K' \in \mathcal{T}} \eta_{K'}$ and (2) refine the marked triangles by bisection. The stopping criterion

$$\text{rel-err} := \frac{|\!|\!| u - u_{\mathcal{T}} |\!|\!|_{DG}}{\|k^{1/2}\nabla u\|_{0,\Omega}} \le \text{ tol}$$

is used, and numerical results with tol $= 0.1$ are reported.

Meshes generated by η_R and ξ are depicted in Figures 2 and 4, respectively. Refinements are centered at origin as expected for efficient estimators. Meshes generated by various a posteriori error estimators for various types of discretizations of

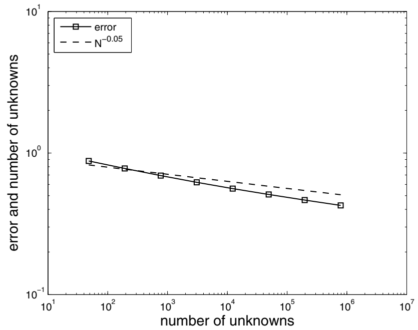

FIG. 1. *Error and number of unknowns for uniform refinements.*

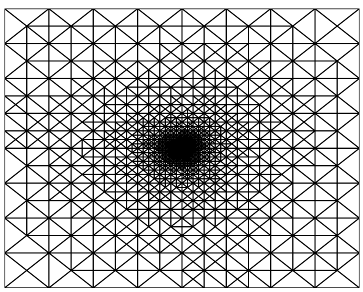

FIG. 2. *Mesh generated by* η_R.

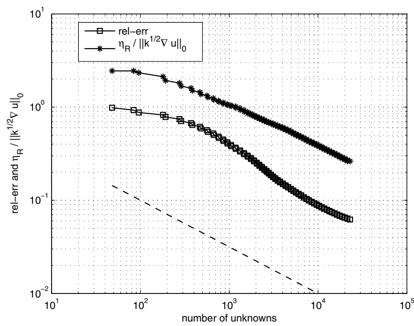

FIG. 3. *Relative error and estimator* $\eta_R/\|k^{1/2}\nabla u\|_{0,\Omega}$.

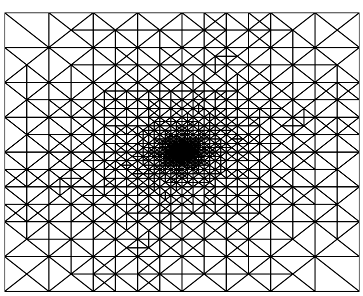

FIG. 4. *Mesh generated by* ξ.

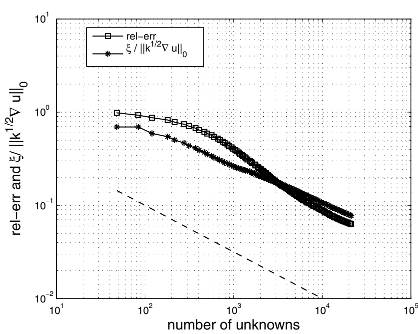

FIG. 5. *Relative error and estimator* $\xi/\|k^{1/2}\nabla u\|_{0,\Omega}$.

this test problem can be found in [14, 15]. As shown in Figures 3 and 5, the slopes of the log(dof)-log(relative error) for the estimators are close to $-1/2$ when there are enough grid points (about several hundreds of unknowns). This implies the optimal decay of the error with respect to the number of unknowns. While the effectivity index, eff-index $:= \frac{\eta}{\|u-u_{\mathcal{T}}\|_{DG}}$, of η_R is about 4, that of ξ is close to 1. This means that the estimator ξ is very accurate.

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