Spectral Properties of Spiral Waves in the Barkley Model

Stephanie Dodson and Björn Sandstede
Division of Applied Mathematics, Brown University

Spiral Waves

- Flow of ions in and out of cardiac cells creates an electric potential, causing the heart to beat.
- Spiral wave patterns cause certain tachycardia rhythms.
- Dangerous rhythms removed with powerful defibrillators.

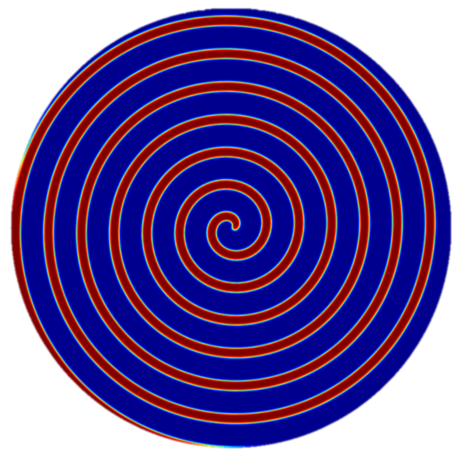

Figure: Spiral wave.

The **Barkley Model** is a reaction-diffusion system for excitable media.

$$\begin{cases} u_t &= \Delta u + \dfrac{1}{\epsilon} u(1-u)\left(u - \dfrac{v+b}{a}\right) \\ v_t &= \delta \Delta v + u - v \end{cases}$$

Parameters a, b, δ, $\epsilon \in \mathbb{R}$ control excitable threshold and fast/slow timescale.

Written as a system in polar coordinates, the model is

$$U_t \;=\; D\Delta_{r,\phi} U + F(U), \qquad U = \begin{pmatrix} u \\ v \end{pmatrix}(r,\phi), \qquad D \;=\; \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}.$$

Rigidly rotating spiral waves, $U^*(r,\psi)$, are stationary solutions in rotating polar frame, $(r,\phi) \to (r,\psi) = (r, \phi - \omega t)$

$$0 \;=\; D\Delta_{r,\psi} U^* + \omega U^*_\psi + F(U^*).$$

Goals

The aim of this research is to analyze the termination of pinned spiral waves under low amplitude forcing which mimics pulses from medical devices [3]. Steps are:

- Understand and illustrate properties of spiral spectra.
- Center manifold reduction for pinned spirals.
- Dynamic model reduction.

Spectra

Temporal Eigenvalues, λ, describe temporal growth:

$$\mathcal{L}U = D\Delta_{r,\psi} U + \omega U_\psi + F'(U^*)U = \lambda U$$

Spatial Eigenvalues are eigenvalues, ν, of the system:

$$\begin{aligned} U_r &= W \\ W_r &= -\left(\frac{W}{r} + \frac{U_{\psi\psi}}{r^2} + D^{-1}\left[\omega U_\psi + F'(U^*)U - \lambda U\right]\right) \end{aligned}$$

On the plane:

- Point spectra
- Essential spectra, Σ_{ess}
  - $(\lambda - \mathcal{L})$ is not Fredholm
  - $\nu \in i\mathbb{R}$

On bounded domain:

- Point spectra
- Absolute spectra, Σ_{abs}
  - Limit of point spectra as domain $\to \infty$
  - Can no longer distinguish stable and unstable spatial eigenvalues

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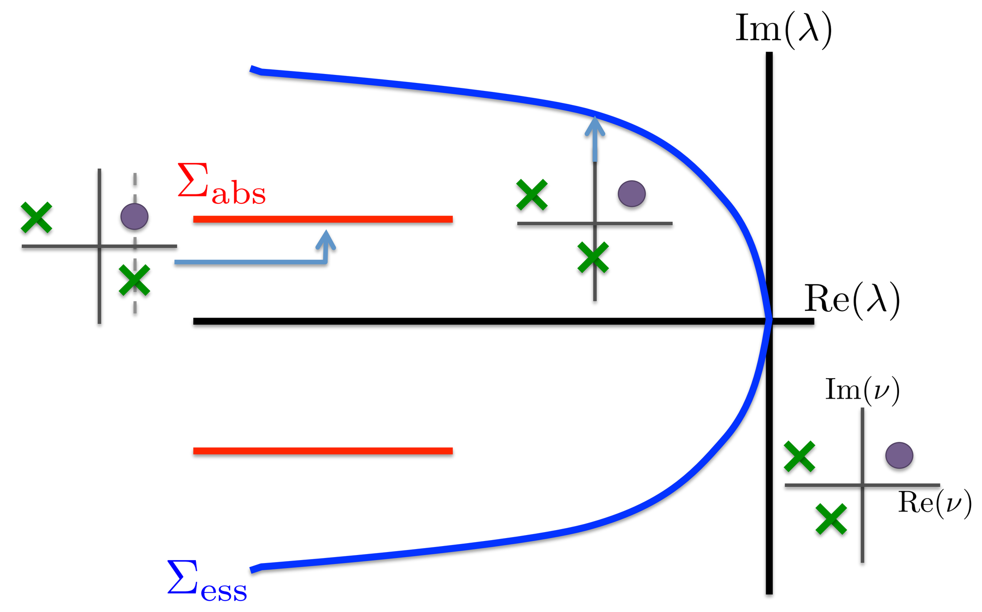

Figure: Cartoon of essential and absolute spectra. Inserts show distribution of spatial eigenvalues.

Spectra from Asymptotic Wave Trains

Asymptotic Wave Trains

$$U_*(r,\psi) \to U_\infty(\kappa r + \psi) = U_\infty(\xi), \quad U_\infty(\xi) = U_\infty(\xi + 2\pi)$$

Stationary solutions of

$$U_t = \kappa^2 D U_{\xi\xi} + \omega U_\xi + F(U).$$

Using the Floquet Ansatz, $U(\xi) = e^{\nu_\infty \xi} V_\infty(\xi)$, $V_\infty(\xi + 2\pi) = V_\infty(\xi)$, the wave train eigenvalue problem becomes

$$\text{(1)} \qquad \lambda_\infty V_\infty = D\left(\kappa\partial_\xi + \nu_\infty\right)^2 V_\infty + \left(\omega\partial_\xi + \nu_\infty\right) V_\infty + F'\left(U_\infty\right) V_\infty.$$

Dispersion relation of spiral, $\lambda_*(\nu_*)$, and wave train, $\lambda_\infty(\nu_\infty)$, are related via

$$\lambda_*(\nu_*) = \lambda_\infty(\nu_\infty) - \omega\nu_\infty + i\omega\ell, \quad \ell \in \mathbb{Z}.$$

- Trace out essential and absolute spectra curves with numerical continuation
- Point spectra requires full curvature of the spiral

Numerical Results, $\delta = 0.2$

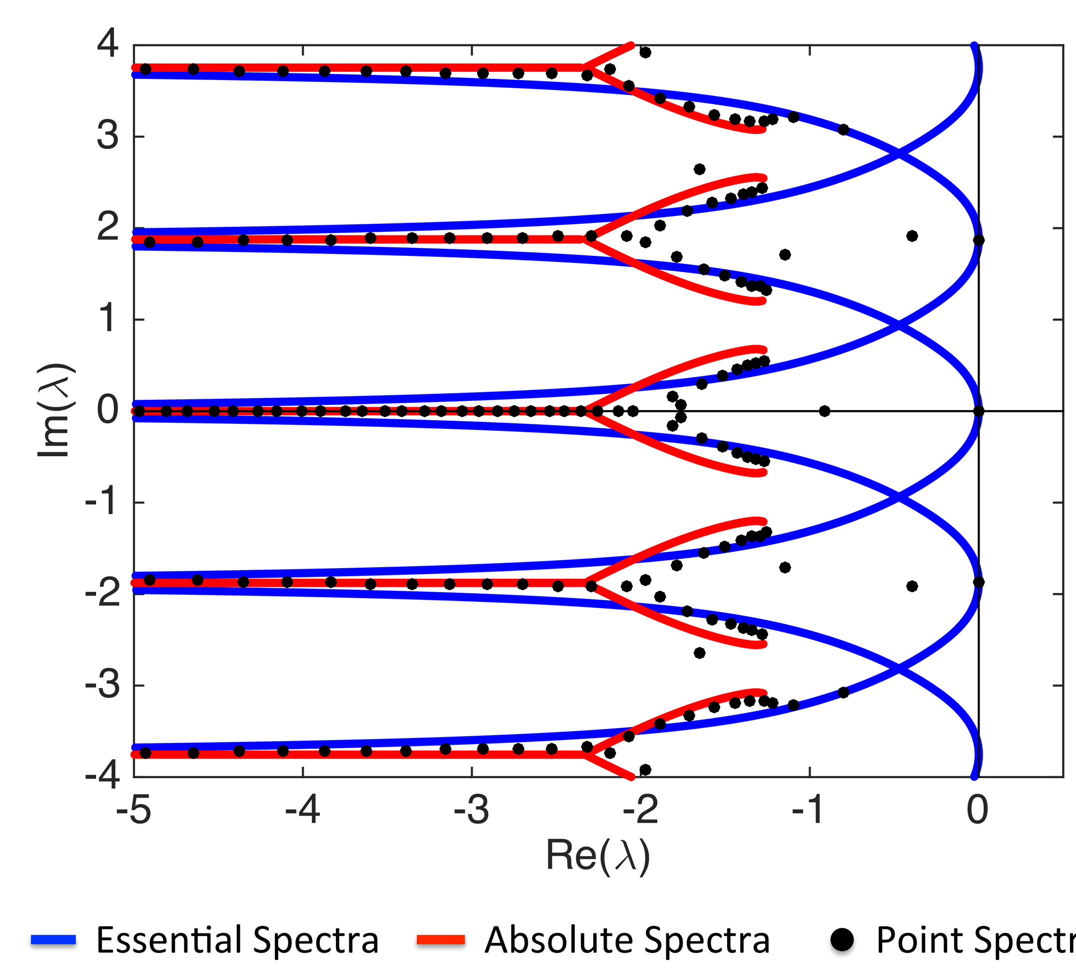

Figure: Essential, absolute, and point spectra for spiral with parameters $\delta = 0.2$, $a = 0.7$, $b = 0.001$, $\epsilon = 0.02$. Point spectra from spiral of radius 20.

Results indicate:

- Σ_{ess} branches meet at ∞.
- Point eigenvalues align along Σ_{abs}.

Results, $\delta = 0$

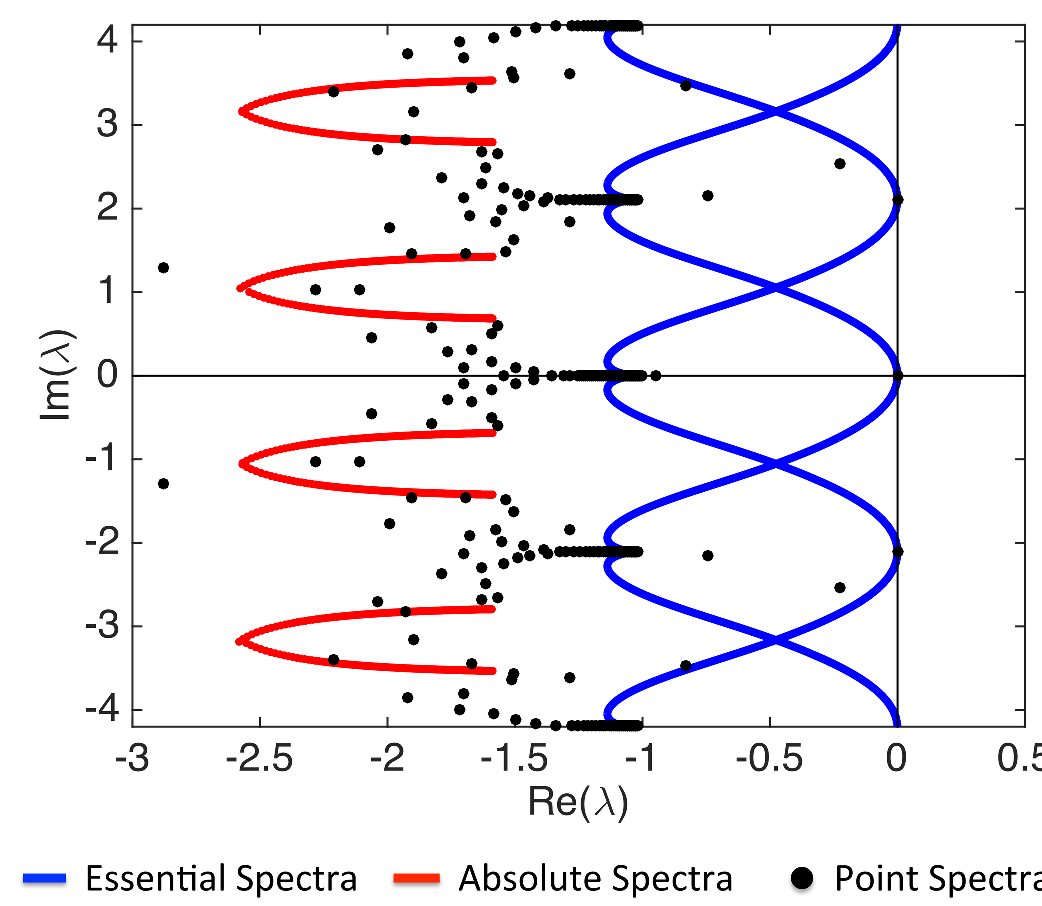

Figure: Essential, absolute, and point spectra for spiral with parameters $\delta = 0.2$, $a = 0.7$, $b = 0.001$, $\epsilon = 0.02$. Point spectra from spiral of radius 20.

- Σ_{ess} branches meet and point eigenvalues cluster at $\lambda_* = -1 + i\omega\ell$.
- Unexpected behavior!

Preliminary Analytic Results

- Using spiral eigenfunctions of the form

$$V_*(r,\psi;\lambda) \;=\; e^{\nu_* r}\, V_\infty(\kappa r + \psi) + \mathcal{O}\left(\frac{1}{r}\right),$$

the spiral eigenvalue problem reduces to exactly (1), except with $\lambda_*(\nu_*)$.

- As $\lambda_* \to -1 + i\ell\omega$, $\ell \in \mathbb{Z}$, we have $\nu_* \to i\gamma$, with $\gamma \in \mathbb{R}$, $\gamma \to \infty$.
- Let $\xi = \gamma x$, $\alpha = 1/\gamma$, and seek eigenfunctions of the form $V(x) = (u, w, v)(x)$,

$$\begin{cases} u' &= w \\ w' &= -\alpha^2 \dfrac{1}{\kappa^2}\left[f_u(U_\infty)u + f_v(U_\infty)v - \lambda_* u\right] - \alpha\dfrac{\omega}{\kappa^2} w - \dfrac{1}{\kappa^2}\left[2\kappa w + u\right] \\ v' &= \dfrac{\alpha}{\omega}\left(v - u + \lambda_* v\right) \end{cases}$$

- Using methods from Geometric Singular Perturbation Theory, u, $w \to 0$, and we are left with the slow dynamics, given by

$$\omega v' \;=\; \left(1 + \lambda_*\right) v.$$

Future Work

- Formalize analytic results for $\delta = 0$ case.
- Investigate faster and more accurate methods of point eigenvalue calculation.
- Extend numerical and analytical spectral results to pinned spirals.
- Determine which spectra are relevant.
- Investigate additional reaction terms.

References

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