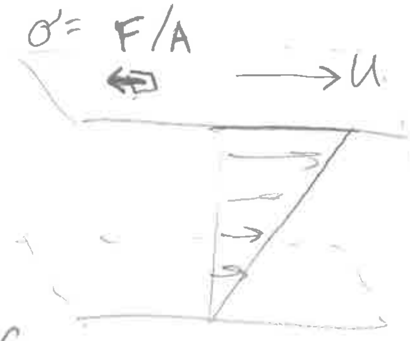
CFD - Lecture 8

- Incompressible Navier-Stokes equations
- Non-dimensionalization & the Reynolds number
- Building a mesh
- HW workshop

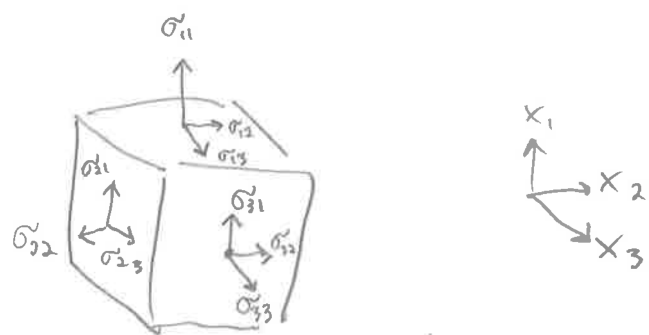
- Today we'll present methods for solving the incompressible Navier - Stokes → first, what are they?

Viscous flow



- For a Newtonian fluid we assume that our fluid stresses are linearly proportional to the local strain rate of the flow

- For a unit box, we can consider the stresses coming from 3 directional components at each face



$$\sigma_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Pushing this through the usual momentum conservation argument, we obtain

$$\partial_t \rho u + \nabla \cdot \rho u^2 = -\nabla p + \nabla \cdot \mu (\nabla u + \nabla u^T)$$

- For slow-moving fluids $Ma = \frac{u}{c} \ll 1$ we can assume that density is constant

- <u>Mass</u> $\rightarrow$

$$\cancel{\partial_t \rho} + \nabla \cdot \rho u = 0$$
$$\rho \nabla \cdot u = 0$$
$$\boxed{\nabla \cdot u = 0}$$

<u>ex</u> Flow in = Flow out

$\nabla \cdot u = 0$

$\Rightarrow \int_\Omega \nabla \cdot u = 0$

$= \oint_{\partial\Omega} u \cdot n = 0$

also $\rho = \text{const.}$

converse

- Rather than coupling pressure, EOS, conservation together, we can instead solve

$$\begin{cases} \partial_t \rho u + \nabla \cdot \rho u^2 = -\nabla p + \nabla \cdot \mu (\nabla u + \nabla u^T) \\ \nabla \cdot u = 0 \end{cases}$$

ρ, μ given, 2 equations + 2 unknowns (u, p)

With some arithmetic this can be written as

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 u \\ \nabla\cdot u = 0 \end{cases}$$

This can be further simplified

Kinetic viscosity $\nu = \frac{\mu}{\rho}$

let $p^* = \frac{p}{\rho}$

Navier - Stokes equations.

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p^* + \nu\nabla^2 u \\ \nabla\cdot u = 0 \end{cases}$$

customary to drop the $p^* \to p$

How to understand → advection vs diffusion

→ pressure acting as Lagrange mult.

Non-dimensionalization

Let u^*, ℓ^* be characteristic length + timescales

$$\bar{x} = \frac{x}{\ell^*} \qquad \bar{t} = t/\ell^* u^* \qquad \bar{u} = \frac{u}{u^*} \qquad \bar{p} = \frac{p\,\cancel{\ell^*}}{u^*}$$

$$\partial_t u = \frac{d\bar{t}}{dt}\frac{\partial u}{\partial \bar{t}} = \frac{d\bar{t}}{dt}\frac{du}{d\bar{u}}\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{1}{\ell^* u^*}u^*\partial_{\bar{t}}\bar{u} = \frac{1}{\ell^*}\partial_{\bar{t}}\bar{u}$$

Continuing with this term by term we obtain

where $\bar{\nabla} = (\partial_{\bar{x}_1}, \partial_{\bar{x}_2}, \partial_{\bar{x}_3})$

$$\partial_{\bar{t}} \bar{u} + \bar{u} \cdot \bar{\nabla} \bar{u} = -\bar{\nabla} \bar{p} + \frac{1}{Re} \bar{\nabla}^2 \bar{u}$$

where $Re = \dfrac{U^* l^*}{\nu}$ is the Reynolds Number

Two limits $Re \downarrow 0$ or $Re \nearrow \infty$

Original experiment by Reynolds

Today we'll do a similar experiment...

First – how to solve NS?

- Projection method (predictor corrector)

1. Semi-discretize in time

$$\frac{u^{n+1} - u^n}{\Delta t} + u^n \cdot \nabla u^{n+1} = -\nabla p^{n+1} + \nu \nabla^2 u^{n+1}$$

$$\nabla \cdot u^{n+1} = 0$$

- What does this matrix look like?

2. Split into non-div-free momentum

$$\frac{u^* - u^n}{\Delta t} + u^n \cdot \nabla u^* = \nu \nabla^2 u^*$$

– note $\nabla \cdot u^* \neq 0$

3. Add a correction such that $\nabla \cdot u^{n+1} = 0$

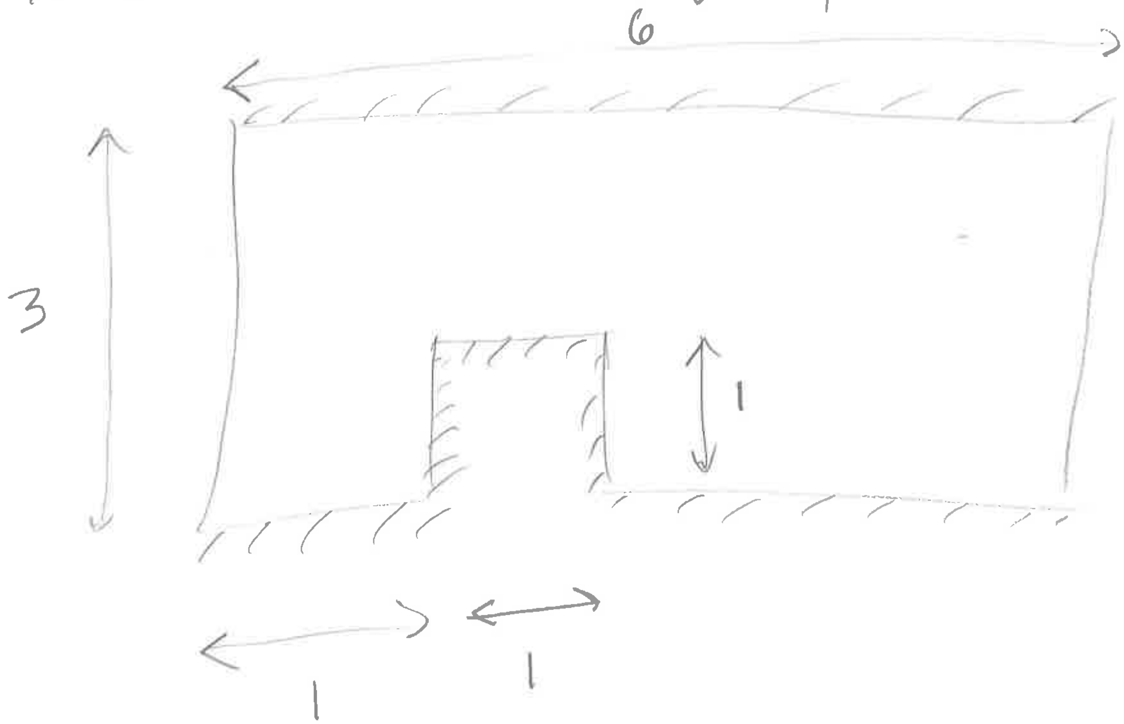
$$\frac{u^{n+1} - u^*}{\Delta t} = -\nabla p^{n+1}$$

$$\nabla \cdot \left(\frac{u^{n+1} - u^*}{\Delta t} \right) = -\nabla^2 p^{n+1} \implies \nabla^2 p^{n+1} = \frac{\nabla \cdot u^*}{\Delta t}$$

- We know how to do all of these things in OpenFOAM...

HW 6

- Implement projection method in OpenFOAM
- Simulate the following problem:



Apply the following BC

<u>Velocity</u>

- inlet – uniform $u = \langle 1, 0 \rangle$
- outlet – $\nabla u \cdot \hat{n} = 0$
- walls – no-slip $u = \langle 0, 0 \rangle$

<u>pressure</u>

- outlet – $p =$ arbitrary const.
- everywhere else – $\partial_n p = 0$

Run a simulation to $t_{final} = 7$ for

$$Re = [0.1, 1, 10, 100, 1000]$$ by varying ν.