1D Euler Equations

CFD - Lecture 6

- Characteristics of Euler eqns
- Shallow water equations
- Shock tube problem
- am119 Euler Foam

$$\partial_t \rho + \partial_x(\rho u) = 0$$
$$(\star) \quad \partial_t(\rho u) + \partial_x(\rho u u + p) = 0$$
$$\partial_t(\rho E) + \partial_x(\rho u E + u p) = 0$$

We'll consider an equivalent EOS for gases to obtain a simpler derivation.

$$p = \rho\left(E - \tfrac{1}{2}u^2\right)(\gamma - 1),$$

here γ is ratio of specific heats

$\boxed{\text{air} \rightarrow \gamma = 1.4}$

Showed last week that $(\star)$ can be put into conservative form

$$u_t + [F(u)]_x = 0$$

$$u = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}, \qquad F(u) = \begin{bmatrix} \rho u \\ \rho u u + p \\ \rho u E + u p \end{bmatrix}$$

Alternatively, we can write in terms of primitive variables

$$q_t + A(q)\, q_x = 0$$

$$(\star\star) \quad q = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} \qquad A = \begin{pmatrix} u & \rho & 0 \\ 0 & u & p \\ 0 & \gamma p & u \end{pmatrix} \Rightarrow \begin{pmatrix} \dot\rho \\ \dot u \\ \dot p \end{pmatrix} = -\begin{pmatrix} u & \rho & 0 \\ 0 & u & p \\ 0 & \gamma p & u \end{pmatrix}\begin{pmatrix} \rho_x \\ u_x \\ p_x \end{pmatrix}$$

- We'll spell out the first two equations

$$0 = \partial_t \rho + \partial_x \rho u = \partial_t \rho + u\partial_x \rho + \rho \partial_x u \quad \checkmark$$

$$0 = \partial_t \rho u + \partial_x \rho u u + \partial_x p = \rho \partial_t u + \underbrace{u \partial_t \rho + u \partial_x \rho u}_{\text{continuity eqn} \to =0} + \rho u \partial_x u + \partial_x p$$

$$0 = \rho \partial_t u + \rho u \partial_x u + \partial_x p$$

$$= \partial_t u + u \partial_x u + \frac{1}{\rho}\partial_x p \quad \checkmark$$

(the third is similar but more tedious)

- We can study the characteristics of (**) by finding the eigenvalues of A

↳ Solve for λ such that $\det(A - \lambda I) = 0$

$$(-\gamma p^2 u + u^3) + (\gamma p^2 - 3u^2)\lambda + 3u\lambda^2 - \lambda^3 = 0$$

- $$\lambda = \Big\{ u,\ u + \underbrace{\sqrt{\tfrac{\gamma p}{\rho}}}_{C = \text{Speed of sound}},\ u - \sqrt{\tfrac{\gamma p}{\rho}} \Big\}$$

$\boxed{\text{real, distinct} \Rightarrow \text{Hyperbolic PDE}}$

- For those without a PDE background, I'll sketch a brief picture of what this tells us. The reference I pulled these notes from is on the website, if people want the technical details.

- Eigenvalues let us diagonalize A

$$I = L \Lambda R$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$

left/right eigenvectors $\Rightarrow$ $LR = RL = I$

So multiplying on the Left, right, and by I

$$q_t = -A q_x$$

$$L q_t R = -L A I q_x R$$

$$= -L A R L q_x R$$

Let $w = L q R$

$$\boxed{w_t = -\Lambda w_x}$$

and we have a set of 3 decoupled PDE

$$\boxed{\partial_t w_i + \lambda_i \partial_x w_i = 0}$$

So we obtain a collection of transport equations where information propagates at a speed $\lambda_i \in \{u, u+c, u-c\}$

Shallow water equations

- Most people don't have experience w/ supersonic jets, but most know what water waves in a bathtub look like

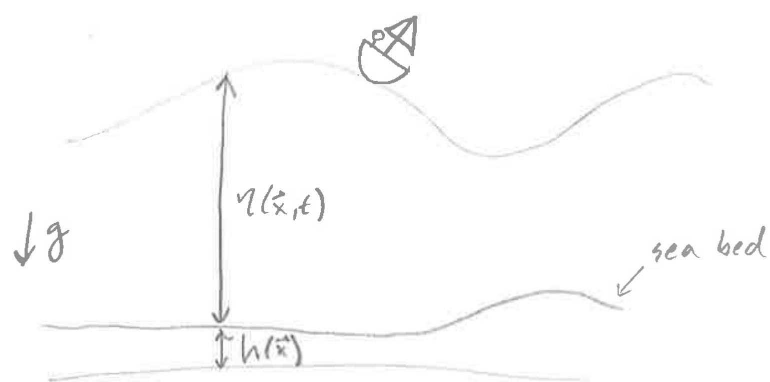
Compressible Euler (1D, primitive) isothermal

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t u + u \partial_x u = -\partial_x p \end{cases}$$

$$p = \rho R T \rightarrow \; = -RT \partial_x \rho$$

Shallow water equations

$$\begin{cases} \partial_t \eta + \partial_x \left((\eta + h) u \right) = 0 \\ \partial_t u + u \partial_x u = -g \partial_x \eta \end{cases}$$



- For $h = 0$ (think bathtub) we can identify

$$\rho \leftrightarrow \eta$$
$$u \leftrightarrow u$$

and it is "equivalent" to talk about water waves or density waves

- For the shallow water equations, we could apply the same eigenvalue analysis to see that

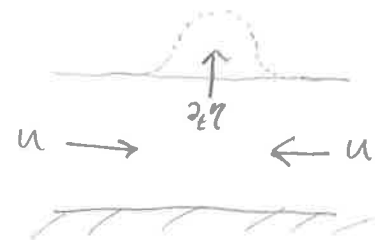
$$\lambda = \left\{ u, \; u \pm \underbrace{\sqrt{g(\eta + h)}}_{\substack{\text{capillary speed} \\ \text{"of sound"}}} \right\}$$

- Easier to physically interpret characteristics of water wave?

- For water waves, wave height holds potential energy

Take $h=0$

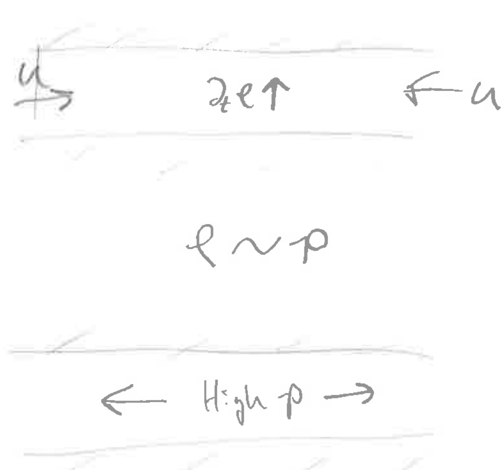
$$\underbrace{\partial_t \eta + u\partial_x \eta}_{\text{advection}} = \underbrace{\eta\,\partial_x u}_{\text{squeezing}}$$



squeezing cartoon

$$\partial_t u + u\partial_x u = \underbrace{-g\,\partial_x \eta}_{\text{restoring force}}$$

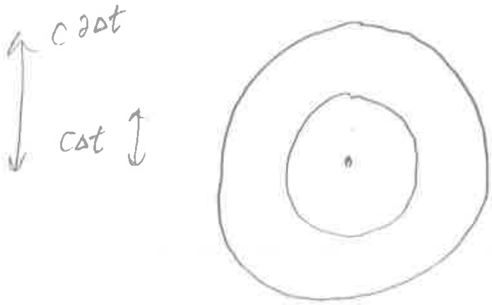
- Euler analogy



Same idea but disturbances propogate as plane waves

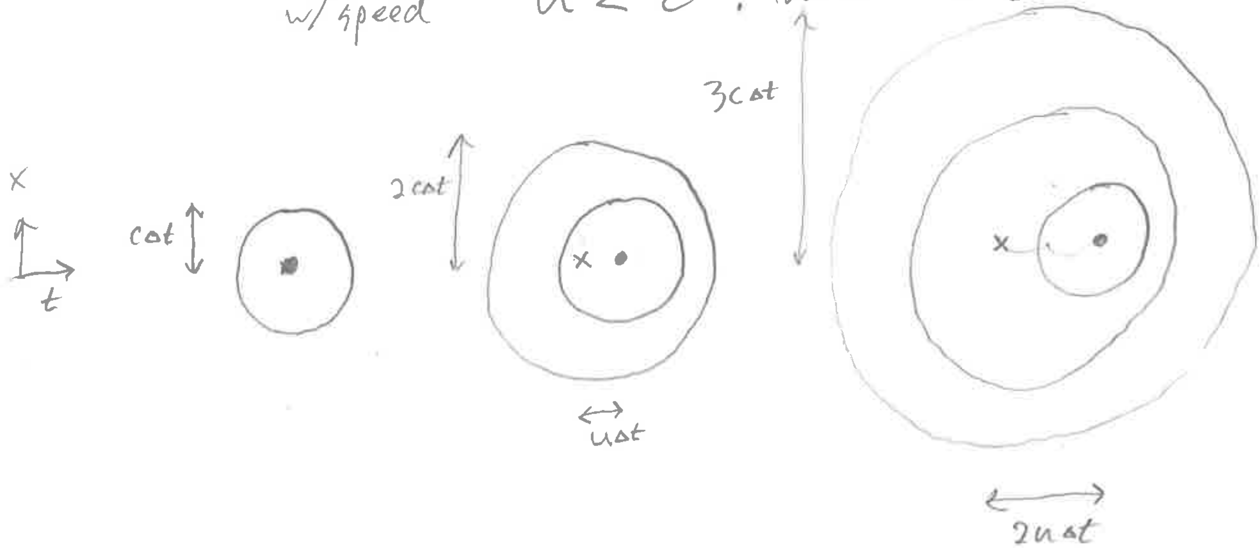
Water waves

- Consider disturbance tapping water every Δt

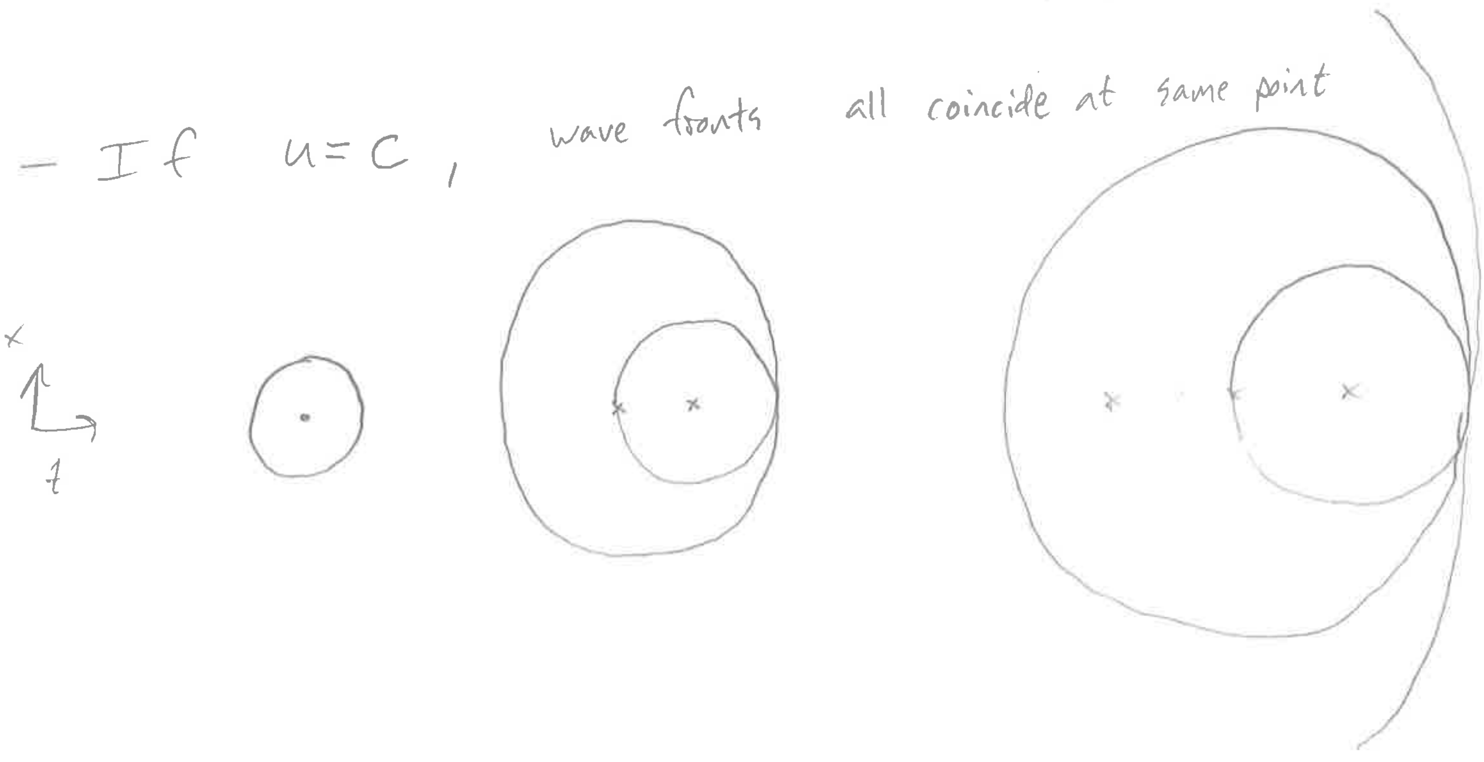


- Each disturbance will propagate at a rate c

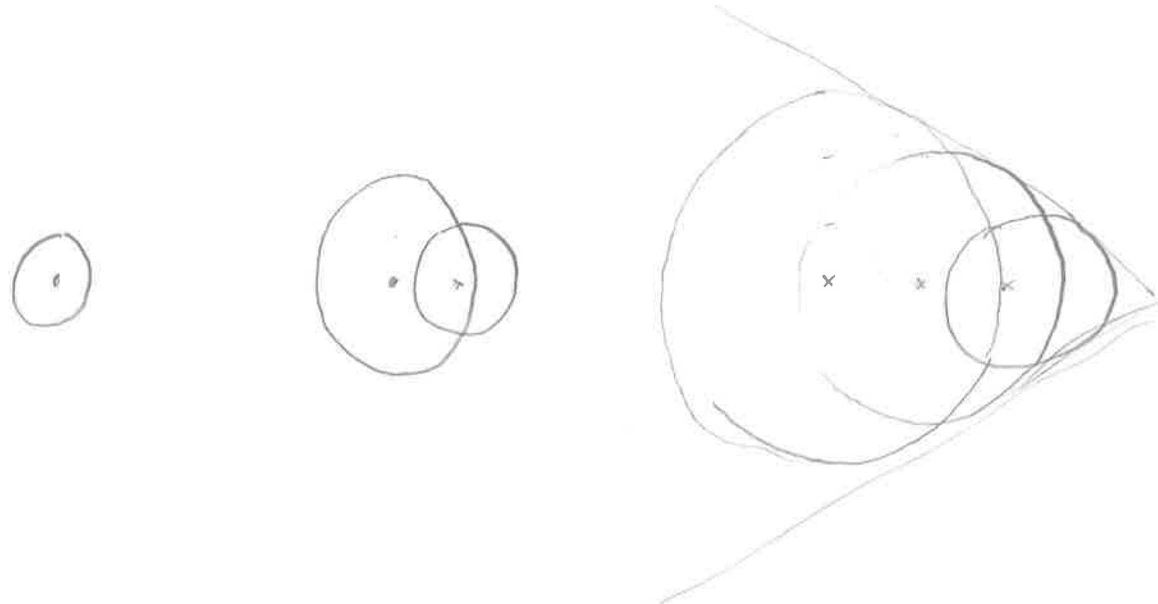
- Now consider disturbance moving to right w/ speed $u < c$. Define $Ma = \frac{u}{c}$.



- If $u = c$, wave fronts all coincide at same point



If $u > c$, waves can't "keep up" w/ disturbance



- For water waves c is slow - think about the speed of surfers

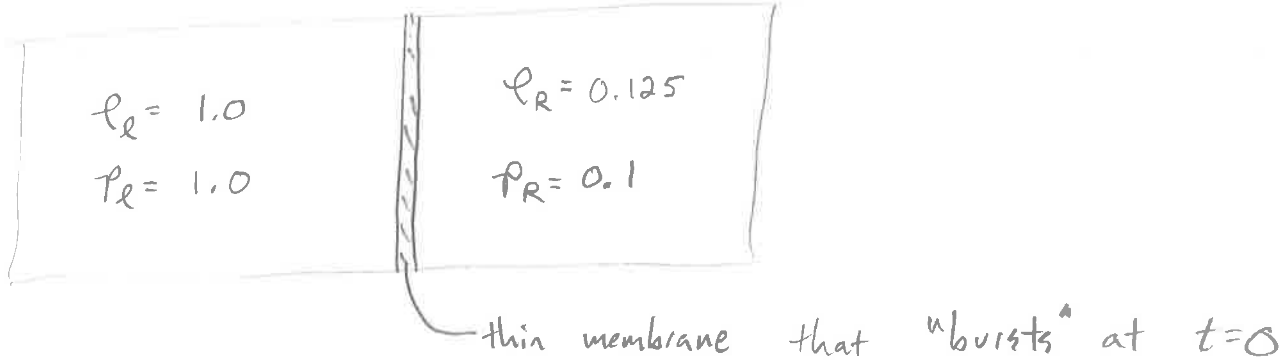
- The propagation cone sketched above is what's commonly refered to as a wake for a ship (pictures on website)

- For compressible gas dynamics, several differences
  - Not so easy to visualize density compression
    - Schlieren photography (pictures on website)
  - Waves propagate differently through ρ, p, u

- Today, we'll look at a test problem for studying how a code can track these discontinuities

Shock tube

- 1D model for flow in a shocktube (Sod 1978, see website)



- While a water wave would propogate only in h, for us we'll get different types of shocks that will propogate from initial discontinuity

Today

Implement in 1D code and in OpenFoam