CFD - Lecture 4

- Conservation of momentum
- Burger's equation
- An introduction to compressible Euler equations
- Foam Tutorial

- First, everybody log in to CCV.

Conservation of Momentum

$$\frac{d}{dt}\left(\text{momentum in system}\right) = \sum \left(\text{Forces acting on system}\right)$$

— think $F = ma$ from high school

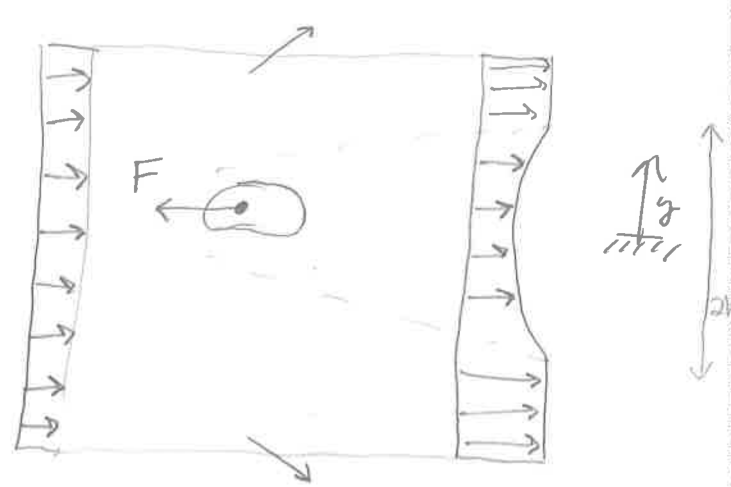
Let ρ = density $[M/L^3]$

u = velocity $[L/T]$

$$\frac{\partial}{\partial t}\int_{\Omega} \rho u \, dx + \underbrace{\oint_{\partial\Omega} (\rho u)\, u\cdot\hat{n}\, dA}_{\text{non-linear}} = \sum_i F_i$$

Ex Drag on a cylinder in a wind tunnel

- Assume
  - uniform velocity at inlet U_∞
  - symmetric velocity at top & bottom
  - measured velocity profile at outlet $\rightarrow u(y)$

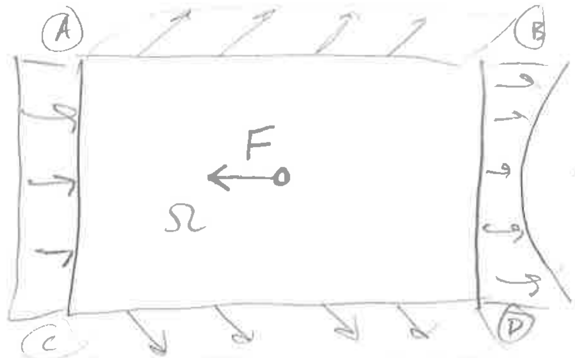


- Because of drag acting on obstacle, velocity at outlet is slower, causing mass flux at top and bottom
- By symmetry, assume equal & opposite

- From conservation, can recover force on obstacle from outlet vel.

- assume $u(y) = \begin{cases} u_\infty & |y| > h \\ u_\infty\left(1 - \frac{1}{2}\left(\frac{y}{h}\right)^2\right) & |y| < h \end{cases}$

$$\cancel{\partial_t \int_\Omega \rho u \, dx} + \int_{\partial\Omega} (\rho u) \cdot (u \cdot \hat{n}) \, dA = -F$$



$$\int_{AC} (\rho u) \, u \cdot \hat{n} \, dA + \int_{BD} (\rho u) \, u \cdot n \, dA = -F$$

$$\rho\left(\int_{-h}^{h} u_\infty(-u_\infty) \, dy + \int_{-h}^{h} u(y)^2 \, dy\right) = -F$$

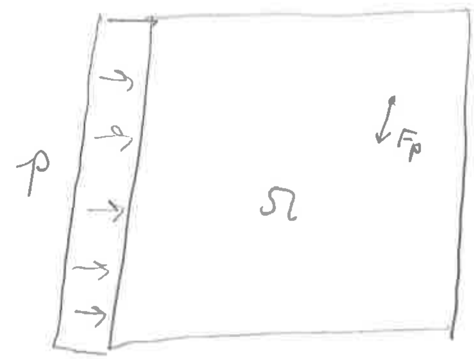
$$\rho\left(-u_\infty^2 \int_{-h}^{h} dy + \int_{-h}^{h} \left[u_\infty\left(1 - \frac{1}{2}\left(\frac{y}{h}\right)^2\right)\right]^2 dy\right) = -F$$

$$\rho\left(-2h u_\infty^2 + \frac{43}{30} h u_\infty^2\right) = -F$$

$$\boxed{F = \frac{17}{30} \rho h u_\infty^2}$$

In general, forces can take form of:
point force, surface force, body force

$$F = F_p + \int_{\partial\Omega} p \, dA + \int_{\Omega} f_B \, d\forall$$



We'll see more about this next week
when we discuss Navier-Stokes derivation

Burgers Equation

- Taking $\sum F = 0$, $\rho = 1$ gives Burgers Eqn up to factor of ½

$$\partial_t u + \partial_x \left(\tfrac{1}{2} u^2\right) = 0$$

- Burgers equation is a model for 1D non-linear advection → exact solutions mean that its easy to prove / demonstrate convergence
- Expand flux term to see

$$\partial_t u + u \partial_x u = 0 \xrightarrow{\text{compare to advection eqn}} \partial_t u + a \partial_x u = 0$$

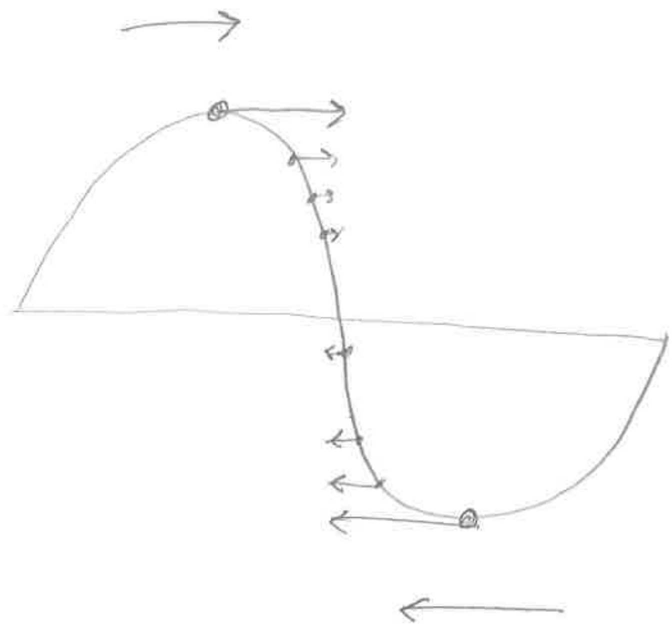
- Non-linear advection

- 1D model problem

- Take $u = \sin x$

- Advection magnitude is non-uniform and depends on solution

- Solve the exact same way, but we'll just be careful about picking the right upstream value



- HW3

Compressible Euler Equations (CEE)

- What we have now is a suite of techniques for handling a variety of physics

CEE
$$
\begin{cases}
\partial_t(\rho u) + \nabla\cdot(\rho u^2) = -\nabla p + \mu \nabla^2 u \\
\partial_t \rho + \nabla\cdot(\rho u) = 0 \\
\rho = f(p, T) \qquad \text{ex} \quad \rho = \dfrac{p}{RT}
\end{cases}
$$

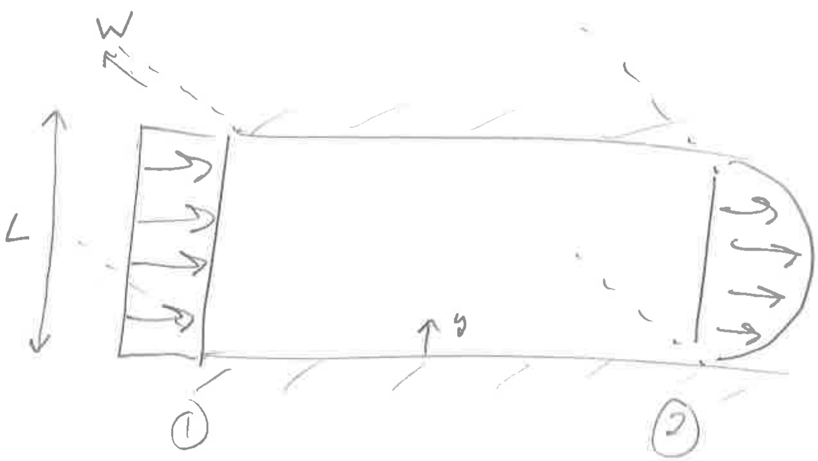
Next week we derive these, but first we'll warm up in Foam

Foam tutorial

- In last class we did flow through a pipe.
- For easy geometry, I'll include calculation for flow between infinite plates here

- $u_{in} = U_\infty$

$u_{out} = C\, y\left(1 - \frac{y}{L}\right)$



$$\int_{\Omega} \cancel{\rho}\, u \cdot n\, dA = 0$$

$$\int_{1} u \cdot \hat{n}\, dA + \int_{2} u \cdot \hat{n}\, dA = 0$$

$dA = W\,dy$

$$-\int_0^L U_\infty W\,dy + \int_0^L C\, y\left(1 - \frac{y}{L}\right) W\,dy = 0$$

$$-U_\infty L + C \int_0^L y\left(1 - \frac{y}{L}\right) dy = 0$$

$$-U_\infty L + \frac{C L^2}{6} = 0 \quad \rightarrow \quad C = \frac{6 U_\infty}{L}$$

$$\boxed{U_{out}(y) = \frac{6 U_\infty}{L^2}\, y(L - y)}$$