CFD - Lecture 3

- HW1
- Integral Conservation Law Vector Calc Refresher
- Finite Volume Method
- OpenFOAM intro

Integral C.L.

- Showed last time

① $\partial_t u + \nabla\cdot\vec{F}(u) = 0 \iff$ ② $\frac{d}{dt}\int_{\Omega} u\,dx = -\int_{\partial\Omega} F(u)\cdot n\,dA \quad \forall \Omega$

For solving analytically — For solving on computer

- Key idea of FVM:
  Define everything through integral forms on each cell of a mesh

- How does ② work? Some engineering examples to get warmed up with vector calc that we'll need...

Conservation of Mass

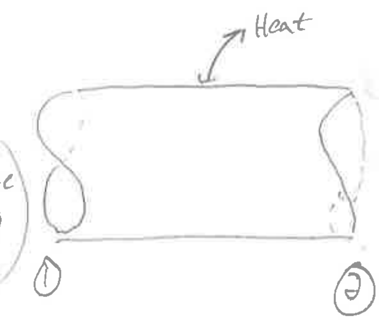
$$\frac{d}{dt}\left(\text{Mass in system}\right) = 0$$

$$\frac{d}{dt}\int_{\Omega} \rho \, dx + \oint_{\partial\Omega} \rho \vec{u}\cdot\hat{n}\, dA = 0 \iff \partial_t \rho + \nabla\cdot\rho\vec{u} = 0$$

$$\left(\text{in C.L form, } \begin{array}{l} u = \rho \\ F(u) = \rho\vec{u} \end{array}\right)$$

- Consider two fluids we all know
  - Air $\Rightarrow \rho = \dfrac{p}{RT}$ to a "good approx."
  - Water $\Rightarrow \rho = \text{const.}$

Ex 1 Consider Flow through a pipe with heat transfer



- Define volumetric flow rate at outlet/inlet (volume per unit time of fluid passing through)

$$Q_2 = \int_{②} \vec{u}\cdot\hat{n}\, dA$$

$$Q_1 = \int_{①} \vec{u}\cdot\hat{n}\, dA$$

- Assume at inlet, $p = p_{ref}$, $T = T_1$, Q_1 given

- If heat transfer occurs within pipe, so that at outlet $p = p_{ref}$, $T = T_2$ (think of an air conditioner)

  what is the outlet flowrate?

- Eqn. (2) w/ steady flow, no flow through walls

Pick Ω as pipe interior

$$\int_{①} \rho \vec{u}\cdot\hat{n}\,dA + \int_{②} \rho \vec{u}\cdot\hat{n}\,dA = 0$$

Assume gas

$$\rho_1 = \frac{p_{ref}}{RT_1} \qquad \rho_2 = \frac{p_{ref}}{RT_2}$$

$$\int_{①} \frac{p_{ref}}{RT_1}\,\vec{u}\cdot\hat{n}\,dA = \int_{②} \frac{p_{ref}}{RT_2}\,\vec{u}\cdot\hat{n}\,dA$$

$$\frac{\cancel{p_{ref}}}{\cancel{R}\,T_1} \underbrace{\int_{①} \vec{u}\cdot\hat{n}\,dA}_{Q_1} = \frac{\cancel{p_{ref}}}{\cancel{R}\,T_2} \underbrace{\int_{②} \vec{u}\cdot\hat{n}\,dA}_{Q_2}$$

$$\boxed{\frac{Q_2}{Q_1} = \frac{T_2}{T_1}}$$

So if we suck out heat to blow cold air in, the flow rate is proportional to the temperature

e.g. if the temp. is twice as hot at the outlet, then twice the volume of gas will be pumped in due to the expansion of gas and conservation of mass.

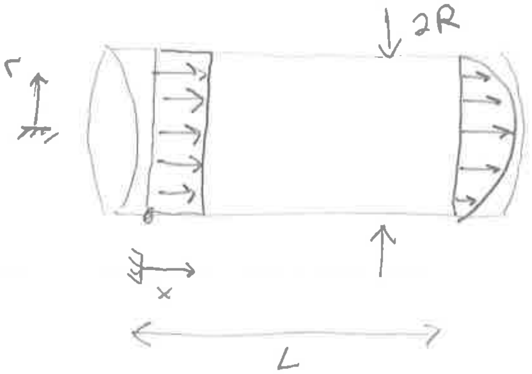
ex Laminar Pipe flow

$\rho = $ const.

- For an initially uniform flow of water through a pipe we measure experimentally that the flow is approximately uniform at the inlet and parabolic further downstream
- (we will show this analytically later in the class)

- If inlet velocity $u(x=0) = U_\infty$ and we assume the parabolic profile $u(x=L) = C\left(1 - \left(\frac{r}{R}\right)^2\right)$

- Calculate C to relate the downstream profile to the upstream velocity magnitude U_∞



Eqn ② w/ steady flow:

$$\int_{x=0} \rho \vec{u}\cdot\hat{n}\, dA + \int_{x=L} \rho \vec{u}\cdot n\, dA = 0$$

$$-\cancel{\rho} U_\infty \int_{x=0} dA + \cancel{\rho} C \int_{x=L} \left(1 - \left(\frac{r}{R}\right)^2\right) dA = 0 \quad , \; dA = 2\pi r\, dr$$

$$-U_\infty \pi R^2 + C \int_{r=0}^{R} \left(1 - \left(\frac{r}{R}\right)^2\right) 2\pi r\, dr = 0$$

$$-U_\infty \cancel{\pi R^2} + C\, \frac{\cancel{\pi R^2}}{2} = 0 \quad \Rightarrow \quad \boxed{C = 2U_\infty}$$

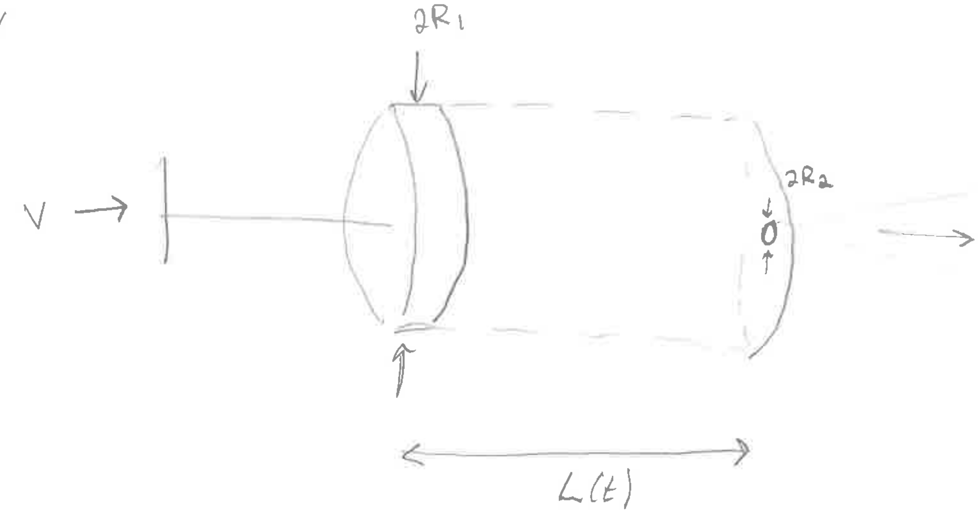
So downstream velocity profile is given by

$$\boxed{u(r) = 2U_\infty\left(1 - \left(\frac{r}{R}\right)^2\right)}$$

ex Piston driven syringe (unsteady domain)

$\rho = \text{const.}$

- Consider syringe filled with water with plunger depressed at a constant velocity V to squirt a jet of water out a small hole. What is the average velocity through the outlet?



- Def Average velocity

$$\overline{u}_{outlet} = \frac{1}{\pi R_2^2} \oint_{outlet} \vec{u} \cdot n \, dA$$

- Take Conservation equation, with $\Omega(t)$ as the interior cylinder with volume $\forall(t) = \pi R_1^2 L(t)$

$$\frac{d}{dt} \int_{\Omega(t)} \rho \, dx + \oint_{outlet} \rho \vec{u} \cdot \hat{n} \, dA = 0$$

$$\cancel{\rho} \left(\frac{d}{dt} \int dx + \oint \vec{u} \cdot \hat{n} \, dA \right) = 0$$

$$\frac{d}{dt} \forall(t) + \pi R_2^2 \, \overline{u}_{outlet} = 0$$

$$\frac{d}{dt} \forall = \pi R_1^2 \frac{dL}{dt} = -\pi R_1^2 V$$

$$-\pi R_1^2 V + \pi R_2^2 \, \overline{u}_{outlet} = 0$$

$$\boxed{\overline{u}_{outlet} = \frac{R_1^2}{R_2^2} V}$$

e.g. mass conservation "amplifies" the velocity of the plunger

We hopefully feel comfortable manipulating integral equations at this point...

Finite Volume Method

- Back to 1D advection - diffusion equation

$$\partial_t u + a\partial_x u - \nu \partial_{xx} u = 0$$

- Write in conservative form

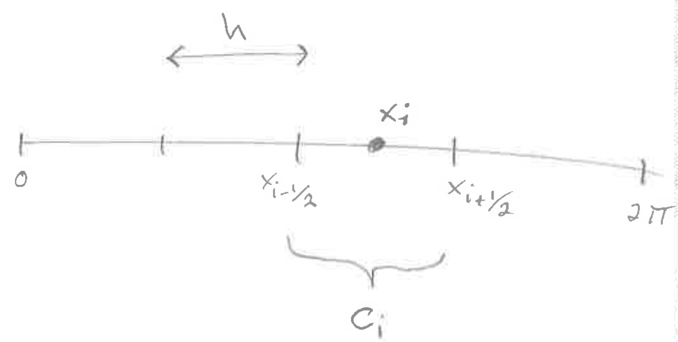
$$\begin{cases} \partial_t u + \nabla \cdot F(u) = 0 \\ F(u) = \vec{a} u - \nu \nabla u \end{cases}$$

- Solve on $[0, 2\pi]$ w/ periodic BC $\rightarrow$ exact sol $u = e^{-\nu t} \sin(x - at)$

- Split domain into N - cells $\rightarrow \Omega = \bigcup_{i=0}^{N-1} C_i$

- Define

$$\forall_i = \int_{C_i} dx \quad \rightarrow \text{cell volume}$$



$$x_i = \frac{1}{\forall_i} \int_{C_i} x \, dx \quad \rightarrow \text{cell barycenter}$$

$$\overline{u}_i = \int_{C_i} u \, dx \quad \rightarrow \text{integral of } u \text{ over cell } C_i$$

- Lemma $\quad u(x_i) = \frac{1}{\forall_i} \overline{u}_i + O(h^2)$

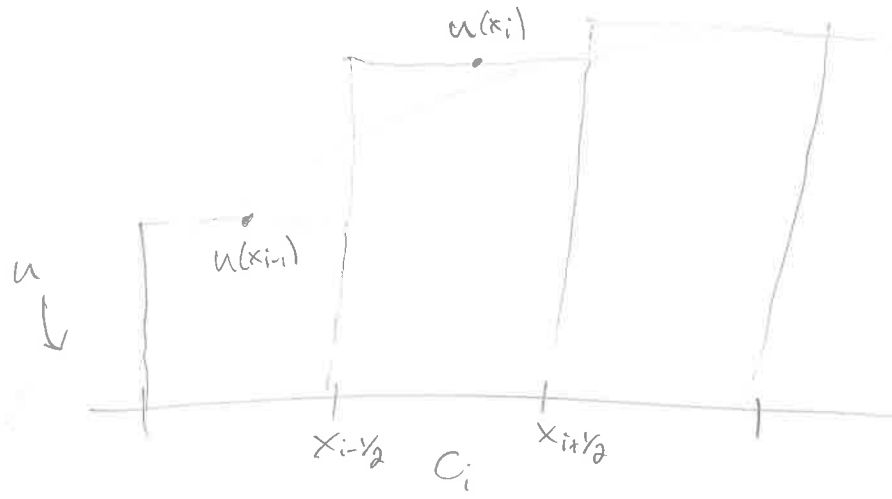
Pf Expand $\quad u(x) = u(x_i) + u'(x_i)(x - x_i) + O(h^2)$

$$\frac{1}{\forall_i}\overline{u}_i = \frac{1}{\forall_i}\int_{C_i} u \, dx = \frac{1}{\forall_i}\left[\int_{C_i} u(x_i) dx + \int u'(x_i) x \, dx - \int u'(x_i) x_i \, dx + O(\forall_i h^2)\right]$$

$$= \frac{1}{\forall_i}\left[\forall_i u(x_i) + \cancel{u'(x_i) \forall_i x_i} - \cancel{u'(x_i) \forall_i x_i} + O(\forall_i h^2)\right]$$

$$= u(x_i) + O(h^2) \quad \blacksquare$$

Idea: u is represented as piece wise const. across each cell



FVM

- First discretize in time

$$\frac{u^{n+1}-u^n}{\Delta t} = -\nabla\cdot F(u^n)$$

- To obtain an equation for each cell, integrate

$$\int_{C_i} \frac{u^{n+1}-u^n}{\Delta t}\,dx = -\int_{C_i} \nabla\cdot F(u^n)\,dx = -\int_{x_{i-1/2}}^{x_{i+1/2}} \partial_x F(u^n)\,dx$$

$$\frac{\bar{u}_i^{n+1}-\bar{u}_i^n}{\Delta t} = -\left(F(u^n)_{i+1/2} - F(u^n)_{i-1/2}\right)$$

$$(\star)\quad \boxed{\forall_i \left(\frac{u_i^{n+1}-u_i^n}{\Delta t}\right) = -\left(F(u^n)_{i+1/2} - F(u^n)_{i-1/2}\right)}$$

So once we know how to approximate the flux terms, we can update using this equation ($\star$) just like the finite difference method

To approximate $F(u) = \underbrace{au} - \nu \partial_x u$,

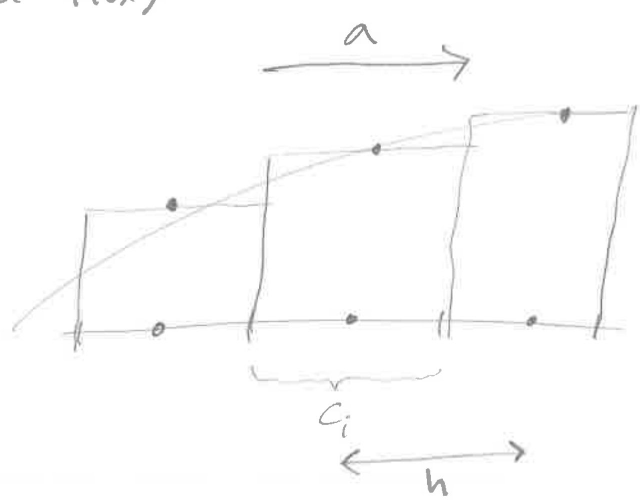
split into $F(u) = F_a(u) + F_d(u)$

(advection) (diffusion)

<u>Advection</u> Pick u from is upstream value

(this is called an upwind flux)

$$\boxed{\begin{aligned} F_a(u^n)_{i+1/2} &= a u_i^n \\ F_a(u^n)_{i-1/2} &= a u_{i-1}^n \end{aligned}}$$



<u>Diffusion</u> Approximate derivative with a central finite difference

(central flux)

$$\boxed{\begin{aligned} F_d(u^n)_{i+1/2} &= -\nu \left(\frac{u_{i+1}^n - u_i^n}{h} \right) \\ F_d(u^n)_{i-1/2} &= -\nu \left(\frac{u_i^n - u_{i-1}^n}{h} \right) \end{aligned}}$$

Good to go! Now we note that for any conservation law on a periodic domain $\Omega = [0, 2\pi]$

$$\partial_t u + \nabla \cdot F = 0 \Rightarrow \frac{d}{dt}\int_\Omega u\, dx = -\int_0^{2\pi} \partial_x F\, dx$$

Fundamental theorem of calc $= -\left(F(2\pi) - F(0)\right)$

Periodicity $= 0$

So that the quantity $\int u\, dx =$ const for all time

In FVM we obtain a discrete analogue. (Try to show this with telescoping series)

$$\frac{d}{dt}\int_0^{2\pi} u\, dx \approx \sum_{i=0}^{N-1} \frac{\bar{u}_i^{n+1} - \bar{u}_i^{n}}{\Delta t} = \sum_{i=0}^{N-1} V_i\left(\frac{u_i^{n+1} - u_i^{n}}{\Delta t}\right)$$

$$= -\sum_{i=0}^{N-1}\left(F(u^n)_{i+1/2} - F(u^n)_{i-1/2}\right)$$

why? $\longrightarrow = 0$

So that $\sum_{i=0}^{N-1} V_i u_i^n =$ const. for all n.