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M. Micheli, P. W. Michor, D. Mumford

Sobolev metrics on diffeomorphism groups and the derived geometry of spaces of submanifolds

Given a finite-dimensional manifold N, the group $\mathrm{Diff}_{\mathcal{S}}(N)$ of diffeomorphisms of N which decrease suitably rapidly to the identity, acts on the manifold $B(M,N)$ of submanifolds of N of diffeomorphism-type M, where M is a compact manifold with $\dim M < \dim N$. Given the right-invariant weak Riemannian metric on $\mathrm{Diff}_{\mathcal{S}}(N)$ induced by a quite general operator $L\colon \mathfrak{X}_{\mathcal{S}}(N) \to \Gamma(T^*N \otimes \mathrm{vol}(N))$, we consider the induced weak Riemannian metric on $B(M,N)$ and compute its geodesics and sectional curvature. To do this, we derive a covariant formula for the curvature in finite and infinite dimensions, we show how it makes O'Neill's formula very transparent, and we finally use it to compute the sectional curvature on $B(M,N)$.

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Dedicated to I. R. Shafarevich on the occasion of his 90th birthday

§ 1. Introduction

It was 46 years ago that Arnold discovered an amazing link between Euler's equation for incompressible non-viscous fluid flow and geodesics in the group of volume-preserving diffeomorphisms $\mathrm{SDiff}(\mathbb{R}^n)$ under the L^2-metric [1]. One goal in this paper is to extend his ideas to a large class of Riemannian metrics on the group $\mathrm{Diff}_{\mathcal{S}}(N)$ of all diffeomorphisms decreasing suitably to the identity of any finite-dimensional manifold N. The resulting geodesic equations are integro-differential equations for fluid-like flows on N determined by an initial velocity field. In previous papers [2]–[4], we looked at the special case where $N = \mathbb{R}^n$ and the metric is a sum of Sobolev norms on each component of the tangent vector, but here we develop the formalism to work in a very general setting.

The extra regularity given by using higher-order norms means that these metrics on the group of diffeomorphisms can induce a metric on many quotient spaces of that group modulo a subgroup. This paper focuses on the space of submanifolds of N diffeomorphic to some M, which we denote by $B(M,N)$. $\mathrm{Diff}_{\mathcal{S}}(N)$ acts on $B(M,N)$ with open orbits, one for each isotopy type of embedding of M in N. The spaces B may be called the *Chow manifolds* of N by analogy with the Chow varieties of algebraic geometry, or *non-linear Grassmannians* because of their

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analogy with the Grassmannian of linear subspaces of a projective space. The key point is that the metrics we study will descend to the spaces $B(M,N)$ so that the map $\mathrm{Diff}_{\mathcal{S}}(N) \to B(M,N)$ (given by the group action on a base point) is a Riemannian submersion. Geodesics from one submanifold to another may be thought of as deformations of one into the other realized by a flow on N of minimal energy.

In the special case where M is a finite set of points, $B(M,N)$ is called the space of *landmark point sets* in N. This has been used extensively by statisticians for example and is the subject of our previous paper [4]. The case $B(S^1,\mathbb{R}^2)$ is the space of all simple closed plane curves and has been studied in many metrics; see [5]–[7] for example. This and the case $B(S^2,\mathbb{R}^3)$ of spheres in 3-space have had many applications to medical imaging, constructing optimal warps of various body parts or sections of body parts from one medical scan to another [8], [9].

The high point of Arnold's analysis was his determination of the sectional curvatures in the group of volume-preserving diffeomorphisms. This has had considerable impact on the analysis of the stability or instability of incompressible fluid flow. A similar formula for the sectional curvature of $B(M,N)$ may be expected to shed light on how stable or unstable geodesics are in this space, for example, whether they are unique and effective for medical applications.

Computing this curvature required a new formula. In general, the induced inner product on the cotangent space of a submersive quotient is much more amenable to calculations than the inner product on the tangent space. The first author found a new formula for the curvature tensor of a Riemannian manifold which uses only derivatives of the former, the dual metric tensor. This result, *Mario's formula*, is proved in § 2. In that section we also define a new class of infinite-dimensional Riemannian manifolds, *robust Riemannian manifolds*, to which Mario's formula and our analysis of submersive quotients applies. We also obtain a transparent new proof of O'Neill's curvature formula. This class of manifolds builds on the theory of *convenient* infinite-dimensional manifolds; see [10]. To facilitate readability, this theory is summarized in an appendix.

In § 3, we describe the diffeomorphism groups of a finite-dimensional manifold N consisting of diffeomorphisms which decrease suitably rapidly to the identity on N if we move to infinity on N; only these admit charts and form a regular Lie group. We shall denote by $\mathrm{Diff}_{\mathcal{S}}(N)$ any of these groups in order to simplify notation, and by $\mathfrak{X}_{\mathcal{S}}(N)$ the corresponding Lie algebra of suitably decreasing vector fields on N. We introduce a very general class of Riemannian metrics given by a positive definite self-adjoint differential operator L from the space of smooth vector fields on N to the space of measure-valued 1-forms. This defines an inner product on vector fields X, Y by

$$\langle X,Y\rangle_L = \int_N (LX,Y).$$

Note that LX paired with Y gives a measure on N and hence can be integrated without assuming that N carries any further structure. Under suitable assumptions, the inverse of L is given by a kernel $K(x,y)$ on $N\times N$ with values in $p_1^*TN \otimes p_2^*TN$. We then describe the geodesic equation in $\mathrm{Diff}_{\mathcal{S}}(N)$ for these metrics. It is especially simple when written in terms of the *momentum.* If $\varphi(t) \in \mathrm{Diff}_{\mathcal{S}}(N)$ is the geodesic, then $X(t) = \partial_t(\varphi)\circ\varphi^{-1}$ is a time-varying vector field on N and its momentum is simply $LX(t)$.

In § 4 we introduce the induced metrics on $B(M, N)$. We give the geodesic equation for these metrics also using momentum. One of the keys to working in this space is to define a convenient set of vector fields and forms on B in terms of auxiliary forms and vector fields on N. In this way, differential geometry on B can be reduced to calculations on N. Lie derivatives on N are especially useful here.

In the final § 5, we compute the sectional curvatures of $B(M, N)$. Like Arnold's formula, we get a formula with several terms, each of which seems to play a different role. The first involves the second derivatives of K and the others are expressed in terms which we call *force* and *stress*. Force is the bilinear version of the acceleration term in the geodesic equation and stress is a derivative of one vector field with respect to the other, that is, half of a Lie bracket, defined in what are essentially local coordinates. For the landmark space case, we proved this formula in our previous paper [4]. We hope that the terms in this formula will be elucidated by further study and analysis of specific cases.

§ 2. A covariant formula for curvature

2.1. Covariant derivative. Let (M, g) be a (finite-dimensional) Riemannian manifold. There will be some formulae which are valid for infinite dimensional manifolds and we will introduce definitions for these below. For each $x \in M$ we also view the metric as a bijective mapping $g_x \colon T_xM \to T_x^*M$. Then g^{-1} is the metric on the cotangent bundle as well as the morphism $T^*M \to TM$. For a 1-form $\alpha \in \Omega^1(M) = \Gamma(T^*M)$, we consider the 'sharp' vector field $\alpha^\sharp = g^{-1}\alpha \in \mathfrak{X}(M)$. If $\alpha = \alpha_i dx^i$, then $\alpha^\sharp = \alpha_i g^{ij}\partial_j$ is just the vector field obtained from α by 'raising indices'. Similarly, for a vector field $X \in \mathfrak{X}(M)$ we consider the 'flat' 1-form $X^\flat = gX$. If $X = X^i\partial_i$, then $X^\flat = X^i g_{ij} dx^j$ is the 1-form obtained from X by 'lowering indices'. Note that

$$\alpha(\beta^\sharp) = g^{-1}(\alpha, \beta) = g(\alpha^\sharp, \beta^\sharp) = \beta(\alpha^\sharp). \tag{1}$$

Our aim is to express the sectional curvature of g in terms of α, β alone. It is important that the exterior derivative satisfies

$$d\alpha(\beta^\sharp, \gamma^\sharp) = (\beta^\sharp)\alpha(\gamma^\sharp) - (\gamma^\sharp)\alpha(\beta^\sharp) - \alpha([\beta^\sharp, \gamma^\sharp]). \tag{2}$$

We recall the definition of the Levi-Civita covariant derivative ∇ and its basic properties:

$$\begin{aligned} 2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned} \tag{3}$$

$$\begin{aligned} (\nabla_X\alpha)(Y) &= X\alpha(Y) - \alpha(\nabla_X Y), \\ g((\nabla_X\alpha)^\sharp, Y) &= (\nabla_X\alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y) \\ &= Xg(\alpha^\sharp, Y) - g(\alpha^\sharp, \nabla_X Y) = g(\nabla_X\alpha^\sharp, Y) \quad \Longrightarrow \end{aligned} \tag{4}$$

$$\begin{aligned} \nabla_X(\alpha^\sharp) &= (\nabla_X\alpha)^\sharp, \\ Xg^{-1}(\alpha, \beta) &= g^{-1}(\nabla_X\alpha, \beta) + g^{-1}(\alpha, \nabla_X\beta), \\ \nabla_{\alpha^\sharp}\beta - \nabla_{\beta^\sharp}\alpha &= g[\alpha^\sharp, \beta^\sharp] = [\alpha^\sharp, \beta^\sharp]^\flat. \end{aligned} \tag{5}$$

From this follows

$$
\begin{aligned}
2(\nabla_{\alpha^\sharp}\beta)(\gamma^\sharp) &= 2g^{-1}(\nabla_{\alpha^\sharp}\beta,\gamma) = 2g((\nabla_{\alpha^\sharp}\beta)^\sharp,\gamma^\sharp) = 2g(\nabla_{\alpha^\sharp}\beta^\sharp,\gamma^\sharp)\\
&= \alpha^\sharp g^{-1}(\beta,\gamma) + \beta^\sharp g^{-1}(\gamma,\alpha) - \gamma^\sharp g^{-1}(\alpha,\beta)\\
&\qquad - g^{-1}(\alpha,[\beta^\sharp,\gamma^\sharp]^\flat) + g^{-1}(\beta,[\gamma^\sharp,\alpha^\sharp]^\flat) + g^{-1}(\gamma,[\alpha^\sharp,\beta^\sharp]^\flat)\\
&= \alpha^\sharp\beta(\gamma^\sharp) + \beta^\sharp\gamma(\alpha^\sharp) - \gamma^\sharp\beta(\alpha^\sharp) - \alpha([\beta^\sharp,\gamma^\sharp]) + \beta([\gamma^\sharp,\alpha^\sharp]) + \gamma([\alpha^\sharp,\beta^\sharp])\\
&= \beta^\sharp\gamma(\alpha^\sharp) - \alpha([\beta^\sharp,\gamma^\sharp]) + \gamma([\alpha^\sharp,\beta^\sharp]) - d\beta(\gamma^\sharp,\alpha^\sharp)\\
&= \alpha^\sharp\gamma(\beta^\sharp) - \beta^\sharp\alpha(\gamma^\sharp) + \gamma^\sharp\alpha(\beta^\sharp) + d\alpha(\beta^\sharp,\gamma^\sharp) - d\beta(\gamma^\sharp,\alpha^\sharp) - d\gamma(\alpha^\sharp,\beta^\sharp).
\end{aligned} \tag{6}
$$

2.2. Theorem (Mario's Formula). *Assume that all 1-forms $\alpha,\beta,\gamma,\delta\in\Omega^1_g(M)$ are closed. Then curvature is given by*

$$
\begin{aligned}
g\big(R(\alpha^\sharp,\beta^\sharp)\gamma^\sharp,\delta^\sharp\big) &= R_1+R_2+R_3,\\
R_1 &= \frac14\big(-\alpha^\sharp\gamma^\sharp\delta(\beta^\sharp) + \alpha^\sharp\delta^\sharp\beta(\gamma^\sharp) + \beta^\sharp\gamma^\sharp\delta(\alpha^\sharp) - \beta^\sharp\delta^\sharp\alpha(\gamma^\sharp)\\
&\qquad - \gamma^\sharp\alpha^\sharp\delta(\beta^\sharp) + \gamma^\sharp\beta^\sharp\delta(\alpha^\sharp) + \delta^\sharp\alpha^\sharp\beta(\gamma^\sharp) - \delta^\sharp\beta^\sharp\alpha(\gamma^\sharp)\big),\\
R_2 &= \frac14\big(-g^{-1}(d(\gamma(\beta^\sharp)),d(\delta(\alpha^\sharp))) + g^{-1}(d(\gamma(\alpha^\sharp)),d(\delta(\beta^\sharp)))\big),\\
R_3 &= \frac14\big(g([\delta^\sharp,\alpha^\sharp],[\beta^\sharp,\gamma^\sharp]) - g([\delta^\sharp,\beta^\sharp],[\alpha^\sharp,\gamma^\sharp]) + 2g([\alpha^\sharp,\beta^\sharp],[\gamma^\sharp,\delta^\sharp])\big).
\end{aligned}
$$

For the numerator of the sectional curvature we get

$$
\begin{aligned}
g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big) &= R_1+R_2+R_3,\\
R_1 &= \frac12\big(\alpha^\sharp\alpha^\sharp(\|\beta\|^2) - (\alpha^\sharp\beta^\sharp+\beta^\sharp\alpha^\sharp)g^{-1}(\alpha,\beta) + \beta^\sharp\beta^\sharp(\|\alpha\|^2)\big)\\
&= \frac12\big(\alpha^\sharp\beta([\alpha^\sharp,\beta^\sharp]) - \beta^\sharp\alpha([\alpha^\sharp,\beta^\sharp])\big),\\
R_2 &= \frac14\big(\|d(g^{-1}(\alpha,\beta))\|^2 - g^{-1}(d(\|\alpha\|^2),d(\|\beta\|^2))\big),\\
R_3 &= -\frac34\big\|[\alpha^\sharp,\beta^\sharp]\big\|^2_g.
\end{aligned}
$$

Recall that the sectional curvature is then

$$
k(\alpha^\sharp,\beta^\sharp) = \frac{g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big)}{\|\alpha\|^2\,\|\beta\|^2 - g^{-1}(\alpha,\beta)^2}.
$$

Proof. We shall need that for a function f we have

$$
\begin{aligned}
(\nabla_{\beta^\sharp}\gamma)^\sharp f &= df\big((\nabla_{\beta^\sharp}\gamma)^\sharp\big) = g^{-1}(df,\nabla_{\beta^\sharp}\gamma) = \beta^\sharp g^{-1}(df,\gamma) - g^{-1}(\nabla_{\beta^\sharp}df,\gamma)\\
&= \beta^\sharp\gamma^\sharp f - \nabla_{\beta^\sharp}df(\gamma^\sharp) = \beta^\sharp\gamma^\sharp f - \frac12\beta^\sharp df(\gamma^\sharp) + \frac12 df^\sharp\gamma(\beta^\sharp) - \frac12\gamma^\sharp\beta(df^\sharp)\\
&= \frac12 df^\sharp\gamma(\beta^\sharp) + \frac12[\beta^\sharp,\gamma^\sharp]f = \frac12 d(\gamma(\beta^\sharp))(df^\sharp) + \frac12[\beta^\sharp,\gamma^\sharp]f.
\end{aligned} \tag{7}
$$

For the three summands in the curvature formula, by multiple uses of formulae (2), (6) and the closedness of $\alpha, \beta, \gamma, \delta$, a straightforward calculation gives us:

$$
\begin{aligned}
&4(\nabla_{\alpha^\sharp}\nabla_{\beta^\sharp}\gamma)(\delta^\sharp) = \\
&\quad = 2\alpha^\sharp(\nabla_{\beta^\sharp}\gamma)(\delta^\sharp) - 2(\nabla_{\beta^\sharp}\gamma)^\sharp\delta(\alpha^\sharp) + 2\delta^\sharp\alpha((\nabla_{\beta^\sharp}\gamma)^\sharp) - 2d(\nabla_{\beta^\sharp}\gamma)(\delta^\sharp, \alpha^\sharp) \\
&\quad = 2\alpha^\sharp(\nabla_{\beta^\sharp}\gamma)(\delta^\sharp) - d(\gamma(\beta^\sharp))(d(\delta(\alpha^\sharp)^\sharp) - [\beta^\sharp, \gamma^\sharp]\delta(\alpha^\sharp) \\
&\qquad\qquad + 2\alpha^\sharp(\nabla_{\beta^\sharp}\gamma)(\delta^\sharp) + 2(\nabla_{\beta^\sharp}\gamma)([\delta^\sharp, \alpha^\sharp]) = \cdots \\
&\quad \cdots = -g^{-1}\big(d(\gamma(\beta^\sharp)), d(\delta(\alpha^\sharp))\big) + g\big([\delta^\sharp, \alpha^\sharp], [\beta^\sharp, \gamma^\sharp]\big) + 2\alpha^\sharp\beta^\sharp\gamma(\delta^\sharp) \\
&\qquad\qquad - 2\alpha^\sharp\gamma^\sharp\delta(\beta^\sharp) + \alpha^\sharp\delta^\sharp\beta(\gamma^\sharp) - [\beta^\sharp, \gamma^\sharp]\delta(\alpha^\sharp) + \delta^\sharp\alpha^\sharp\beta(\gamma^\sharp),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&-4(\nabla_{\beta^\sharp}\nabla_{\alpha^\sharp}\gamma)(\delta^\sharp) = +g^{-1}\big(d(\gamma(\alpha^\sharp)), d(\delta(\beta^\sharp))\big) - g\big([\delta^\sharp, \beta^\sharp], [\alpha^\sharp, \gamma^\sharp]\big), \\
&\qquad - 2\beta^\sharp\alpha^\sharp\gamma(\delta^\sharp) + 2\beta^\sharp\gamma^\sharp\delta(\alpha^\sharp) - \beta^\sharp\delta^\sharp\alpha(\gamma^\sharp) + [\alpha^\sharp, \gamma^\sharp]\delta(\beta^\sharp) - \delta^\sharp\beta^\sharp\alpha(\gamma^\sharp), \\
&-2(\nabla_{[\alpha^\sharp, \beta^\sharp]}\gamma)(\delta^\sharp) = \\
&\qquad = -[\alpha^\sharp, \beta^\sharp]\gamma(\delta^\sharp) + \gamma^\sharp\delta([\alpha^\sharp, \beta^\sharp]) - \delta^\sharp\gamma([\alpha^\sharp, \beta^\sharp]) - d[\alpha^\sharp, \beta^\sharp]^\flat(\gamma^\sharp, \delta^\sharp) \\
&\qquad = -[\alpha^\sharp, \beta^\sharp]\gamma(\delta^\sharp) + g\big([\alpha^\sharp, \beta^\sharp], [\gamma^\sharp, \delta^\sharp]\big).
\end{aligned}
$$

We can now compute the curvature (remember that $d\alpha = d\beta = \cdots = 0$):

$$
\begin{aligned}
4g\big(R(\alpha^\sharp, \beta^\sharp)\gamma^\sharp, \delta^\sharp\big) &= 4\delta\big(R(\alpha^\sharp, \beta^\sharp)\gamma^\sharp\big) = 4\delta\big(\nabla_{\alpha^\sharp}\nabla_{\beta^\sharp}\gamma^\sharp - \nabla_{\beta^\sharp}\nabla_{\alpha^\sharp}\gamma^\sharp - \nabla_{[\alpha^\sharp, \beta^\sharp]}\gamma^\sharp\big) \\
&= 4\big(\nabla_{\alpha^\sharp}\nabla_{\beta^\sharp}\gamma - \nabla_{\beta^\sharp}\nabla_{\alpha^\sharp}\gamma - \nabla_{[\alpha^\sharp, \beta^\sharp]}\gamma\big)(\delta^\sharp) \\
&= -g^{-1}\big(d(\gamma(\beta^\sharp)), d(\delta(\alpha^\sharp))\big) + g^{-1}\big(d(\gamma(\alpha^\sharp)), d(\delta(\beta^\sharp))\big) \\
&\qquad + g\big([\delta^\sharp, \alpha^\sharp], [\beta^\sharp, \gamma^\sharp]\big) - g\big([\delta^\sharp, \beta^\sharp], [\alpha^\sharp, \gamma^\sharp]\big) + 2g\big([\alpha^\sharp, \beta^\sharp], [\gamma^\sharp, \delta^\sharp]\big) \\
&\qquad - \alpha^\sharp\gamma^\sharp\delta(\beta^\sharp) + \alpha^\sharp\delta^\sharp\beta(\gamma^\sharp) + \beta^\sharp\gamma^\sharp\delta(\alpha^\sharp) - \beta^\sharp\delta^\sharp\alpha(\gamma^\sharp) \\
&\qquad - \gamma^\sharp\alpha^\sharp\delta(\beta^\sharp) + \gamma^\sharp\beta^\sharp\delta(\alpha^\sharp) + \delta^\sharp\alpha^\sharp\beta(\gamma^\sharp) - \delta^\sharp\beta^\sharp\alpha(\gamma^\sharp).
\end{aligned}
$$

For the sectional curvature expression, this simplifies (as always, for closed 1-forms) to the expression in the theorem. The two versions of R_1 correspond to each other, using $d\alpha = 0$ and $d\beta = 0$.

2.3. Mario's formula in coordinates. The formula for sectional curvature becomes especially transparent if we expand it in coordinates. Assume that $\alpha = \alpha_i dx^i$, $\beta = \beta_i dx^i$, where the coefficients α_i, β_i are *constants*, whence α, β are closed. Then $\alpha^\sharp = g^{ij}\alpha_i\partial_j$, $\beta^\sharp = g^{ij}\beta_i\partial_j$. Substituting these in the terms of the right-hand side of Mario's formula for sectional curvature, we get

$$
\begin{aligned}
\text{2nd deriv. terms} = 2R_1 &= 2\alpha^\sharp\alpha^\sharp(\|\beta\|^2) + 2\beta^\sharp\beta^\sharp(\|\alpha\|^2) - 2(\alpha^\sharp\beta^\sharp + \beta^\sharp\alpha^\sharp)g^{-1}(\alpha, \beta) \\
&= 2\alpha_i g^{is}(\alpha_j g^{jt}(\beta_k\beta_l g^{kl})_{,t})_{,s} + 2\beta_i g^{is}(\beta_j g^{jt}(\alpha_k\alpha_l g^{kl})_{,t})_{,s} \\
&\qquad - 2\alpha_i g^{is}(\beta_j g^{jt}(\beta_k\alpha_l g^{kl})_{,t})_{,s} - 2\beta_i g^{is}(\alpha_j g^{jt}(\alpha_k\beta_l g^{kl})_{,t})_{,s} \\
&= 2(\alpha_i\beta_k - \alpha_k\beta_i)(\alpha_j\beta_l - \alpha_l\beta_j)g^{is}(g^{jt}g^{kl}_{,t})_{,s}, \\
\text{1st deriv. terms} = 4R_2 &= \|d(g^{-1}(\alpha, \beta))\|^2 - g^{-1}\big(d(\|\alpha\|^2), d(\|\beta\|^2)\big) \\
&= (\alpha_i\beta_j g^{ij})_{,s}g^{st}(\alpha_l\beta_k g^{kl})_{,t} - (\alpha_i\alpha_j g^{ij})_{,s}g^{st}(\beta_k\beta_l g^{kl})_{,t} \\
&= -\frac{1}{2}(\alpha_i\beta_k - \alpha_k\beta_i)(\alpha_j\beta_l - \alpha_l\beta_j)g^{ij}_{,s}g^{st}g^{kl}_{,t},
\end{aligned}
$$

$$\begin{aligned}\text{Lie bracket} &= [\alpha^\sharp,\beta^\sharp] = \big(\alpha_i g^{is}(\beta_k g^{kt})_{,s} - \beta_i g^{is}(\alpha_k g^{kt})_{,s}\big)\partial_t\\ &= (\alpha_i\beta_k - \alpha_k\beta_i) g^{is} g^{kt}_{,s}\partial_t,\\ \text{Lie bracket term} = 4R_3 &= -3g\big([\alpha^\sharp,\beta^\sharp],[\alpha^\sharp,\beta^\sharp]\big)\\ &= -3(\alpha_i\beta_k-\alpha_k\beta_i)(\alpha_j\beta_l-\alpha_l\beta_j) g^{is} g^{kp}_{,s} g_{pq} g^{jt} g^{lq}_{,t}.\end{aligned}$$

Hence we have the coordinate version for the three terms in sectional curvature:

$$g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big) = (\alpha_i\beta_k-\alpha_k\beta_i)(\alpha_j\beta_l-\alpha_l\beta_j)\big(R_1^{ijkl}+R_2^{ijkl}+R_3^{ijkl}\big),$$
$$R_1^{ijkl} = \frac{1}{2} g^{is}(g^{jt}g^{kl}_{,t})_{,s}, \qquad R_2^{ijkl} = -\frac{1}{8} g^{ij}_{,s} g^{st} g^{kl}_{,t},$$
$$R_3^{ijkl} = -\frac{3}{4} g^{is} g^{kp}_{,s} g_{pq} g^{jt} g^{lq}_{,t}.$$

Note that the usual contravariant metric tensor g_{ij} occurs in only one place, everything else being derived from the covariant metric tensor g^{ij}. Note that the first term R_1 can be split into a pure second derivative term $R_{11} = g^{is}g^{jt}g^{kl}_{,st}$ plus a first derivative term $R_{12} = g^{is}g^{jt}_{,s}g^{kl}_{,t}$.

There is also a version of Mario's formula which is, in a sense, intermediate between the coordinate-free version and the coordinate version. The main thing that coordinates allow you to do is to take derivatives using the associated flat connection. In the case of this formula, this introduces auxiliary vector fields X_α and X_β playing the role of 'locally constant' extensions of the values of $\alpha^\sharp$ and $\beta^\sharp$ at the point $x \in M$ where the curvature is being calculated and for which the 1-forms α, β appear to be locally constant too. More precisely, assume we are given X_α and X_β such that:

1) $X_\alpha(x) = \alpha^\sharp(x)$, $X_\beta(x) = \beta^\sharp(x)$;

2) then $\alpha^\sharp - X_\alpha$ is zero at x and hence has a well-defined derivative $D_x(\alpha^\sharp - X_\alpha)$ lying in $\mathrm{Hom}(T_xM, T_xM)$; for a vector field Y we have $D_x(\alpha^\sharp - X_\alpha).Y_x = [Y, \alpha^\sharp - X_\alpha](x) = \mathcal{L}_Y(\alpha^\sharp - X_\alpha)|_x$; the same holds for β;

3) $\mathcal{L}_{X_\alpha}(\alpha) = \mathcal{L}_{X_\alpha}(\beta) = \mathcal{L}_{X_\beta}(\alpha) = \mathcal{L}_{X_\beta}(\beta) = 0$;

4) $[X_\alpha, X_\beta] = 0$.

Locally constant 1-forms and vector fields satisfy these conditions. Using these forms and vector fields, we then define

$$F(\alpha,\beta) := \frac{1}{2} d(g^{-1}(\alpha,\beta)),$$

a 1-form on M called the *force*, and

$$\mathcal{D}(\alpha,\beta)(x) := D_x(\beta^\sharp - X_\beta).\alpha^\sharp(x) = d(\beta^\sharp - X_\beta).\alpha^\sharp(x),$$

a tangent vector at x called the *stress*.

Then, in the notation above,

$$g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big)(x) = R_{11} + R_{12} + R_2 + R_3,$$
$$R_{11} = \frac{1}{2}\big(\mathcal{L}^2_{X_\alpha}(g^{-1})(\beta,\beta) - 2\mathcal{L}_{X_\alpha}\mathcal{L}_{X_\beta}(g^{-1})(\alpha,\beta) + \mathcal{L}^2_{X_\beta}(g^{-1})(\alpha,\alpha)\big)(x),$$
$$R_{12} = \langle F(\alpha,\alpha), \mathcal{D}(\beta,\beta)\rangle + \langle F(\beta,\beta), \mathcal{D}(\alpha,\alpha)\rangle - \langle F(\alpha,\beta), \mathcal{D}(\alpha,\beta) + \mathcal{D}(\beta,\alpha)\rangle,$$

$$R_2 = \big(\|F(\alpha,\beta)\|^2_{g^{-1}} - \big\langle F(\alpha,\alpha)), F(\beta,\beta)\big\rangle_{g^{-1}}\big)(x),$$
$$R_3 = -\frac{3}{4}\|\mathcal{D}(\alpha,\beta) - \mathcal{D}(\beta,\alpha)\|^2_{g_x}.$$

The reformulation of R_1 follows from the calculation

$$\begin{aligned}\alpha^\sharp\alpha^\sharp(\|\beta\|^2)(x) &= X_\alpha\alpha^\sharp(\|\beta\|^2)(x) = X_\alpha X_\alpha(\|\beta\|^2)(x) + X_\alpha(\alpha^\sharp - X_\alpha)(\|\beta\|^2)(x)\\ &= \mathcal{L}_{X_\alpha}\mathcal{L}_{X_\alpha}(g^{-1}(\beta,\beta))(x) + \langle D_x(\alpha^\sharp - X_\alpha).X_\alpha(x), d\|\beta\|^2)(x)\rangle\\ &= \mathcal{L}^2_{X_\alpha}(g^{-1})(\beta,\beta)(x) + \langle D_x(\alpha^\sharp - X_\alpha).X_\alpha(x), d\|\beta\|^2)(x)\rangle,\end{aligned}$$

and similar result for the other terms. The reformulation of R_3 comes from the calculation

$$\begin{aligned}[\alpha^\sharp,\beta^\sharp](x) &= (X_\alpha\circ\beta^\sharp)(x) - (X_\beta\circ\alpha^\sharp)(x)\\ &= (X_\alpha\circ(\beta^\sharp - X_\beta))(x) - (X_\beta\circ(\alpha^\sharp - X_\alpha)(x)\\ &= D_x((\beta^\sharp - X_\beta).X_\alpha(x) - D_x(\alpha^\sharp - X_\alpha).X_\beta(x).\end{aligned}$$

2.4. Infinite-dimensional manifolds. The main focus of this paper are the infinite-dimensional manifolds of diffeomorphisms of a finite-dimensional N, of the embeddings of one finite-dimensional M into another N and of the set of submanifolds F of a manifold N. These are infinite-dimensional and can be realized in multiple ways depending on the degree of smoothness imposed on the diffeomorphism/embedding/submanifold. The first two have realizations as Hilbert manifolds but the last has not. Moreover, the group law on the Hilbert manifold version of the group of diffeomorphisms is not differentiable. If one desires to carry over finite-dimensional techniques to the infinite-dimensional setting, it works much more smoothly to use the Fréchet space of C^∞ functions decreasing rapidly at infinity as the base vector space for charts of these spaces. But then its dual is not Fréchet, so one needs a bigger category for charts on bundles. The best setting has been developed by one of the authors and his collaborators [10] and uses 'c^∞-open' subsets in arbitrary 'convenient' locally convex topological vector spaces for charts. This theory and some of the reasons why it works are summarized in the appendix. For our purposes, complete locally convex topological vector spaces (which are always convenient) suffice and, on them, 'c^∞-open' just means open.

To extend Mario's formula to infinite-dimensional manifolds then, let (M,g) be a so-called 'weak Riemannian manifold' [10]: a convenient manifold M and a smooth map

$$g\colon TM\times_M TM\longrightarrow\mathbb{R}$$

which is a positive-definite symmetric bilinear form g_x on each tangent space T_xM, $x\in M$. For a convenient manifold we have to choose what we mean by 1-forms carefully. For each $x\in M$ the metric defines a mapping $g_x\colon T_xM\to T^*_xM$ (which we denote by the same symbol g_x). In the case of a Riemannian–Hilbert manifold, this is bijective and has an inverse but otherwise is only injective, whence the term 'weak metric'. The image $g(TM)\subset T^*M$ is called the *g-smooth cotangent bundle*. Then g^{-1} is the metric on the g-smooth cotangent bundle as well as the morphism $g(TM)\to TM$. Now define $\Omega^1_g(M) := \Gamma(g(TM))$ and $\alpha^\sharp = g^{-1}\alpha\in\mathfrak{X}(M)$,

$X^\flat = gX$ are as above. The exterior derivative is now defined by

$$d\alpha(\beta^\sharp, \gamma^\sharp) = (\beta^\sharp)\alpha(\gamma^\sharp) - (\gamma^\sharp)\alpha(\beta^\sharp) - \alpha([\beta^\sharp, \gamma^\sharp]).$$

We have $d\colon \Omega^1_g(M) \to \Omega^2(M) = \Gamma(L^2_{\text{skew}}(TM;\mathbb{R}))$ since the embedding $g(TM) \subset T^*M$ is a smooth fibre-linear mapping. Note that on an infinite-dimensional manifold M there are many choices of differential forms but only one of them is suitable for analysis on manifolds. These are discussed in [10, § 33]. Here we consider subspaces of these differential forms.

Further requirements need to be imposed on (M, g) for our theory to work. Since it is an infinite-dimensional weak Riemannian manifold, the Levi-Civita covariant derivative might not exist in TM. The Levi-Civita covariant derivative exists if and only if the metric itself admits gradients with respect to itself in the following senses. The easiest way to express this is locally in a chart $U \subset M$. Let V_U be the vector space of constant vector fields on U. Then we assume that there are smooth maps $\text{grad}_1 g$ and $\text{grad}_2 g$ from $U \times V_U$ to V_U, quadratic in V_U such that

$$\left.\begin{aligned} D_{x,Z}\, g_x(X,X) &= g_x(Z, \text{grad}_1\, g(x)(X,X)) \\ D_{x,X}\, g_x(X,Z) &= g_x(\text{grad}_2\, g(x)(X,X), Z) \end{aligned}\right. \quad \text{for all } Z. \tag{8}$$

(If we express this globally we also get derivatives of the vector fields X and Z.) This allows to use (3) to get the covariant derivative. Then the rest of the derivation of Mario's formula goes through and the final formula for curvature holds in both the finite- and infinite-dimensional cases. There are situations where the covariant derivative exists but not both gradients; see [11] and the corresponding extension [12, Appendix] to the real line.

Some constructions to be done shortly encounter a second problem: they lead to vector fields whose values do not lie in T_xM, but in the Hilbert space completion $\overline{T_xM}$ with respect to the inner product g_x. To manipulate these as in the finite-dimensional case, we need to know that $\bigcup_{x\in M} \overline{T_xM}$ forms a smooth vector bundle over M. More precisely, choose an atlas $(U_\alpha, u_\alpha\colon U_\alpha \to E)$ of M, where the sets $U_\alpha \subset M$ form an open cover of M, each $u_\alpha\colon U_\alpha \to E$ is a homeomorphism of U_α onto the open subset $u_\alpha(U_\alpha)$ of the convenient vector space E which models M, and $u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}\colon u_\beta(U_\alpha \cap U_\beta) \to u_\alpha(U_\alpha \cap U_\beta))$ is a smooth diffeomorphism. The mappings $x \mapsto \varphi_{\alpha\beta}(x) = du_{\alpha\beta}(u_\beta^{-1}(x)) \in L(E, E)$ then form the cocycle of transition functions $\varphi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(E)$ which define the tangent bundle TM. We then assume that the local expressions of each Riemannian metric g_x on E are equivalent weak inner products and hence define Hilbert space completions which are quasi-isometric via extensions of the embeddings of E (in each chart). Let us call one such Hilbert space $\mathcal{H}$. We then require that all transition functions $\varphi_{\alpha\beta}(x)\colon E \to E$ extend to bounded linear isomorphisms $\mathcal{H} \to \mathcal{H}$ and that each $\varphi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to L(\mathcal{H}, \mathcal{H})$ is again smooth.

These two properties will be sufficient for all the constructions we need so we make them into a definition.

Definition. A convenient weak Riemannian manifold (M, g) will be called a *robust* Riemannian manifold if the following conditions hold.

1) The Levi-Civita covariant derivative exists. Equivalently, the metric g_x admits gradients in the above two senses.

2) The completions $\overline{T_xM}$ form a vector bundle as described above.

Note that a Hilbert manifold is automatically robust. We can make the relationship between robust manifolds and Hilbert manifolds more explicit if we introduce another definition, that of a *pre-Hilbert manifold* similar to the notion of a pre-Hilbert topological vector space.

DEFINITION. A robust Riemannian manifold (M, g) is said to be *pre-Hilbert* if there exists an atlas $(U_\alpha, u_\alpha\colon U_\alpha \to E)$ for which:

1) each $u_\alpha(U_\alpha)$ is contained in the Hilbert norm interior of its closure in $\mathcal{H}$, which we denote $u_\alpha(U_\alpha)^{\mathcal{H}}$;

2) all chart-change maps $u_{\alpha\beta}$ extend to smooth mappings between the open subsets $u_\alpha(U_\alpha)^{\mathcal{H}}$ and hence define a completion $M \subset M^{\mathcal{H}}$ which is a Hilbert manifold.

Note that in this definition the atlas must be properly chosen: for example its open sets U_α must be open in the weak topology defined by path lengths. More precisely, for any weak Riemannian manifold M, the inner products g_x assign a length to every smooth path in M and we get a distance function $d(x, y)$ as the infimum of lengths of paths joining x and y (which might however be zero for some $x \neq y$). The topology defined by path lengths is usually much weaker than the strong topology given by the definition of M.

These distinctions are well illustrated by the spaces we will discuss below. Firstly, manifolds of smooth mappings like $\operatorname{Emb}(M, N)$ with their canonically induced Sobolev metrics of order $s > \dim M/2$ do admit completions $\operatorname{Emb}^s(M, N)$ to Hilbert manifolds and hence are pre-Hilbert; see [10, § 42.1] for the explicit chart changes. But their quotient manifolds $B(M, N) = \operatorname{Emb}(M, N)/\operatorname{Diff}(M)$ are only robust in general because the second condition fails. The extensions of the chart-change maps are homeomorphisms but not differentiable. This is due to the fact that the Sobolev completions $\operatorname{Diff}^s(M)$ of $\operatorname{Diff}(M)$ of order $s > \dim M/2$ are smooth manifolds themselves, but only topological groups: right translations are still smooth, while left translations and inversions are only continuous (and not even Lipschitz). So the action of $\operatorname{Diff}^s(M)$ on $\operatorname{Emb}^s(M, N)$, after Sobolev completion, has aspects which are only continuous, and thus $B^s(M, N) = \operatorname{Emb}^s(M, N)/\operatorname{Diff}^s(M)$ is only a topological manifold in general. This phenomenon also appears in the chart changes of the canonical atlas of $B(M, N)$; see [10, § 44.1] for an explicit formula of the chart change and the role of inversion in $\operatorname{Diff}^s(M)$ in it.

2.5. Covariant curvature and O'Neill's formula: finite-dimensional case. Let $p\colon (E, g_E) \to (B, g_B)$ be a Riemannian submersion between finite-dimensional manifolds, that is, for each $b \in B$ and $x \in E_b := p^{-1}(b)$ the g_E-orthogonal splitting

$$T_x E = T_x(E_{p(x)}) \oplus T_x(E_{p(x)})^\perp =: T_x(E_{p(x)}) \oplus \operatorname{Hor}_x(p)$$

has the property that $T_x p\colon (\operatorname{Hor}_x(p), g_E) \to (T_b B, g_B)$ is an isometry. Each vector field $X \in \mathfrak{X}(E)$ is decomposed as $X = X^{\text{hor}} + X^{\text{ver}}$ into horizontal and vertical parts. Each vector field $\xi \in \mathfrak{X}(B)$ can be lifted uniquely to a smooth horizontal field $\xi^{\text{hor}} \in \Gamma(\operatorname{Hor}(p)) \subset \mathfrak{X}(E)$. O'Neill's formula says that for any two horizontal vector fields X, Y on E and any $x \in E$, the sectional curvatures of E and B are related by

$$g_{p(x)}\big(R^B(p_*(X_x), p_*(Y_x))p_*(Y_x), p_*(X_x)\big) = g_x\big(R^E(X_x, Y_x)Y_x, X_x\big) + \frac{3}{4}\|[X, Y]^{\text{ver}}\|_x^2.$$

Comparing Mario's formula on E and B gives an immediate proof of this fact. We start with a lemma.

Lemma. *If $\alpha \in \Omega^1(B)$ is a 1-form on B, then the vector field $(p^*\alpha)^\sharp$ is horizontal and we have $Tp\circ(p^*\alpha)^\sharp = \alpha^\sharp \circ p$. Therefore $(p^*\alpha)^\sharp$ equals the horizontal lift $(\alpha^\sharp)^{\text{hor}}$. For each $x \in E$ the mapping $(T_xp)^*\colon (T^*_{p(x)}B, g_B^{-1}) \to (T^*_xE, g_E^{-1})$ is an isometry.*

Proof. All this holds because for $X_x \in T_xE$ we have

$$\begin{aligned} g_E((p^*\alpha)^\sharp_x, X_x) &= (p^*\alpha)_x(X_x) = \alpha_{p(x)}(T_xp.X_x) = \alpha_{p(x)}(T_xp.X_x^{\text{hor}}) \\ &= g_E((p^*\alpha)^\sharp_x, X_x^{\text{hor}}), \\ g_B\big(T_xp(p^*\alpha)^\sharp_x, T_xp.X_x\big) &= g_E((p^*\alpha)^\sharp_x, X_x^{\text{hor}}) = \alpha_{p(x)}(T_xp.X_x) \\ &= g_B(\alpha^\sharp_{p(x)}, T_xp.X_x). \end{aligned}$$

More generally we have

$$g_E^{-1}(p^*\alpha, p^*\beta) = g_E((p^*\alpha)^\sharp, (p^*\beta)^\sharp) = g_B(\alpha^\sharp, \beta^\sharp)\circ p = p^*g_B^{-1}(\alpha,\beta).$$

Consequently, we get for 1-forms α, β on B:

$$\begin{aligned} d\|p^*\alpha\|^2_{g_E^{-1}} &= dp^*\|\alpha\|^2_{g_B^{-1}} = p^*d\|\alpha\|^2_{g_B^{-1}}, \\ (p^*\beta)^\sharp\|p^*\alpha\|^2_{g_E^{-1}} &= (p^*d\|\alpha\|^2_{g_B^{-1}})((\alpha^\sharp)^{\text{hor}}) = p^*(\beta^\sharp\|\alpha\|^2_{g_B^{-1}}). \end{aligned}$$

In the following computation we use

$$\|[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]^{\text{hor}}\|^2_{g_E} = p^*\|[\alpha^\sharp, \beta^\sharp]\|^2_{g_B}.$$

We take Mario's formula (§ 2.2) and apply it to the closed 1-forms $p^*\alpha$, $p^*\beta$ on E, where α, β are closed 1-forms on B. Using the results above, we get

$$\begin{aligned} &4g_E\big(R((p^*\alpha)^\sharp, (p^*\beta)^\sharp)(p^*\beta)^\sharp, (p^*\alpha)^\sharp\big) \\ &\quad = \|d(g_E^{-1}(p^*\alpha, p^*\beta))\|^2_{g_E^{-1}} - g_E^{-1}\big(d(\|p^*\alpha\|^2_{g_E^{-1}}), d(\|p^*\beta\|^2_{g_E^{-1}})\big) \\ &\qquad - 3\|[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]^{\text{hor}}\|^2_{g_E} - 3\|[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]^{\text{ver}}\|^2_{g_E} \\ &\qquad + 2(p^*\alpha)^\sharp(p^*\alpha)^\sharp(\|p^*\beta\|^2_{g_E^{-1}}) + 2(p^*\beta)^\sharp(p^*\beta)^\sharp(\|p^*\alpha\|^2_{g_E^{-1}}) \\ &\qquad - 2((p^*\alpha)^\sharp(p^*\beta)^\sharp + (p^*\beta)^\sharp(p^*\alpha)^\sharp)g_E^{-1}(p^*\alpha, p^*\beta) \\ &\quad = p^*\|d(g_B^{-1}(\alpha,\beta))\|^2_{g_B^{-1}} - p^*g_B^{-1}\big(d(\|\alpha\|^2_{g_B^{-1}}), d(\|\beta\|^2_{g_B^{-1}})\big) - 3p^*\|[\alpha^\sharp, \beta^\sharp]\|^2_{g_B} \\ &\qquad - 3\|[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]^{\text{ver}}\|^2_{g_E} \\ &\qquad + 2p^*\big(\alpha^\sharp\alpha^\sharp(\|\beta\|^2_{g_B^{-1}})\big) + 2p^*\big(\beta^\sharp\beta^\sharp(\|\alpha\|^2_{g_B^{-1}})\big) - 2p^*\big((\alpha^\sharp\beta^\sharp + \beta^\sharp\alpha^\sharp)g_B^{-1}(\alpha,\beta)\big) \\ &\quad = 4p^*g_B\big(R^B(\alpha^\sharp, \beta^\sharp)\beta^\sharp, \alpha^\sharp\big) - 3\|[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]^{\text{ver}}\|^2_{g_E}, \end{aligned}$$

which is a short proof of O'Neill's formula.

2.6. Covariant curvature and O'Neill's formula. Let $p\colon (E, g_E) \to (B, g_B)$ be a Riemann submersion between infinite-dimensional robust Riemann manifolds, that is, for each $b \in B$ and $x \in E_b := p^{-1}(b)$, the tangent mapping $T_xp\colon (T_xE, g_E) \to (T_bB, g_B)$ is a surjective metric quotient map, so that

$$\|\xi_b\|_{g_B} := \inf\big\{X_x \in T_xE\colon T_xp.X_x = \xi_b\big\}. \tag{9}$$

The infimum need not be attained in T_xE but will be in the completion $\overline{T_xE}$. The orthogonal subspace $\{Y_x\colon g_E(Y_x, T_x(E_b)) = 0\}$ has therefore to be taken in $\overline{T_xE}$.

If $\alpha_b = g_B(\alpha_b^\sharp, \quad) \in g_B(T_bB) \subset T_b^*B$ is an element in the g_B-smooth dual, then $p^*\alpha_b := (T_xp)^*(\alpha_b) = g_B(\alpha_b^\sharp, T_xp \quad)\colon T_xE \to \mathbb{R}$ is in T_x^*M but in general it is not an element in the smooth dual $g_E(T_xE)$. It is, however, an element of the Hilbert space completion $\overline{g_E(T_xE)}$ of the g_E-smooth dual $g_E(T_xE)$ with respect to the norm $\| \cdot \|_{g_E^{-1}}$, and the element $g_E^{-1}(p^*\alpha_b) =: (p^*\alpha_b)^\sharp$ is in the $\| \cdot \|_{g_E}$-completion $\overline{T_xE}$ of T_xE. We can call $g_E^{-1}(p^*\alpha_b) =: (p^*\alpha_b)^\sharp$ the *horizontal lift* of $\alpha_b^\sharp = g_B^{-1}(\alpha_b) \in T_bB$.

In the following we discuss the manifold E and write g instead of g_E. The metric g_x can be evaluated at elements of the completion $\overline{T_xE}$. Moreover, for any smooth sections $X, Y \in \Gamma(\overline{TE})$ the mapping $g(X,Y)\colon M \to \mathbb{R}$ is still smooth. Indeed, this is a local question, so let E be c^∞-open in a convenient vector space V_E. Since the evaluations on $X \otimes Y$ form a set of bounded linear functionals on the space $L^2_{\text{sym}}(\overline{V_M}; \mathbb{R})$ of bounded symmetric bilinear forms on $\overline{V_M}$ which recognize bounded subsets, it follows that g is smooth as a mapping $M \to L^2_{\text{sym}}(\overline{V_M}; \mathbb{R})$, by the smooth uniform boundedness theorem; see [10].

Lemma. *If α is a smooth 1-form on an open subset U of B with values in the g_B-smooth dual $g_B(TB)$, then $p^*\alpha$ is a smooth 1-form on $p^{-1}(U) \subset E$ with values in the $\| \cdot \|_{g_E^{-1}}$-completion of the g_E-smooth dual $g_E(TE)$. Thus also $(p^*\alpha)^\sharp$ is smooth from E into the g_E-completion of TE, and it has values in the subbundle g_E-orthogonal to the vertical bundle in the g_E-completion. We may continuously extend T_xp to the $\| \cdot \|_{g_E^{-1}}$-completion, and then we have $Tp \circ (p^*\alpha)^\sharp = \alpha^\sharp \circ p$. Moreover, the Lie bracket of two such forms, $[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]$, is defined. The exterior derivative $d(p^*\alpha)$ is defined and is applicable to vector fields with values in the completion, like $(p^*\beta)^\sharp$.*

That the Lie bracket is defined is also a non-trivial statement: we have to differentiate in directions which are not tangent to the manifold.

Proof of the lemma. This is a local question, and so we may assume that $U = B$ and $p^{-1}(U) = E$ are c^∞-open subsets in convenient vector spaces V_B and V_E, respectively, so that all tangent bundles are trivial. By definition, $\alpha^\sharp = g_B^{-1} \circ \alpha\colon B \to B \times V_B$ is smooth. We have to show that $(p^*\alpha)^\sharp = g_E^{-1} \circ p^*\alpha$ is a smooth mapping from E into the $\| \cdot \|_{g_E}$-completion of V_E. By the smooth uniform boundedness theorem (see [10]) it suffices to check that the composition with each bounded linear functional in a set $\mathcal{S} \subset V_E'$ is smooth, where $\mathcal{S} \subseteq V_E'$ is a set of linear functionals on V_E which recognizes bounded subsets of V_E. For this property, functionals of the form $g_E(v, \cdot\,)$ for $v \in V_E$ suffice. But

$$x \mapsto (g_E)_x(v, (p^*\alpha)^\sharp|_x) = p^*\alpha|_x(v) = \alpha|_x(T_xp.v)$$

is obviously smooth.

We can continuously extend the metric quotient mapping T_xp to the $\| \cdot \|_{g_E}$-completion and get a mapping $T_xp\colon \overline{T_xE} \to \overline{T_bB}$, where $b = p(x)$. For a second form $\beta \in \Gamma(g_B(TB))$, we then have

$$\begin{aligned} g_B(\beta^\sharp|_b, T_xp.(p^*\alpha)^\sharp|_x) &= (\beta_b(T_xp.(p^*\alpha)^\sharp|_x) = (p^*\beta)|_x((p^*\alpha)^\sharp|_x) \\ &= g_E^{-1}((T_xp)^*\beta, (T_xp)^*\alpha) = g_B(\beta_b, \alpha_b) = g_B(\beta^\sharp|_b, (\alpha^\sharp \circ p)(x)), \end{aligned}$$

which implies that $Tp \circ (p^*\alpha)^\sharp = \alpha^\sharp \circ p$.

For the Lie bracket of two such forms, $[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]$, we can again assume that all bundles are trivial. Then

$$[(p^*\alpha)^\sharp, (p^*\beta)^\sharp](x) = d((p^*\beta)^\sharp)(x)((p^*\alpha)^\sharp) - d((p^*\alpha)^\sharp)(x)((p^*\beta)^\sharp),$$

$$\begin{aligned} d((p^*\beta)^\sharp)(x)((p^*\alpha)^\sharp) &= d(g_E^{-1}\circ (Tp)^*\circ\beta\circ p)(x)((p^*\alpha)^\sharp) \\ &= d(g_E^{-1}\circ (Tp)^*\circ\beta)(b).T_xp.(p^*\alpha)^\sharp = d(g_E^{-1}\circ (Tp)^*\circ\beta)(b).\alpha^\sharp(p(x)). \end{aligned}$$

So the Lie bracket is well defined.

By assumption, the metric $g = g_E$ admits gradients with respect to itself as in (8) § 2.4. In a local chart we have

$$\begin{aligned} D_{x,Z}g_x(X,X) &= g_x(Z, \operatorname{grad}_1 g(x)(X,X)) \\ D_{x,Z}g_x(Z,X) &= g_x(\operatorname{grad}_2 g(x)(Z,Z),X) \end{aligned} \tag{10}$$

for $X, Z \in V_E$. We can then take $X \in \overline{V_E}$ in the upper left expression of (10) and thus also in the right hand side. Then the upper right term allows to take $Z \in \overline{V_E M}$ also. This carries over to the lower expression.

Thus the local expressions of the Christoffel symbols of the Levi-Civita covariant derivative extend to sections of the completed tensor bundle $\overline{TE}$, and therefore the Levi-Civita covariant derivative extends to smooth sections of $\overline{TE}$ which are differentiable in directions in $\overline{TE}$ like $(p^*\alpha)^\sharp$. Thus expressions like $\nabla^E_{(p^*\alpha)^\sharp}(p^*\beta)^\sharp$ make sense and are again of the same type so that one can iterate. Thus the curvature expression $g_E\big(R((p^*\alpha)^\sharp, (p^*\beta)^\sharp)(p^*\alpha)^\sharp, (p^*\beta)^\sharp\big)$ makes sense. Moreover, all operations used in the proof in § 2.2 work again, so this result holds. The proof in § 2.5 works and we can conclude the following result.

Theorem 2.1. *Let $p\colon (E, g_E) \to (B, g_B)$ be a Riemann submersion between infinite-dimensional robust Riemann manifolds. Then for 1-forms $\alpha, \beta \in \Omega^1_{g_B}(B)$ O'Neill's formula holds in the form*

$$\begin{aligned} &g_B\big(R^B(\alpha^\sharp, \beta^\sharp)\beta^\sharp, \alpha^\sharp\big) \\ &\qquad = g_E\big(R^E((p^*\alpha)^\sharp, (p^*\beta)^\sharp)(p^*\beta)^\sharp, (p^*\alpha)^\sharp\big) + \frac{3}{4}\|[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]^{\mathrm{ver}}\|^2_{g_E}. \end{aligned}$$

§ 3. The diffeomorphism group $\operatorname{Diff}_{\mathcal{S}}(N)$

3.1. Diffeomorphism groups. Let N be one of the following manifolds.

1) N is a compact manifold. Then let $\operatorname{Diff}(N)$ be the regular Lie group [10, § 38] consisting of all smooth diffeomorphisms of M.

2) N is $\mathbb{R}^n$. We let $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ denote the group of all diffeomorphisms of $\mathbb{R}^n$ which decay rapidly towards the identity. This is a regular Lie group (for $n = 1$ this is proved in [3, § 6.4]; the proof there works for arbitrary n). Its Lie algebra is the space $\mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n)$ of rapidly decreasing vector fields, with the negative of the usual bracket as Lie bracket.

3) More generally, (N, g) is a non-compact Riemannian manifold of bounded geometry; see [13]. It is a complete Riemannian manifold and all covariant derivatives of the curvature are bounded with respect to g. Then there is a well-developed

theory of Sobolev spaces on N. Let H^∞ denote the intersection of all Sobolev spaces, which consists of smooth functions (or sections). Even on $N = \mathbb{R}$ the space H^∞ is strictly larger than the subspace $\mathcal{S}$ of all rapidly decreasing functions (or sections), which can be defined by the condition that the Riemannian norm of all iterated covariant derivatives decreases faster than the inverse of any power of the Riemannian distance. There is almost no information available on the space $\mathcal{S}$ for a general Riemannian manifold of bounded geometry. For the following we let $\mathcal{S}$ denote either H^∞ or the space of rapidly decreasing functions. We let $\operatorname{Diff}_{\mathcal{S}}(N)$ denote the group of all diffeomorphisms which decay rapidly towards the identity (or differ from the identity by H^∞). It is a regular Lie group with Lie algebra the space $\mathfrak{X}_{\mathcal{S}}(N)$ of rapidly decreasing vector fields with the negative of the usual bracket. In [3, § 6.4] this was proved for $N = \mathbb{R}$, but a similar proof works for the general case discussed here.

In general, we need to impose some boundary conditions near infinity for groups of diffeomorphisms on a non-compact manifold N: the full group $\operatorname{Diff}(N)$ of all diffeomorphisms with its natural compact C^∞-topology is not locally contractible, so it does not admit any atlas of open charts.

For uniformity of notation, we shall denote by $\operatorname{Diff}_{\mathcal{S}}(N)$ any of these regular Lie groups. Its Lie algebra is denoted by $\mathfrak{X}_{\mathcal{S}}(N)$ in each of these cases, with the negative of the usual bracket as Lie bracket. We shall also denote by $\mathcal{O} = C^\infty \cap \mathcal{S}'$ the space of smooth functions in the dual space $\mathcal{S}'$ (to be specific, this is the space $\mathcal{O}_M$ in the sense of Laurent Schwartz when $N = \mathbb{R}^n$).

3.2. Riemann metrics on the diffeomorphism group. Motivated by the concept of robust Riemannian manifolds and by [14, Ch. 12] we will construct a right-invariant weak Riemannian metric by assuming that we have a Hilbert space $\mathcal{H}$ together with two bounded injective linear mappings

$$\mathfrak{X}_S(N) = \Gamma_{\mathcal{S}}(TN) \xrightarrow{j_1} \mathcal{H} \xrightarrow{j_2} \Gamma_{C_b^2}(TN), \tag{11}$$

where $\Gamma_{C_b^2}(TN)$ is the Banach space of all C^2 vector fields X on N which are globally bounded together with $\nabla^g X$ and $\nabla^g \nabla^g X$ with respect to g, such that $j_2 \circ j_1 \colon \Gamma_{\mathcal{S}}(TN) \to \Gamma_{C_b^2}(TN)$ is the canonical embedding. We also assume that j_1 has dense image.

Dualizing the Banach spaces in equation (11) and using the canonical isomorphisms (which we call L and K) between $\mathcal{H}$ and its dual $\mathcal{H}'$, we get the diagram

$$\begin{array}{ccc}
\Gamma_{\mathcal{S}}(TN) & & \Gamma_{\mathcal{S}'}(T^*N) \\
\big\downarrow{\scriptstyle j_1} & & \big\uparrow{\scriptstyle j_1'} \\
\mathcal{H} & \underset{K}{\overset{L}{\rightleftarrows}} & \mathcal{H}' \\
\big\downarrow{\scriptstyle j_2} & & \big\uparrow{\scriptstyle j_2'} \\
\Gamma_{C_b^2}(TN) & & \Gamma_{M^2}(T^*N)
\end{array} \tag{12}$$

Here we have written $\Gamma_{\mathcal{S}'}(T^*N)$ for the dual of the space of smooth vector fields $\Gamma_{\mathcal{S}}(TN) = \mathfrak{X}_{\mathcal{S}}(N)$. We call these 1-*co-currents* as 1-currents are elements in the dual of $\Gamma_{\mathcal{S}}(T^*N)$. It contains smooth measure-valued cotangent vectors on N (which we will write as $\Gamma_{\mathcal{S}}(T^*N \otimes \mathrm{vol}(N)))$ as well as the bigger subspace of second derivatives of finite measure-valued 1-forms on N, which we have written as $\Gamma_{M^2}(T^*N)$ and which is part of the dual of $\Gamma_{C_b^2}(TN)$. In what follows, we will have many momentum variables with values in these spaces.

The restriction of L to $\mathfrak{X}_{\mathcal{S}}(N)$ via j_1 gives us a positive-definite weak inner product on $\mathfrak{X}_{\mathcal{S}}(N)$. It may be defined by a distribution-valued kernel, which we also write as L:

$$\langle\,\cdot\,,\,\cdot\,\rangle_L \colon \mathfrak{X}_{\mathcal{S}}(N) \times \mathfrak{X}_{\mathcal{S}}(N) \to \mathbb{R}$$

is defined by

$$\langle X, Y\rangle_L = \langle j_1 X, j_1 Y\rangle_{\mathcal{H}} = \iint_{N\times N} \big(X(y_1)\otimes Y(y_2), L(y_1,y_2)\big),$$

where $L \in \Gamma_{\mathcal{S}'}(\mathrm{pr}_1^*(T^*N)\otimes \mathrm{pr}_2^*(T^*N))$.

Extending this weak inner product right-invariantly over $\mathrm{Diff}_{\mathcal{S}}(N)$, we get a robust weak Riemannian manifold in the sense of § 2.4.

In the case (called the *standard case* below) when $N = \mathbb{R}^n$ and

$$\langle X, Y\rangle_L = \int_{\mathbb{R}^n} \langle (1 - A\Delta)^l X, Y\rangle\, dx,$$

we have

$$L(x,y) = \left(\frac{1}{(2\pi)^n}\int_{\xi\in\mathbb{R}^n} e^{i\langle \xi, x-y\rangle}(1 + A|\xi|^2)^l\, d\xi\right)\sum_{i=1}^n (du^i|_x \otimes dx)\otimes(du^i|_y\otimes dy),$$

where $d\xi$, dx and dy denote the Lebesgue measure and (u^i) are linear coordinates on $\mathbb{R}^n$. Here $\mathcal{H}$ is the space of Sobolev H^l-vector fields on N.

Note that given an operator L with appropriate properties, we can reconstruct the Hilbert space $\mathcal{H}$ with the two bounded injective mappings j_1, j_2.

Construction of the reproducing kernel K. The inverse map K is even nicer as it is given by a C^2-tensor, the reproducing kernel. To see this, note that $\Gamma_{M^2}(T^*N)$ contains the measures supported at one point x defined by an element $\alpha_x \in T_x^*N$. Then $j_2(K(j_2'(\alpha_x)))$ is given by a C^2-vector field K_{α_x} on N which satisfies

$$\langle K_{\alpha_x}, X\rangle_{\mathcal{H}} = \alpha_x(j_2X)(x) \quad \text{for all} \quad X\in\mathcal{H}, \quad \alpha_x \in T_x^*N. \tag{13}$$

The map $\alpha_x \mapsto K_{\alpha_x}$ is weakly C_b^2. Thus by [10, Theorem 12.8] this mapping is strongly Lip^1 (that is, differentiable and the derivative is locally Lipschitz, for the norm on $\mathcal{H}$). Since $\mathrm{ev}_y \circ K \colon T_x^*N \ni \alpha_x \mapsto K_{\alpha_x}(y) \in T_yN$ is linear we get a corresponding element $K(x,y) \in L(T_x^*N, T_yN) = T_xN \otimes T_yN$ with $K(y,x)(\alpha_x) = K_{\alpha_x}(y)$.

Using (13) twice, we have (omitting j_2)

$$\beta_y.K(y,x)(\alpha_x) = \big\langle K(\,\cdot\,,x)(\alpha_x), K(\,\cdot\,,y)(\beta_y)\big\rangle_{\mathcal{H}} = \alpha_x.K(x,y)(\beta_y)$$

so that

1) $K(x,y)^\top = K(y,x) \colon T_y^*N \to T_xN$;
2) $K \in \Gamma_{C_b^2}(\mathrm{pr}_1{}^*\, TN \otimes \mathrm{pr}_2^*\, TN)$.

Moreover, the operator K defined directly by integration

$$K\colon \Gamma_{M^2}(T^*N) \to \Gamma_{C_b^2}(TN), \qquad K(\alpha)(y_2) = \int_{y_1 \in N} (K(y_1, y_2), \alpha(y_1)),$$

is the same as the inverse K of L. In fact, by definition, they agree on sections in $\Gamma_{C^2}(T^*M)$ with finite support and these are weakly dense. Hence they agree everywhere.

We will sometimes use the abbreviations $\langle \alpha | K |$, $| K | \beta \rangle$ and $\langle \alpha | K | \beta \rangle$ for the contraction of the vector values of K in its first and second variable against 1-forms α and β. Often these are measure-valued 1-forms so that, after contracting, there remains a measure in that variable which can be integrated.

Thus the C^2-tensor K determines L and hence $\mathcal{H}$ and hence the whole metric on $\mathrm{Diff}_{\mathcal{S}}(N)$. It is tempting to start with the tensor K, assuming that it is symmetric and positive definite in a suitable sense. But rather subtle conditions on K are required in order that its inverse L be defined on all infinitely differentiable vector fields. For example, if $N = \mathbb{R}$, the Gaussian kernel $K(x, y) = e^{-|x-y|^2}$ does not give such an L.

In the standard case we have

$$K(x, y) = K_l(x - y) \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y^i}, \qquad K_l(x) = \frac{1}{(2\pi)^n} \int_{\xi \in \mathbb{R}^n} \frac{e^{i\langle \xi, x \rangle}}{(1 + A|\xi|^2)^l} d\xi,$$

where K_l is given by a classical Bessel function of differentiability class C^{2l}.

3.3. The zero compressibility limit. Although the family of metrics above does not include the case originally studied by Arnold (the L^2-metric on volume-preserving diffeomorphisms), it does include metrics which have this case as a limit. Taking $N = \mathbb{R}^n$ and starting with the standard Sobolev metric, we can add a divergence term with a coefficient B:

$$\langle X, Y \rangle_L = \int_{\mathbb{R}^n} \big(\langle (1 - A\Delta)^l X, Y \rangle + B.\operatorname{div}(X) \operatorname{div}(Y) \big)\, dx.$$

Note that as B approaches ∞, the geodesics will tend to lie on the cosets with respect to the subgroup of volume-preserving diffeomorphisms. And when, in addition, A approaches zero, we get the simple L^2-metric used by Arnold. This suggests that, as in the so-called 'zero-viscosity limit', we should be able to construct geodesics in Arnold's metric, that is, solutions of Euler's equation, as limits of geodesics for this larger family of metrics on the full group.

The resulting kernels L and K are no longer diagonal. To L, we must add

$$B \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{(2\pi)^n} \int_{\xi \in \mathbb{R}^n} e^{i\langle \xi, x-y \rangle} \xi_i . \xi_j \, d\xi \right) (du^i|_x \otimes dx) \otimes (du^j|_y \otimes dy).$$

It can be checked that the corresponding kernel K will have the form

$$K(x, y) = K_0(x - y) \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y^i} + \sum_{i=1}^{n} \sum_{j=1}^{n} (K_B)_{,ij}(x - y) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y^j},$$

where K_0 is the kernel as above for the standard norm of order l, and K_B is a second radially symmetric kernel on $\mathbb{R}^n$ depending on B.

3.4. The geodesic equation. According to [1], the geodesic equation on any Lie group G with a right-invariant metric is given as follows. Let $g(t)$ be a path in G and let $u(t) = \dot g(t).g(t)^{-1} = T(\mu^{g(t)^{-1}})\dot g(t)$ be the right logarithmic derivative, a path in its Lie algebra $\mathfrak{g}$. Here $\mu^g \colon G \to G$ is right translation by g. Then $g(t)$ is a geodesic if and only if

$$\partial_t u = -\operatorname{ad}_u^\top u,$$

where the transpose $\operatorname{ad}_X^\top$ is the adjoint of $\operatorname{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ with respect to the metric on $\mathfrak{g}$.

In our case the Lie algebra of $\operatorname{Diff}_{\mathcal{S}}(N)$ is the space $\mathfrak{X}_{\mathcal{S}}(N)$ of all rapidly decreasing smooth vector fields with Lie bracket (we write $\operatorname{ad}_X Y$) the negative of the usual Lie bracket: $\operatorname{ad}_X Y = -[X,Y]$. Then a smooth curve $t \mapsto \varphi(t)$ of diffeomorphisms is a geodesic for the right-invariant weak Riemannian metric on $\operatorname{Diff}_{\mathcal{S}}(N)$ induced by the weak inner product $\langle\,\cdot\,,\,\cdot\,\rangle_L$ on $\mathfrak{X}_{\mathcal{S}}(N)$ if and only if

$$\partial_t u = -\operatorname{ad}_u^\top u$$

as above. Here the time-dependent vector field u is now given by $\partial_t\varphi(t) = u(t)\circ\varphi(t)$, and the transpose $\operatorname{ad}_X^\top$ by

$$\langle \operatorname{ad}_X^\top Y, Z\rangle_L = \langle Y, \operatorname{ad}_X Z\rangle_L = -\langle Y, [X,Z]\rangle_L.$$

The inner product is weak; existence of $\operatorname{ad}_X^\top$ implies condition 1) for robustness of the weak Riemannian manifold $(\operatorname{Diff}_{\mathcal{S}}(N), \langle\,\cdot\,,\,\cdot\,\rangle_L)$; it is equivalent to the fact that the dual mapping $\operatorname{ad}_X^* \colon \mathfrak{X}_{\mathcal{S}}(N)' \to \mathfrak{X}_{\mathcal{S}}(N)'$ maps the smooth dual $L(\mathfrak{X}_{\mathcal{S}}(N))$ to itself. We also have $L \circ \operatorname{ad}_X^\top = \operatorname{ad}_X^* \circ L$. Using *Lie derivatives*, the computation of ad_X^* is especially simple. Namely, for any section ω of $T^*N \otimes \operatorname{vol}$ and vector fields $\xi, \eta \in \mathfrak{X}_{\mathcal{S}}(N)$, we have

$$\int_N (\omega, [\xi,\eta]) = \int_N (\omega, \mathcal{L}_\xi(\eta)) = -\int_N (\mathcal{L}_\xi(\omega), \eta),$$

whence $\operatorname{ad}_\xi^*(\omega) = +\mathcal{L}_\xi(\omega)$. Thus the Hamiltonian version of the geodesic equation on the smooth dual $L(\mathfrak{X}_{\mathcal{S}}(N)) \subset \Gamma_{C^2}(T^*N \otimes \operatorname{vol})$ becomes

$$\partial_t \alpha = -\operatorname{ad}_{K(\alpha)}^* \alpha = -\mathcal{L}_{K(\alpha)}\alpha$$

or, keeping track of everything,

$$\begin{aligned} \partial_t\varphi &= u \circ \varphi, & \partial_t\alpha &= -\mathcal{L}_u\alpha, \\ u &= K(\alpha) = \alpha^\sharp, & \alpha &= L(u) = u^\flat. \end{aligned}$$

One can also derive the geodesic equation from the conserved momentum mapping $J \colon T\operatorname{Diff}_{\mathcal{S}}(N) \to \mathfrak{X}_{\mathcal{S}}(N)'$ given by $J(g,X) = L \circ \operatorname{Ad}(g)^\top X$, where $\operatorname{Ad}(g)X = Tg \circ X \circ g^{-1}$. This means that $\operatorname{Ad}(g(t))u(t)$ is conserved, and $0 = \partial_t \operatorname{Ad}(g(t))u(t)$ leads quickly to the geodesic equation. It is remarkable that the momentum mapping exists if and only if $(\operatorname{Diff}_{\mathcal{S}}(N), \langle\,\cdot\,,\,\cdot\,\rangle_L)$ is a robust weak Riemannian manifold.

§ 4. The differentiable Chow manifold (alias the non-linear Grassmannian)

4.1. The differentiable Chow manifold as a homogeneous space for $\mathrm{Diff}_{\mathcal{S}}(N)$ and the induced weak Riemannian metric. Let M be a compact manifold with $\dim(M) < \dim(N)$. The space of submanifolds of N diffeomorphic to M will be called $B(M,N)$. In the case when $m = 0$ and $N = \mathbb{R}^D$ (that is, M is a finite set of, say p, points in Euclidean D-space), the space $B(M,N)$ is what we called the space $\mathcal{L}^p(\mathbb{R}^D)$ of landmark points in our earlier paper [4].

$B(M,N)$ can be viewed as a quotient of $\mathrm{Diff}_{\mathcal{S}}(N)$. If we fix a base submanifold $F_0 \subset N$ diffeomorphic to M, then we get a map of $\mathrm{Diff}_{\mathcal{S}}(N)$ into $B(M,N)$ given by $\varphi \mapsto \varphi(F_0)$. The image will be an open subset $B_0(M,N)$ of $B(M,N)$, which is the quotient of $\mathrm{Diff}_{\mathcal{S}}(N)$ by the subgroup of diffeomorphisms which map F_0 to itself. We will study $B(M,N)$ using this approach and without further comment replace the full space $B(M,N)$ by this component $B_0(M,N)$.

The normal bundle to $F \subset N$ may be defined as $TB^{\perp} \subset TN|_B$, using an auxiliary Riemann metric on N. But we want to avoid this auxiliary metric, so we shall define the normal bundle as the quotient $\mathrm{Nor}(F) := TN|_F/TF$ over F. Then its dual bundle, the conormal bundle, is $\mathrm{Nor}^*(F) = \mathrm{Annihilator}(TF) \subset T^*N|_F$, a subbundle not a quotient. The tangent space $T_F B(M,N)$ to $B(M,N)$ at F can be identified with the space of all smooth sections $\Gamma_{\mathcal{S}}(\mathrm{Nor}(F))$ of the normal bundle.

A simple way to construct local coordinates on $B(M,N)$ near a point $F \in B(M,N)$ is to trivialize a neighbourhood of $F \subset N$. To be precise, assume we have a tubular neighbourhood, that is, an isomorphism Φ:

$$\begin{array}{ccc} B(M,N) & & \mathrm{Nor}(F) \\ \cup & & \cup \\ U_B & \xrightarrow{\ \Phi\ } & U_N \\ \cup & & \cup \\ F & = & \text{0-section} \end{array}$$

from an open neighbourhood U_B of F in N to an open neighbourhood U_N of the 0-section in the normal bundle $\mathrm{Nor}(F)$. Assume moreover that Φ is the identity on F and its normal derivative along F induces the identity map on $\mathrm{Nor}(F)$. The map Φ induces a local projection $\pi\colon U_B \to F$ and a partial linear structure in the fibres of this projection. Then we get an open set $U_\Phi \subset B(M,N)$ consisting of submanifolds $F' \subset U_B$ which intersect the fibres of π normally at exactly one point. Under Φ, these submanifolds are all given by smooth sections of $\mathrm{Nor}(F)$ which lie in U_N. If we call this set of sections U_Γ, we have a chart

$$B(M,N) \supset U_\Phi \cong U_\Gamma \subset \Gamma_{\mathcal{S}}(\mathrm{Nor}(F)).$$

We define a Riemannian metric on $B(M,N)$ following the procedure used for $\mathrm{Diff}_{\mathcal{S}}(N)$. For any $F \subset N$, we decompose $\mathcal{H}$ into

$$\begin{aligned} \mathcal{H}_F^{\mathrm{vert}} &= j_2^{-1}\big(\{X \in \Gamma_{C_b^2}(TN)\colon X(x) \in T_xF, \text{ for all } x \in F\}\big), \\ \mathcal{H}_F^{\mathrm{hor}} &= \text{perpendicular complement of } \mathcal{H}_F^{\mathrm{vert}}. \end{aligned}$$

It is then easy to check that we get the diagram

$$
\begin{array}{ccccc}
\Gamma_{\mathcal{S}}(TN) & \xhookrightarrow{j_1} & \mathcal{H} & \xhookrightarrow{j_2} & \Gamma_{C^2_b}(TN) \\
\downarrow{\scriptstyle \text{res}} & & \downarrow & & \downarrow{\scriptstyle \text{res}} \\
\Gamma_{\mathcal{S}}(\operatorname{Nor}(F)) & \xhookrightarrow{j_1^f} & \mathcal{H}_F^{\text{hor}} & \xhookrightarrow{j_2^f} & \Gamma_{C^2_b}(\operatorname{Nor}(F))
\end{array}
$$

Since this is an orthogonal decomposition, L and K take $\mathcal{H}_F^{\text{vert}}$ and $\mathcal{H}_F^{\text{hor}}$ into their own duals and, as before, we get

$$
\begin{array}{ccc}
\Gamma_{\mathcal{S}}(\operatorname{Nor}(F)) & & \Gamma_{\mathcal{S}'}(\operatorname{Nor}^*(F)) \\
\big\downarrow{\scriptstyle j_1} & & \big\uparrow{\scriptstyle j_1'} \\
\mathcal{H}_F^{\text{hor}} & \underset{K_F}{\overset{L_F}{\rightleftarrows}} & (\mathcal{H}_F^{\text{hor}})' \\
\big\downarrow{\scriptstyle j_2} & & \big\uparrow{\scriptstyle j_2'} \\
\Gamma_{C^2_b}(\operatorname{Nor}(F)) & & \Gamma_{M^2}(\operatorname{Nor}^*(F))
\end{array}
$$

K_F is just the restriction of K to this subspace of $\mathcal{H}'$ and is given by the kernel:

$$K_F(x_1, x_2) := \text{image of } K(x_1, x_2) \in \operatorname{Nor}_{x_1}(F) \otimes \operatorname{Nor}_{x_2}(F)), \qquad x_1, x_2 \in F.$$

This is a C^2-section over $F \times F$ of $\operatorname{pr}_1^* \operatorname{Nor}(F) \otimes \operatorname{pr}_2^* \operatorname{Nor}(F)$. We can identify the space of horizontal vector fields $\mathcal{H}_F^{\text{hor}}$ as the closure of the image under K_F of measure-valued 1-forms supported by F and with values in $\operatorname{Nor}^*(F)$. A dense set of elements in $\mathcal{H}_F^{\text{hor}}$ is given by either taking the 1-forms with finite support or taking smooth 1-forms. In the first approach, $\mathcal{H}_F^{\text{hor}}$ is the closure of the span of the vector fields $\big|K_F(\,\cdot\,, x)\big|\alpha_x\big\rangle$, where $x \in F$ and $\alpha_x \in \operatorname{Nor}_x^*(F)$. In the smooth case, fix a volume form κ on M and a smooth covector $\xi \in \Gamma_{\mathcal{S}}(\operatorname{Nor}^*(F))$. Then $\xi.\kappa$ defines a horizontal vector field h as follows:

$$h(x_1) = \int_{x_2 \in F} \big|K_F(x_1, x_2)\big|\xi(x_2).\kappa(x_2)\big\rangle.$$

The horizontal lift h^{hor} of any $h \in T_F B(M, N)$ is then

$$h^{\text{hor}}(y_1) = K(L_F h)(y_1) = \int_{x_2 \in F} \big|K(y_1, x_2)\big|L_F h(x_2)\big\rangle, \qquad y_1 \in N.$$

Note that all elements of the cotangent space $\alpha \in \Gamma_{\mathcal{S}'}(\operatorname{Nor}^*(F))$ can be pushed up to N by $(j_F)_*$, where $j_F \colon F \hookrightarrow N$ is the inclusion, and this identifies $(j_F)_*\alpha$ with a 1-co-current on N.

Finally, the induced homogeneous weak Riemannian metric on $B(M, N)$ is given as follows:

$$\langle h, k\rangle_F = \int_N (h^{\text{hor}}(y_1), L(k^{\text{hor}})(y_1)) = \int_{y_1 \in N} \big(K(L_F h)(y_1), (L_F k)(y_1)\big)$$

$$= \int_{(y_1,y_2)\in N\times N} (K(y_1,y_2),(L_F h)(y_1)\otimes(L_F k)(y_2))$$
$$= \int_{(x_1,x_1)\in F\times F} \big\langle L_F h(x_1)\big|K_F(x_1,x_2)\big|L_F h(x_2)\big\rangle.$$

With this metric, the projection from $\mathrm{Diff}_{\mathcal{S}}(N)$ to $B(M,N)$ is a submersion. The inverse co-metric on the smooth cotangent bundle $\bigsqcup_{F\in B(M,N)}\Gamma(\mathrm{Nor}^*(F)\otimes \mathrm{vol}(F))\subset T^*B(M,N)$ is much simpler and easier to handle:

$$\langle\alpha,\beta\rangle_F = \iint_{F\times F}\big\langle\alpha(x_1)\big|K_F(x_1,x_2)\big|\beta(x_1)\big\rangle.$$

It is simply the restriction to the co-metric on the Hilbert subbundle of $T^*\,\mathrm{Diff}_{\mathcal{S}}(N)$ defined by $\mathcal{H}'$ to the Hilbert subbundle of the subspace $T^*B(M,N)$ defined by $\mathcal{H}'_F$.

Because they are related by a submersion, the geodesics on $B(M,N)$ are the horizontal geodesics on $\mathrm{Diff}_{\mathcal{S}}(N)$, as described in the last displayed set of formulae in § 3.4. We have two variables: a family $\{F(t)\}$ of submanifolds in $B(M,N)$ and a time-varying momentum $\alpha(t,\,\cdot\,)\in\mathrm{Nor}^*(F(t))\otimes\mathrm{vol}(F(t))$ which lifts to the horizontal 1-co-current $(j_{F(t)})_*(\alpha(t,\,\cdot\,))$ on N. Then the horizontal geodesic on $\mathrm{Diff}_{\mathcal{S}}(N)$ is given by the same equations as before:

$$\partial_t(F(t)) = \mathrm{res}_{\mathrm{Nor}(F(t))}(u(t,\,\cdot\,)),$$
$$u(t,x) = \int_{(F(t))_y}\big|K(x,y)\big|\alpha(t,y)\big\rangle\in\mathfrak{X}_{\mathcal{S}}(N),$$
$$\partial_t\big((j_{F(t)})_*(\alpha(t,\,\cdot\,))\big) = -\mathcal{L}_{u(t,\,\cdot\,)}((j_{F(t)})_*(\alpha(t,\,\cdot\,))).$$

This is a complete description for geodesics on $B(M,N)$, but it is not very clear how to compute the Lie derivative of $(j_{F(t)})_*(\alpha(t,\,\cdot\,))$. One can unwind this Lie derivative via a torsion-free connection, but we turn to a different approach which will be essential for working out the curvature of $B(M,N)$.

4.2. Auxiliary tensors on $B(M,N)$**.** Our goal is to reduce calculations on the infinite-dimensional space $B(M,N)$ to calculations on the finite-dimensional space N. To do this, we construct a number of useful tensors on $B(M,N)$ from tensors on N and compute the standard operations on them. These will enable us to get control of the geometry of $B(M,N)$. Let m be the dimension of M and n the dimension of N. For $F\in B(M,N)$, let $j_F\colon F\hookrightarrow N$ be the embedding. We will assume for simplicity that M is orientable, so that $\mathrm{vol}(M)\cong\Omega^m(M)$.

1) We denote by ℓ the left action

$$\ell\colon\ \mathrm{Diff}_{\mathcal{S}}(N)\times B(M,N)\to B(M,N)$$

given by $\ell(\varphi,F)$ or $\ell^F(\varphi)=\varphi(F)$. For a vector field $X\in\mathfrak{X}_{\mathcal{S}}(N)$, let B_X be the infinitesimal action (or fundamental vector field) on $B(M,N)$ given by $B_X(F) = T_{\mathrm{Id}}(\ell^F)X$ with its flow $\mathrm{Fl}^{B_X}_t(F)=\mathrm{Fl}^X_t(F)$. The fundamental vector field mapping of a left action is a Lie algebra anti-homomorphism, and the Lie bracket on $\mathrm{Diff}_{\mathcal{S}}(N)$ is the negative of the usual Lie bracket on $\mathfrak{X}_{\mathcal{S}}(N)$, so we have $[B_X,B_Y]=B_{[X,Y]}$. The set of these vectors $\{B_X(F)\colon X\in\mathfrak{X}_{\mathcal{S}}(N)\}$ equals the whole tangent space $T_FB(M,N)$.

2) Note that $B(M,N)$ is naturally a submanifold of the vector space of m-currents on N:

$$B(M,N) \hookrightarrow \Omega^m_{\mathcal{S}}(N)' = \Gamma_{\mathcal{S}'}(\Lambda^m TN), \qquad \text{via } F \mapsto \Big(\omega \mapsto \int_F \omega\Big).$$

Any $\alpha \in \Omega^m(N)$ is a linear coordinate on $\Gamma_{\mathcal{S}'}(\Lambda^m TM)$ and this restricts to the function $B_\alpha \in C^\infty(B(M,N),\mathbb{R})$ given by $B_\alpha(F) = \int_F \alpha$. If $\alpha = d\beta$ for $\beta \in \Omega^{m-1}(N)$, then

$$B_\alpha(F) = B_{d\beta}(F) = \int_F j_F^* d\beta = \int_F dj_F^*\beta = 0$$

by Stokes' theorem.

For $\alpha \in \Omega^m(N)$ and $X \in \mathfrak{X}_{\mathcal{S}}(N)$ we can evaluate the vector field B_X on the function B_α:

$$\begin{aligned} B_X(B_\alpha)(F) = dB_\alpha(B_X)(F) = \partial_t|_0 B_\alpha(\mathrm{Fl}^X_t(F)) &= \int_F j_F^* \mathcal{L}_X \alpha = B_{\mathcal{L}_X(\alpha)}(F) \\ \text{as well as } = \int_F j_F^*(i_X d\alpha + d i_X \alpha) &= \int_F j_F^* i_X d\alpha = B_{i_X(d\alpha)}(F). \end{aligned}$$

If $X \in \mathfrak{X}_{\mathcal{S}}(N)$ is tangent to F along F, then $B_X(B_\alpha)(F) = \int_F \mathcal{L}_{X|F} j_F^* \alpha = 0$.

More generally, a pm-form α on N^k determines a function $B^{(p)}_\alpha$ on $B(M,N)$ by the formula $B^{(p)}_\alpha(F) = \int_{F^p} \alpha$. Using this for $p = 2$, we find that for any m-forms α, β on N, the inner product of B_α and B_β is given by

$$g_B^{-1}(B_\alpha, B_\beta) = B^{(2)}_{\langle \alpha | K | \beta \rangle}.$$

3) For $\alpha \in \Omega^{m+k}(N)$ we denote by B_α the k-form in $\Omega^k(B(M,N))$ given by the skew-symmetric multi-linear form

$$(B_\alpha)_F(B_{X_1}(F), \dots, B_{X_k}(F)) = \int_F {j_F}^*(i_{X_1 \wedge \dots \wedge X_k} \alpha).$$

This is well defined: if one of the X_i is tangential to F at a point $x \in F$, then ${j_F}^*$ pulls back the resulting m-form to 0 at x.

Note that any smooth cotangent vector a to $F \in B(M,N)$ is equal to $B_\alpha(F)$ for some closed $(m+1)$-form α. Smooth cotangent vectors at F are elements of $\Gamma_{\mathcal{S}}(F, \mathrm{Nor}^*(F) \otimes \Omega^m(F))$. Fix a nowhere-zero global section κ of $\Omega^m(F)$. Then $\frac{a}{\kappa}$ is the differential of a unique function f on the normal bundle to F which is linear on each fibre. Let φ be a local isomorphism from a neighbourhood of F in N to a neighbourhood of the 0-section in this normal bundle, and let ρ be a function on the normal bundle which is equal to one near the 0-section and has support in this neighbourhood. Take $\alpha = d(f.\kappa \circ \varphi)$ (extended by zero). It is easy to see that this does the trick.

Likewise, a $(pm+k)$-form $\alpha \in \Omega^{pm+k}(N^p)$ determines a k-form on $B(M,N)$ as follows. First, for $X \in \mathfrak{X}_{\mathcal{S}}(N)$ let $X^{(p)} \in \mathfrak{X}(N^p)$ be given by

$$\begin{aligned} X^{(p)}_{(n_1,\dots,n_p)} := (X_{n_1} \times 0_{n_2} \times \dots \times 0_{n_p}) + (0_{n_1} \times X_{n_2} \times 0_{n_3} \times \dots \times 0_{n_p}) + \cdots \\ \cdots + (0_{n_1} \times \dots \times 0_{n_{p-1}} \times X_{n_p}). \end{aligned}$$

Then we put

$$(B^{(p)}_\alpha)_F(B_{X_1}(F),\dots,B_{X_k}(F))=\int_{F^p} j_{F^p}{}^*(i_{X_1^{(p)}\wedge\dots\wedge X_k^{(p)}}\alpha).$$

This is just B applied to the submanifold $F^p\subset N^p$ and to the special vector fields $X^{(p)}$. Thus all properties of B continue to hold for $B^{(p)}$; in particular, item 4) below holds with $X^{(p)}$ in place of X.

4) We have $i_{B_X}B_\alpha=B_{i_X\alpha}$ because

$$\begin{aligned}\bigl(i_{B_{X_1}}B_\alpha\bigr)\bigl(B_{X_2},\dots,B_{X_k}\bigr)(F)&=B_\alpha\bigl(B_{X_1},B_{X_2},\dots,B_{X_k}\bigr)(f)\\&=\int_F j_F{}^*\bigl(i_{X_k}\dots i_{X_2}(i_{X_1}\alpha)\bigr)=B_{i_{X_1}\alpha}\bigl(B_{X_2},\dots,B_{X_k}\bigr)(F).\end{aligned}$$

For the exterior derivative we have $dB_\alpha=B_{d\alpha}$ for any $\alpha\in\Omega^{m+k}(N)$. Namely,

$$\begin{aligned}(dB_\alpha)(B_{X_0},\dots,B_{X_k})(F)&=\sum_{i=0}^k(-1)^iB_{X_i}(B_\alpha(B_{X_0},\dots,\widehat{B_{X_i}},\dots,B_{X_k}))(F)\\&\qquad+\sum_{i<j}(-1)^{i+j}B_\alpha(B_{[X_i,X_j]},B_{X_0},\dots,\widehat{B_{X_i}},\dots,\widehat{B_{X_j}},\dots,B_{X_k})(F)\\&=\sum_{i=0}^k(-1)^i\int_F j_F^*i_{X_i}di_{X_0\wedge\dots\wedge\widehat{X_i}\wedge\dots\wedge X_k}\alpha\\&\qquad+\sum_{i<j}(-1)^{i+j}\int_F j_F^*i_{[X_i,X_j]\wedge X_0\wedge\dots\widehat{X_i}\dots\widehat{X_j}\dots\wedge X_k}\alpha\\&=\int_F j_F{}^*\biggl(\sum_{i=0}^k(-1)^i\mathcal{L}_{X_i}i_{X_k}\cdots\widehat{i_{X_i}}\cdots i_{X_0}\\&\qquad-\sum_{i<j}(-1)^ii_{X_0\wedge\dots\wedge\widehat{X_i}\wedge\dots\wedge X_{j-1}\wedge[X_i,X_j]\wedge X_{j+1}\wedge\dots\wedge X_k}\biggr)\alpha\\&=\int_F j_F{}^*\sum_{i=0}^k(-1)^i\biggl(\mathcal{L}_{X_i}i_{X_k}\cdots\widehat{i_{X_i}}\cdots i_{X_0}\\&\qquad-\sum_{j=i+1}^k i_{X_k}\cdots i_{X_{j+1}}[\mathcal{L}_{X_i},i_{X_j}]i_{X_{j-1}}\cdots\widehat{i_{X_i}}\cdots i_{X_0}\biggr)\alpha\\&=\int_F j_F{}^*\biggl(\sum_{i=0}^k(-1)^ii_{X_k}\cdots i_{X_{i+1}}\mathcal{L}_{X_i}i_{X_{i-1}}\cdots i_{X_0}\alpha\biggr)\\&=\int_F j_F{}^*\biggl(\sum_{i=0}^k(-1)^ii_{X_k}\cdots i_{X_{i+1}}(d\,i_{X_i}+i_{X_i}d)i_{X_{i-1}}\cdots i_{X_0}\alpha\biggr)\\&=\int_F j_F{}^*\biggl(\sum_{i=0}^k(-1)^ii_{X_k}\cdots i_{X_{i+1}}d\,i_{X_i}\cdots i_{X_0}\\&\qquad+\sum_{i=0}^k(-1)^ii_{X_k}\cdots i_{X_i}di_{X_{i-1}}\cdots i_{X_0}\biggr)\alpha\\&=0+\int_F j_F{}^*i_{X_k}\cdots i_{X_0}d\alpha=B_{d\alpha}(B_{X_0},\dots,B_{X_k})(F).\end{aligned}$$

Finally we have $\mathcal{L}_{B_X} B_\alpha = B_{\mathcal{L}_X \alpha}$ since

$$\mathcal{L}_{B_X} B_\alpha = (i_{B_X}\, d + d\, i_{B_X}) B_\alpha = B_{(i_X d + d i_X)\alpha} = B_{\mathcal{L}_X \alpha}.$$

Note that these identities generalize the results in item 2).

5) For $\alpha \in \Omega^{m+1}(N)$ we pull back to $\operatorname{Diff}_{\mathcal{S}}(N)$ the 1-form B_α on $B(M,N)$ where $\varphi_0(F_0) = F$:

$$\begin{gathered}
\big((\ell^{F_0})^* B_\alpha\big)_{\varphi_0}(X \circ \varphi_0) = \big((\ell^F)^* B_\alpha\big)_{\mathrm{Id}}(X) = (B_\alpha)_F(B_X(F)) = \int_F j_F^* i_X\, \alpha, \\
\big((\ell^F)^* B_\alpha\big)_{\mathrm{Id}} = \alpha|_F =: \mu(\alpha, F) = \mu_\alpha(F) = \mu^F(\alpha) \in \mathfrak{X}_{\mathcal{S}}(N)', \\
\mu\colon \Omega^{m+1}(N) \times B(M,N) \to \mathfrak{X}_{\mathcal{S}}(N)', \\
\mu(\alpha, F) \text{ is a 1-co-current with support along } F.
\end{gathered}$$

The mapping $\mu\colon \Omega^{m+1}(N) \times B(M,N) \to \mathfrak{X}_c(N)'$ is smooth, $\mu^F\colon \Omega^{m+1}(N) \to \mathfrak{X}_c(N)'$ is bounded linear, and the differential of $\mu_\alpha\colon B(M,N) \to \mathfrak{X}_{\mathcal{S}}(N)'$ is computed as follows:

$$\begin{aligned}
\langle d(\mu_\alpha)(B_X(F)), Y\rangle &= \langle D_{F,B_X} \mu(\alpha, F), Y\rangle = D_{F,B_X} \langle \mu(\alpha, F), Y\rangle = \partial_t|_0 \langle \alpha_{\mathrm{Fl}^X_t(F)}, Y\rangle \\
&= \partial_t|_0 \int_{\mathrm{Fl}^X_t(F)} j_{\mathrm{Fl}^X_t(F)}{}^* i_Y \alpha = \partial_t|_0 \int_{\mathrm{Fl}^X_t(F)} (\mathrm{Fl}^X_t \circ j_F \circ (\mathrm{Fl}^X_t|_F)^{-1})^* i_Y \alpha \\
&= \partial_t|_0 \int_{\mathrm{Fl}^X_t(F)} (\mathrm{Fl}^X_t|_F)^{-1})^* j_F{}^* (\mathrm{Fl}^X_t)^* i_Y \alpha \\
&= \partial_t|_0 \int_F j_F{}^* (\mathrm{Fl}^X_t)^* i_Y \alpha = \int_F j_F{}^* \mathcal{L}_X (i_Y \alpha) \\
&= \int_F j_F{}^* (i_{[X,Y]} \alpha + i_Y \mathcal{L}_X \alpha) = \langle \mu(\alpha, F), \mathcal{L}_X Y\rangle + \langle \mu(\mathcal{L}_X \alpha, F), Y\rangle.
\end{aligned}$$

This means that

$$d\mu_\alpha(B_X(F)) = \mu(\alpha, F) \circ \mathcal{L}_X + \mu(\mathcal{L}_X \alpha, F) = -\mathcal{L}_X \mu(\alpha, F) + \mu(\mathcal{L}_X \alpha, F), \tag{14}$$

where $\mathcal{L}_X \mu(\alpha, F)$ denotes the Lie derivative of 1-currents. There are two interpretations of formula (14):

$$d\mu_\alpha(B_X) = -\mathcal{L}_X \circ \mu_\alpha + \mu_{\mathcal{L}_X \alpha}, \qquad d\mu_\alpha(B_X(F)) = -(\mathcal{L}_X \mu^F)(\alpha).$$

We shall also need the mapping $\mu\colon \Omega^m(N) \times B(M,N) \to C^\infty_c(N)'$ with values in the linear space of distributions (without the density part) on N which is given by

$$\langle \mu(\gamma, F), f\rangle = \int_F f.\gamma = \int_F j_F{}^*(g\gamma).$$

The distribution $\mu(\gamma, F)$ is again bounded linear in $\gamma \in \Omega^m(N)$, and its derivative with respect to F is again given by (14), with the same proof as above.

§ 5. Geodesics and curvature on $B(M, N)$

We want to use the auxiliary tensors of the last section to derive formulae for geodesics and curvature on $B(M)$, using Mario's formula to compute the curvature. The basic idea is to write a smooth covector a at a point $F \in B(M, N)$ as B_α, where α is an $(m+1)$-form on N. As always, for any $(m+1)$-form α on N, $B^\sharp_\alpha$ is the (C^2) vector field on $B(M, N)$ which is dual to the smooth 1-form B_α. At each point $F \in B$, $B^\sharp_\alpha$ lifts horizontally to a tangent vector at the identity to $\operatorname{Diff}_{\mathcal{S}}(N)$, which is given by the vector field

$$\mu(\alpha, F)^\sharp = \int_N |K|\mu(\alpha, F)\rangle \in \mathfrak{X}_{C^2}(N),$$

so that $B_{\mu(\alpha,F)^\sharp}(F) = B^\sharp_\alpha(F)$; see item 5) in § 4.2.

With these covectors, we consider next the force introduced in § 2.3. We have

$$2F(\alpha, \beta) = d(\langle B_\alpha, B_\beta\rangle) = d\big(B^{(2)}_{\langle\alpha|K|\beta\rangle}\big) = B^{(2)}_{d(\langle\alpha|K|\beta\rangle)}.$$

But $\langle\alpha|K|\beta\rangle$ is a $2m$-form on $N \times N$ and d can be split into two parts, $d_1 + d_2$, acting on the first and second factors. Evaluating this 1-form at F and taking its inner product with $B_X, X \in \mathfrak{X}_{\mathcal{S}}(N)$, we get

$$\Big(B^{(2)}_{d(\langle\alpha|K|\beta\rangle)}(F), B_X(F)\Big) = \iint_{F\times F} j_{F\times F}{}^* i_{X^{(2)}}(d(\langle\alpha|K|\beta\rangle))$$
$$= \iint_{F\times F} j_{F\times F}{}^*\big((i_X)_1(d_1(\langle\alpha|K|\beta\rangle)) + (i_X)_2(d_2(\langle\alpha|K|\beta\rangle))\big)$$

because $F \times F$ has type (m, m) and the integrand must have the same type,

$$= \int_F j_F^* i_X d\big(i_{\mu(\beta,F)^\sharp}(\alpha) + i_{\mu(\alpha,F)^\sharp}(\beta)\big),$$

whence

$$2F(\alpha, \beta) = B^{(2)}_{d(\langle\alpha|K|\beta\rangle)} = B_\gamma, \qquad \gamma = \mathcal{L}_{\mu(\beta,F)^\sharp}(\alpha) + \mathcal{L}_{\mu(\alpha,F)^\sharp}(\beta).$$

Here the superscript 2 on X means that $X^{(2)}$ is the vector field on $N \times N$ given by $0 \times X + X \times 0$, whereas on B, because $d(\langle\alpha|K|\beta\rangle)$ is a $(2m+1)$-form on $N \times N$, we must apply $B^{(2)}$, not B, to it. Thus we define the force F using operations on the finite-dimensional manifold N by putting

$$F_N(\alpha, \beta, F) := \big(\text{image in } \operatorname{Nor}^*(F) \otimes \operatorname{vol}(F)\big)\bigg(\frac{1}{2}(\mathcal{L}_{\mu(\beta,F)^\sharp}(\alpha) + \mathcal{L}_{\mu(\alpha,F)^\sharp}(\beta))\bigg).$$

The term 'force' comes from the fact that the geodesic acceleration is given by $F(\alpha, \alpha)$. In our case, we find that the geodesic equation on $B(M, N)$ can be extended to an equation in the variables $F(t) \in B(M, N)$ and $\alpha(t, \cdot)$, a time-varying $(m+1)$-form on N:

$$\partial_t(F(t)) = (\text{res to } \operatorname{Nor}(F))u,$$
$$u = \mu(\alpha, F)^\sharp = \int_{F_t(y)} |K(\,\cdot\,, y)|\alpha(y)\rangle,$$
$$\partial_t(\alpha) = F(\alpha, \alpha, F) = \mathcal{L}_u(\alpha).$$

Moving to curvature, fix F. Then we claim that for any two smooth covectors a, b at F, we can construct not only two closed $(m+1)$-forms α, β on N as above, but also two commuting vector fields X_α, X_β on N in a neighbourhood of F such that:

1) $B_\alpha(F) = a$ and $B_\beta(F) = b$;
2) $B_{X_\alpha}(F) = a^\sharp$ and $B_{X_\beta}(F) = b^\sharp$;
3) $\mathcal{L}_{X_\alpha}(\alpha) = \mathcal{L}_{X_\alpha}(\beta) = \mathcal{L}_{X_\beta}(\alpha) = \mathcal{L}_{X_\beta}(\beta) = 0$;
4) $[X_\alpha, X_\beta] = 0$.

We can do this using a local isomorphism of N with the normal bundle to F in N as above. This gives a projection π of a neighbourhood of F in N to F and a partial linear structure on its fibres. Then for α and β use $(m+1)$-forms $\kappa\wedge\omega$, where κ is a pullback of an m-form on F and ω is a 1-form constant along the fibres; and for X_α and X_β use vector fields which are tangent to the fibres of π and constant with respect to the linear structure on them.

We are now in a position to use the version of Mario's formula given in § 2.3. As it stands, this formula calculates curvature using operations on $B(M,N)$. What we want to do is to write everything using forms and fields on N instead. We first need an expression for the stress $D(\alpha,\beta)$ in this formula. Using the notation in item 2) of § 2.3, we have

$$D(\alpha,\beta,F) = D_{F,B_{X_\alpha}(F)}(B^\sharp_\beta - B_{X_\beta}) = [B_{X_\alpha}, B^\sharp_\beta - B_{X_\beta}](F) = [B_{X_\alpha}, B^\sharp_\beta](F).$$

In order to compute the Lie bracket, we apply it to a smooth function B_γ on $B(M,N)$, where $\gamma\in\Omega^m(N)$. Then we have, using § 4.2 repeatedly,

$$(\mathcal{L}_{B^\sharp_\beta} B_\gamma)(F) = (\mathcal{L}_{B_{\mu(be,F)^\sharp}} B_\gamma)(F) = B_{\mathcal{L}_{\mu(be,F)^\sharp}\gamma}(F),$$

$$\begin{aligned}(\mathcal{L}_{B_{X_\alpha}}\mathcal{L}_{B^\sharp_\beta} B_\gamma)(F) &= (\mathcal{L}_{B_{X_\alpha}} B_{\mathcal{L}_{\mu(\beta,F)^\sharp}\gamma})(F)\\ &= B(\mathcal{L}_{D_{F,B_{X_\alpha}}\mu(\beta,F)^\sharp}\gamma)(F) + B(\mathcal{L}_{X_\alpha}\mathcal{L}_{\mu(\beta,F)^\sharp}\gamma)(F),\end{aligned}$$

$$(\mathcal{L}_{B^\sharp_\beta}\mathcal{L}_{B_{X_\alpha}} B_\gamma)(F) = (\mathcal{L}_{B_{\mu(\beta,F)^\sharp}} B_{\mathcal{L}_{X_\alpha}\gamma})(F) = B(\mathcal{L}_{\mu(\beta,F)^\sharp}\mathcal{L}_{X_\alpha}\gamma)(F),$$

$$\begin{aligned}D_{F,B_{X_\alpha}}\mu(\beta,F)^\sharp &= D_{F,B_{X_\alpha}}\int_N |K|\mu(\beta,F)\rangle = \int_N |K| D_{F,B_{X_\alpha}}\mu(\beta,F)\rangle\\ &= \int_N \big|K\big|(-\mathcal{L}_{X_\alpha}\mu(\beta,F) + \mu(\mathcal{L}_{X_\alpha}\beta,F))\big\rangle \quad \text{by (14)}\\ &= \int_N \big|\mathcal{L}_{0\times X_\alpha}K\big|\mu(\beta,F)\big\rangle + \mu(\mathcal{L}_{X_\alpha}\beta,F)^\sharp,\end{aligned}$$

$$\begin{aligned}([B_{X_\alpha}, B_{\mu(be,F)^\sharp}]B_\gamma)(F) &= (\mathcal{L}_{B^\sharp_\beta}B_\gamma - \mathcal{L}_{B^\sharp_\beta}\mathcal{L}_{B_{X_\alpha}}B_\gamma)(F)\\ &= B(\mathcal{L}_{D_{F,B_{X_\alpha}}\mu(\beta,F)^\sharp}\gamma)(F) + B(\mathcal{L}_{[X_\alpha,\mu(\beta,F)^\sharp]}\gamma)(F)\\ &= (\mathcal{L}_{B(D_{F,B_{X_\alpha}}\mu(\beta,F)^\sharp + [X_\alpha,\mu(\beta,F)^\sharp])}B_\gamma)(F),\end{aligned}$$

$$\begin{aligned}[B_{X_\alpha}, B_{\mu(be,F)^\sharp}](F) &= B(D_{F,B_{X_\alpha}}\mu(\beta,F)^\sharp + \mathcal{L}_{X_\alpha}\mu(\beta,F)^\sharp)\\ &= B\Big(\int_N \big|\mathcal{L}_{0\times X_\alpha}K\big|\mu(\beta,F)\big\rangle + \mu(\mathcal{L}_{X_\alpha}\beta,F)^\sharp + \int_N \big|\mathcal{L}_{X_\alpha\times 0}K\big|\mu(\beta,F)\big\rangle\Big)\\ &= B\Big(\int_N \big|\mathcal{L}_{X_\alpha^{(2)}}K\big|\mu(\beta,F)\big\rangle\Big) + 0.\end{aligned}$$

Thus we define the stress on N by putting

$$D(\alpha,\beta,F)(x) = \big(\text{restr. to Nor}(F)\big)\Big(-\int_{y\in F}\big|\mathcal{L}_{X_\alpha^{(2)}}(x,y)K(x,y)\big|\beta(y)\rangle\Big).$$

Next consider the second derivative terms in R_{11}. A typical term works out as follows:

$$\begin{aligned} B_{X_\alpha}B_{X_\alpha}(\langle B_\beta,B_\beta\rangle) &= \mathcal{L}_{B_{X_\alpha}}\mathcal{L}_{B_{X_\alpha}}(\langle B_\beta,B_\beta\rangle) = B_{\mathcal{L}_{X_\alpha^{(2)}}\mathcal{L}_{X_\alpha^{(2)}}\langle\beta|K|\beta\rangle} \\ &= B_{\langle\beta|\mathcal{L}_{B_{X_\alpha^2}}\mathcal{L}_{X_\alpha^{(2)}}K|\beta\rangle}. \end{aligned}$$

Slightly extending the Lie bracket notation, we can write

$$\langle\beta|\mathcal{L}_{X_\alpha^{(2)}}\mathcal{L}_{X_\alpha^{(2)}}K|\beta\rangle = \big\langle\beta|[X_\alpha^{(2)},[X_\alpha^{(2)},K]]|\beta\big\rangle.$$

Analogous formulae hold for the other terms.

Finally, putting everything together, we find the formula for curvature:

$$\begin{aligned} \langle R_{B(M,N)}(B_\alpha^\sharp,B_\beta^\sharp)B_\beta^\sharp,B_\alpha^\sharp\rangle(F) &= R_{11}+R_{12}+R_2+R_3, \\ R_{11} = \frac{1}{2}\iint_{F\times F}\Big(\big\langle\beta\big|\mathcal{L}_{X_\alpha^{(2)}}\mathcal{L}_{X_\alpha^{(2)}}K\big|\beta\big\rangle &+ \big\langle\alpha\big|\mathcal{L}_{X_\beta^{(2)}}\mathcal{L}_{X_\beta^{(2)}}K\big|\alpha\big\rangle \\ &- 2\big\langle\alpha\big|\mathcal{L}_{X_\alpha^{(2)}}\mathcal{L}_{X_\beta^{(2)}}K\big|\beta\big\rangle\Big), \\ R_{12} = \int_F\Big(\langle D(\alpha,\alpha,F),F(\beta,\beta,F)\rangle &+ \langle D(\beta,\beta,F),F(\alpha,\alpha,F)\rangle \\ &- \langle D(\alpha,\beta,F)+D(\beta,\alpha,F),F(\alpha,\beta,F)\rangle\Big), \\ R_2 &= \|F(\alpha,\beta,F)\|_{K_F}^2 - \big\langle F(\alpha,\alpha,F)),F(\beta,\beta,F)\big\rangle_{K_F}, \\ R_3 &= -\frac{3}{4}\|D_N(\alpha,\beta,F)-D_N(\beta,\alpha,F)\|_{L_F}^2. \end{aligned}$$

In the case of landmark points, where $m=0$, $N=\mathbb{R}^D$ and K is diagonal, it is easy to check that our force and stress and the above formula for curvature are exactly the same as those given in our earlier paper [4]. In that paper the individual terms are studied in special cases giving some intuition about them.

§ 6. Appendix on convenient calculus: calculus beyond Banach spaces

The traditional differential calculus works well for finite-dimensional vector spaces and for Banach spaces. For more general locally convex spaces we sketch here the convenient approach as explained in [15] and [10]. The main difficulty is that composition of linear mappings stops being jointly continuous at the level of Banach spaces, for any compatible topology. We use the notation of [10] and this is the main reference for the whole appendix.

6.1. Convenient vector spaces and the c^∞-topology. Let E be a locally convex vector space. A curve $c\colon \mathbb{R} \to E$ is said to be *smooth*, or C^∞, if all derivatives exist and are continuous; this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, but only on its associated bornology (system of bounded sets).

E is called a *convenient vector space* if one of the following equivalent conditions is satisfied (called c^∞-completeness):

1) for any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann) integral $\int_0^1 c(t)dt$ exists in E;

2) a curve $c\colon \mathbb{R} \to E$ is smooth if and only if $\lambda \circ c$ is smooth for all $\lambda \in E'$, where E' is the dual consisting of all continuous linear functionals on E;

3) any Mackey-Cauchy sequence (that is, $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E; this is visibly a weak completeness requirement.

The final topology with respect to all smooth curves is called the c^∞-topology on E and is denoted by $c^\infty E$. For Fréchet spaces it coincides with the given locally convex topology, but on the space $\mathcal{D}$ of test functions with compact support on $\mathbb{R}$ it is strictly finer.

6.2. Smooth mappings. Let E, F and G be convenient vector spaces, and let $U \subset E$ be c^∞-open. Here is the key definition that makes everything work: a mapping $f\colon U \to F$ is said to be *smooth*, or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$.

The main properties of smooth calculus are the following.

1) *For mappings on Fréchet spaces this notion of smoothness coincides with all other reasonable definitions. Even on* $\mathbb{R}^2$ *this is non-trivial.*

2) *Multilinear mappings are smooth if and only if they are bounded.*

3) *If* $f\colon E \supseteq U \to F$ *is smooth, then the derivative* $df\colon U \times E \to F$ *is smooth, and also* $df\colon U \to L(E, F)$ *is smooth, where* $L(E, F)$ *denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.*

4) *The chain rule holds.*

5) *The space* $C^\infty(U, F)$ *is again a convenient vector space, where the structure is given by the obvious injection*

$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R},U),\ell\in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \qquad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ *carries the topology of compact convergence in each derivative separately.*

6) *The exponential law holds: for* c^∞*-open* $V \subset F$,

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

is a linear diffeomorphism of convenient vector spaces. Note that this is the main assumption of variational calculus where a smooth curve in a space of functions is assumed to be just a smooth function in one variable more.

7) *A linear mapping* $f\colon E \to C^\infty(V, G)$ *is smooth (bounded) if and only if* $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ *is smooth for each* $v \in V$*. This is called the smooth uniform boundedness theorem* [10, § 5.26].

8) *The following canonical mappings are smooth*:

$$\begin{aligned} \mathrm{ev}\colon C^\infty(E,F)\times E\to F, &\qquad \mathrm{ev}(f,x)=f(x),\\ \mathrm{ins}\colon E\to C^\infty(F,E\times F), &\qquad \mathrm{ins}(x)(y)=(x,y),\\ (\quad)^\wedge\colon C^\infty(E,C^\infty(F,G)) &\to C^\infty(E\times F,G),\\ (\quad)^\vee\colon C^\infty(E\times F,G) &\to C^\infty(E,C^\infty(F,G)),\\ \mathrm{comp}\colon C^\infty(F,G)\times C^\infty(E,F) &\to C^\infty(E,G),\\ C^\infty(\quad,\quad)\colon C^\infty(F,F_1)\times C^\infty(E_1,E) &\to C^\infty(C^\infty(E,F),C^\infty(E_1,F_1)),\\ (f,g) &\mapsto (h\mapsto f\circ h\circ g),\\ \prod\colon \prod C^\infty(E_i,F_i) &\to C^\infty\Big(\prod E_i,\prod F_i\Big). \end{aligned}$$

Smooth mappings are always continuous in the c^∞-topology, but there are smooth mappings which are not continuous in the given topology of E. This is unavoidable and not so horrible as it might appear at first sight. For example, the evaluation $E\times E^*\to\mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth.

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M. Micheli
Université Paris Descartes, France
E-mail: mariomicheli@gmail.com

P. W. Michor
Fakultät für Mathematik, Universität Wien, Austria
E-mail: Peter.Michor@univie.ac.at

D. Mumford
Division of Applied Mathematics, Brown University,
Providence, RI, USA
E-mail: David_Mumford@brown.edu