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Stuff It!

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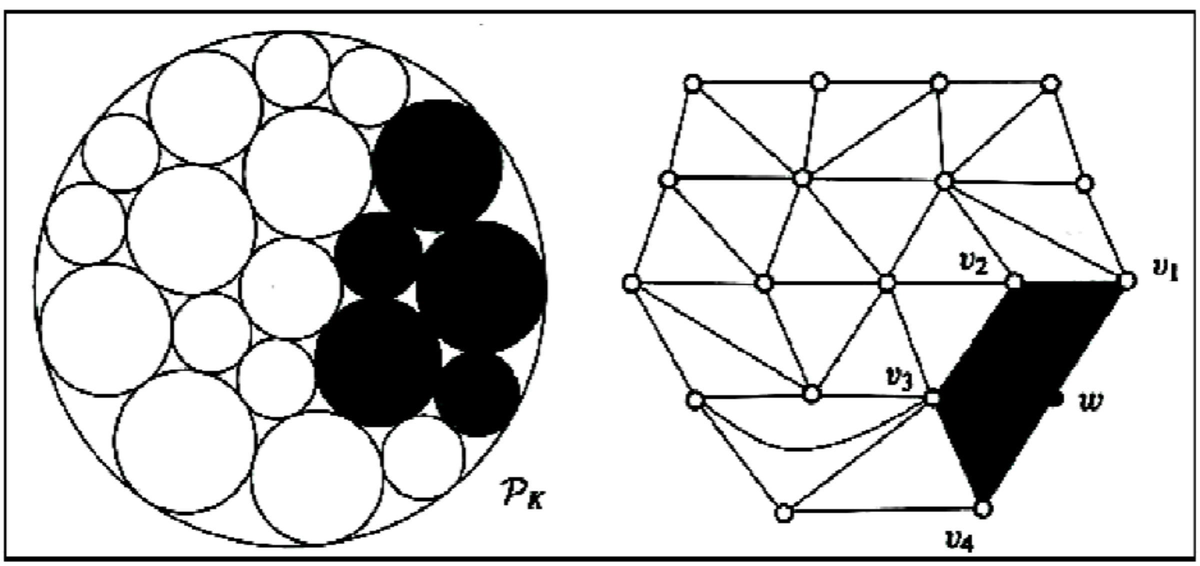
Introduction to Circle Packing: The Theory of Discrete Analytic Functions. Kenneth Stephenson. xii + 356 pp. Cambridge University Press, 2005. $60.

In the past, books about science and math were written in many different styles. For example, Galileo published his most famous books as dialogues between Simplicio (a rather dense traditional man who is a proponent of Ptolemaic/Aristotelian theory) and two more enlightened citizens—Sagredo (a modern Copernican) and Salviati (an unbiased representative of the educated public). Today, unfortunately, scientific exposition on the research level and especially mathematics writing have fallen into a trap—a rigid professional code dictates the structure and style in which new results are to be communicated. In mathematics, this tradition forces the author to write a strictly linear sequence of definitions, propositions, theorems, corollaries and lemmas, giving little hint of where they came from, where he or she is going with them and above all why. This strangely limited but pure style traces back to Euclid and to his modern-day reincarnation, the French mathematics collective that wrote under the pseudonym "Bourbaki."

What is remarkable about *Introduction to Circle Packing* is that its author, Kenneth Stephenson, a professor of mathematics at the University of Tennessee in Knoxville, is working hard to break out of this mold. I share his ambition, having written a book three years ago that is not altogether unrelated to his.



On the left is a large circle packed with 19 smaller ones. The graph on the right shows the associated configuration for that packing. The vertex marked w corresponds to the dark circle c_w, and the neighboring vertices v_1, v_2, v_3 and v_4 correspond to the shaded circles touching c_w.
From *Introduction to Circle Packing*

The landscape of mathematics books may be roughly described as consisting of three domains: (a) professional books written in the style described above; (b) popular books that are intended to present important mathematical ideas but which, being constrained to have no equations, are limited to anecdotes and vague analogies; and (c) a small smattering of things in between. Examples of works in the "popular" category are *Chaos,* by James Gleick (1987), and *Stalking the Riemann Hypothesis,* by Daniel N. Rockmore (2005). Both have the laudable goal of conveying the excitement of an active and important area of mathematical research, but neither book, in my view, ever actually explains what the research is really about. The problem is not that Gleick and Rockmore do not try. The problem is that you can't explain real math without asking a little bit more of the reader.

Stephenson's book deals with another truly beautiful new development in math. Moreover, it is one in which the stunning first-level results involve only elementary geometry. The simplest and most basic result is this: Imagine packing many small circles inside one big one. If you do this as efficiently as possible (producing what is called a "maximal" packing), the small circles will touch one another as much as possible, and the only gaps will be regions that are bounded by three mutually tangent circles (and thus look like triangles with curved sides). An example is shown in the illustration on this page: On the left side, 19 circles are packed in one big circle. Any such a "packing" of a big circle by smaller ones has an associated configuration: It is a graph. The configuration for this circle packing is shown to the right of it. Note how the black vertex marked *w* corresponds to the dark circle c_w, and the neighboring vertices v_1, v_2, v_3 and v_4 correspond to the lightly shaded circles touching c_w. In general, to make a configuration, you take one vertex for each circle, join two with an edge when the circles touch and fill in a triangle whenever three circles are mutually tangent. The wonderful result is that if you start with any such configuration (technically, a topological triangulation of the disk), then there is a nearly unique way to assign radii to the circles so that there is a packing with this configuration. (The only nonuniqueness comes from the possibility of moving the circle packing by the three-dimensional group of Möbius transformations.) What is remarkable here is that a discrete object, the configuration, is determining the whole geometry of the circles.

Paul Koebe discovered and proved this remarkable fact in 1936, but it lay dormant until it was picked up first by E. M. Andreev, then by William Thurston, Kenneth Stephenson, Zheng-Xu He and Oded Schramm. What a wonderful example of how international mathematics is: The ball bounced from Germany to Russia to the United States, China and Israel. Circle packing is now a full-fledged area of investigation, which this book seeks to popularize.

How successful is Stephenson in explaining his ideas to a larger audience? The book has four parts, which get harder as you go along. Chapters 1 through 7, which include all of Part I and most of Part II, might be accessible to someone with nothing more than a good high-school math background who is willing to read slowly and put in some time to feel his or her way into the ideas. This material is written formally, with proofs. Stephenson does use some technical math terms, such as *homotopic* and *homeomorphic.* But he explains and illustrates them. He says, excusing his style, Let me share something with you. Mathematicians *write* linearly—I am *compelled* to put this stuff here! On the other hand, if I were *reading* this, I would glance briefly, then skip ahead to the main show, returning only when I found myself in a bind.

Fortunately, Stephenson is one of a new breed of pure mathematicians, growing in number, who love to combine experiment with theory. This means he has computer code to carry out these packings and investigate their properties. And the book is interlaced with experiments—some successful, some not, some which worked one day but not the next when pushed further. His immense enthusiasm for this subject comes through on every page.

After Chapter 7, things get more heavily mathematical in many places. As a whole, the book is suitable for graduate students in math. In addition to making these ideas accessible, Stephenson has a second agenda: to make other mathematicians believe that circle packings are the basis for a discrete version of complex analysis that is every bit as rich and deep as traditional complex analysis, which deals with continuous variables. To make this point clear, the last part of the book, chapters 19 through 23, makes explicit the links between discrete and continuous complex analysis.

At this point, the book is really for mathematicians only. But along the way, there are many beautiful examples and discussions that are not very technical, such as the discrete Belyi functions covered in sections 16.3 and 23.1. And although it looks like heavy math from the outside, the argument about hyperbolic and parabolic uniqueness given in section 8.3 (which introduces geometric tools based on winding numbers) is explained simply and slowly and is quite breathtaking. With proper preparation, an honors high-school math class could understand it.

Three years ago I wrote a book, *Indra's Pearls,* with Caroline Series and Dave Wright**,** on a different aspect of complex analysis—the fractal shapes generated by infinite discrete groups of conformal maps. Like Stephenson, we tried to start very slowly and explain the basic concepts on a high-school level, using many pictures. But as we went on, our excitement took over, and we wrote a book that only a graduate student or an avid hobbyist or groupie is going to read all the way through to the end.

I'm not sure what the best way is to find a middle ground between standard research monographs and journalistic reports from the math front. But it is, I feel, important to try to bridge the undeniable gap between those who are math-literate (physical scientists, economists, computer scientists and engineers) and the greater educated public. I am glad to see a new genre of math books with both pictures and formulae in them, written so that a reader who is willing to put in a little hard thinking can come away really understanding something wonderful in mathematics.