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# Assessing the accuracy of ancient eclipse predictions

**David Mumford**

**Abstract.**

Multiple studies have addressed the question of whether ancient eclipse predictions correctly predicted known past eclipses. My approach instead is to generate thousands of synthetic eclipses, starting from random choices for the longitude of the conjunction, node and perigee, then comparing the magnitude of resulting eclipses with ancient and modern predictions. Writing code for ancient algorithms requires a detailed analysis of the text, not found in most commentaries. Working with Qu Anjing, Jayant Shah and Mark Schiefsky, we hope to do this with the Almagest, multiple Indian siddhantas and Chinese calendars. This paper will describe the results for the Yuan dynasty *Shoushili*. These raise the question of how the writers of this calendar understood its remarkably accurate algorithms for lunar parallax and whether and when secret geometric models were known to Chinese astronomers.

## §1. Introduction

All ancient cultures struggled to master the cycles of the heavens, creating calendars predicting the behavior of the sun and the moon. Among the observed phenomena, one of the hardest to predict were the eclipses. Total or near total solar eclipses were rare but certainly the most terrifying and portentious experiences, hence their prediction was a matter of great importance. We have records of many eclipse observations and many attempts at predictions from Babylon, Greece, India, China and Mayan cultures.

We can chart humanities' progress in understanding the paths of the sun and moon in stages:

(1) Keeping records of days, synodic months, tropical years leading to knowledge of their length and longer cycles (e.g. Metonic cycle).
(2) Discovery of the bi-yearly times where especially lunar eclipses occur at full moons, leading to an appreciation of the lunar nodes. This is, for example, the high point of Mayan discoveries.

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(3) More precise measurements of new/full moons, solstices/equinoxes leading to the recognition of the *anomalies*, periodic acceleration and deceleration of the moon and sun in their tracks, the perihelion and perigee. This is found in Seleucid Babylon, Hipparchus in Greece (c.150 BCE), the Paitamaha Siddhanta in India (c. 400 CE) and the Chinese calendar *Huangjili* [皇極曆] written by Liu Zhou [劉焯], (604 CE).
(4) The discovery of lunar parallax. Without knowing of this, solar eclipse prediction was doomed to be very inaccurate. This crucial discovery is found in Hipparchus again, in Aryabhata's siddhanta in India (499 CE) and in Yi Xing's [一行] calendar in China (728 CE).
(5) Of lesser importance for eclipse prediction, closer studies of lunar motion reveal layer upon layer of sublety. Evection is described in Ptolemy, (140 CE) and Manjula and Vatesvara in India (932 CE). After Newton's work, more 3-body effects of the Earth-Moon-Sun emerged as did the 2-body effects of Venus and Jupiter on the moon and the influence of the Earth's oblateness.

Going back at least to Hipparchus, Aryabhata and the Chinese calendars of the Han dynasty, quite complex algorithms were created to make eclipse predictions. Many of these have been edited and some translated into modern Western languages and attempts have been made to assess their accuracy by comparing them with the actual eclipses of that culture and time as reconstructed from modern algorithms. However a full analysis using the tools of modern mathematics does not seem to me to have been made.

I want to propose that this requires two steps. The first is to convert the written description of the algorithms to unambiguous computer code. All too often, when you seek to implement a sequence of operations described in a long ancient text, ambiguities arise. Moreover, when put in computer code, the differences between algorithms becomes much clearer. For example, Indian algorithms introduced iterative loops that were unknown to earlier algorithms. Secondly, to assess accuracy, we need a large supply of hypothetical eclipses: this can be provided by *randomly* synthesizing celestial configurations for all the variable points that affect the sun's and moon's orbits and then seeing what sort of eclipse, if any, is predicted by both the ancient and the modern algorithms. This is the project on which this paper is a first report. Not being a real scholar nor someone who can read Sanskrit, classical Chinese or Greek, I am very fortunate to be working with Jayant Shah on the Indian material, with Qu Anjing on the Chinese material and with Mark Schiefsky on the Greek material. I thank them all for many suggestions and explanations. Often reading the primary source is the only way to resolve ambiguities and this has been essential for this project.

## §2. Models, old and new

Celestial models are all about periodic effects. Originally, these were modeled by epicycles. But it's really the same ideas that re-appear in modern form as Fourier series or, more precisely, in the theory of almost periodic functions. Thus if we consider motion in a plane and put a complex coordinate on the plane, epicycle motion is described by the formula:

$$p(t) = \sum_k a_k e^{i\omega_k t}.$$

If one term is dominant, i.e. $|a_1| \gg |a_k|$, if $k > 1$, then, taking logs and expanding, you get a formula of the form:

$$\text{longitude} = \omega_1 t + \sum_{\vec{A}} c_{\vec{A}} \sin\left(\sum_k A_k(\omega_k - \omega_1)t + \phi_{\vec{A}}\right).$$

This is, in fact, exactly the way one school of modern modelers writes its predictions. For the moon, the prediction takes the form:

$$\text{longitude} = \omega_1 t + \sum_{i,j,k,\ell} c_{i,j,k,\ell} \sin(i.D + j.L + k.L' + \ell.F)$$

where $\omega_1 t$ is the mean moon and $D, L, L', F$ are also angles, linearly increasing as a function of time, namely the angles between the mean moon and mean sun (the mean elongation), the mean moon and perigee, the mean sun and perihelion and finally the mean moon and ascending node. We need to be careful about what longitude means as the moon's orbit has a small inclination to the ecliptic. The modern approach is to use the words longitude and latitude as spherical coordinates relative to the ecliptic and its pole. Thus the longitude of the moon is the angle of its projection to the ecliptic to the chosen ecliptic base point and the elongation is the angle from this projection to the sun.

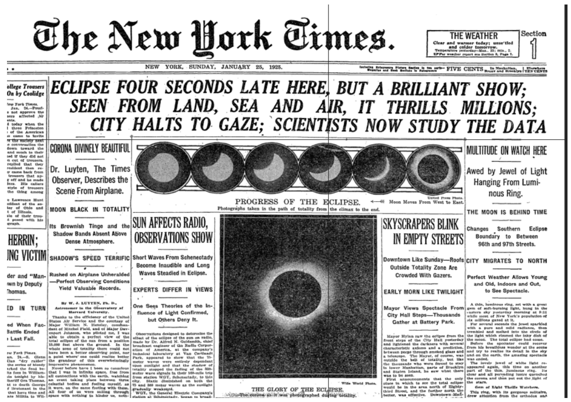
Even in 1925, however, the best modern formulas made errors of several arc-seconds: see figure 1. A recent version, the *Éphéméride Lunaire Parisenne* due to Michelle Chaprint-Touzé and Jean Chapront ([2]) has 20,560 terms for the longitude of the moon of which 19,136 relate to the effects of the other planets on the moon and so use also the angle between the mean moon and the mean longitude of that planet. For example, Venus can shift the moon's longitude by as much as 14 arc-seconds. But the 1023 terms involving only $D, L, L', F$ give arc-minute accuracy. To illustrate how far we have come, here are ELP's main terms for the longitude, latitude and distance of the moon:

| | | |
|---|---|---|
| lunar long. $=$ | $+13.176^\circ t$ | *mean motion* |
| | $+6.289^\circ \sin(L)$ | *eqn of center* |
| | $+1.274^\circ \sin(2D-L)$ | *evection* |
| | $+0.658^\circ \sin(2D)$ | *variation* |
| | $+0.214^\circ \sin(2L)$ | *2nd order Keplerian effect* |
| | $-0.185^\circ \sin(L')$ | *annual effect* |
| | $-0.114^\circ \sin(2F)$ | *correct for projection to ecliptic* |
| | $+$ terms with coeff. $\leq 0.06^\circ$ | |
| lunar lat. $=$ | $+5.128^\circ \sin(F)$ | *inclination of lunar orbit* |
| | $+0.559^\circ \sin(L).\cos(F)$ | *equation of center* |
| | $+0.173^\circ \sin(2D-F)$ | *wobble of lunar orbit* |
| | $+$ terms with coeff. $\leq 0.06^\circ$ | |
| dist to moon $=$ | $+385001$ km. | *mean distance* |
| | $-20905\cos(L)$ | *equation of center* |
| | $-3699\cos(2D-L)$ | *wobble* |
| | $-2956\cos(D)$ | *wobble 2nd part* |
| | $-570\cos(2L)$ | *2nd order Keplerian effect* |
| | $+$ terms less than $570$ km. | |

Here 'Keplerian' effects refer to describing his elliptical motion in spherical coordinates but the word 'wobble' above just means the sun causes the plane of moon's orbit to vary, a tricky effect depending on elongation that has no technical name. Fortunately, in assessing ancient predictions, I only need arc-minute accuracy and so I chose to use ELP for the modern predictions, taking only the terms in $D, L, L', F$. Then in generating random eclipses, we need not choose the longitude for instance of Venus but need only the longitude of the conjunction itself, of the perigee, the perihelion and the ascending node. Note also that at eclipses, $D$ is very nearly zero and $F$ cannot be too big either. The implementation of ELP that I use is in Matlab and is part of Eran Ofek's astronomy & astrophysics package for Matlab ([5]).

Because we are looking ancient predictions, it is important to see if the elements of the lunar and solar orbits have changed substantially going back a thousand years. Using ELP, the changes are estimated in [8]. The elements of the lunar orbit are found on p.669 of this paper and have not changed very much: going back to year 0, the semi-major axis of the lunar orbit changes only by about 0.08 kilometers, its eccentricity by about $3 \times 10^{-7}$ and its inclination to the ecliptic of date by about $1.6 \times 10^{-3}$ seconds. The largest change is that the speed of the lunar perigee's motion changes by about $-1.5$ seconds/year.

One effect of the modern predictions is that what we called the 'wobble' of the lunar orbit has an important influence on its inclination to the ecliptic. This figure has a major effect on the eclipse limits – how close a conjunction



The New York Times.

THE WEATHER

Section 1

NEW YORK, SUNDAY, JANUARY 25, 1925.

FIVE CENTS

ECLIPSE FOUR SECONDS LATE HERE, BUT A BRILLIANT SHOW; SEEN FROM LAND, SEA AND AIR, IT THRILLS MILLIONS; CITY HALTS TO GAZE; SCIENTISTS NOW STUDY THE DATA

CORONA DIVINELY BEAUTIFUL

Dr. Luyten, The Times Observer, Describes the Scene From Airplane.

MOON BLACK IN TOTALITY

Its Brownish Tinge and the Shadow Bands Absent Above Dense Atmosphere.

SHADOW'S SPEED TERRIFIC

Rushed on Airplane Unheralded —Perfect Observing Conditions Yield Valuable Records.

By W. J. LUYTEN, Ph. D., Astronomer in the Observatory of Harvard University.

PROGRESS OF THE ECLIPSE. ← Moon Moves From West to East.

Photographs taken in the path of totality from the climax to the end.

SUN AFFECTS RADIO, OBSERVATIONS SHOW

Short Waves From Schenectady Become Inaudible and Long Waves Steadied in Eclipse.

EXPERTS DIFFER IN VIEWS

One Sees Theories of the Influence of Light Confirmed, but Others Deny It.

THE GLORY OF THE ECLIPSE.

The corona as it was photographed during totality.

SKYSCRAPERS BLINK IN EMPTY STREETS

Downtown Like Sunday—Roofs Outside Totality Zone Are Crowded With Gazers.

EARLY MORN LIKE TWILIGHT

Mayor Views Spectacle From City Hall Steps—Thousands Gather at Battery Park.

MULTITUDE ON WATCH HERE

Awed by Jewel of Light Hanging From Luminous Ring.

THE MOON IS BEHIND TIME

Changes Southern Eclipse Boundary to Between 96th and 97th Streets.

CITY MIGRATES TO NORTH

Perfect Weather Allows Young and Old, Indoors and Out, to See Spectacle.

Fig. 1. The first page of the New York Times, Jan.25, 1925 – the current best predictions still had substantial errors!

must be to a node for an eclipse to occur. By computing the angle between the ecliptic and the tangent plane to the moon's orbit at any point in time, this inclination becomes a well defined function of time. This is also worked out from ELP in the above cited paper, p.671:

$$\begin{array}{lll} \text{inclination } i = & 5.1567^\circ & \text{mean motion} \\ & +0.1351^\circ \cos(2D-2F) & \text{biannual wobble, sun to node} \\ & -0.0111^\circ \cos(2D) & \text{monthly variation} \\ & +0.0104^\circ \cos(2F) & \text{monthly variation} \\ & +\cdots & \text{small terms, coeff } < 0.01^\circ \end{array}$$

The bottom line is that at eclipses, $D$ and $F$ are small and the inclination is close to $5.1567 + 0.1351 - 0.0111 + 0.0104 = 5.291$ degrees. A nice plot is found in [4], figure 4–10, p.46.

## §3. The *Shoushili*

In China, going back to the Han dynasty, the emperor issued an almanac every year, something like our ephemerides, including dates of new moons, eclipses and information for determining propitious and unpropitious times for rituals, planting etc. To compute these, the Bureau of Astronomy wrote computational treatises called *li* [曆] in Chinese, usually translated as "calendars", every 50 years or so. Such a calendar would be issued if the predictions of the old one were getting worse so that a new epoch with new constants was needed or to reinforce the legitimacy of a new dynasty, They also sometimes embodied increased knowledge of the basic math behind the observed bodies, the sun, moon and planets and might then be completely rewritten.

The *Shoushili* [授時曆], or "The Granting the Seasons Calendar" has been translated by Nathan Sivin in [9]. It was issued in 1280 CE by Kubulai Khan. Kubulai had actually commanded two groups to write calendars, one group made up of Muslim astronomers and one of Chinese. So as not to rock the boat too much, he shrewdly adopted the *Shoushili*, written by Chinese, one that followed the Chinese tradition to a great extent. It is arguably the best calendar before Matteo Ricci succeeded in making his way into the Emperor Wanli's [萬曆] circle in 1601 and Western science began to make inroads.

The *Shoushili* is notable in having extraordinarily accurate values of synodic and nodal months (the time between new moons and the time between two lunar crossings of the ascending node): they were accurate to about 1/3 and 1/4 second respectively. The first requires long observations of the dates of new moons and the second presumably records of ancient eclipses which had been recorded in the dynastic annals as early as the 8th century BCE. The nodical month, translated by Sivin as the "Crossing Terminal Constant" is given as 27 days, 2122.24 'parts', where each part is 1/10,000th of a day[1]. The modern value, in these terms, is 27 days, 2122.2082 parts. From the 8th century BCE to the time of writing the *Shoushili*, there had been about 25,000 lunar crossings of the ascending node, so an early eclipse observation accurate to within 2 hours would suffice for achieving this accuracy. In any case, starting from positions of the sun, moon and nodes at a given recent epoch, these two periods allow you to extrapolate and compute the distance of every *mean* conjunction from the nearest node. This was the first step of their prediction.

The second step is to correct for the equation of center, the observed fact that neither the sun nor the moon moves at constant speed around its orbit ('pacing the tread of the sun/moon' in Sivin's translation). The resulting apparent position is given by adding to their mean position a periodic piecewise cubic function of time that closely approximates the sine function. The major error made here was that the solar correction was taken to be about 25% larger than it actually is. The maximum correction in the *Shoushili* is $2.367°$ while modern observations give it as $1.914°$. Interestingly, a similar error was made by Ptolemy and by the Indian astronomers. I suspect the reason was quite simple: the equation of center was estimated by finding the date and time of the solstices and the equinoxes and fitting the equation of center to the lengths of the four seasons between these events. To estimate the date and time of the

[1] As Sivin explains on p.68, pp.87–89, [9], these were written as something like "2 thousands, 1 hundred, 2 tens, 2 units, 24 hundredths".

solstices is not so hard: one observes either the length of the day or the maximum height of the sun the sun for several weeks before and after the solstice. It forms a roughly parabolic curve and one seeks the extreme point (max or min). The best way to do this is to average two dates, before and after the solstice when the length or heights are the same. For the equinox, however, you need to discover when sunrise and sunset divide the day in two equal parts or when their positions on the horizon are exactly opposite. Here the difficulty is that refraction prolongs the day: the sun is still visible even when it is geometrically slightly below the horizon. The early astronomers were unaware of this effect and I believe this accounts reasonably well for the error.

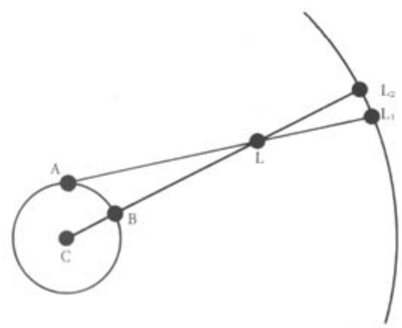



Fig. 2. The parallax of the moon: seen from point A, the position L of the moon against the sky shifts towards the horizon from where it appears at point B.

The third and really difficult step is to account for lunar parallax (and a possible solar parallax if the sun is assumed not too distant). The essential fact to bear in mind is that Chinese calendars *never describe geometric models of the positions of the celestial bodies in space.* In fact, to my knowledge, discussions of whether the earth is round or flat are confined to occasional philosophical speculations and thus no serious estimate of the earth's size appears in any surviving manuscripts before the coming of the Jesuits. Without the model shown in figure 2, how would parallax be understood? Yet without such a correction, eclipse predictions must necessarily be wildly inaccurate. Somehow, the *Shoushili* came up with a reasonably accurate first approximation of the parallax correction! At the end of this paper, we will present one theory of what might have been going on among the Chinese astronomers.

From a modern perspective, the paralactic correction needs to steps: first, estimate the angle $\theta$ between the observer's zenith and the moon seen from the center of the earth. The moon should then be displaced by an angle approximately equal to $C.\sin(\theta)$ towards the horizon. The constant $C$ depends on the ratio of the distance of the moon to the earth's radius and is about one degree. Secondly, this displacement is written as a vector sum of a displacement in longitude to a displacement in latitude. The latitude correction brings the moon closer to or further from the sun which sits on the ecliptic and the longitude correction changes the time when the sun and moon are closest, the time of the eclipse if it occurs. Qu Anjing (lecture at Harvard, Jan. 2014) has shown that the latitude correction in degrees is very close to the following:

$$\begin{aligned}
\Delta(\text{lunar lati.}) &\approx \frac{180H}{\pi\tan(\iota)} \cdot (\cos(\epsilon).\sin(\phi) - \sin(\epsilon).\cos(\phi).\sin(\lambda+h)) \\
&= k_0(\epsilon,\phi,\iota,H) \\
&\quad - k(\epsilon,\phi,\iota,H)\,(\sin(\lambda).\cos(h)+\cos(\lambda).\sin(h))
\end{aligned}$$

$$\begin{aligned}
\text{where } H &= \text{(radius earth)}/\text{(distance earth to moon)} \\
\iota &= \text{inclination moon's orbit to ecliptic} \\
\epsilon &= \text{inclination of the ecliptic} \\
\phi &= \text{latitude of the observer} \\
\lambda &= \text{Right ascension of sun, i.e. date} \\
h &= \text{hour angle of sun, or time of day relative to noon}
\end{aligned}$$

And then the *Shoushili*'s formula is:

$$\Delta(\text{lunar latitude}) = 6.065^\circ - 4.395^\circ \left( si(\lambda).\Big(1-\frac{|h|}{d/2}\Big) + si(\lambda+90^\circ).\frac{h}{6}) \right)$$

where $si(x) = 4\frac{x}{\pi}(1-\frac{x}{\pi})$ for $0 \le x \le \pi$, and its negative for $\pi \le x \le 2\pi$, a piecewise quadratic approximation to sine and $d$ is the day length at the date of the possible eclipse. Working out Qu's $k_0$ and $k$ for Beijing, we get $6.2^\circ$ and $3.23^\circ$. Thus the *Shoushili* formula is an approximation of Qu's formula obtained by replacing the trig functions of $h$ by piecewise linear functions and the trig functions of $\lambda$ by quadratic approximations.

As for the longitude correction, this is handled by a simple additive factor in the hour angle $h$ of the conjunction, zero at noon and otherwise pushing the conjunction towards sunrise/sunset:

$$\text{adjusted hour angle} = h + (\tfrac{5}{24})^2 \cdot h \cdot (12 - |h|).$$

Here is the whole *Shoushili* eclipse prediction algorithm in pseudo-code:

```
1. Generate random longitudes in [0,360] for origin at
   vernal equinox.  MM=MS mean moon/sun, PG for the
   moon’s perigee, Nd for the node within 15 - of MM and
   a sign bit AD (asc or desc), perihelion PH assumed at
   winter solstice 270 degrees.
2. Use cubic polynomials for dS and dM, the equations of
   center of sun and moon.
3. Define DistNd := (MM-Nd)*(vM/(vM-Vn) + dS;
              DT := (TropYr/360)*MS + (dS-dM)/vM˜;
   where vM˜ is the velocity of the moon corrected for its
   position relative to perigee using the moon’s eqn table.
4. Compute the hour angle of the eclipse by:
              HrAng := 24*(DT - round(DT));
   and its correction due to parallax by:
              HrAng := HrAng*(1+(5/24)^2*(12-abs(HrAng));
5. Compute:
              DayLen := 12 + 2.766*sin(MS - 2.367);
              if abs(HrAng) > DayLen/2, NIGHT&NO_VIS_ECL
6. With a parabolic approx psin to sin, define the
   parallax correction:
      DistNd := DistNd + AD*(-6.065 + 4.395*(
        {psin(MS)*(1-abs(HrAng)/(DayLen/2)) +
         psin(MS+90 -)*(HrAng/6)});
7. Define YY := +1 if (AD=+1 & MM>Nd) OR (AD=-1 & MM<Nd),
          else -1;
8. EclipseMag := max(0,1-abs(DistNd)/(7+YY));
```

## §4. Simulation results

The analysis proceeded as follows: we ran 20,000 trials, choosing random longitude of mean conjunction, random perigee and a node, randomly ascending or descending and within 15 degrees of the conjunction. We fix the perihelion at the winter solstice as in the *Shoushili* (which is in fact very accurate, NASA [3] gives the value 270.585 for the year 1280 CE). In each trial, we ran both ELP and the *Shoushili* algorithms. To see if an eclipse was predicted, a simple summary statistic is a sort of signed magnitude. Take the time when the sun and moon are closest and compute the ratio of (i) the latitude of the moon minus the latitude of the sun by (ii) the sim opt the observed radius of the moon and the radius of the sun. If this figure has absolute value less than one if and only if there is a partial eclipse and the eclipse gets more total as it approaches zero. We call this figure the *normalized approach*.

For an eclipse to be predicted, we need to examine both the normalized approach and whether this approach occurs during daylight. So as not to penalize borderline eclipses, we define a *strong prediction* as one where the absolute value of normalized approach is less than 0.9 and the time of the nearest approach is at least one hour after sunrise and one hour before sunset. A *weak prediction* is one where the absolute value of the normalized approach is less than one and this occurs during daylight. Then we count all strong *Shoushili* predictions as correct if they are at least weakly predicted buy modern algorithms. We have false positive errors if not. And a strong modern prediction which is not even a weak *Shoushili* prediction is called a false negative error.

Of there 20,00 trials, 2257 were Chinese strong eclipse predictions, out of which 185 were false positives and a further 14 were strong modern predictions but not even weak *Shoushili* predictions. This means the rate of false positives is 185/2257 or about 8% while the rate of false negatives (unpredicted eclipses) was only 14/2257 or less than 1%. Clearly for the writers of the calendar, it was much safer to predict an eclipse that did not occur than to not predict one that did occur. Figure 3 shows a plot of the Chinese normalized approach ($y$-axis) against the modern normalized approach ($x$-axis) for 5000 trials. Every trial that resulted in a strong modern or strong Chinese prediction is plotted, using a black circle if the other gave at least a weak prediction, red crosses for false negatives (in this run, slightly more than average) and the green stars for false positives. The diagonals represent predictions differing by $\pm 0.25$. The standard deviation of the difference, Chinese minus modern normalized approach, for all trials with either a Chinese or modern strong prediction was 0.18. Note the tilt of the plotted points is slightly less than $45°$, i.e. the Chinese predictions are on average somewhat smaller than their true values. This reflects the fact that their eclipse limits were set a bit too high (to lessen the frequency of false negatives). How about the accuracy of the time of occurrence? A plot of the Chinese and modern predictions for the hour of the maximum eclipse is shown in figure 4. The standard deviation of their difference over all trials with either a Chinese or modern strong prediction was just over 30 minutes.

How about the equations of center? Plots of the Chinese solar and lunar equations of center versus the modern values are shown in figure 5. By modern values here we take the true longitude seen from the earth's center minus the mean longitude. Regressing against a linear function, we find the slope of the solar data is 1.28 showing again the overestimation of the solar equation of center. The standard deviation for the lunar equation of center is $0.33°$ reflecting the absence of higher order Keplerian terms and the annual effect in the ELP equations given above. If we assume that errors in predicted eclipses due to errors in the equations of center and due to errors in parallax are independent,

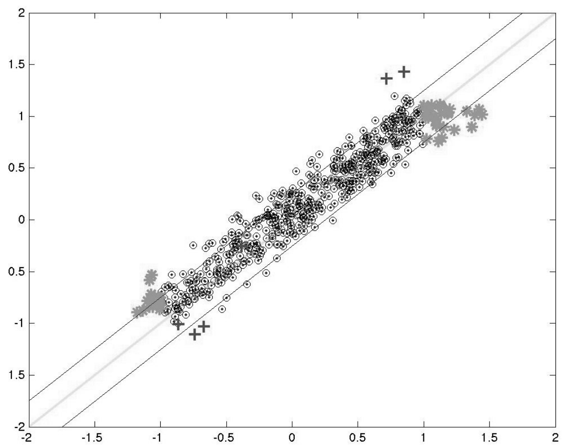



Fig. 3. *Shoushili* predicted normalized approach vs. true normalized approach for 572 simulated eclipses. See text for explanations.

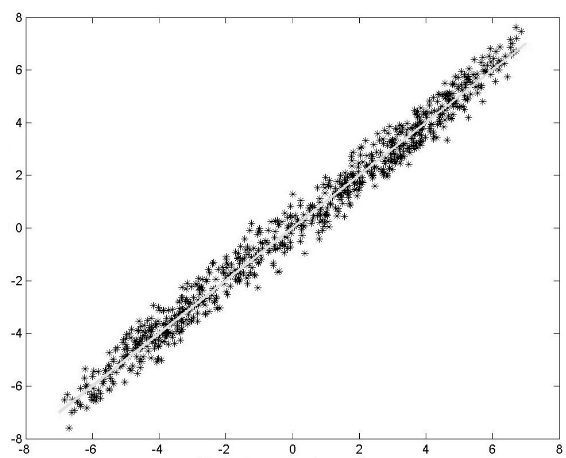



Fig. 4. *Shoushili* predicted hour angle vs. true hour angle for 572 simulated eclipses. The scale is in hours before or after noon.

their variances will add and we can estimate the relative weight of these two sources of error by running both the *Shoushili* and ELP with and without any parallax. We find that about 12% of the variance between Chinese and modern values is due to the equation of center and 88% due to parallax. Another indication of this is that the errors increase for eclipses near sunrise and sunset hence near the horizon, decreasing for eclipses near noon hence nearer the zenith.

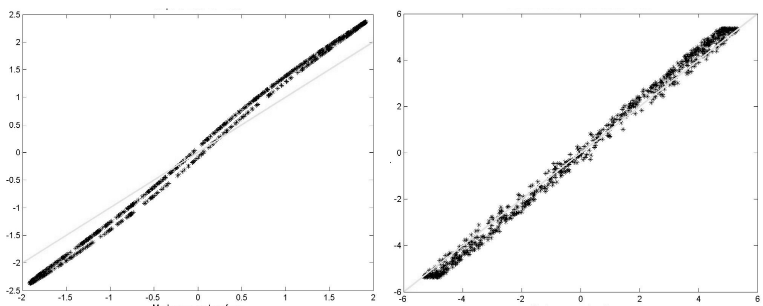


Fig. 5. Left: *Shoushili* predicted solar equation of center vs. true solar equation of center for 572 simulated eclipses. Right: same for lunar equations of center.

## §5. Shah's results on the Tantrasangraha

I hope to extend this analysis to many other algorithms: Ptolemy's *Almagest*, earlier Chinese calendars, esp. the *Dayanli* [大衍曆], or the *Great Expansion* calendar of Yi Xing (728 CE) and several Indian siddhantas. Indian siddhantas date from the fifth century CE but reached their most accurate level in the work of Kerala astronomer-mathematician Nilakantha who wrote the *Tantrasangraha* in 1501 CE, available in a recent edited English translation [6]. Jayant Shah has carried out a similar analysis on this siddhanta, putting its algorithms into matlab code [7]. This section reports on the results of his code.

In stark contrast to the Chinese approach, all Indian siddhantas are based on explicit geometric models. Some aspects of their models were refined over time but others were rigidly fixed by tradition. Thus a fundamental feature of their models is that all celestial bodies, sun, moon and planets, are assumed to be moving at the same mean speed. Thus their relative distances can be inferred from their periods. One consequence is that sun as well as the moon has a non-negligible parallax that affects eclipses. Unfortunately this magnifies the solar parallax by a factor of 30 introducing a substantial error. On the other hand, a strength of algorithms, unique to India, is their use of iteration to find solutions of geometric models with quite complex constraints. Their use of iterative algorithms can be traced all the way back to Panini's Sanskrit grammar (variously dated to the 4th or 5th century BCE). An example is that their standard epicycle models have an epicycle with variable radius in which the radius is required to have a fixed ratio to the distance of the planet to the earth, a constraint that is readily solved by iteration.

On the whole, their results are more definitely better than those in the *Shoushili*. Comparisons between the Tantrasangraha and modern predictions of (a) distance to node at maximum eclipse, (b) normalized latitude, (c) hour angle of maximum eclipse and (d) the solar equation of center are shown in figure 6. As in the Chinese algorithm, more eclipses were predicted than actually occurred: in fact there were no missed strong eclipses in our simulations but the error rate for false positives was 5%.(as compared to 8% for the *Shoushili*). The standard deviation for the error of the predicted nodal distances is $0.48^\circ$ compared to $1.12^\circ$ for the *Shoushili*. The plot for the error in predicted magnitude shows an interesting error: it is tilted with slope about 0.9 instead of 1. The cause of this is another instance of how traditional models in India are sometimes accepted without further study. In this case, the inclination of the moon's orbit to the ecliptic was always taken as $4.5^\circ$ whereas its correct value should have been about $5.3^\circ$. Making this inclination so small means that eclipses are predicted for conjunctions further from the node and all eclipses are predicted to have magnitudes correspondingly larger (or normalized latitude too small). Giving the sun observable parallax as well as the moon creates a further but much smaller error. The standard deviation of the error in hour angle is astonishingly a bit less than 15 minutes. The solar equation of center is again a bit too large as in the *Shoushili*. The division of the rror between equation of center and parallax works out about the same as in the *Shoushili*: about 12% for equation of center and 88% for parallax.

## §**6. Speculation**

Accurate predictions of eclipses require first of all understanding the lunar nodes, the points in the sky where the sun's and moon's orbits intersect. For example, without any surviving indication of a geometric understanding, it is still clear that the Maya knew there were two times of year near which a solar eclipse might occur at a new moon and a lunar eclipse at a full moon. We now call these the times when the sun crosses one of the nodes. The next step is to understand that neither the sun nor the moon move at constant speed in their orbits. In the *Shoushili* this is described as 'pacing the tread of the sun/moon'. This is also a phenomenon that doesn't immediately call out for a geometric

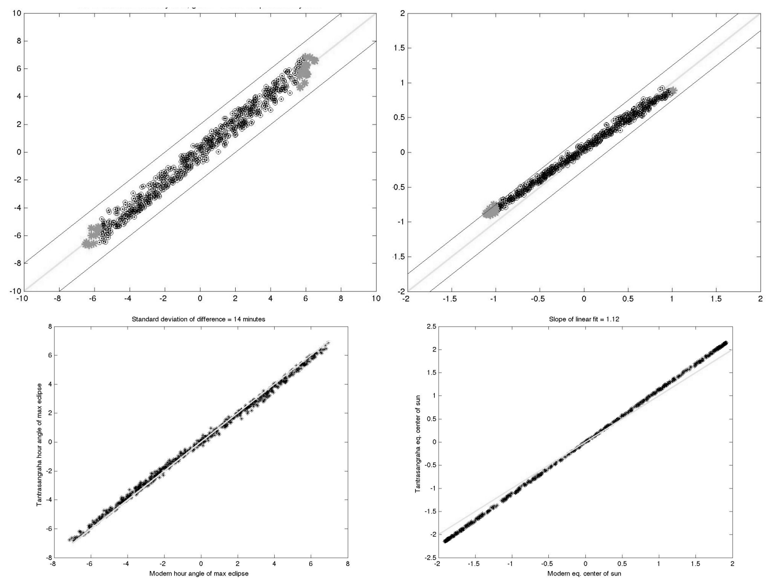



Fig. 6. Top left: Tantrasangraha predictions for distance of observer's conjunction to the node vs. the modern; top right: same for normalized latitude; bottom left: same for hour angle; bottom right: same for sun's equation of center.

model, although if one has a prior conviction that celestial motion should be uniform and circular, then an epicycle model is a very natural explanation. Every culture has some such prior beliefs that constrain their understanding of the world.

The final and crucial step is taking account of lunar parallax: that the moon appears not where it is expected to be but displaced by up to a degree towards the horizon. How can such a strange phenomenon be understood without a geometric model? We have seen that this is the most important factor in eclipse prediction. We have also seen that the *Shoushili* does quite a good job in allowing for this. This was first noticed in China by Zhang Zixin around 550 CE after 30 years of observations[2]. Accurate predictions first appear in Chinese calendars in the Tang dynasty in the calendar of Yi Xing, the *Dayanli* of 728 CE. Where did Yi Xing's understanding come from?

[2]Personal communication from Qu Anjing.

It is essential to realize that predicting eclipses was particularly important in China: if the calendar failed to predict a noticeable eclipse, it could be taken as an indication that the Emperor did not have the 'Mandate of Heaven'. In the Tang dynasty, the silk road was open and very active with many merchants connecting China with far flung kingdoms. In particular, Buddhism, that first spread from India to China in the Han dynasty, was attracting adherents in China and there was a strong desire to import and translate Buddhist manuscripts from India. It seems likely that word got to the Emperor that the Indians could predict eclipses very successfully because several Indian astronomers were invited to the court in Chang'an [長安] (now Xi'an [西安]) in the 7th century CE. Gautama Siddha was born in Chang'an in the late 7th century CE to one of these Indian families and he translated various Indian siddhantas into Chinese and composed a calendar, the *Jiuzhili* [九執暦] in 718 CE. Although his calendar was not adopted by the Emperor Xuanzong [玄宗], one can guess that it strongly influenced Yi Xing's calendar which was officially adopted ten years later.

All this makes a nice story and would seem to explain who knew what when. But it conceals a major question: why is there no reference in any Chinese manuscripts of the time to the underlying geometric models of the Indo-Greek astronomical tradition? The Indian formulas for parallax were based on the model of a round earth of a definite size (circumference = 3300 yojanas) and a moon circling it at a definite distance (mean = 34,380 yojanas), resulting in parallax that had a maximum value of about one degree for the moon near the horizon. But no estimate of the earth's circumference is given in any surviving Chinese manuscript before the arrival of the Jesuits in the 17th century. The traditional model of the Middle Kingdom was a flat square sitting under the round dome of the sky. From Han times, the Chinese had built armillary spheres showing the heavens rotating above the earth which is placed in the center. This was developed in several somewhat different cosmologies and in some places the earth was referred to as a 'yoke' inside the egg shell of the heavens. But if, indeed, the earth was round, no estimate was given anywhere for its size.

In fact, a surviving 1136 CE stone map of greater China, the *Yujitu* [禹迹図], shows the land from Inner Mongolia in the north to Hanoi in the south with inscribed perpendicular NS and EW grid lines stated to be *everywhere 100 li apart*. But two NS lines that are 100 *li* [里] apart at its southern limit should be only 77 li apart at its northern edge. In fact, the map is distorted to conform to the rules of Pei Xiu [裴秀] who prescribed in the 3rd century CE what are, to all intents and purposes, Cartesian coordinates for maps. Figure 7, from my paper with Alexander Akin [1], shows how this map must be warped so that the true NS lines are vertical and the true EW lines are horizontal. North-south lines are easy to lay off using either the north star or a compass, yet in this map they are off in places by dozens of degrees. In other words, this monumental map was distorted to conform to the assumption of a flat earth.

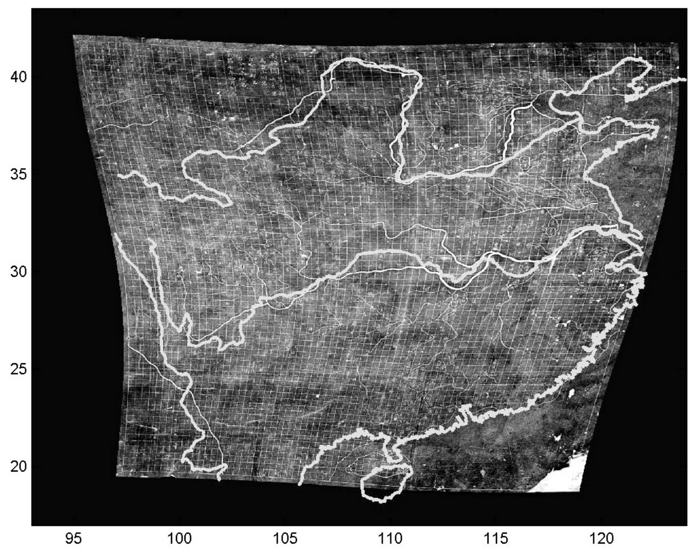



Fig. 7. The white lines and white grid are a rubbing from the original carved stone *Yujitu*. It has been warped so that 45 landmarks are at their correct positions with respect to the longitude/latitude axes. The yellow lines are the true coastline, Yellow, Yangtse and Mekong rivers. Note how far from vertical the original NS lines become.

The story gets more curious when you consider one of the main tasks that the Emperor gave to Yi Xing. The use of 8 foot gnomons to measure the height of the sun goes back to pre-Han times and is described in the *Zhoubi Suanjing* [周髀算経]. Moreover it was understood that the height of the sun, e.g. at midday of the summer solstice, varied systematically as you moved north or south. Yi Xing was asked to measure the Middle Kingdom in the north-south direction by making accurate measurements of the gnomon shadow in 20 places. A major innovation that he made was to realize that north-south distances were proportional to the change in the angle of elevation, not the change in the cotangent of the angle of elevation (that is, the shadow length of the gnomon). And to do this, he made a table of the tangent function. It is hard to believe that if he did this and if knew and read the works of the Indian astronomers in Chang'an, he did not understand that the earth is round and did not use his measurements to confirm the Indian estimate its size. *But he never wrote this down!*

After discussions with Qu Anjing and other Chinese friends, I'd like to suggest a conjecture. The tradition of a flat earth, going back to Confucian times, was something no one could easily contradict. But Yi Xing understood this was false and that the earth's true size and shape were essential for drafting future calendars and indeed for cartographers also. Now many personnel in the Bureau of Astronomy came from a small number of families and it quite possible that this model was passed down as esoteric knowledge within these families. Later we encounter the extraordinary polymath Shen Kua [沈括] (1031–1095 CE) who headed the Bureau of Astronomy for some years sand who later negotiated treaties based on accurate maps and eventually made a single huge, now lost, map for the emperor some 50 years before the *Yujitu* was carved. Surely the knowledge of the true size of the earth came down to him. But, like Galileo, to keep his head even when exiled in disfavor, he held his tongue. The final act was how in the time of Kublai Khan, the Chinese drafters of the *Shoushili* worked quite independently of the Muslim astronomers and still, without a hint of how such strange formulae might arise, produced an accurate estimate of parallax. Just as the tenacious Chinese tradition had blocked Indian ideas from general adoption in the Tang, the pattern repeated itself by blocking Islamic ideas from adoption in the Yuan. Transmission of ideas, even scientific ones, depends on both sides being open to it.

## References

[ 1 ] A. Akin and D. Mumford. "Yu laid out the lands": georeferencing the Chinese Yujitu [Map of the Tracks of Yu] of 1136. Cartography and Geographic Information Science, **39**, pp. 154–169 (2012).

[ 2 ] M. Chapront-Touzé and J. Chapront, The lunar ephemeris ELP-2000, *Astronomy and Astrophysics*, **124**, pp. 50–62 (1983).

[ 3 ] The NASA website http://aom.giss.nasa.gov/cgi-bin/srorbpar.cgi

[ 4 ] Fred Espenak and Jean Meeus. Five Millennium Catalog of Solar Eclipses – Revised. NASA/TP-2009-214174

[ 5 ] E. Ofek, URL: http://webhome.weizmann.ac.il/home/eofek/matlab/, paper submitted in 2014.

[ 6 ] K. Ramasubramanian and M. S. Sriram. *Tantrasangraha of Nilakantha Somayaji*, Hindustan Book Agency (2011).

[ 7 ] J. Shah. Medieval Eclipse Prediction: A Parallel Bias in Indian and Chinese Astronomy, *Ganita Bharati*, **37**, pp. 163–178 (2015).

[ 8 ] J. Simon, P. Bretagnon, J. Chapront, M. Chaprint-Touzé, G. Francou and J. Laskar. Numerical expressions for precession formulae and mean elements for the Moon and the planets. *Astronomy and Astrophysics*, **282**, pp. 663–683 (1994).
[ 9 ] N. Sivin, Granting the Seasons, Springer-Verlag (2009).

*Division of Applied Math, Brown University*
*E-mail address*: david_mumford@brown.edu