“Yu laid out the lands:” georeferencing the Chinese *Yujitu* [Map of the Tracks of Yu] of 1136

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ABSTRACT: In an historically contextualized overview of the earliest extant version of the *Yujitu*, engraved on stone under the Liu Yu regime but believed to be a copy of a Northern Song map, the paper analyzes factors that complicate its georeferencing. The paper introduces a new algorithm for nonlinear georeferencing, applying it to 45 points and finding that the placement of sites on the north-south axis must have been based on latitudinal observation. Despite the *Yujitu's* startlingly modern appearance, the paper finds that there are areas in which it reflects a loyalty to classical texts.

KEYWORDS: GIS, georeferencing, historical cartography, China, Song dynasty, *Yugong*

Introduction

In the year 1136 CE, a remarkable map of China was carved into the face of an upright monument on the grounds of a school in Xi'an. Called the *Yujitu*, the Map of the Tracks of Yu, it claimed descent from records of the legendary Yu, a Xia Dynasty sage-king credited with taming floods using feats of engineering that gave the known world its contemporary form in the third millennium BCE. At the time of this map's engraving, what we now think of as China existed not as a single entity but as two mighty empires, the Song in the south and the Jin in the north, as well as the Western Xia in the northwest, and a number of smaller short-lived regimes. Only a decade earlier the Jurchen invaders of the Jin had struck into the heart of Song territory, forcing the Song to move its capital south while losing an enormous swath of territory.

The *Yugong*, a text discussed in some detail below, is the earliest canonical geographical work in the Confucian tradition, dating at least to the turn of the fifth century BCE, and generations of scholars searched for ways to lay out the features it names upon maps of the world as it was later understood. The *Yujitu* is the most monumental extant artifact of this tradition of *Yugong* scholarship. Its form, engraved on a rectangular stone plaque, enabled students and visitors to use paper and ink to make detailed rubbings to be taken away for later reference. The stone thus functioned as something like an immovable printing block, remaining in Xi'an while copies of its map found their way further afield. Harvard University holds one such rubbing made from the original stone, which shows more detail than any previously published version. It is possible that many more such stone tablets with versions of this map once existed at other schools around the empire; another nearly identical example, somewhat clumsier in execution, survives from 1142.

The map is about one meter square and, roughly speaking, represents China and its environs: north to what is now Inner Mongolia, south to Vietnam, east to the Pacific Ocean and west to what is now Xinjiang and Burma. What is extraordinary is that (a) it is overlaid on a precise rectangular grid of equally spaced horizontal and vertical lines and that (b) all the major features of the coastline, the Yellow, Yangtze and other rivers are immediately recognizable to the viewer familiar with their appearance in modern maps. The grid itself consists of 71 vertical lines

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and 74 horizontal lines, running east-west and north-south. The squares formed in the resulting grid are described in the map's scale as representing 100 *li* to a side (about 34 kilometers, using the value of the *li* that we infer below for the time when this map was carved). On the other side of the same stone monument is a very different map, which appears like a relic of another era although it was carved six months later. This map, the Huayitu, depicts the spatial relationship of various "barbarian" peoples beyond the sphere of Sinitic civilization. Despite the comparative crudeness of this latter map, it is actually more accurate in its depiction of southwestern rivers, for reasons discussed later in this paper.

In many respects the *Yujitu* can trace its ancestry to the work of Pei Xiu, who is credited with systematizing a grid-based system of scale for maps. Pei (224-271), whose cartographic principles are cited in countless gazetteers and whose systematic approach earned him Joseph Needham's coronation as the father of "scientific cartography," also used the *Yugong* and other records to draw a famous silk map, the *Yugong diyutu* 禹貢地域圖, in eighteen sheets, now long lost. He laid out six principles for map making, including the use of a rectangular grid of equally spaced parallel lines. In this way, he said, ratios of distances, right angles and acute angles, curves and straight lines in the real world would be correctly represented on the map. He went on to say that derived distances could then be measured using right triangles (the Chinese equivalent of Pythagoras's theorem). Thus all distances in the world are proportional to the corresponding distance on the map, which we call today the Euclidean plane metric. As we know now, this is possible only for flat surfaces (technically, surfaces with zero Gaussian curvature). But at the scale used for most early local maps the earth's curvature would not cause noticeable problems.

However, the southern limit of the *Yujitu* lies at about latitude 18° and the northern limit at about 43°, a vast swath of territory which poses a problem when depicted on a map: north-south (i.e. meridian) lines should converge as one moves north because of the earth's curvature, their separation being proportional to the cosine of the latitude. Given that cos(43°)/cos(18°) is about 0.77, true meridian lines in a map covering such an area are not even roughly a constant distance apart. What then are the vertical grid lines of the *Yujitu*? If these were intended to be true north-south lines, it is hard to believe that the Chinese empirical data did not reveal the fallacy of requiring them to be equally spaced over such a large distance. There is, to our knowledge, no surviving record that this problem was noted, even though the concept of a round earth did appear as one possibility in a number of texts, beginning with the second century scholar Zhang Heng, who famously described the earth as resembling a crossbow pellet surrounded by the heavens like the suspended yolk of an egg.

The purpose of this paper is to make a careful analysis of the *Yujitu*, exploring first its historical context and the factors which complicate modern georeferencing, then proceeding with such a georeferencing based on a recently developed algorithm, in order to assess the accuracy of the map and, in particular, of seeing what compromises were made to approximate a traditional flat earth map over so large an area. The results of this analysis suggest that the *Yujitu* was drawn with quite accurate east-west grid lines closely corresponding to equidistant constant latitude lines, but that its 'north-south' lines diverge far from meridians and in so doing they maintain a roughly constant spacing from each other. Thus Pei's principle that the map should correctly represent right angles was abandoned in order to maintain his other requirements. The georeferencing also allows us to see where the map is quite spatially accurate and where it is not. An important function of the algorithm applied here is its utility in separating large-scale deviations, such as a shift in north-south alignment discussed below, from local inaccuracies. Finally, we posit some conclusions about what this suggests about Chinese traditions in cartography and astronomy.

Some Details of the Features Represented in the *Yujitu*

The most explicitly geographical work of the Confucian canon is the *Yugong*, one of the texts collected in the *Shangshu* 尚書, or Book of Documents. It begins: "Yu laid out the lands. Going along the mountains, he cut down the trees. He determined the high mountains and the great rivers" (Karlgren 1950: 12). Though modern scholars differ on the dating of this work, Edward Shaughnessy suggests that the *Yugong* chapter was composed as late as the Qin dynasty (Shaughnessy 1993: 378.)

Figure 1. The Sage King Yu, as depicted in the late Ming *Tushu Bian* of Zhang Huang.

As an ostensible record of the sage-king Yu's suppression of the floodwaters that ravaged the world in the 21st century BCE, the *Yugong* also served for centuries as a classical outline of topography and regional division, providing a dense enumeration of rivers and landscape features (Figure 1). Its brevity (Karlgren's English translation occupies less than five pages) has not prevented scholars of two millennia from turning out countless pages in their quest to visualize more clearly the lay of the land and the administrative order of the "Nine Regions" (*jiuzhou* 九州) described therein. As the preface to one late Ming dynasty work on the subject states, "The *Yugong* is the ancestor of all geographical treatises, past and present" (Ai Nanying, *Yugong tuzhu* p. 1a). A wide range of *Yugong*-related works from the Song to the Qing may be found in the 7-volume collection *Lidai Yugong wenxian jicheng*.

Figure 2. Full view of the *Yujitu* rubbing held at Harvard University.

The earliest extant maps based on the *Yugong* come from the 12th century. Of these, the oldest is the *Yujitu* discussed in this paper. A reproduction of a rubbing of the map, held at Harvard University, is shown in Figure 2.

The *Yujitu* takes as its base layer a river network charted on a regular grid of squares representing 100 *li* to a side. This network is liberally plotted with mountain names. It is not entirely a physiographic map, however, as constellations of cities extend over its surface. If we take a close look at Zhejiang, for example, we see these different types of features - rivers, mountains and towns of prefectural level and higher - all in the same style and character size.

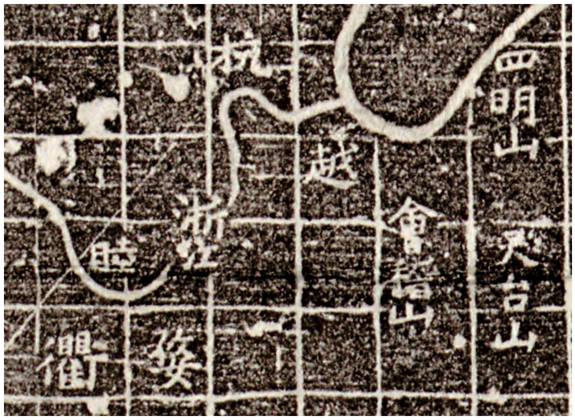


Figure 3. Close up of Zhejiang on the *Yujitu*. At upper right is Siming Mountain, with Hangzhou directly to its west and the Zhejiang (now Qiantang) river, for which the province is named, wending its way to the sea, all labeled in the same font without the use of symbols.

A small square at the upper left lists the map's contents as: "Names of mountains and rivers from the *Yugong*, names of provinces and prefectures from past and present, and mountain and river names from past and present." Thus it takes the names of natural features recorded in the ancient text of the *Yugong*, which was believed to cover the entire physical sphere of classical Civilization, then plots these places alongside towns and cities from the beginning of the dynastic period up to the present. It does all of this without representing any political or administrative boundaries from the time it was engraved. This depiction of the historical sphere of civilization, carved in an age when the world it depicts was riven by fragmentation into competing states and political instability, might have been intended at another level as a wistful call for the reunification of this space under a single virtuous and stable dynasty.

In many respects, the *Yujitu* was designed to represent the same types of features as Pei's *Yugong diyutu*, cited above. Pei's preface to that map has been preserved and reads in part, "Referring back to antiquity, I have examined according to the *Yugong* the mountains and lakes, the courses of the rivers, the plateaus and plains, the slopes and marshes, the limits of the nine ancient provinces and the sixteen modern ones, taking account of commanderies and fiefs, prefectures and cities, and not forgetting the names of places where the ancient kingdoms concluded treaties or held meetings; and lastly, inserting the roads, paths and navigable waters, I have made this map in eighteen sheets." (Needham 1959: 540). Rather than representing a single era, Pei Xiu's maps depicted places and events from different moments in history. Whether these were all superimposed on regional maps, or separately indicated in individual vignettes, is unclear. Both of these techniques were commonly used later in the imperial period, the former frequently adapted for decorative wall maps or teaching materials (such as maps highlighting all of the former capitals), the latter form being more suited to books such as the *Lidai dili zhizhang tu* 歷代地理指掌圖 where pages could mark temporal installments in a series showing changes over the years (Akin 2009).

Before outlining the methodology of our analysis, it is important to note how the process of georeferencing the *Yujitu* highlights the necessity of reckoning with the ambiguities of 12th century data in a manner elided by scholars who have only emphasized the map's hydrological accuracy. The nature of this map, originally intended not only as a tool for teaching geography but as a reference for situating the toponyms that students would encounter in their readings of the Classics and later dynastic histories, makes it difficult to tally its culturally shaded intentions with the objective expectations of GIS software. It was not meant simply to be an accurate map of features as they stood when it was engraved; it mapped out the legacy of a culture on the face of the land, and made concessions to this historical legacy in ways that "distort" the map from our perspective as georeferencers, or "in-form" it, as we might say if we were students of the Confucian canon preparing for an exam on the Classics. The situation is similar to that encountered in the late 15th century world maps of Henricus Martellus. European cartographers of the time had Latin translations of Ptolemy's *Geographike Hyphegesis* but had also begun obtaining data from more recent explorations by Portuguese sailors. Martellus's maps mark an attempt to merge this new data with Ptolemy's. Where there was conflict, it was hard to contradict an ancient authority.

Georeferencing a map such as this one differs from the cases examined in most discussions of historical maps and GIS. For example, the 19th and 20th century historical maps discussed in the chapter on "Historical Maps in GIS" by David Rumsey and Meredith Williams in *Past Time, Past Place: GIS for History* include specific data about boundary lines, travel routes, and so on, which already depended on advanced survey technology (Rumsey and Williams 2002). Data of this type do not appear on the *Yujitu*. In a more general sense, though, these authors' observation that "Historical maps capture the attitudes of those who made them and represent worldviews of their time" applies as much to the *Yujitu* as to a map of Lewis and Clark's trek.

The courses of rivers change, the most obvious examples including the Yellow River and the Huai, so georeferencing based on river features in flat alluvial plains is far more difficult than simply linking points on the rivers as depicted on the *Yujitu* with their modern equivalents. It takes research to determine whether a mismatch means inaccuracy, or a shift in the river. Establishing links based on river features is thus not optimal, except for places where the river's course is bounded by significant hills or mountains on each side or some other immobile channel. Thus the points chosen for linkage to modern coordinates are city sites, mountains, constrained points along rivers and coastal sites lying outside of the river deltas.

Ever since Edouard Chavannes published a report on this stone engraving (Chavannes 1903), and particularly after Joseph Needham illustrated it in his paradigm-shifting discussion of science in ancient China (Needham 1959:

547), numerous scholars have cited the *Yujitu* as a technical marvel. Certain aspects of its accuracy have been discussed; Needham writes that "comparison of the network of river systems with a modern chart shows at once the extraordinary correctness of the pattern," while Cao Wanru briefly discusses its scale, based on distances between just three points, Kaifeng, Luoyang and Shangqiu (Cao 1990: 5 & 21). Cordell Yee, in a challenge to Needham's emphasis on mathematical and technical progress in the Chinese cartographic tradition, delves into this and other Chinese maps for their cultural significance (Yee 1994). This is a map, however, that simply cries out to be georeferenced. The network of river courses it depicts against a regular graticule immediately strikes the contemporary viewer as a surprisingly modern map. This could be a rather superficial impression, however; how do we test it?

Our analysis is based on choosing specific landmarks shown in this map that can also be readily identified with modern sites and their coordinates. We have chosen 45 points on the map which can be fairly well identified on modern maps. Twenty-five of these are cities or towns, nine are points along rivers, six are mountains, four are features on the coast and one is a lake. For each of these points, we can identify their true latitude and longitude with reasonable certainty as well as fix their coordinates on the *Yujitu* using its own grid. To obtain latitude and longitude, for the cities and towns we have used the Chinese Historical GIS published online by Harvard University (*http://www.fas.harvard.edu/~chgis/*) and for physical features we have used Google Earth. We list all our landmarks in a table below along with both sets of coordinates These landmarks are shown in Figure 4 on the *Yujitu*.

Note that the map is fairly well covered by these landmarks. The Gansu corridor was well explored and gives some reliable landmarks in the Northwest. In the Southwest unfortunately we were unable to find useful landmarks between the sharp bend of the Yangtze near Panzihua and Hanoi. Dian Lake is marked on the map but it is grossly out of place. Using it when georeferencing creates a major distortion of the map in the Southwest; it is perhaps more useful to as-

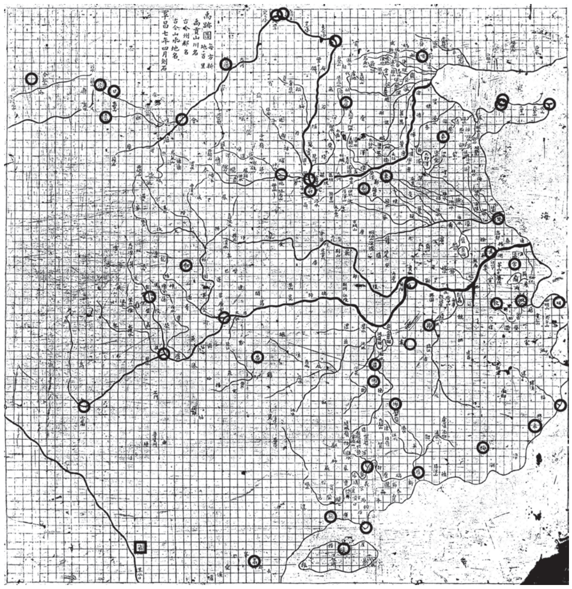


Figure 4. The 45 landmarks chosen on the *Yujitu*.

sume Yunnan had not been accurately surveyed and not to include it when analyzing the map.

It is important to note that although the Chinese GIS provides extremely precise coordinates for city sites, down to six decimal places, these figures are not based on actual measurement but on the derived center point of polygons. In many cases we have found them to be somewhat inaccurate, though at the scale of the *Yujitu* the difference is negligible. In cases where the characters on the *Yujitu* naming sites cover a large part of a grid square, we have generally taken the center of the name as the intended location, but in cases where the characters have been shifted to the side, for instance, moved away from a river for legibility (much as modern maps do), we have made a considered guess of the intended location. For example Tiande Jun (Bayan Nur) is squeezed between the Yellow River and the top of the map and seems to have been placed to the East of its intended location where there was more room on the stone. The case of the Heishui, a river depicted as beginning in the northwest and reappearing in the southwest, is discussed below.

Latitude and the East-West Gridlines

From some point in the first millennium BCE, it was known to Chinese observers that the sun appeared lower in the sky as one went north, and higher as one traveled south. A standard vertical pole, the gnomon, was used to measure the length of the sun's shadow. A manual said to have origins in the Zhou Dynasty, the *Zhou bi suan jing* (Zhou Classic of Gnomon Calculations), survives in a Han Dynasty version. Most notable in this book is the use of the comparison of the sun's shadow at mid-day at two different locations a known distance apart on the same meridian, a test corresponding to Eratosthenes' experiment. However, unlike Eratosthenes, the change of the length of the shadow was modeled by assuming the sun sat at some moderate height above a flat earth, rather than a much more distant sun illuminating a spherical earth. Thus the Chinese deduced from these measurements the putative height of the sun over the flat earth, while the Greeks deduced the radius of a spherical earth from the same measurements. In fact, working out the trigonometry, the formulas are nearly the same. Suppose the gnomon has height h, the two locations are a distance d apart and the shadows have length s_1 and s_2. Then we can compare the *Zhou bi* with Eratosthenes mathematically like this:

Zhou bi: height of sun $= d/\left(\frac{s_1}{h} - \frac{s_2}{h}\right)$

Eratosthenes: radius of Earth =

$$d/\left(\arctan\left(\frac{s_1}{h}\right) - \arctan\left(\frac{s_2}{h}\right)\right)$$

Thus a value of 80,000 *li* for the height of the sun was derived. The *Zhou bi* formula was then inverted, showing that the length of the gnomon's shadow varies linearly with distance as you move north or south. In this way, the length of the shadow could be used not only to measure north/south distances, but as a sort of north/south "coordinate" which could be assigned to every location.

However, this north/south "coordinate" was modified in the Tang Dynasty. By this time Indian thinkers had absorbed and further developed the Greek model of a round earth around which the sun, moon and planets rotated with epicycles. Many parts of their geometric models were quite accurate, such as the size of the earth and the distance from the earth to the moon. The ratio of these last two distances was essential to computing the parallax of the moon and hence accurately predicting solar eclipses – something of great value to the Chinese emperors for proving their legitimacy as sons of Heaven attuned with the cycles of the cosmos. As part of the cultural exchange between Buddhist South Asia and China that took place during the Tang period, Indian mathematics made significant inroads among Tang thinkers. The Buddhist monk Yi Xing 一行 was charged by Emperor Xuanzong with recalibrating latitudinal measurements in order, among other things, to compute solar eclipses more accurately. In 721-725 CE, Yi Xing had precise measurements made of the shadow lengths of the sun at both the summer and winter solstices, as well as the height of the pole star, at thirteen locations between 17° and 51° north. Yi Xing recognized that north-south distances were not proportional to shadow length in these observations, but rather to the angle between the sun (or the North star) and the zenith. In section V of his calendar, the *Da Yan Li*, entitled "Method of Pacing the Gnomon-Clepsydra," he gives recipes which show that as one goes north, the position of the sun, moon, planets and stars all shift from north to south. Celestial objects in the north rise higher, those in the south sink lower amounting to a rotation of the sky around the observer's east-west axis. He calls the angle of rotation the number of "degrees" one has moved north. Thus he has shifted from the inaccurate measurement of north-south distances by the gnomon's noon shadow to the correct measurement, that of the angle between the sun at midday and the zenith. He gives a conversion table between these two measurements that is, in effect, a table of the tangent function. As far as we can determine, no one at the time seems to have taken the further step of explaining this by the hypothesis that the earth was round. (See Ang 1979 for translation and analysis of Yi Xing's work).

This history of observationally-based measurements of north-south distances is reflected in the *Yujitu*, which is generally quite accurate in its north-south placement of landmarks. In Figure 5 we have graphed the north/south *Yujitu* coordinates of all landmarks against their true latitude.

The figure shows a least squares fit of the *Yujitu* coordinates as a linear function of modern latitudes. The fit is very good, with the standard

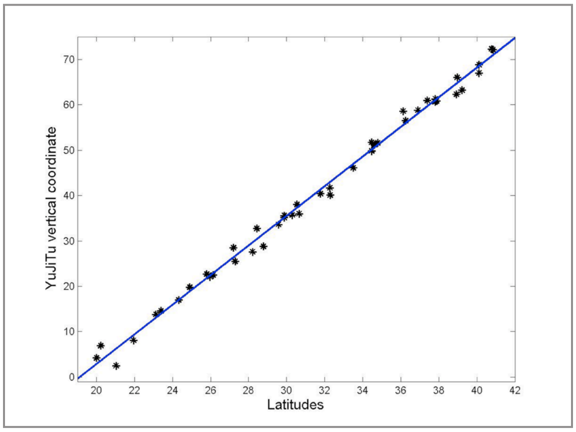


Figure 5. A plot of the *Yujitu's* north/south coordinate against their modern latitude for the 45 landmarks.

deviation of the residuals being 1.52 squares of the *Yujitu's* grid or 0.47 degrees (52 kilometers). This suggests that the angles of heavenly bodies could be measured accurately to within half a degree by the instruments of the time. The latitudes of the southern landmarks, Hanoi and the Leizhou Peninsula, seem to be less accurate, indicating that surveys were not as good in the south. As noted above, we have excluded Dian Lake as a landmark because it appears grossly out of place and thus does not fit in this regression line at all well. One can use the slope of the least squares line to relate the Chinese length unit *li* to kilometers. This gives one *li* on the map as equal to approximately 340 meters, close to the estimated Tang Dynasty value of 323 meters. In conclusion, the horizontal lines on the *Yujitu* are indeed quite close to true equally spaced lines of constant latitude.

A similar plot of the *Yujitu* vertical coordinate against the tangent of latitude does not fit quite

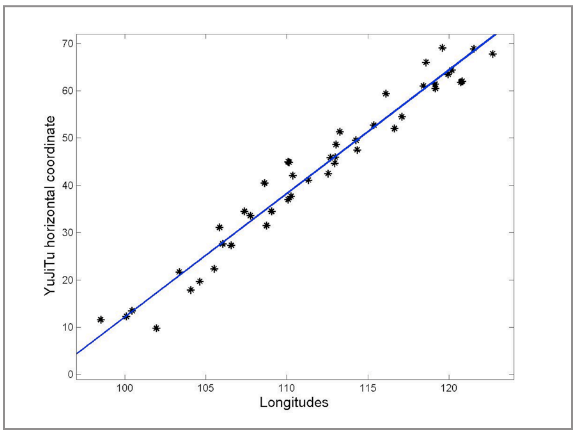


Figure 6. A plot of the *Yujitu's* east/west coordinates against longitude for the 45 landmarks.

as well, confirming that the angular altitudes of heavenly bodies, not gnomon shadow lengths, were used to position features in the north-south axis. The standard deviation is 18% larger.

We can also make a similar plot for longitude versus the *Yujitu's* east/west coordinate. This is shown in Figure 6. There is a large scatter; in fact, the standard deviation of the residuals here is 3.35 grid squares, more than double that for latitude. Clearly the placement of locations on the east-west axis needs more analysis, so for this, we turn to georeferencing. The following section is concerned with the mathematics behind the computation of optimal warping.

A New Algorithm for Georeferencing

There is a long tradition in biology, starting from the famous book *On Growth and Form* by D'Arcy Thompson, of comparing the shapes of animals by computing a warping of the plane or space containing them or their two-dimensional projections (Thompson 1961). The warping of skull surfaces in Figure 7 is actually the same process as the nonlinear georeferencing used for fitting historical maps to modern coordinates.

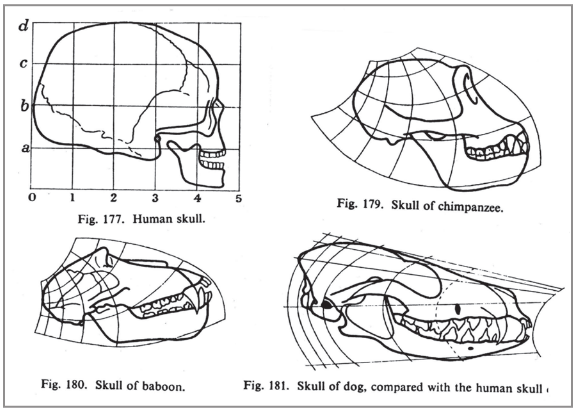


Figure 7. Skulls of two primates and a dog warped to a human skull, from D'Arcy Thompson, *On Growth and Form*.

Thompson's ideas were further developed by a group of statisticians, David Kendall, Kanti Mardia and others, who were studying data made up of sets of landmark points: characteristic points on shapes of interest. Ulf Grenander and Michael Miller introduced a method for computing the optimal warping of the ambient Euclidean space that matches corresponding landmarks based on modeling the warping as resulting from a fluid flow in all of space and the differential geometric idea of geodesics. Miller's group has used this extensively to register pairs of MRI scans and other medical images as well as to estimate average normal body configurations. (The work of the statistical community is summarized in Dryden and Mardia 1998. See Younes 2010 for the mathematical approach to shape and especially medical images, which started with Ulf Grenander and Michael Miller).

Nonlinear georeferencing in cartography has usually been done with splines. Although this is generally quite satisfactory, it is important to realize that splines are based on the idea of splicing together multiple low order polynomials to create smooth functions. This approach does not take into account the constraint that one seeks a *one-to-one* warping, which is called a *diffeomorphism* in mathematics. That is, one seeks a differentiable map with an *inverse* that is then automatically differentiable also. Whether one is comparing two skulls or two maps, one wants to relate them by stretching a rubber sheet but not letting the sheet double back on itself. To incorporate this constraint requires quite a different type of mathematics.

Mathematically, a warping is a smooth invertible map ϕ: $\mathrm{R}^n \to \mathrm{R}^n$ (where R^n stands for Euclidean n-dimensional space). The set of all such warpings is called the diffeomorphism group of R^n. The basic idea is to seek not just one warping but a time varying set of warping maps ϕ_t, starting when $t = 0$ at a map which moves nothing and ending when $t = 1$ at a map which carries one set of landmarks to the other set. Think of this as a fluid flow that carries the original landmarks with it, bringing them to the position of the new landmarks. At any instant of time t, each point x in space (not just the landmarks) is moving in some direction, given by a vector $v(x,t)$. Formally this means:

$$\frac{\partial \phi_t}{\partial t}(x) = v\,(\phi_t(x),\, t) \tag{1}$$

We then measure how large the warping is by introducing a norm on vector fields and integrating this norm with respect to t. The simplest norm on vector fields $v(x)$ is given by:

$$\|v\|^2 = \int_{\mathrm{R}^n} \langle (I - c^2\Delta)^k v(x),\, v(x) \rangle \, dx \tag{2}$$

where Δ stands for the Laplace operator and c is a constant whose units are length in the space in which x lies: this penalizes the size of the individual vectors $v(x)$ and their low order x-derivatives (more derivatives as k gets bigger). Then the size of the final warping is defined as:

$$\mathrm{size}(\phi) = \min_{\text{paths } \{\phi_t\}} \int_0^1 \|\, v\,(g,t)\, \|\, dt \tag{3}$$

where the minimum is taken over all paths starting at the identity and ending at the given warping ϕ. The minimizing path is known as a *geodesic* in the diffeomorphism group for the metric defined by the above norm on vector fields.

If we anchor this now to landmark points, we can simplify everything. Let $\{x_1,\ldots,x_N\}$ be one set of N landmark points and let $\{y_1,\ldots,y_N\}$ be another. Then what we really want is to find the warping ϕ of smallest size which satisfies $\phi(x_i) = y_i$. We can find this in two steps. The first is to solve the infinitesimal version: suppose we are given v_i, a vector at the point x_i, for all i. Then it's not hard to find the vector field $v(x)$ of least norm such that $v(x_i) = v_i$. It is given by the so-called *Green's function* $G(x)$ of the operator $(I - c^2\Delta)^k$ by means of the formula:

$$v(x) = G(x - x_1)u_1 + \ldots + G\,(x - x_N)u_N \tag{4}$$

where the vectors u_i, called the *momenta*, are chosen so that v has the right values at x_i: $v(x_i) = v_i$. This expression is central in the theory: it shows how infinitesimal movements of the landmarks extend to vector fields on the whole ambient space. One introduces the N by N matrix $\boldsymbol{G}$ with i, j^{th} entry $G(x_i - x_j)$. Then the u_i and the norm of v are given like this:

$$u_i = \sum_j (\boldsymbol{G}^{-1})_{ij} v_j,\ \|v\|^2 = \sum_i \langle u_i, v_i \rangle = \sum_{i,j=1}^{N} (\boldsymbol{G}^{-1})_{ij} \langle v_i, v_j \rangle \tag{5}$$

Finally, instead of having to deal with solving for geodesics in the very large diffeomorphism group, we can now deal only with computations involving the landmarks, extending to warpings using equation (4). This means we use equation (5) to define a norm for vectors on the space of all N-tuples $\{v_i\}$ and need only compute geodesic paths $\{x_i(t)\}$ in the much smaller space of landmark points. Once these are found, we extrapolate its tangent vectors to vector fields $v(x,t)$ on the ambient space using (4) and integrate to find the path $\{\phi_t\}$ in the diffeomorphism group.

In many cases, however, the landmark points are not known exactly, perhaps because some noise was involved in their measurement, and one wants to solve for a warping ϕ such that the differences $\phi(x_i) - y_i$ are small. The simplest way to do this is to solve for paths $\{\phi_t\}$ and $\{x_i(t)\}$ minimizing:

$$\int_0^1 \left(\| v(g,t) \|^2 + \lambda^{-2} \sum_i \| v(x_i(t), t) - dx_i/dt \|^2 \right) dt \qquad (6)$$

Here λ measures the standard deviation of the allowed velocity noise and $\lambda = 0$ is the case of exact matching. This has the effect that we seek geodesics on landmark space with the modified metric allowing for noise $(\boldsymbol{G} + \lambda^2 \boldsymbol{I})^{-1}$.

A few comments on this algorithm:

a. First of all, there need not be a single optimal warping of the ambient space carrying one set of landmark points to another. This is because the space of landmark points has some *positive curvature*, hence geodesics need not be unique. This occurs for example when two points change position. Then they must move around each other and they have a choice: two ways to do this in 2-space, infinitely many ways in higher space.

b. To actually compute geodesics, there is the *shooting method* where one integrates forward the geodesic differential equation and progressively refines its initial direction to make the geodesic come closer to the desired endpoint. And there are *minimization methods* where one chooses any path to the desired endpoint but moves the whole path to minimize its length. In our experiments we have used a very effective variant of the latter following Liu *et al.* (2010).

c. A key question in using this algorithm, however, is what parameters must be set to solve for the warping. There are three: c, λ and k. c determines how far the effect of moving one landmark point extends, so as to drag nearby landmark points with it and λ and c together determine how exactly the warping matches the landmarks. This can be measured by standard deviation:

$$\text{mismatch} = \sqrt{\sum_{i=1}^{N} \| \phi(x_i) - y_i \|^2 / N} \qquad (7)$$

One must experiment with c and λ so the mismatch is similar to the best estimate of the noisiness in the data. One must also experiment with c so that the warping neither overfits the data with unnecessary wrinkles nor overly smoothes fluctuations. This last is what statisticians call a *bias/variance* trade-off. We will give examples of how this works below when we apply the algorithm to the *Yujitu*.

d. As for k, it determines how many derivatives are penalized in measuring the size of a warping. It seems best to keep it as small as possible because higher derivatives are not physically meaningful and also lead to very ill conditioned metrics. If you are working in n-space, k must be greater than (n+1)/2 to make the math work. In our case, we have n=2 and we could use k=2. But k=5/2 leads to the very simple Green's function G given by:

$$\boldsymbol{G}(x) = (1 + |x/c|)\, e^{-|x/c|} \qquad (8)$$

so we used this.

Georeferencing the *Yujitu* Via Landmark Space Geodesics

For the *Yujitu* we have 45 points (x_i, y_i) given in the intrinsic *Yujitu* coordinate system with x_i ranging from 0 to 70 and y_i ranging from 0 to 73 and the corresponding 45 points (u_i, v_i) given by their true longitude u_i and latitude v_i. These points appear in table 1.

We first need to center and rescale the *Yujitu* coordinates to approximately register the maps before warping. We do this by first rescaling the *Yujitu* using the slope of least squares fit of its north/south coordinate with the true latitude and then translating the *Yujitu* coordinates so their mean equals the mean of their longitude and latitude. In formulas, let $(x(0), y(0))$ and $(u(0), v(0))$ be the means, let α be the inverse of the slope of the regression line in Figure 2, that is the number of degrees per grid line. α equals 0.306 degrees. Then we change coordinates on the *Yujitu* via:

$$\tilde{x}_i = \alpha x_i - \alpha x(0) + u(0), \tilde{y}_i = \alpha y_i - \alpha y(0) + v(0) \qquad (9)$$

YJT Inscription	Pinyin transliteration	Modern site used as equivalent	LATITUDE	LONGITUDE	YJT VERTICAL COORDINATE	YJT HORIZONTAL COORDINATE
成都	Chengdu		30.65	104.08	36	17.9
杭州	Hangzhou		30.29	120.17	35.7	64.35
登州	Dengzhou		37.80	120.74	60.6	61.8
梅州	Meizhou		24.32	116.11	17	59.4
廣州	Guangzhou		23.13	113.26	13.8	51.3
思州	Sizhou	Siyangzhen	27.21	108.75	28.5	31.5
瓊州	Qiongzhou		20.01	110.36	4.2	42.1
漢陽	Hanyang	Wuhan	30.54	114.26	38	49.5
開封	Kaifeng		34.79	114.34	51.6	47.5
順州	Shunzhou	Shunyi	40.12	116.65	67	52
甘州	Ganzhou	Zhangye	38.93	100.45	62.3	13.5
常州	Changzhou		31.78	119.95	40.4	63.5
楚州	Chuzhou		33.50	119.14	46.1	61.4
歙州	Shezhou	Shexian	29.87	118.43	35.1	61
劍州	Jianzhou	Jiange	32.29	105.52	40.1	22.4
明州	Mingzhou	Ningbo	29.87	121.54	35.6	68.9
潭州	Tanzhou	Changsha	28.20	112.98	27.6	46
欽州	Qinzhou		21.95	108.62	8.1	40.5
郴州	Chenzhou		25.80	113.03	22.7	48.6
筠州	Yunzhou	Gao'an	28.43	115.37	32.7	52.7
真州	Zhenzhou	Yizheng	32.27	119.18	41.7	60.5
天德軍	Tiande Jun	Bayan Nur	40.77	107.39	72.3	34.5
泉州	Quanzhou		24.91	118.59	19.8	66
安南府	Annan Fu	Hanoi	21.03	105.85	2.5	31.1
太原	Taiyuan		37.87	112.56	60.8	42.5
祁連山	Qilian Shan		39.22	98.51	63.2	11.6
賀蘭山	Helan Shan		38.96	106.03	66	27.6
泰山	Tai Shan	East Marchmont	36.26	117.10	56.5	54.5
華山	Hua Shan	West Marchmont	34.46	110.08	49.8	37
嵩山	Song Shan	Center Marchmont	34.48	112.96	49.9	44.7
衡山	Heng Shan	South Marchmont	27.29	112.70	25.5	45.9
青海	Qinghai	Koko Nor	36.90	100.10	58.8	12.3
COAST AND RIVER LANDMARKS						
Penglai cape			37.82	120.82	61.3	62
Wending peninsula cape			37.39	122.70	60.9	67.8
Mouth of Min River near Fuzhou			26.13	119.60	22.5	69.1
Leizhou peninsula cape			20.23	110.14	6.9	44.9
Sharp bend in Yangtze near Panzhihua			25.97	101.94	22.1	9.8
Confl. Yangtze, Min Rivers at Yibin			28.77	104.63	28.8	19.7
Confl. Yangtze, Jialing R. at Chongqing			29.57	106.58	33.6	27.3
Confl. Yellow, Huangshui near Lanzhou			36.12	103.36	58.6	21.7
NW corner Ordos Loop			40.84	107.75	72.1	33.6
NE corner Ordos Loop			40.11	111.34	68.8	41.1
Confluence Jing, Wei Rivers, West of Xi'an			34.46	109.06	51.7	34.5
Confluence Yellow, Wei Rivers			34.62	110.26	51.2	37.7
Pearl River tributaries: Yu+Qian->Xi at Guiping			23.40	110.09	14.6	45

Table 1.

and seek an optimal diffeomorphism ϕ such that $\phi(\tilde{x}_i, \tilde{y}_i) = (u_i, v_i)$. Then we need to choose the parameters. We have seen above that the mismatch of the north/south *Yujitu* coordinate was approximately 1.53 grids, or 0.47 degrees. Assuming the east/west coordinate error has about the same value, we estimate that a reasonable two-dimensional mismatch should be around $\sqrt{2} \cdot 0.47 \approx 0.66$ degrees. On the other hand, we seek a diffeomorphism that does not show bumps and twists due to over-fitting the irregularities in our sampling of landmark points but remains sensitive to large scale systematic differences between the *Yujitu* and modern maps.

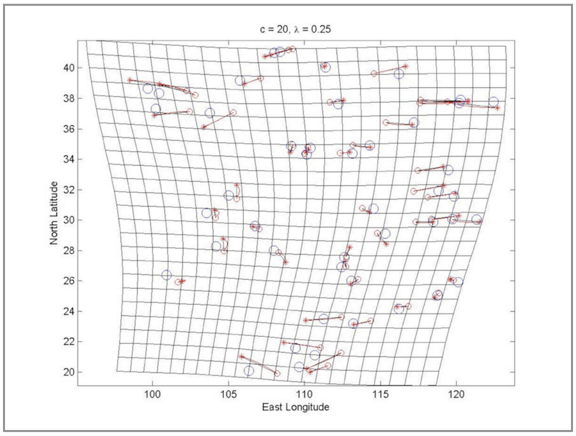
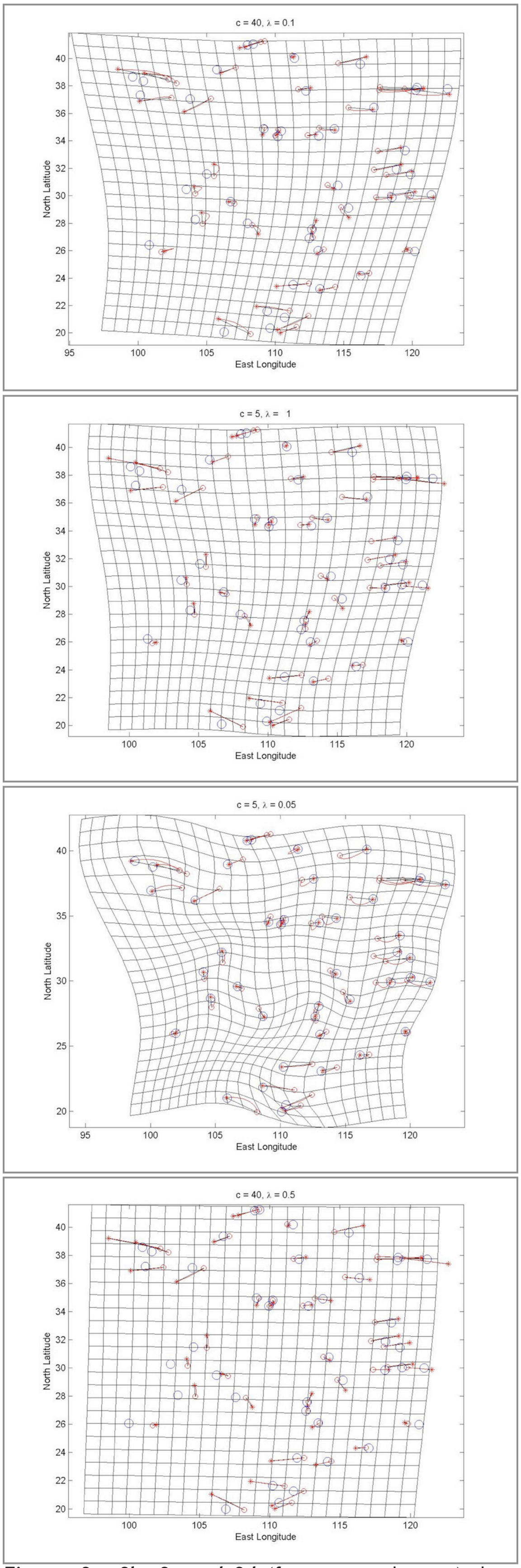


Figure 8. Georeferencing the *Yujitu's* grid lines to modern latitude and longitude using the 45 landmarks.

In Figures 8 and 9 we show the georeferencing given by the computed diffeomorphism for five different pairs of values (c, λ). We display this using modern longitude and latitude as axes. We show with small circles the 45 *Yujitu* landmarks (x_i, y_i) in rescaled and translated coordinates; we show with stars the 45 modern landmarks (u_i, v_i); and we show with big circles the georeferenced *Yujitu* $\phi(x_i, y_i)$ landmarks. Finally we show with solid lines the georeferenced *Yujitu* grid, taking every third horizontal and vertical line for clarity, starting at the second grid line in from the bottom and left because there are no landmarks near these edges. The best diffeomorphism was found with c equal to 20 degrees, λ equal to 0.25, giving a mismatch of 0.64 degrees. This is shown in figure 8. Alternate choices of these values are shown in figure 9. The values c=40, λ=0.1 and c=5, λ=1 gave mismatches of 0.64 and 0.67 respectively with resulting warps which are very



Figures 9a, 9b, 9c and 9d (from top to bottom) show alternate georeferencing with various vales of c and λ.

similar to what we took as the optimal. We conclude that the warping is not very sensitive to *c* so long as the mismatch is the same. On the other hand, when the mismatch and *c* are varied together, dramatically different warpings result. Thus we also show $c = 5$ and $\lambda = 0.05$ which means that more exact matches are required and the warping does not propagate over very long distances. The result is a very wiggly warping that clearly is incorporating many small errors in the *Yujitu* and treating them as significant. This is what statisticians refer to as an error due to seeking low bias at the expense of high variance in the model. Finally we show $c = 40$, $\lambda = 0.5$. Here we get only a small nearly affine warping which clearly does not capture major systematic aspects of the data. This is what statisticians refer to as an error due to seeking a low variance model at the expense of high bias (i.e. the model is biased to be very smooth).

The most obvious conclusion is that the horizontal grid lines of the *Yujitu* are nearly flat, running east/west with good accuracy and equal spacing; meanwhile, the vertical grid lines wave and flare at the top of the map, betraying their role as markers of distance across the surface (as described in the map's key) rather than north/south meridian lines. To test this, we can display the georeferenced grid lines using east/west distance rather than longitude with the same scale as the vertical latitude coordinate. Degrees of latitude all have the same length, namely 111.325 kilometers (except for a minor correction of less than 1% due to the oblateness of the earth), and this is also the length of a degree of longitude along the equator. Taking this as a scale, the legend would make the *Yujitu* square. But what should we take as the point from which to measure east/west distances? None of the georeferenced vertical grid lines are particularly straight, so we take the middle or median vertical grid line and measure distance from this. The result is shown in Figure 10. With these coordinates the *Yujitu's* gridlines nearly form a square, confirming that the requirement for 100 *li* spacing was the primary consideration in the layout of the map. In other words, it was based on the flat-earth model of Pei Xiu and his tradition of maps based on grids formed by perpendicular lines. Clearly the legend's assertion that the vertical grid lines were equally spaced 100 *li* apart took precedence. One effect of the distortion necessarily introduced is that the horizontal and vertical grid lines represent contours on the earth's surface that do not form a right angle when they meet. This is particularly evident in figure 9 in the region of 116 E / 27 N, for example.

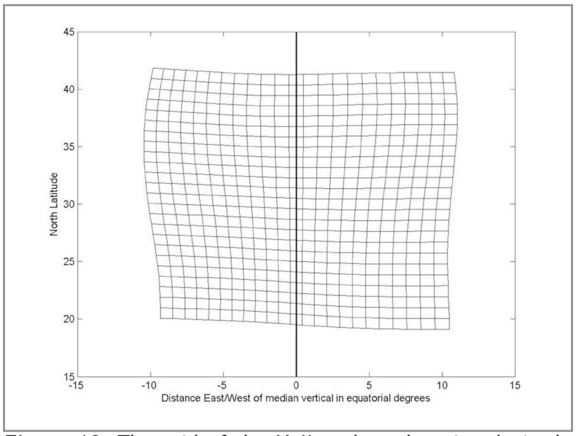


Figure 10. The grid of the *Yujitu* plotted against latitude and the east/west distance from its middle vertical line.

Assuming the horizontal grid lines do go east/west and that the vertical grid lines intersect them with constant spacing but perpendicularity is not respected, the cartographers who made the *Yujitu* had the freedom to shear their grid. That is, they could have fixed any one of the vertical lines and laid out the remaining verticals at constant distances to the east and west. In fact, inspecting the Eastern part of Figure 8 suggests that these vertical grid lines were laid out to follow more closely the coast which in the south actually trends nearly due west. This could, for example, have been done so as to fit the whole of China most economically in a square map.

It should be noted that no two dimensional map can represent distances on a spherical surface accurately in all directions. Even if you lay out the map to make the edges of each grid square 100 *li*, the diagonals of these grid squares cannot all have length $100\sqrt{2}$ *li*. The grid squares will inevitably become more and more skewed as the map covers more territory, with one diagonal smaller, one larger than $100\sqrt{2}$ *li*. Once we have georeferenced the grid lines in the *Yujitu* using our landmarks, we can georeference the whole map. This result is shown in Figure 11.

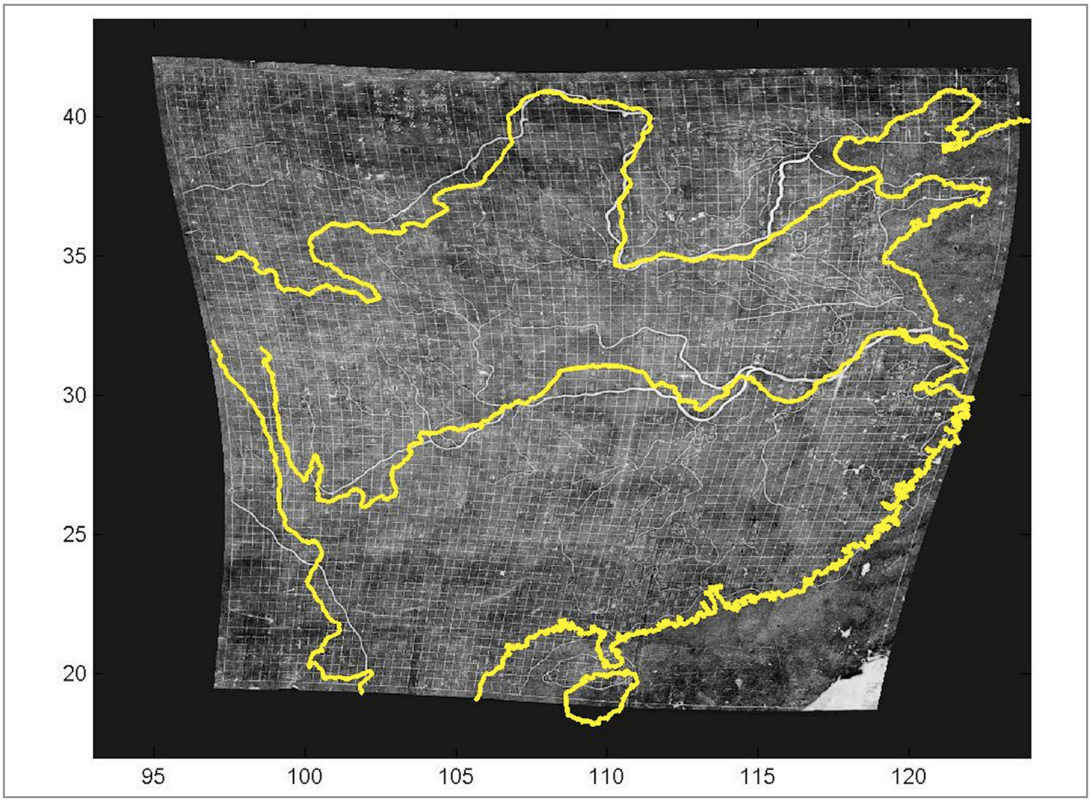


Figure 11. The entire *Yujitu* is shown georeferenced in latitude/longitude coordinates, with the modern coastline, Yellow, Yangtze and Mekong rivers superimposed.

Conclusion

The entire map has been warped by the same diffeomorphism used to bring the landmarks close to their true latitude and longitude and is shown there with the true coastline and modern courses of the Yellow, Yangtze and Mekong rivers. Several things are immediately evident. Firstly, the *Yujitu* correctly shows the flow of the lower course of the Yellow River before it shifted dramatically; this is not a flaw in representation but a major change that took place in the natural world. On the other hand, Hainan Island and the adjacent coastline to its West are greatly distorted: Hainan is shown running westerly very close to the Vietnamese coastline rather than southwesterly with the large Tonkin Gulf between it and Vietnam. Devices used for measuring distances on land were not helpful at sea, so this could be due to misjudgment of distance - or simply an effort to squeeze the island onto a square map. Moreover, the northeastern frontier has also been pulled down so that the northern bay of the Bohai Gulf disappears, possibly so that the mouth of the Liao River remains on the map. (Note that the name of the Liao River is given on the map but the river itself is not depicted. For an introduction to Chinese survey methods and methods of calculating distance, see Needham 1959: 569-579).

There are some blatant errors on the map; for example, Ya prefecture on the island of Hainan is placed at the opposite end of the island from its true site. Other contemporary sources have it in the right place (including the *Huayitu*, which was carved on the reverse side of the very same stone as the *Yujitu*, only six months later). Another landmark which is shifted dramatically is Dian Lake in the southwest which is shown only one east/west grid line higher than Hanoi although it should be about 12 lines higher on the map to be consistent with the map's scale. Such errors should be noted in any discussion of the map's accuracy.

The cartographer(s) who composed the *Yujitu* treated the *Yugong* as something of a sacred text; information already thought to be incorrect at the time the *Yujitu* was carved remains in place, perhaps because of the cartographer's misgivings about challenging a work from the Confucian canon. As Cao Wanru notes, the source of the Yangtze River appears in a way that hints at contemporary knowledge while not daring to entirely refute the *Yugong's* assertion that it begins in the Min Mountains (Cao 1990: 5 & 21). Also, rather than having the Yellow River originate in the Kunlun Mountains, a common belief when this map was made, it is shown as beginning at a place called Jishi, the traditional origin given in the *Yugong*. A georeferencing based on river courses would pose a dilemma because the map shows courses already known by many to be incorrect at the time it was made. Any discussion of "accuracy" would have to incorporate this intellectual and cultural context.

The compilers of the *Yujitu* found ways to tiptoe around cases where the *Yugong* is particularly obscure. One visually striking example in the southwest is the Heishui, or Black River, which is only vaguely described in the *Yugong* as starting at Sanwei Mountain 三危山 and flowing into the southern sea. The *Yujitu* shows Sanwei mountain in the far northwest, in what is now Gansu, and then the river conveniently goes off the map, swinging back into the southwest before entering the sea in the south. At the time the *Yugong* was composed not much was known about the far west, but by the 12th century it was understood that there was a river basin containing the Mekong and other rivers flowing down from the Himalayas through southeast Asia, where Buddhist monks and state-sponsored diplomats had by this time traveled. The compilers of this map, in a bit of georeferencing of their own, decided to equate the southern portion of the fleetingly described Heishui in the *Yugong* with one of these rivers. How exactly it could get from central Asia down to the south was a problem, but one that could be evaded by taking its course off the edge of the map.

We have no way of knowing whether the Heishui recorded in the *Yugong* was originally based on vague knowledge of a real river, or on mythology (if real, it most likely would have been a river closer to the interior of "China proper"). By the time the *Yujitu* was carved, geographical knowledge had expanded dramatically, which made it necessary to portray the river, which simply *had* to exist because it was mentioned in the Confucian canon, as flowing out far beyond the limits of the surveyed world to reappear as one of the rivers flowing into southeast Asia. (For further discussion of efforts by Chinese scholars to place the Heishui, see Akin (2009)).

We have presented several aspects of the *Yujitu* that have not been noted before, in the arenas of both cultural history and mathematical analysis. The map's distortion, which shows that location along the vertical axis is substantially more accurate than along the horizontal axis, suggests powerfully that placement of sites on the north-south axis was based on observation of the sun's angle from a substantial number of locations. These observations created an equivalent of latitude measurement. While it has long been known that Yi Xing took such measurements during the Tang dynasty, we have no indication of the sources used for generating this map, whether they were collected from records available at the time or collected anew. This accuracy on the north-south axis does not apply to all features on the map, as we have noted above in the cases of features like Dian Lake, but in the North, for example, places with similar latitude which were as much as 2000 kilometers apart are accurately located on the same east-west line. This accuracy would have been extraordinarily difficult to achieve if based only on terrestrial surveys and measurement of distance on roads. In measuring east-west locations, there was no celestial observation that could determine whether two locations were on the same meridian, so the placement of sites on the east-west axis was substantially less accurate, as shown in figure 7. However *distances* along the east-west axes, presumably based on terrestrial surveys, are quite accurate as shown in figure 10.

Despite the strikingly modern appearance of the map, and its survey-based elements, we have also shown that the *Yujitu* served a particular cultural goal that we overlook if we examine only the ways in which it agrees with modern cartography. The *Yujitu's* modern appearance masks the fact that it actually clings to a textual tradition rather than incorporating the most accurate data available, as has been shown in the case of the Heishui and other rivers. Because the map was intended to reflect "The traces of Yu," its compilers strove to match the testimony of the *Yugong* even when that document became difficult to reconcile with currently available data. This was not true of all maps at the time; the *Huayitu*, another map carved on the reverse of the very same stone only months later, presents a very different picture of the rivers in the southwest, indicating several courses rather than the single stream of the mythical Heishui (Needham 1959: 292-293).

Our analysis leaves many important questions unanswered. One of these must be, what was the relationship of the *Yujitu* to the cartographic milieu of the day? It appears to us as a stunning artifact rearing suddenly out of the mists of obscurity, but clearly this was no orphan work, composed at one go; the gathering of the information it represents would have taken the collaboration of many people. Perhaps a team worked together on a new cartographic project, or it may be the case that a number of existing works, even Pei Xiu's work before it was lost, were gathered and synthesized by one or more scholars. The resolution of this mystery may await the discovery of new evidence as yet unknown.

The *Yujitu* illustrates the tenacity of a flat earth view. Even if its compositors realized that the earth was round this knowledge was not imparted in its projection, which attempts to follow the principles that Pei Xiu devised for maps of smaller regions. With the use of a compass and sighting tube an observer of the time had the means to determine that the north/south lines as depicted on the map deviated significantly from true north/south, with the southeastern coastline veering as much as 18 degrees off true. Having little evidence of the map's development, however, we have even less understanding of whether such a technique might have been applied or its results comprehended. Further research, most likely to be found in textual records rather than maps, may reveal whether there were efforts to find true long-distance orientation between distant points during this period.

Only a generation earlier, before the Jin invasion, the polymath Shen Kua (1031-1095) is said to have made a map with unprecedented accuracy. Shen, who had traveled extensively as both an official and a diplomat, spent at least two years after his impeachment completing a monumental map with 43 panels drawn at a scale of 1:900,000 (the *Yujitu* has a scale of about 1:3,000,000). Most unfortunately, this map is now lost. He is said to have perfected the use of the compass for measuring directions, devising a system of recording directions as multiples of 15º, so one would expect his map to have been informed by reasonably good directional data (Sivin 1970: 380). Could there have been any relationship between Shen's map and the content of the later *Yujitu*? Alas, this question must remain unanswered for the time being.

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