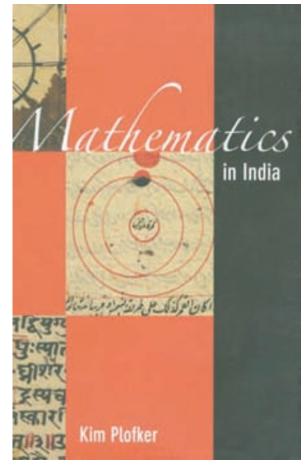
Mathematics in India

Reviewed by David Mumford



Mathematics in India
Kim Plofker
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Did you know that Vedic priests were using the so-called Pythagorean theorem to construct their fire altars in 800 BCE?; that the differential equation for the sine function, in finite difference form, was described by Indian mathematician-astronomers in the fifth century CE?; and that "Gregory's" series $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$ was proven using the power series for arctangent and, with ingenious summation methods, used to accurately compute π in southwest India in the fourteenth century? If any of this surprises you, Plofker's book is for you.

Her book fills a huge gap: a detailed, eminently readable, scholarly survey of the full scope of Indian[1] mathematics and astronomy (the two were inseparable in India) from their Vedic beginnings to roughly 1800. There is only one other survey, Datta and Singh's 1938 *History of Hindu Mathematics*, recently reprinted but very hard to obtain in the West (I found a copy in a small specialized bookstore in Chennai). They describe in some detail the Indian work in arithmetic and algebra and, supplemented by the equally hard to find *Geometry in Ancient and Medieval India* by Sarasvati Amma (1979), one can get an overview of most topics.[2] But the drawback for Westerners is that neither gives much historical context or explains the importance of astronomy as a driving force for mathematical research in India. While Western scholars have been studying traditional Indian mathematics since the late eighteenth century and Indian scholars have been working hard to assemble and republish surviving Sanskrit manuscripts, a widespread appreciation of the greatest achievements and the unique characteristics of the Indian approach to mathematics has been lacking in the West. Standard surveys of the history of mathematics hardly scratch the surface in telling this story.[3] Today, there is a resurgence of activity in this area both in India and the West. The prosperity and success of India has created support for a new generation of Sanskrit scholars to dig deeper into the huge literature still hidden in Indian libraries. Meanwhile the shift in the West toward a multicultural perspective has allowed us Westerners to shake off old biases and look more clearly at other traditions. This book will go a long way to opening the eyes of all mathematicians and historians of mathematics to the rich legacy of mathematics to which India gave birth.

The first episode in the story of Indian mathematics is that of the *Śulba-sūtras*, "The rules of the cord", described in section 2.2 of Plofker's book.[4]

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[1] *The word "India" is used in Plofker's book and in my review to indicate the whole of the Indian subcontinent, including especially Pakistan, where many famous centers of scholarship, e.g., Takshila, were located.*

[2] *For those who might be in India and want to find copies, Datta and Singh's book is published by Bharatiya Kala Prakashan, Delhi, and Amma's book by Motilal Banarsidass, Delhi. An excellent way to trace the literature is through Hayashi's article "Indian mathematics" in the AMS's CD* History of Mathematics from Antiquity to the Present: A Selective Annotated Bibliography *(2000).*

[3] *The only survey that comes close is Victor Katz's* A History of Mathematics.

[4] *There are multiple ways to transcribe Sanskrit (and Hindi) characters into Roman letters. We follow the precise scholarly system, as does Plofker (cf. her Appendix A) which uses diacritical marks: (i) long vowels have a bar over them; (ii) there are "retroflex" versions of t, d, and n where the tongue curls back, indicated by a dot beneath the letter; (iii) h, as in th, indicates aspiration, a breathy sound, not the English "th"; and (iv) the "sh" sound is written either as ś or as ṣ (the two are distinguishable to Indians but not native English speakers).*

These are part of the "limbs of the Vedas", secular compositions[5] that were orally transmitted, like the sacred verses of the Vedas themselves. The earliest, composed by Baudhāyana, is thought to date from roughly 800 BCE. On the one hand, this work describes rules for laying out with cords the sacrificial fire altars of the Vedas. On the other hand, it is a primer on plane geometry, with many of the same constructions and assertions as those found in the first two books of Euclid. In particular, as I mentioned above, one finds here the earliest explicit statement of "Pythagorean" theorem (so it might arguably be called Baudhāyana's theorem). It is completely clear that this result was known to the Babylonians circa 1800 BCE, but they did not state it as such—like all their mathematical results, it is only recorded in examples and in problems using it. And, to be sure, there are no justifications for it in the *Śulba-sūtras* either—these sutras are just lists of rules. But Pythagorean theorem was very important because an altar often had to have a specific area, e.g., two or three times that of another. There is much more in these sutras: for example, Euclidean style "geometric algebra", very good approximations to $\sqrt{2}$, and reasonable approximations to π.

Another major root of Indian mathematics is the work of Pāṇini and Piṅgala (perhaps in the fifth century BCE and the third century BCE respectively), described in section 3.3 of Plofker's book. Though Pāṇini is usually described as the great grammarian of Sanskrit, codifying the rules of the language that was then being written down for the first time, his ideas have a much wider significance than that. Amazingly, he introduced abstract symbols to denote various subsets of letters and words that would be treated in some common way in some rules; and he produced rewrite rules that were to be applied recursively in a precise order.[6] One could say without exaggeration that he anticipated the basic ideas of modern computer science. One wishes Plofker had described Pāṇini's ideas at more length. As far as I know, there is no exposition of his grammar that would make it accessible to the non-linguist/Sanskrit scholar. P. P. Divakaran has traced the continuing influence of the idea of recursion on Indian mathematics,[7] leading to the thesis that this is one of the major distinctive features of Indian mathematics.

[5]*Technically, they are called* smṛti *("remembered text") as opposed to* śruti *("heard", i.e., from divine sources).*

[6]*To get a glimpse of this, see Plofker, p. 54; F. Staal, "Artificial languages across sciences and civilization",* J. Indian Philosophy*, pp. 89–141 (esp. sections 11–12), 2006; or B. Gillon, "Aṣṭādhyāyī and linguistic theory",* J. Indian Philosophy*, pp. 445–468, 2007.*

[7]*"Notes on Yukti-Bhāṣā: Recursive methods in Indian mathematics", forthcoming in a book entitled* Studies in the History of Mathematics in India.

Piṅgala, who came a few centuries later, analyzed the prosody of Sanskrit verses. To do so, he introduced what is essentially binary notation for numbers, along with Pascal's triangle (the binomial coefficients). His work started a long line of research on counting patterns, including many of the fundamental ideas of combinatorics (e.g., the "Fibonacci" sequence appears sometime in 500-800 CE in the work of Virahānka). There is an interesting treatment of this early period of Indian mathematics in Frits Staal's excellent recent book *Discovering the Vedas*,[8] ch.14. For example, Staal traces recursion back to the elaborate and precise structure of Vedic rituals.

After this period, unfortunately, one encounters a gap, and very little survives to show what mathematicians were thinking about for more than 500 years. This was the period of Alexander's invasion, the Indo-Greek Empire that existed side by side with the Mauryan dynasty including Aśoka's reign, and the Indo-Scythian and Kushan empires that followed. It was a period of extensive trade between India and the West, India and China. Was there an exchange of mathematical ideas too? No one knows, and this has become a rather political point. Plofker, I believe, does a really good job discussing the contentious issues, stating in section 4.6 the "consensus" view but also the other points of view. She states carefully the arguments on both sides and lets the reader take away what he or she will. She deals similarly with the early influences from the Middle East in section 2.5 and of the exchanges with the Islamic world in Chapter 8.

For my part, I follow my late colleague David Pingree, who trained a whole generation of scholars in ancient mathematics and astronomy. He argues that the early version of Greek astronomy, due to Hipparchus, reached India along with Greek astrology. The early Indian division of the ecliptic into twenty-eight *Nakṣatras*, (the moon slept with a different wife every night in each trip around the ecliptic) was replaced by the Greek zodiac of twelve solar constellations and—more to the point—an analysis of solar, lunar, and planetary motion based on epicycles appears full-blown in the great treatise, the *Āryabhaṭīya* of Āryabhaṭa, written in 499 CE. But also many things in the Indian treatment are totally different from the Greek version. Their treatment of spherical trigonometry is based on three-dimensional projections, using right triangles *inside* the sphere,[9] an approach

[8]*Penguin Books, 2008.*

[9]*A basic formula in the Gola section of the* Āryabhaṭīya *is that if P is a point on the ecliptic with longitude λ, then the declination δ of P is given by $\sin(\delta) = \sin(\lambda).\sin(i)$, i the inclination of the ecliptic. If I understand it right, later writings suggest this was proven by considering the planar right triangle given by P, P_1, P_2, where P_1 is the orthogonal projection of P onto the plane of the equator (inside the sphere!) and P_2 is its projection onto the line*

which I find much simpler and more natural than Ptolemy's use of Menelaus's theorem. Above all, as mentioned above, they found the finite difference equation satisfied by samples $\sin(n.\Delta\theta)$ of sine (see Plofker, section 4.3.3). This seems to have set the future development of mathematics and astronomy in India on a path totally distinct from anything in the West (or in China).

It is important to recognize two essential differences here between the Indian approach and that of the Greeks. First of all, whereas Eudoxus, Euclid, and many other Greek mathematicians created pure mathematics, devoid of any actual numbers and based especially on their invention of indirect *reductio ad absurdum* arguments, the Indians were primarily applied mathematicians focused on finding algorithms for astronomical predictions and philosophically predisposed to reject indirect arguments. In fact, Buddhists and Jains created what is now called Belnap's four-valued logic claiming that assertions can be true, false, neither, or both. The Indian mathematics tradition consistently looked for *constructive arguments and justifications and numerical algorithms*. So whereas Euclid's *Elements* was embraced by Islamic mathematicians and by the Chinese when Matteo Ricci translated it in 1607, it simply didn't fit with the Indian way of viewing math. In fact, there is no evidence that it reached India before the eighteenth century.

Secondly, this scholarly work was mostly carried out by Brahmins who had been trained since a very early age to memorize both sacred and secular Sanskrit verses. Thus they put their mathematics not in extended treatises on parchment as was done in Alexandria but in very compact (and cryptic) Sanskrit verses meant to be memorized by their students. What happened when they needed to pass on their sine tables to future generations? *They composed verses of sine differences*, arguably because these were much more compact than the sines themselves, hence easier to set to verse and memorize.[10] Because their tables listed sines every 3.75 degrees, these first order differences did not closely match the sine table read backward; but the second differences were almost exactly a small negative multiple of the sines themselves, and this they noticed.

There are several excellent recent books that give more background on these early developments. The mathematical sections of the *Āryabhaṭīya* with the seventh century commentary on it by Bhāskara (I) and an extensive modern commentary, all entitled *Expounding the Mathematical Seed*, has been published[11] in English by Agathe Keller. One hopes she will follow this with an edition of the astronomical chapters. And Glen van Brummelen has written a cross-cultural study of the use of trigonometry, entitled *The Mathematics of the Heavens and the Earth*,[12] which compares in some detail Greek and Indian work.

Chapters 5 and 6 of Plofker's book, entitled *The Genre of Medieval Mathematics* and *The Development of "Canonical" Mathematics*, are devoted to the sixth through twelfth centuries of Indian mathematical work, starting with Āryabhaṭa and ending with Bhāskara (also called Bhāskara II or Bhāskaracharya, distinguishing him from the earlier Bhāskara). This was a period of intense mathematical-astronomical activity from which many works have survived, and I want to touch on some of its high points. We find already in the seventh century the full arithmetic of negative numbers in Brahmagupta's *Brāhma-sphuṭa-siddhānta* (see Plofker, p. 151). This may sound mundane but, surprisingly, nothing similar appears in the West until Wallis's *Algebra* published in 1685.[13] And in the *Bakhshālī* manuscript, an incredibly rare birch bark manuscript unearthed by a farmer's plow in 1881, we find algebraic equations more or less in the style of Viète, Fermat, and Descartes. It is incomplete and neither title nor author survives, but paleographical evidence suggests that it was written between the eighth and twelfth centuries, and Hayashi argues that its rules and examples date from the seventh century.[14] The manuscript puts equations in boxes, like our displayed formulas. On p. 159 of Plofker's book, she gives the example from bark fragment 59. The full display in the original is below.

0	5	yu	mū	0	‖	sā	0	7	+	mū	0
1	1			1	‖		1	1			

Here the 0's (given by solid dots in the manuscript) stand for unknowns, the 1's (given by the sigma-like subscripted symbols in the manuscript)

through Υ, the intersection of the equator and the ecliptic. It is immediate to derive the formula using this triangle.

[10] *Using $R = 3438$, the number of minutes in a radian to the nearest integer, and $\Delta\theta = 3.75$ degrees, they calculated $R \cdot \sin(n\Delta\theta)$ to the nearest integer.*

[11] *Springer-Verlag, 2008, for an obscene price of $238!!*

[12] *Princeton University Press, 2009.*

[13] *For example, both Cardano and Harriot were unsure whether to make $(-1)\cdot(-1)$ equal to -1 or $+1$.*

[14] *See the fully edited and commented edition by Takao Hayashi,* The Bakhshali Manuscript: An Ancient Indian Mathematical Treatise, *John Benjamin Pub. Co., 1995.*

are just denominators, the *yu* means 5 is added to the unknown on its left, + sign signifies that 7 is subtracted, *mū* means square root, and *sā* is a pronoun indicating that the 1st and 3rd unknowns are equal. The whole thing has the modern equivalent:

$$\sqrt{x+5} = w, \quad \sqrt{x-7} = z.$$

This is to be solved in integers, giving $x = 11$. Note, however, that the Bakhshālī manuscript does not solve its problems using manipulations of its equations. In Brahmagupta's treatment of algebra in the *Brāhma-Sphuṭa-Siddhānta*, distinct colors are used to represent distinct unknowns (see Plofker's discussion of his Chapter 18, pp. 149–157).

The use of negative numbers to represent points on a line to the left of a base point appears in Bhāskara's *Līlāvatī*. This twelfth-century book, described in section 6.2.1 of Plofker's book, is arguably the most famous of all Indian texts on mathematics. Given the fact that *Līlāvatī* literally means "beautiful" or "playful" and that many verses are addressed to "the fawn-eyed one", the conjecture made by a Persian translator that the book was written to explain mathematics to Bhāskara's daughter seems quite reasonable.

Another basic tool which appears in all Indian manuscripts is what they called the pulverizer. This is an extension of the Euclidean algorithm, the idea of starting with two positive integers and repeatedly subtracting the lesser from the greater. They go further than Euclid in using this to systematically write down all solutions of first-order integer equations $ax + by = c$. It seems unlikely that the Greek algorithm, embedded in the *Elements* in highly abstract form, was transmitted to India, hence more likely that the idea was discovered independently in India. In fact, Indian astronomers had a very pressing application for this algorithm. Although they had, in fact, abandoned almost all of the ancient Vedic astronomy, they were not happy doing this and they retained one startling idea from that tradition: the vast epochs into which the past was divided, the yugas, all had to begin with one spectacular conjunction of the sun, the moon, and all the planets. To ascertain when the present yuga began and thus put future predictions on a sound basis, they had to solve such integer equations involving the periods of the heavenly bodies.

There are other high points of the work of this period. One of them is Brahmagupta's formula for the area of a quadrilateral inscribed in a circle. How he discovered this is a fascinating question. No justification has been found in any manuscripts earlier than the Kerala work (see below). It can be derived from Pythagorean theorem and simple geometry but only with substantial algebraic computation. Did Brahmagupta use algebra, manipulations of algebraic equations, to find it or not? That he gives many quite complex auxiliary results on cyclic quadrilaterals suggests he played with such quadrilaterals extensively.[15]

Indian work on Pell's equation in the general form $x^2 - Ny^2 = c$ also goes back to Brahmagupta. He discovered its multiplicative property—solutions for c_1 and c_2 can be "multiplied" to give one for c_1c_2 (Plofker, pp. 154-156). A complete algorithm, known as the "cyclic method", for constructing a solution to the basic equation $x^2 - Ny^2 = 1$ was discovered by Jayadeva, whose work is dated indirectly to the eleventh century (Plofker, pp. 194-195). Note again the emphasis on construction instead of indirect proofs of existence, which are the staple of our treatment of the subject.[16] Why such a focus on this equation? One idea is that if x, y is a solution, then x/y is a good approximation to $\sqrt{N}$.

The discovery of the finite difference equation for sine led Indian mathematicians eventually to the full theory of calculus for polynomials and for sine, cosine, arcsine, and arctangent functions, that is, for everything connected to the circle and sphere that might be motivated by the applications to astronomy. This work matured over the thousand-year period in which the West slumbered, reaching its climax in the work of the Kerala school in the fourteenth to sixteenth centuries. I won't describe the full evolution but cannot omit a mention of the discovery of the formula for the area and volume of the sphere by Bhāskara II. Essentially, he rediscovered the derivation found in Archimedes' *On the Sphere and the Cylinder I*. That is, he sliced the surface of the sphere by equally spaced lines of latitude and, using this, reduced the calculation of the area to the integral of sine. Now, he knew that cosine differences were sines but, startlingly, he integrates sine by summing his tables! He seems well aware that this is approximate and that a limiting argument is needed but this is implicit in his work. My belief is that, given his applied orientation, this was the more convincing argument. In any case, the argument using the discrete fundamental theorem of calculus is given a few centuries later by the Kerala school, where one also finds explicit statements on the need for a limiting process, like: "The greater the number [of subdivisions of an arc], the more accurate the circumference [given by the length of the inscribed polygon]" and "Here the arc segment has to be imagined to be as small as one wants… [but] since one has to explain [it] in a certain [definite] way, [I] have said [so far] that a quadrant has twenty-four chords."

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[15] *Added in proof: I just received a copy of S. Kichenassamy's article "Brahmagupta's derivation of the area of a cyclic quadrilateral",* Historia Mathematica, *2009.*

[16] *See, e.g., Artin's* Algebra, *Prentice-Hall, 1991, pp. 434-437.*

Chapter 7 of Plofker's book is devoted to the crown jewel of Indian mathematics, the work of the Kerala school. Kerala is a narrow fertile strip between the mountains and the Arabian Sea along the southwest coast of India. Here, in a number of small villages, supported by the Maharaja of Calicut, an amazing dynasty[17] of mathematicians and astronomers lived and thrived. A large proportion of their results were attributed by later writers to the founder of this school, Madhava of Sangamagramma, who lived from approximately 1350 to 1425. It seems fair to me to compare him with Newton and Leibniz. The high points of their mathematical work were the discoveries of the power series expansions of arctangent, sine, and cosine. By a marvelous and unique happenstance, there survives an informal exposition of these results with full derivations, written in Malayalam, the vernacular of Kerala, by Jyeṣṭhedeva perhaps about 1540. This book, the *Gaṇita-Yukti-Bhāṣā*, has only very recently been translated into English with an extensive commentary.[18] As a result, this book gives a unique insight into Indian methods. Simply put, these are recursion, induction, and careful passage to the limit.

I want to give one example in more detail, the derivation of the power series expansion for sine. It seems most transparent to explain the idea of the proof in modern form and then to indicate how Jyeṣṭhadeva's actual derivation differed from this. The derivation is based on the integral equation for sine:

$$\begin{aligned}\theta - \sin(\theta) &= \int_0^\theta (1-\cos(\beta))d\beta \\ &= \int_0^\theta \left(\int_0^\beta \sin(\alpha)d\alpha\right) d\beta = (K * \sin)(\theta) \\ &\qquad \text{where } K(x,y) = \max(0, x-y).\end{aligned}$$

Jyeṣṭhadeva uses a finite difference form of this equation using discrete samples of sine: he subdivides the arc $[0,\theta]$ into n "arc-bits" of size $\Delta\theta = \theta/n$ and, choosing a big radius R (like 3438, see above), he works with sampled "Rsines" $B_k = R \cdot \sin(k\Delta\theta)$ and also the "full chord" of the arc-bit: $2R\sin(\Delta\theta/2)$. Then, based on the formula for the second difference of sines which goes back to Aryabhata, he derives:

$$\begin{aligned}\theta - \sin(\theta) \approx bB_1 - B_n = (2\sin(\Delta\theta/2))^2 \\ \cdot ((B_1 + \cdots + B_{n-1}) + (B_1 + \cdots + B_{n-2}) \\ + \cdots + (B_2 + B_1) + B_1).\end{aligned}$$

[17] *The names we know form an essentially linear sequence of teacher and student (sometimes son).*

[18] *Translated and edited by the late K. V. Sarma, with notes by K. Ramasubramanian, M. D. Srinivas, and M. S. Sriram, and published in India in 2008 by the Hindustan Book Agency at a price of \$31 and distributed in the West by Springer for \$199, a 640% markup.*

Note that the right-hand side is exactly the finite difference version of the double integral in the calculus version. Here is how Jyeṣṭhadeva expressed this formula (p. 97 of Sarma's translation; here "repeated summation" stands for the sum of sums on the right):

> Here, multiply the repeated summation of the Rsines by the square of the full chord and divide by the square of the radius. … In this manner we get the result that when the repeated summation of the Rsines up to the tip of a particular arc-bit is done, the result will be the difference between the next higher Rsine and the corresponding arc. Here the arc-bit has to be conceived as being as minute as possible. Then the first Rsine difference will be the same as the first arc-bit. Hence, if multiplied by the desired number, the result will certainly be the desired arc.

He is always clear about which formulas are exact and which formulas are approximations. In the modern approach, one has the usual iterative solution to the integral equation $\theta - K*\theta + K*K*\theta - \cdots$, and you can work out each term here using the indefinite integrals $\int_0^y x^n dx = \frac{y^{n+1}}{n+1}$ resulting in the power series for sine. Jyeṣṭhadeva does the same thing in finite difference form, starting with $\sin(\theta) \approx \theta$, and recursively improving the estimate for sine by resubstitution into the left-hand side of the above identity. Instead of the integral of powers, he needs the approximate sum of powers, i.e., $\sum_{k=1}^n k^p = \frac{k^{p+1}}{p+1} + O(k^p)$, and he has evaluated these earlier (in fact, results like this go way back in Indian mathematics). Then repeated resubstitution gives the usual power series $\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots$ for sine. I consider this argument to be completely correct, but I am aware that it is *not* a rigorous proof by modern standards. It can, however, be converted into such a proof by anyone with basic familiarity with ϵ, δ-techniques. I hope I have whetted your appetite enough so you will want to feast on the riches laid out in Plofker's book and the *Gaṇita-Yukti- Bhāṣā* itself.

It is very tempting to read the history of mathematics as a long evolution toward the present state of deep knowledge. I see nothing wrong with understanding the older discoveries in the light of what we know now—like a contemporary metallurgist analyzing ancient swords. Needham, the great scholar of Chinese science, wrote "To write the History of Science we have to take modern science as the yardstick—that is the only thing we can do—but modern science will change and the end is not yet." Nevertheless, it is much more satisfying, when reading ancient works, to know as much as possible about the society in which these mathematicians worked, to know what mathematics was used for in their society, and how they themselves lived.

Chapter 1 in Plofker's book is an extensive introduction that gives vital background on the history and traditions from which all the Indian work sprang. A series of extremely helpful appendices provide basic facts about Sanskrit, a glossary of Sanskrit terms, and a list of the most significant Indian mathematicians with whatever basic facts about them are known (often distressingly little). In places she gives some literal translations such as those of numbers in the colorful concrete number system that uses a standard list of sets with well-known cardinalities, e.g., "In a *kalpa*, the revolutions of the moon are equal to five skies [0], qualities [3], qualities, five, sages [7], arrows [5]", which means 5,753,300,000 lunar months with the digits described backwards starting from the one's place. It is becoming more and more recognized that, for a good understanding of ancient writings, one needs *both* an extremely literal translation and one paraphrased so as to be clear in modern terms. In Sanskrit a literal translation of the complex compound words sometimes gives additional insight into the author's understanding (as well as the ambiguities of the text), so one wishes there were more places in this book where Plofker gave such literal translations without using modern expressions.

I have not touched on the astronomical side of the story. Suffice it to say that almost all treatises from the sixth century on deal with both astronomy and mathematics. To follow these, one needs a bit of a primer in geocentric astronomy, a vanishing specialty these days, and Plofker provides a very handy introduction in section 4.1. Just as in Indian mathematics, there is a steady increase in sophistication over the centuries, culminating in dramatic advances in Kerala. Most strikingly, Nīlakaṇṭha in the fifteenth century proposed a model in which the planets were moving in eccentric and inclined circles with respect to the mean sun moving in its ecliptic orbit—a "virtually" heliocentric model remarkably better than Ptolemy's.

It is high time that the full story of Indian mathematics from Vedic times through 1600 became generally known. I am not minimizing the genius of the Greeks and their wonderful invention of pure mathematics, but other peoples have been doing math in different ways, and they have often attained the same goals independently. Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge. In India, where concrete applications were never far from theory, justifications were more informal and mostly verbal rather than written. One should also recall that the European Enlightenment was an orgy of correct and important but semirigorous math in which Greek ideals were forgotten. The recent episodes with deep mathematics flowing from quantum field and string theory teach us the same lesson: that the muse of mathematics can be wooed in many different ways and her secrets teased out of her. And so they were in India: read this book to learn more of this wonderful story!