Why I am a Platonist

David Mumford

Like the previous authors of this ongoing debate,[1] I have to begin by clarifying what "Platonism" means to me. Here's my phrase:

> The belief that there is a body of *mathematical objects, relations and facts about them* that is independent of and unaffected by human endeavors to discover them.

This is essentially Davies' first flavor of Platonism, but in his article he isn't content with my phrase "there is" a body of objects etc., but feels he must characterize this belief as existence in a realm *outside* or *beyond* space-time. I think using these prepositions already implies certain philosophical, specifically ontological, assumptions. Hersh is more tolerant, merely adding the qualification that this body of objects, etc. is *objective*, which still puts a special ontological spin on the belief. Mazur seems closest to my simple statement above when he appropriates Huck Finn's words saying that this body of objects etc. *just happened* (instead of being invented by people). "Just happened" implies that, one way or another, they are there, without further characterization of how they exist or especially 'where' they exist.

Probably most mathematicians get a gut feeling that math is "out there" from their personal experiences struggling to understand some mathematical situation, to prove or disprove some theorem. But this is such a slippery subjective argument that I want to take a somewhat different tack. I want to say why studying the History of Mathematics makes mathematics seem to me to be universal and unchanging, invariant across time and space. Historians are disposed to dismiss amateurs like me as being naïve by imposing their modern point of view on ancient writings and not understanding the cultural influences, the proper historical context in which the work was done. I would counter: is a metallurgist imposing modern biases when he/she analyzes the metallic content of an ancient weapon, using the periodic table? It really all depends on whether you accept the Platonic universal view of mathematical truth or not. If you accept this view, using modern mathematics to analyze writings from other times and places is no different from the metallurgist's using modern knowledge of metals. So let me present my reading of several historical writings which seem to me to shout out that all mathematicians are working on one and the same body of truths.

1 E.B.Davies, *Let Platonism Die*, this Newsletter, June 2007; Reuben Hersh, *On Platonism*, and Barry Mazur, *Mathematical Platonism and its Opposites*, this Newsletter, June 2008.

2 *The Works of Archimedes*, edited by T.L.Heath, Dover.

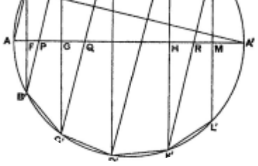
Universality: Exhibit I – Archimedes

I picked up the Dover paperback *The Works of Archimedes* sometime as an undergrad at Harvard. He is said to have wanted his gravestone carved with the figure of a sphere inside a cylinder tangent to it along the equator – certainly he felt one of his crowning achievements was calculating the surface area and the volume of the sphere. You open "On the Sphere and the Cylinder I" and you find this assumption (I have slightly changed Heath's translation[2] to conform with contemporary mathematical usage):

> *Curves in the plane having the same endpoints have different lengths whenever both are concave in the same direction and one is included between the other and the straight line with the same endpoints; and the curve which is included is the shorter.*
>
> *Surfaces with a common planar boundary have different areas whenever both are concave in the same direction and one surface is included between the other and the plane containing their common boundary, and the surface which is included has smaller area.*

I was astonished by these axioms. The reason was that they are so elegant, so exactly right for the arguments which follow, which depend on a whole series of estimates using these assumptions. Today, we would say that the fact that he doesn't *prove* them is a shortcoming! OK, but anyone (for instance the slave boy who finds the diagonal of a unit square to be √2 in Plato's *Republic*) with any experience of the world, finds them evident. Finding exactly the right way to pin something down was a thing I had found in my personal experience to be one of the most satisfying and beautiful aspects of math. Clearly, I thought, he worked like the best mathematicians I knew – no allowance for the years was needed.

Then there is a second breathtaking thing in this "paper" of Archimedes. You find the complicated diagram below in his proof of Proposition 22. Here L is any point on the upper hemi-circle and $A,B,C,D,\ldots,K,L$ is an equal subdivision of the arc between A and L. His reasoning with the sphere has led him to a point where he needs to *add* the lengths of all the line segments BB', CC', $DD',\ldots,KK'$, LL'. He notes that all the slivers of triangles like PCG, $GC'Q$, etc are similar, so this sum is a multiple of



$$AF+FP+PG+GQ+\ldots+HR+RM$$

What is he doing? If θ is the angle subtended by the arc AL and the radius of the circle is 1, then this sum equals 1-cos(θ) and he is evaluating a Riemann sum of

$$\int_0^\theta \sin(\phi)\, d\phi ,$$

the indefinite integral of sine,[3] which, as we know, is equal to 1–cos(θ). He is some 2000 years ahead of the time when this was rediscovered in the West (but actually not very far ahead of the time when Indian Mathematicians discovered it). No historian will convince me that his idea is not the same of mine when looking at this mathematical proposition. For lack of space, I will not go into the highly abstract Book V of Euclid's *Elements*. As I read it, it is pure Dedekind and Bourbaki, an abstract analysis of geometric arithmetic from first principles. Its culmination is the assertion that there is a well-defined operation of addition on the equivalence classes of line segment pairs which define ratios.

As Littlewood said to Hardy, the Greek mathematicians spoke a language modern mathematicians can understand, they were not clever schoolboys but were "fellows of a different college". They were working and thinking the same way as Hardy and Littlewood. There is nothing whatsoever that needs to be adjusted to compensate for their living in a different time and place, in a different culture, with a different language and education from us. We are all understanding the same abstract mathematical set of ideas and seeing the same relationships.

Universality: Exhibit II – Madhava and Gregory

My second set of examples involves how very different cultures, at different times and places, often converge to identical results. The conventional History of Mathematics hypothesizes a single line of development, from Babylonians to Greeks to Arabs to Renaissance Europe to the Enlightenment to today. But from a more multicultural perspective, one finds that both Indian and Chinese mathematics developed largely as distinct streams, with some possible exchanges. This allows one to study of whether or not the 'same' mathematics was discovered independently by very different cultures. In my work on this in the last few years, my overall conclusion is that sometimes the order of discovery changes but there is a strong tendency to converge.

A very striking example is the formula:

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$$

This is known as Gregory's formula in the West, from its discovery by James Gregory in 1672. But it had also been written down sometime around 1550 in the *Ganita-Yukti-Bhasa*,[4] which is an exposition by Jyesthadeva of the results attained by the school of Madhava in Kerala, India during the previous two centuries. One can compare the histories which led up to each discovery and it is striking what distinct routes the two cultures took. In India, a Leibnizian version of the calculus of trigonometric functions using finite differences goes back as far as Aryabhata c.500 CE. Why did they look at finite differences? Apparently *in order to facilitate memorizing tables of sines*! Whatever its roots and in spite of many political upheavals, there is a more or less continuous development of mathematical ideas in India, from Aryabhata through Jyesthadeva. They were led to sum powers of integers, then to integrate powers and finally to power series expansions of sine, cosine and arctan (which gives the above formula). In contrast, in Europe, there is a total break during the dark ages, then a revival in which Euclid played a dominating role in defining what mathematics ought to be. Interestingly, for both Gregory and the Kerala mathematicians, the question of the irrationality of pi was a major driving force: both believed it to be true (this was explicitly asserted by Nilakantha in India[5]) but neither could find a proof.

In my own research, I have been fascinated by the example of negative numbers. It is a little known fact that negative numbers were not universally accepted in Europe until the creation of abstract algebra in the mid 19th-century. As late as 1843, Augustus de Morgan[6] could say

> *It is not our intention to follow the earlier algebraists through their different uses of negative numbers. These creations of algebra retained their existence, in the face of the obvious deficiency of rational explanation which characterized every attempt at their theory.*

In Britain especially, a controversy raged during the 18th century about the acceptability of negative numbers. In contrast, negative numbers were incorporated into counting boards from something like 200 BCE in China. Red rods (the auspicious color) were used for positives, black (very inauspicious) was used for negatives. Likewise, the rules for the arithmetic of negatives are explicitly stated by Brahmagupta in India c.650 CE.[7] In other words, there was a deep cultural division between the East where negative numbers were accepted from the beginning and the West, where, under Euclid's influence, arithmetic remained the calculus of lengths and areas, both automatically positive quantities. But, as a Platonist, I feel there is only one true science of mathematics and so, indeed, these different cultures eventually passed the same milestones as they dug deeper.

3 The Riemann sum is, of course, not equal to the integral: this is why he has a constant multiplier. He proves rigorously in the end that the multiplier tends to 1 as the subdivision gets finer.

4 *Ganita-Yukti-Bhasa*, translated and edited by K.V.Sarma, K. Ramasubramanian, M.D.Srinivas and M.S.Sriram, Hindustan Book Agency, 2008.

5 The Kerala mathematicians had most of the ingredients which came together in Lambert's 1761 proof of the irrationality if pi. They did not have the idea of continued fractions, although they did know the Euclidean algorithm for gcd's. They also never used and were quite averse to proof by contradiction.

6 Article on *Negative and Imaginary Numbers*, in the *Penny Cyclopedia*, 1843.

7 For the Chinese material, see *The Nine Chapters on the Mathematical Art: Companion and Commentary*, Shen Kangshen, John N. Crossley, and Anthony W. -C. Lun, Oxford University Press, 1999. For the Indian material, see *Mathematics in India, 500 BCE–1800 CE*, Chapter 5, Kim Plofker, Princeton University Press, 2009.

Is mathematics the unique occupant of the Platonic realm?

As mathematicians, we have been trained to seek the most general setting for any theorem. Only when we find this do we feel we have understood the real nature of a result. So if we believe that mathematical truth is universal and independent of culture, shouldn't we ask whether this is uniquely the property of mathematical truth or whether it is true of more general aspects of cognition? In fact, "Platonism" comes from Plato's *Republic*, Book VII and there you find that he proposes "an intellectual world", a "world of knowledge" where all things pertaining to reason and truth and beauty and justice are to be found in their full glory (cf. http://classics.mit.edu/Plato/republic.8.vii.html).

The ethical realm of laws and moral plays a central role in Plato's Republic. Hersh uses the example of the "divine right of kings" to ridicule the idea that ethical principles can have a universal Platonic existence. But cannot we imagine that humanity can discover deeper ethical principles over the centuries, just as our mathematics discovers deeper theorems? Jefferson's phrase "all men are created equal" has a good claim to be a universal ethical principle, not contingent on one specific culture's beliefs, and even applicable to other actual or potential civilizations in our galaxy.

Another hint of universality, I suggest, is that all human languages can be translated into each other with only occasional difficulties. This seems a quite non-trivial fact to me and suggests considerable universality to all the concepts we use in thinking. My cousin wrote a children's book[8] in which the first sentence was "Albert John was a loyal cat". Loyalty is a highly abstract notion, yet no parent reading this to a child would consider the concept of loyalty to be a major challenge to the child's understanding. All children at a rather early age seem to access this concept. Doesn't this suggest that the concept of loyalty has a universal existence, applicable to any society of intelligent beings?

Concepts in general are slippery things: they come with illustrative examples, with typical properties (but usually with exceptions, e.g. most but not all birds fly), with links to more general, less general and sibling concepts. In order to put some order into the world of concepts, people have made *graphs* out of concepts for a long time. Roget wrote his thesaurus in 1852: taking each word as a vertex, his thesaurus puts edges between any two closely related concepts. A major structure in early AI was the *semantic net* which introduced a variety of directed edges such as *is-a* links, as in "a robin is-a bird". Statisticians studying AI introduced *causal Bayesian networks* where edges modeled causal effects. Case grammar equips concepts with multiple slots, such as the temporal and spatial location of an act and filling these makes a grammatical *parse graph*. Grenander has proposed a general framework for such graphical models.[9] There is no definitive formalism for all these graphical structures but it seems clear that such graphs are part of the life and what gives structure to the objects of the world of knowledge.

The idea that such a spider web is the key to understanding the Platonic world is very familiar in mathematics. Everyone has heard of the bizarre circumlocutions used in books on the Foundations of Mathematics to define natural numbers, e.g. 5. Frege's approach was to make 5 equal to the set of all sets with 5 elements in them. This loose use of sets led to those nasty paradoxes. Von Neumann's idea was to make 5 the specific set with 5 elements $\{0,1,2,3,4\}$. Doing this recursively, starting from $0=\{\}$ (the empty set), one finds the rather bizarre definition:

$$5 = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}\}\},$$
$$\{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}\}, \{\{\}, \{\{\}\}\}\}\}\}$$

This seems pretty *ad hoc*! Children certainly prefer their fingers as a model set. I think it is much better to follow the standard idea of defining natural numbers by axioms for the successor relation and proving that all models of these axioms are isomorphic. More generally, it is widely accepted that *categories* are the natural setting for all mathematical objects: a specific mathematical object ought to be defined as an object of some type in a category, *unique up to unique isomorphisms*. In other words, mathematical objects don't exist as specific things, but are pure structure. They can only be defined in their own terms. I would argue that categories are simply a mathematical example of the class of cognitive graphs which connect Platonic concepts in general.

Brian Davies argues that we should study fMRI's of our brains when think about 5, about Gregory's formula or about Archimedes' proof and that these scans will provide a scientific test of Platonism. But the startling thing about the cortex of the human brain is how uniform its structure is and how it does not seem to have changed in any fundamental way within the whole class of mammals.[10] This suggests that mental skills are all developments of much simpler skills possessed, e.g. by mice. What is this basic skill? I would suggest that it is the ability to convert the analog world of continuous valued signals into a discrete representation using concepts and to allow these activated concepts to interact via their graphical links. The ability of humans to think about math is one result of the huge expansion of memory in *homo sapiens*, which allows huge graphs of concepts and their relations to be stored and activated and understood at one and the same time in our brains.

The discrete representation does have a cortical instantiation, a reduction to physical effects which are continuous at a micro level but honed to produce a sharp flip-flop digital behavior (e.g. in the production of a neural spike) on a larger scale. But this does not mean that this discrete concept-based parse of the world is not as true and fundamental a reality as that of the physical neurons. The

8 Ruth Silcock, *Albert John Out Hunting*, Viking Kestrel Books, 1980.

9 Ulf Grenander, *General pattern Theory*, Oxford University Press, 1993.

10 I have martialled the evidence for this in several articles: On the Computational Architecture of the Neocortex, *Biological Cybernetics*, 1991 and Neuronal Architectures for Pattern-theoretic Problems, in *Large Scale Neuronal Theories of the Brain*, C. Koch editor, MIT Press, 1994.

firing of a neuron and the occurrence of a concept-based situation are totally different sorts of existence. How do I personally make peace with what Hersh calls "*the fatal flaw*" of dualism? I like to describe this as there being two orthogonal sides of reality. One is blood flow, neural spike trains, etc.; the other is the word 'loyal', the number 5, etc. But I think the latter is just as real, is not just an epiphenomenon and that mathematics provides its anchor.

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