The Invention of Algebra as Reification

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Mathematics in Ancient Times

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Outline

I. What is “algebra”? What do I mean by reification?
II. Illustrate reification with a modern example:
France adds sheaf cohomology to Italian alg.geom.
I. Babylon and China: algebra without symbols
II. Diophantus and Brahmagupta: independent reification of the unknown
III. The irregular evolution of algebra to its 17th century final form
IV. Final remarks: reification of numbers themselves, in calculus and the critique of William of Ockham.

I. Introductory remarks

- What does the word “algebra” refer to?
- Why is algebra such a shock to14 year olds?
- Reification: transforming some cluster of examples into a named, manipulable thing
- “x” is *not* a number but may become one, giving sense to “x is even”, $x+2$, $x^2=2$
- My hypothesis: this is the true point of departure for algebra
- Hard for us to grasp why “x” is a big deal, so here’s a recent example of reification

II. A glimpse of algebraic geometry and cohomology

- an *algebraic variety X* is the locus of zeros of a set of polynomials $f_k(x_1, x_2, \cdots, x_n)$ (irreducible, smooth)
- a rational function *f* on *X* is the restiction to *X* of a rational function $f = p(x_1, \cdots, x_n)/q(x_1, \cdots, x_n)$
- such an *f* has zeros and poles of various orders n_i on codim 1 subvarieties $D_i \subset X$
- a *divisor D* on *X* is some linear combination of codim 1 subvarieties, e.g. $(f) = \sum_i n_i D_i$, the divisor of *f*
- the key player is the vector space

$$\mathcal{L}(D) = \{\text{rat'l fcns } f \mid (f) + D \geq 0\}$$

that is, rational fcns with bounds on their poles, maybe prescribed zeros called a *linear system*.

- The Italian school (Castelnuovo, Enriques, Severi) were geometers and preferred to deal with linear systems geometrically:

$$|D| = \{\text{set of positive divisors } (f) + D\}$$

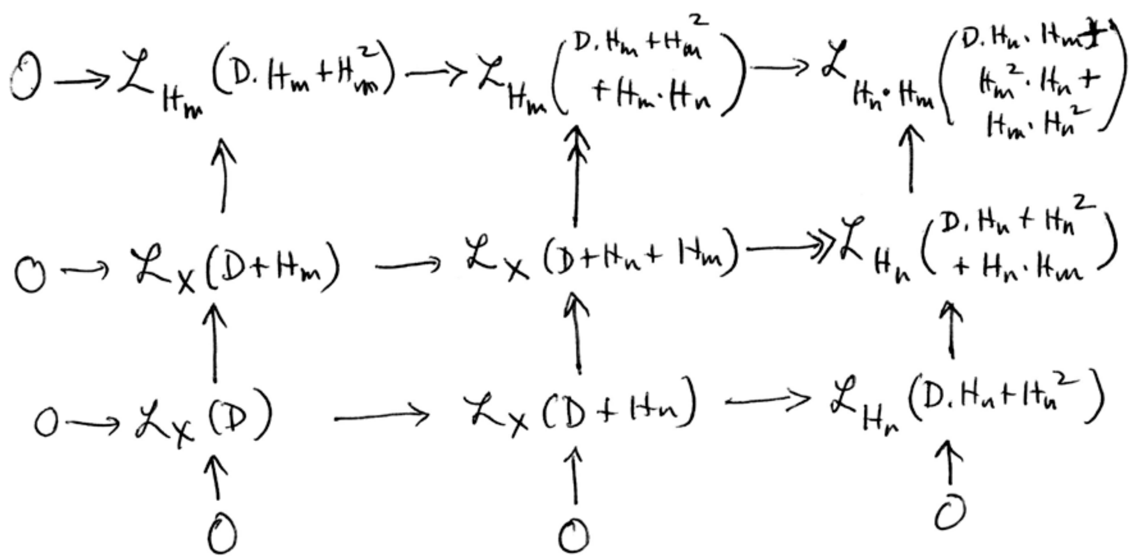
$$= \text{ proj. space of } \mathcal{L}(D)$$

- But if *X* is a surface, *C* a curve on *X*, one seeks to compute the dimension of linear systems by:

$$0 \to \mathcal{L}_X(D - C) \to \mathcal{L}_X(D) \to \mathcal{L}_C(D.C)$$

- The last map is not always *onto*! and the Italian school dealt with the *number* of linear conditions on $f \in \mathcal{L}_C(D.C)$ needed to "lift" it to $\mathcal{L}_X(D)$
- The French school (H.Cartan, J.-P. Serre) said look at the *cokernel* $\mathcal{L}_C(D.C) / \text{image}(\mathcal{L}_X(D))$

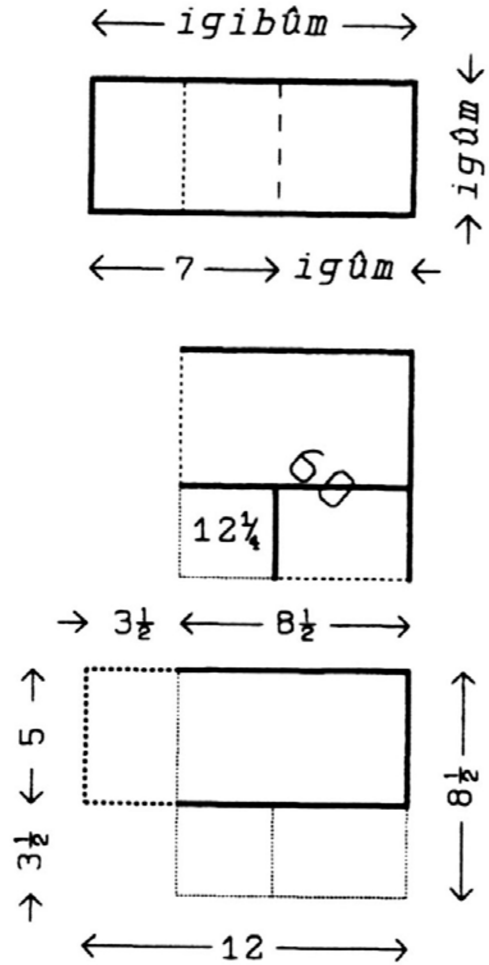
- If *D* has high enough degree, this cokernel depends only on *D*-*C* and is written $H^1(X, D-C)$
- The cokernel is quite abstract but, once H^1 is reified, it is seen to generalize to H^k, manipulated almost mechanically using *exact sequences* and applied in virtually every proof.
- e.g. that H^1(D) is "unique up to canonical isomorphism" comes from "diagram chasing":



IIIa. Algebra without symbols : Babylon:

- All problems are concrete applied math, dealing with rounded measurements contrived for simple answer.
- Tablets read like hand-held calculator code: see next slide

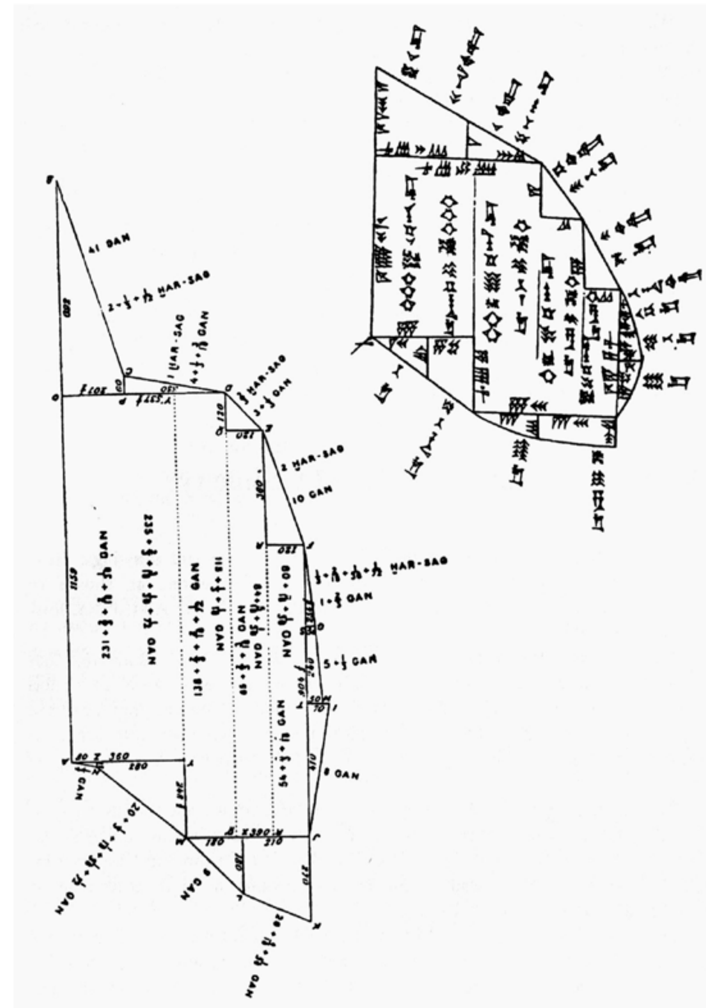
The igibum over the igum, 7 it goes beyond | $x\text{-}y = a$ (7); $x.y = b$ (60)
igum and igibum what?
You, 7 which the igibum | $t = a$
over the igum goes beyond
to two break: 3° 30´ | $t = t/2$
3° 30´ together with 3° 30´
make hold: 12° 15´ | $t = t^2$
to 12° 15´ which comes up for you
1` the surface append: 1` 12° 15´ | $t = t+b$
The equalside of 1` 12° 15´ what? 8° 30´ | $t = \sqrt{t}$
8° 30´ and 8° 30´ its counterpart, lay down | $t_1 = t_2 = t$
3° 30´ the made-hold
from one tear out | $t_1 = t_1 - a/2$
to one append | $t_2 = t_2 + a/2$
the first is 12, the second is 5 | $x = t_2$
12 is the igibum, 5 the igum | $y = t_1$



YBC 6967, Translation by Høyrup, pp.55-57

and his geometric reconstruction of its logic. Note that the computer code is exactly the quadratic formula.

- BUT land diagrams are not exact!
- What seems to have been reified are the abstract rectangles, with cut and paste operations but no names
- Other algebra problems require, e.g. rectangle where one side = # workers, other = # days, area = loads moved.
- A transition from geometry to CS pure procedural perspective



IIIb. Algebra without symbols: China -- consistently algorithmic.

"Nine Chapters", Chapter 8, "Rectangular Arrays" is all about solving systems of linear equations by Gaussian elimination.

<u>Problem 1:</u> Now given 3 bundles top grade paddy, 2 bundles medium grade, 1 bundle low grade. Yield: 39 dou of grain. 2 bdles top, 3 bdles medium, 1 bdle low. Yield 34 dou. 1 bdle top, 2 bdles medium, 3 bdles low. Yield 26 dou. Tell: how much paddy does one bundle of each grade yield?

$$3T + 2M + L = 39$$
$$2T + 3M + L = 34$$
$$T + 2M + 3L = 26$$

Left: the problem in our version

Below: Chinese version and use of *Fangcheng rule*

1	**2**	**3**
2	**3**	**2**
3	**1**	**1**
26	**34**	**39**

➡

1	**0**	**3**
2	**5**	**2**
3	**1**	**1**
26	**24**	**39**

➡

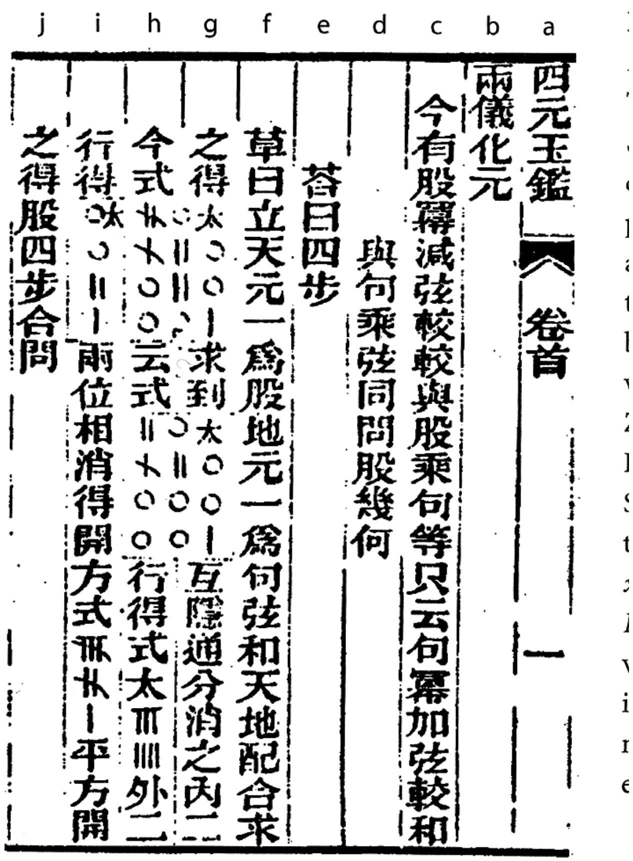
0	**0**	**3**
4	**5**	**2**
8	**1**	**1**
41	**24**	**39**

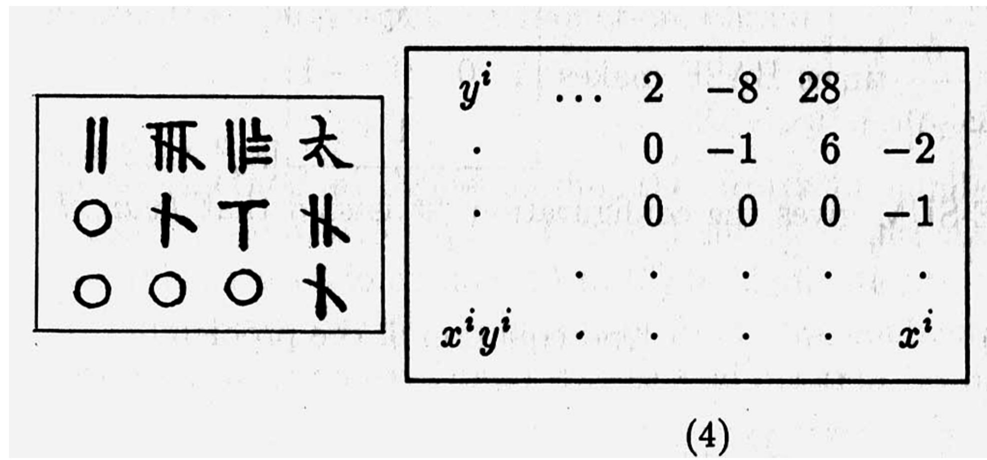
➡

0	**0**	**3**
0	**5**	**2**
36	**1**	**1**
99	**24**	**39**

Song dynasty algebra: Zhu Shijie (1303 CE)

extends the array method to the solution of simultaneous multi-variable polynomial equations – <u>without any symbols for unknowns</u>! (In West: Bezout)



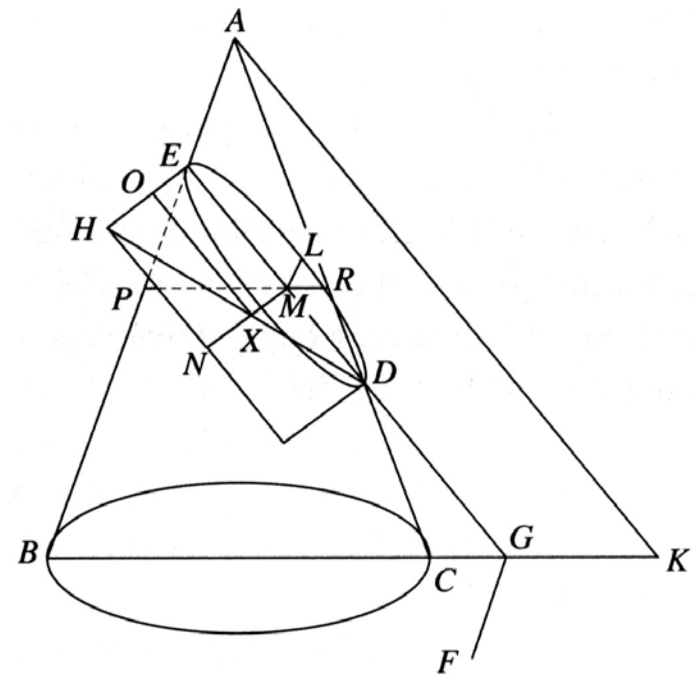


$$2y^3 - xy^2 - 8y^2 + 6xy - x^2 + 28y - 2x$$

An example of a polynomial, top left the Chinese sticks; top right their interpretation; bottom the modern form

IVa. Symbolism in Greece:
its origin using letters for indeterminate points

Apollonius's derivation of the geom. version of the equation of an ellipse



<u>Proposition I.13</u>

Let there be a cone whose vertex is the point A and whose base is the circle BC and let it be cut ….

We are dealing with "equations" and displays here but *the symbols* are points, not numbers; the "equations" are stated geometric constraints of collinearity, equal distance.

Diophantus (?50 CE? – ?350 CE?)

An anomaly in the Greek tradition with algebraic formulas and pure math algebraic problems

$$x^2 3x12[\text{cnst}]9 = [\text{square}]$$

$$3x^2 + 12x + 9 = y^2$$

- All symbols are abbreviations, e.g. Δ^Y means *dynamis* for x^2, ⋀ means λειπειν, for minus
- One symbol for an unknown, ς , which is believed to be short for *arithmos* – so all problems must be reduced to an equation with one unknown.
- He states al Khwarizmi’s 2 rules for simplifying equations.

IV.39: *To find three numbers such that the difference of the greatest and the middle has to the difference of the middle and the least a given ratio, and further such that the sum of any two is a square.*

He then (arbitrarily) takes the ratio to be 3:1 and the sum of the middle and least to be 4, making the three numbers

$2-x, 2+x, 7x+2,$ with $0<x<2,$ $6x+4,$ $8x+4$ squares

Now his main *ansatz*: shrewd choices for the square

using a z (which he can't do) set the square equal to $z+2$:

$6x+4=(z+2)^2,$ so the 4's cancel, $6x=z^2+4z.$

Then $\frac{9}{4}(8z+4)=3z^2+12z+9$ must be a square.

Take this equal to $(u.z-3)^2,$ so the 9's cancel.

Then $3z+12=u^2z-6u$ or $z=\dfrac{6u+12}{u^2-3}.$

Finally, set $u=5,$ etc.

Twice he must make a substitution and change to a new variable! while he has only one symbol for an unknown

“So I am led to make the 3 *dynameis* (x^2) <and> 12 *arithmous* (x) <and> 9 units equal to a square <number>. I form the square from 3 units wanting **some <number of>** *arithmous* (x); and the *arithmos* (x) comes from **some number** taken six times and augmented by 12 <units>, that is, the <quantity> of the 12 units of the equalization, and divided by the excess of the square formed from **the number** on the <quantity> 3 of the *dynameis* (x^2) in the equalization. Therefore I am led to find **a number** which when taken six times and augmented by 12 units, and divided by the excess that the square on it exceeds the 3 units, makes the quotient (*parabolê*) less than 2 units.”

(translation by Jean Christianidis)

After this the bold **“number”** now becomes a new *arithmos.* Note “equalization” means he is rearranging the equation.

IVb. Symbolism in India: its origin using compound neologisms for elements in grammar, prosody

- Example 1: words are defined in Pānini's sūtra 1.4.14:

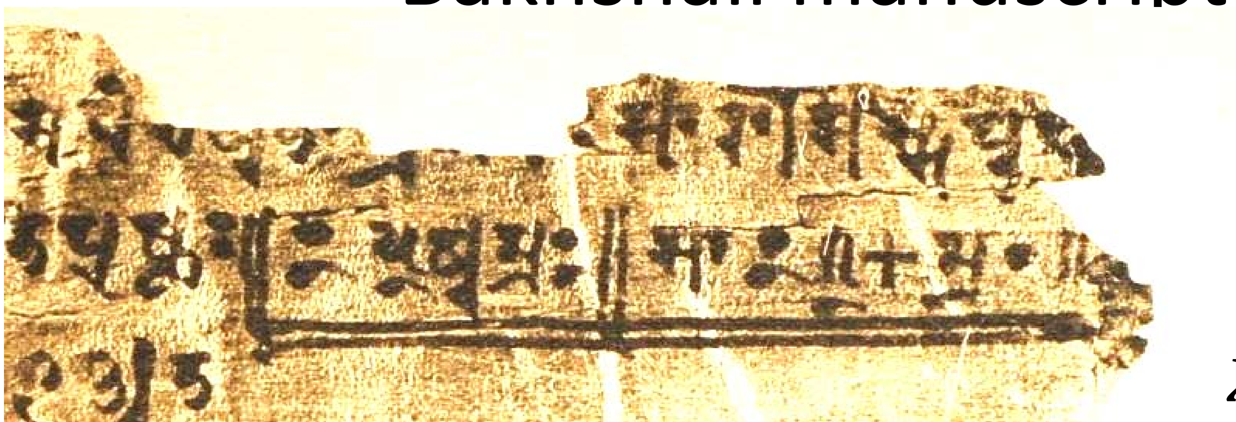
  suptiṅantaṃ padam

  "a word (*padam)* is what ends in *sup* or in *tiṅ.*"
- All nominal endings are put in a long list. It starts with *su* and ends with *p. sup* therefore refers to all nouns.
- All verbal endings are put in a long list. It starts with *ti* and ends with *ṅ. tiṅ* therefore refers to all verbs.
- Example 2: Pingala's sūtra

  dvikau glau

"Two times the pair G (=abbrev. for 'guru'), L (=abbrev. for 'laghu'),

Bakhshali manuscript I



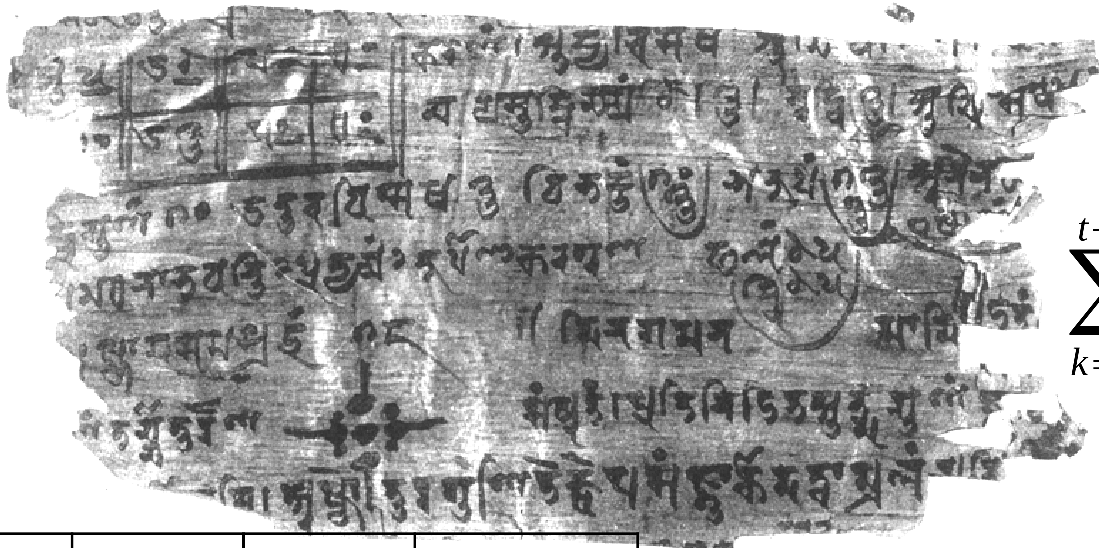
0	5	*yu*	*mū*	0	*sā*	0	7	+	*mū*	0
1	1			1		1	1			1

$$x^2 + 5 = \square$$

$$x^2 - 7 = \square$$

- As in Diophantus, one symbol for the unknown, a small filled circle, *śūnya sthāna,* the empty place
- Many symbols and words in text are abbreviations.
- ‘+’ for minus.
- An exuberant collection of idiosyncratic compact expressions, e.g. for iterated sums and products.

Bakshali manuscript II



5 RECTO

$$\sum_{k=0}^{t-1}(5+6k)=\sum_{k=0}^{t-1}(10+3k)$$

ā 5 1	u 6 1	pa • 1	dha • 1
ā 10 1	u 3 1	pa • 1	dha • 1

Answer: t = 4⅓!
In other similar problems, t is even irrational. Thus the sum, quadratic in t, is being interpolated. It seems that variables are implicitly assumed to be any real number.

Note the "QED" type sign, for end of sutra.

Brahmagupta to Bhaskara II: the full development of algebraic machinery

Symbols for half a dozen or more variables and all necessary operations

- yā→yāvat-tāvat="as many as so many";
- cā→ cālaca=black;
- nī →nīlaca=blue;

variables

- rū →rūpa=number; (for a constant)
- dot over symbol → negative;
- new line → equals;
- bh →product;
- c →square root;

Variables always stand for integers and goal is to find all integer solutions of an equation.

In Bhaskara II, we find a big bag of tricks:

"*If thou be conversant with operations of algebra, tell the number of which the biquadrate less double the sum of the square and 200 times the simple number is a myriad less one*." (Vija-Ganita, V.138)"

$$\text{ya v v } 1 \text{ ya v } \dot{2} \text{ ya } \dot{4}00 \text{ ru } 0$$

$$\text{ya v v } 0 \text{ ya v } 0 \text{ ya } 0 \text{ ru } 9999$$

$$x^4 - 2x^2 - 400x = 9999$$

He suggests the natural idea is to add $400x+1$ to make LHS a square: a dead end! "Hence ingenuity is called for." No instead add $4x^2 + 400x + 1$, getting:

$$\left(x^2+1\right)^2 = \left(2x+100\right)^2, \text{ hence}$$

$$x^2+1 = \pm\left(2x+100\right), \text{ hence } x = 11. \text{ Ignores } - \text{ sign.}$$

Another trick:

"Example from ancient authors: the square of the sum of two numbers, added to the cube of their sum, is equal to twice the sum of their cubes. Tell the numbers, mathematician!" (VII.178)

"The quantities are to be so put by the intelligent algebraist, as that the solution may not run into length. They are accordingly put ya 1 ca $\dot{1}$ and ya 1 ca 1

To solve $(x+y)^2+(x+y)^3=2\left(x^3+y^3\right)$,

make the substitution $x=u+v,\ y=u-v$

What is NOT done: systematically consider solutions of all cubic, quartic, etc equations. Like Diophantus, it looks more like play. The real stuff is astronomy.

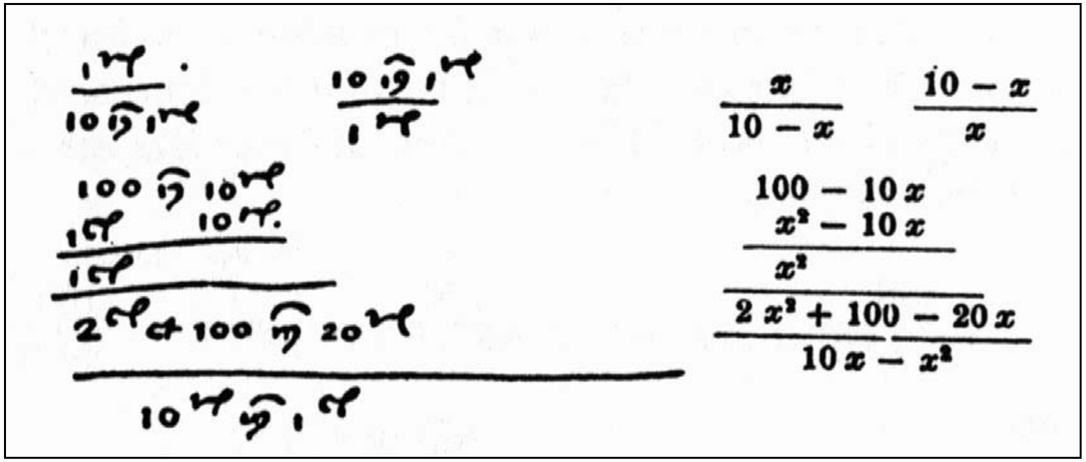
Four hallmarks of reification in algebra

- Use of symbols for unknowns
- Rules to simplify equations
- Substitutions of expressions for variables
- Displays to separate prose from algebraic statements

V. The irregular evolution of algebra to its 17th century final form

- Brahmagupta's BSS taken to Caliph al Mansur in 770 CE
- Al Khwarizmi (c.790-c.850) writes his treatise *Hisab al-jabr w'al-muqabala* on calculating by completion and reduction with little more than Babylonian algebra, no symbols or formulas, the word *"thing"* or *"root"* used for the unknown in some (but not all) places.
- Diophantus is translated into Arabic c.900 CE but again no algebraic symbols or formulas are used.
- Leonardo of Pisa learns Arab arithmetic and algebra and writes *Liber Abaci* in 1202, with marginal arrays of numbers but without symbols or formulas

Regiomontanus (1436-1476) read Diophantus in Greek and now uses symbols for *x*



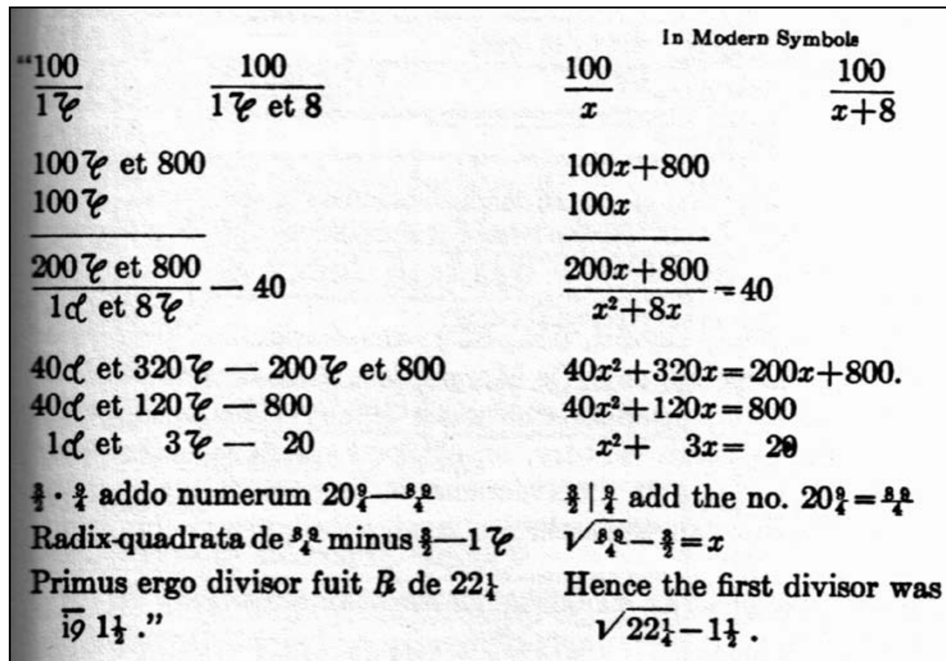
$$\frac{x}{10-x} \qquad \frac{10-x}{x}$$

$$\frac{\begin{array}{c}100-10x\\ x^2-10x\end{array}}{x^2}$$

$$\frac{2x^2+100-20x}{10x-x^2}$$

From letters written c. 1460-1470 with algebraic computations.

(from Cajori)



	In Modern Symbols
"$\frac{100}{1ƺ}$ $\frac{100}{1ƺ \text{ et } 8}$	$\frac{100}{x}$ $\frac{100}{x+8}$
100ƺ et 800	$100x+800$
100ƺ	$100x$
$\frac{200ƺ \text{ et } 800}{1ȼ \text{ et } 8ƺ}$ — 40	$\frac{200x+800}{x^2+8x}=40$
40ȼ et 320ƺ — 200ƺ et 800	$40x^2+320x=200x+800.$
40ȼ et 120ƺ — 800	$40x^2+120x=800$
1ȼ et 3ƺ — 20	$x^2+3x=20$
$\frac{3}{2}\cdot\frac{3}{2}$ addo numerum $20\frac{9}{4}$—$\frac{89}{4}$	$\frac{3}{2}\,\vert\,\frac{3}{2}$ add the no. $20\frac{9}{4}=\frac{89}{4}$
Radix-quadrata de $\frac{89}{4}$ minus $\frac{3}{2}$—1ƺ	$\sqrt{\frac{89}{4}}-\frac{3}{2}=x$
Primus ergo divisor fuit ℞ de $22\frac{1}{4}$ $\overline{\text{i9}}$ $1\frac{1}{2}$."	Hence the first divisor was $\sqrt{22\frac{1}{4}}-1\frac{1}{2}$.

Cardano

unknown is “*rem ignotam, quam vocamus positionem*”, abbreviated to *pos.*

QVÆSTIO VIII.

Fac ex 6. tres partes, in continua proportione, quarum quadrata primæ & secundæ iuncta simul faciant 4. ponemus primam 1. positionem, quadratum eius est 1. quadratum, residuum igitur ad 4. est quadratum secundæ quantitatis, id est 4. m̄. 1. quadrato, huius radicem, & 1 positionem detrahe ex 6. habebis tertiam quantitatem, vt vides, quare ducta prima in teriam, ha-

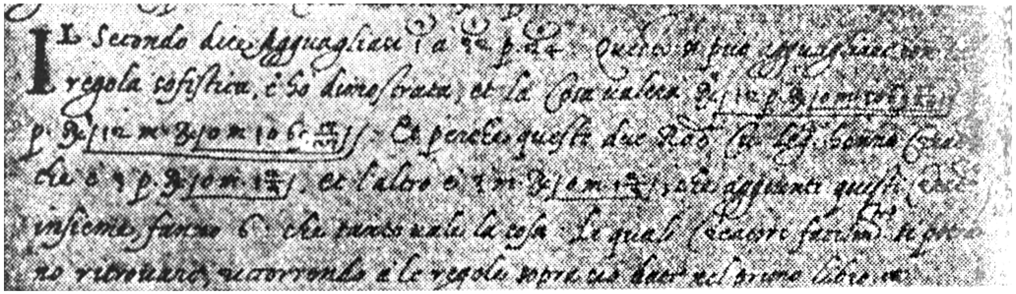
1. pos. | v. ℞. 4. m̄. 1. quad. | 6. m̄. 1. pos. m̄. ℞. v. 4. m̄. 1. quad.

6. pos. m̄. 1. quad. m̄. ℞. v. 4. quad. m̄. 1. quad. quad.

$$1.\text{pos.} \,|\, \text{v.}R.4.\overline{\text{m}}.1.\text{quad.} \,|\, 6.\overline{\text{m}}.1.\text{pos.}\overline{\text{m}}.R.\text{v.}4.\overline{\text{m}}.1.\text{quad}$$

$$1.x \quad | \sqrt{4 - 1.x^2} \quad | 6 - 1.x - \sqrt{4 - 1.x^2}$$

Bombelli

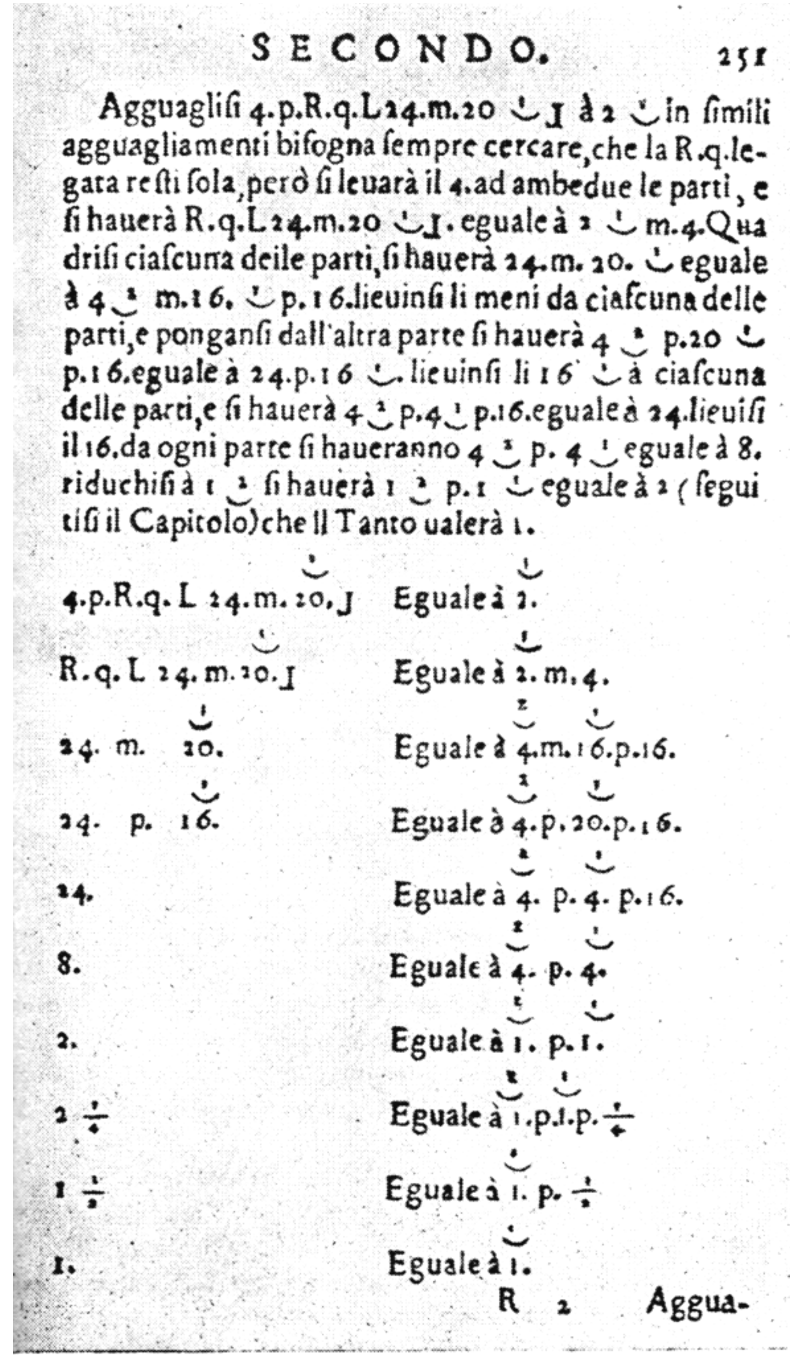


Top: a letter – note his use of partial boxes as in the Bakhshali ms. to set off formulas and scope. Right: Displayed formulas in his *L'Algebra* (1572) solves:

$$4.\,p.R.q.[24.m.\overset{\overset{1}{\cup}}{20}.] \text{ equals } \overset{\overset{1}{\cup}}{2}$$

$$4 + \sqrt{24 - 20x} = 2x$$

Answer: $x = 1$



Agguaglisi 4.p.R.q.L24.m.20 $\overset{1}{\smile}$] à 2 $\overset{1}{\smile}$ in simili agguagliamenti bisogna sempre cercare, che la R.q.legata resti sola, però si leuarà il 4.ad ambedue le parti, e si hauerà R.q.L24.m.20 $\overset{1}{\smile}$]. eguale à 2 $\overset{1}{\smile}$ m.4.Quadrisi ciascuna delle parti, si hauerà 24.m.20. $\overset{1}{\smile}$ eguale à 4 $\overset{2}{\smile}$ m.16. $\overset{1}{\smile}$ p.16.lieuinsi li meni da ciascuna delle parti, e pongansi dall'altra parte si hauerà 4 $\overset{2}{\smile}$ p.20 $\overset{1}{\smile}$ p.16.eguale à 24.p.16 $\overset{1}{\smile}$. lieuinsi li 16 $\overset{1}{\smile}$ à ciascuna delle parti, e si hauerà 4 $\overset{2}{\smile}$ p.4 $\overset{1}{\smile}$ p.16.eguale à 24.lieuisi il 16.da ogni parte si haueranno 4 $\overset{2}{\smile}$ p. 4 $\overset{1}{\smile}$ eguale à 8. riduchisi à 1 $\overset{2}{\smile}$ si hauerà 1 $\overset{2}{\smile}$ p.1 $\overset{1}{\smile}$ eguale à 2 (seguitisi il Capitolo) che il Tanto ualerà 1.

4.p.R.q.L 24.m. 20 $\overset{1}{\smile}$.]	Eguale à 2 $\overset{1}{\smile}$.
R.q.L 24.m. 20 $\overset{1}{\smile}$.]	Eguale à 2 $\overset{1}{\smile}$.m.4.
24. m. 20 $\overset{1}{\smile}$.	Eguale à 4 $\overset{2}{\smile}$.m.16 $\overset{1}{\smile}$.p.16.
24. p. 16 $\overset{1}{\smile}$.	Eguale à 4 $\overset{2}{\smile}$.p.20 $\overset{1}{\smile}$.p.16.
24.	Eguale à 4 $\overset{2}{\smile}$. p. 4 $\overset{1}{\smile}$. p.16.
8.	Eguale à 4 $\overset{2}{\smile}$. p. 4 $\overset{1}{\smile}$.
2.	Eguale à 1 $\overset{2}{\smile}$. p.1 $\overset{1}{\smile}$.
2 $\frac{1}{4}$	Eguale à 1 $\overset{2}{\smile}$.p.1 $\overset{1}{\smile}$.p. $\frac{1}{4}$
1 $\frac{1}{2}$	Eguale à 1 $\overset{1}{\smile}$. p. $\frac{1}{2}$
1.	Eguale à 1 $\overset{1}{\smile}$.

Descartes: the equation of a hyperbola

tipliant la ſeconde par la troiſieſme on produit $\frac{ab}{c}y - ab$, qui eſt eſgale à $xy + \frac{b}{c}yy - by$ qui ſe produit en multipliant la premiere par la derniere. & ainſi l'equation qu'il falloit trouuer eſt.

$$yy \infty cy - \frac{cx}{b}y + ay - ac.$$

de laquelle on connoiſt que la ligne E C eſt du premier genre, comme en effect elle n'eſt autre qu'vne Hyperbole.

Note: Viete, Fermat and Descartes finally use letters for parameters instead of random simple numbers. Above the fully abstract formula:

$$y^2 = cy - \frac{cx}{b}y + ay - ae$$

VI. Concluding remarks

- Numbers themselves were arguably the first mathematical reificiation
- In calculus, the ideas first of a *function*, then of an *operator*, a map from functions to functions, must be treated as things.
- Mathematicians tend to be Platonists and accept such reification easily; but there are skeptics, e.g. William of Ockham vs. Scotus, Aquinas, ….
- Perhaps it is not too much to say that some new *reification* has been the central element in all revolutions in the history of math