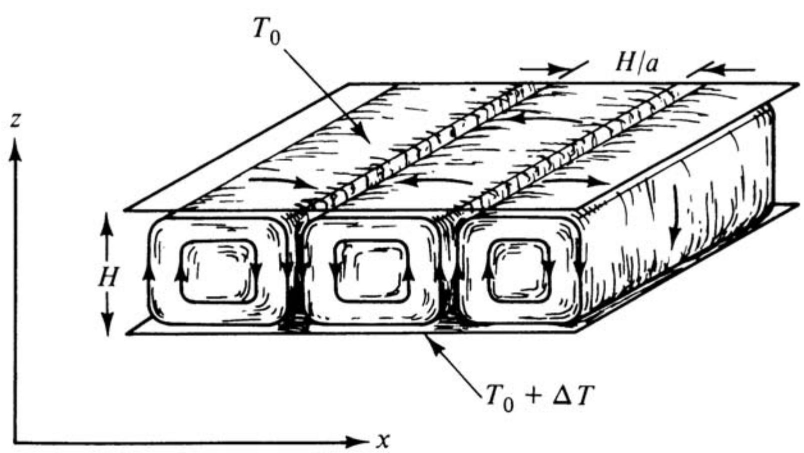
Chapter Fifteen: Butterflies and Tornadoes

Lorenz's equation

Edward Lorenz, a mathematically trained meteorologist at MIT, was interested in the foundations of long-range weather forecasting[1]. With the advent of computers in the 1950's, it had become popular to try to predict the weather by numerical analysis of the equations governing the atmosphere's evolution. The results, unfortunately, were rather poor. A statistical approach looked promising but Lorenz was convinced that statistical methods in use at the time, especially prediction by linear regression, were essentially flawed because the evolution equations are very far from being linear. To test his ideas, he decided to compare different methods applied to a simplified non-linear model for the weather. The number and complexity of the equations was a critical issue because of the limited computing power available in those days – Lorenz's computer had 16K internal memory and could only do 60 multiplications per second!

After experimenting with several examples, Lorenz focused on what are called 'convection rolls'. Solar radiation heats the surface of the earth, hence it heats the atmosphere immediately above the earth. But higher in the stratosphere, the air is cooled by radiation back out into space. But hotter air is less dense hence seeks to rise and denser cooler air seeks to fall. Then they get in each other's way and cause a traffic jam. Sometimes out of this jam comes results in a nice agreement to move up and down in regular ways, called convection rolls (and sometimes not). But the simplest convection rolls look like these[2]:



To create a mathematically tractable situation, he assumed the motion of the air and its temperature had the simplest possible form of the type shown in the figure, which is given by the formulas (using the notation of the figure):

[1] This discussion is taken from Marcelo Vaina's article "What's New on Lorenz Strange Attractors?".
[2] Figure from E. Atlee Jackson, 'Perspectives of nonlinear dynamics', vol. 2, p.141.

$$
\begin{aligned}
\text{horizontal vel.} &= -c\sin(\pi ax/H)\cdot\cos(\pi z/H)\cdot X(t)\\
\text{vertical vel.} &= ca\cos(\pi ax/H)\cdot\sin(\pi z/H)\cdot X(t)\\
\text{temperature} &= T_0 + (1 - z/H)\Delta T +\\
&\qquad + d\sqrt{2}\cos(\pi ax/H)\cdot\sin(\pi z/H)\cdot Y(t) - d\sin(2\pi z/H)\cdot Z(t)
\end{aligned}
$$

where, as in the figure, H is the height of the stratosphere, T_0 is the temperature of the stratosphere, ΔT is the increase in temperature at the earth's surface, a is the ratio of the horizontal and vertical dimension of the convection roll and c,d are messy constants involving gravity and the nature of air. Note that (v_x, v_z) describes a roll exactly as in the figure, turning over at a variable rate $X(t)$. In the expression for $T(t)$, there is a linear gradient of temperature if $Z=Y=0$, but on the whole more heat near the earth if $Z>0$ and less if $Z<0$, while Y puts a temperature asymmetry in the ascending and descending columns of air.

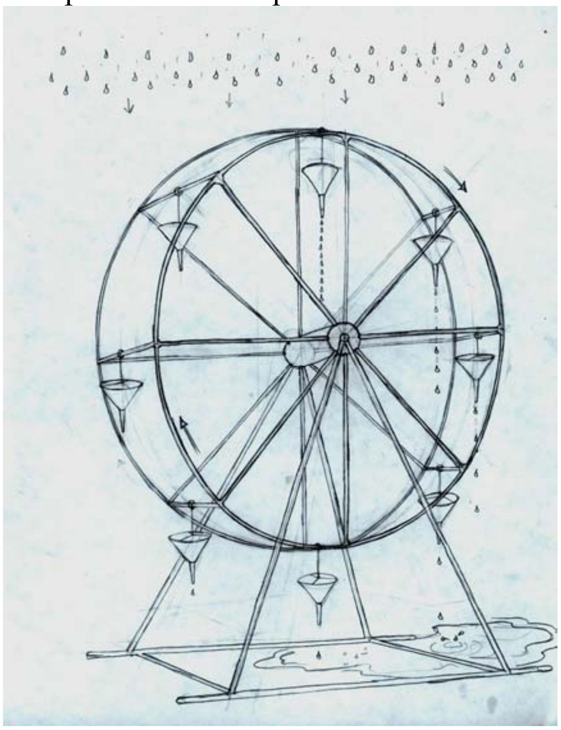
Lorenz substituted these expressions into a big partial differential equation – like the vibrating string equation – incorporating radiative cooling, gravitational force and other effects. He found that to satisfy this more complete model, his three variables X,Y,Z simply had to satisfy three simple ordinary differential equations:

$$
\begin{aligned}
\frac{dX}{dt} &= -\sigma(X - Y)\\
\frac{dY}{dt} &= -Y + X(r - Z)\\
\frac{dZ}{dt} &= -bZ + XY
\end{aligned}
$$

where $r=1/\pi d$, $b=4/(1+a^2)$, σ depends on the nature of air. This equation has been the focus of a huge amount of attention because it does many things, for different values of the parameters. Lorenz chose $\sigma=10$, $b=8/3$ and $r=28$ for various reasons – don't ask me why. BUT this is a remarkable system.

The Water Wheel

There is a quite simple mechanical model which obeys these equations exactly and which was also introduced by Lorenz. This model is a Ferris wheel whose seats are replaced by leaky buckets. Meanwhile the rain is pouring down, filling the buckets to replace the leaking water. The assumption is that the lower buckets are partially shadowed by the upper ones, so that they fill faster at the top than the bottom but symmetrically on the left and right. As above we let $X(t)$ be the speed of rotation, $Y(t)$ the sum of the weights of the buckets times their *horizontal* distance from the hub and $r\text{-}Z(t)$ the sum of their weights times their *vertical* distance from the hub where r is the value of this sum when the wheel is locked and buckets reach an equilibrium between the rain and their leaks. A little math suffices to show that (X,Y,Z) satisfy Lorenz's equations if the units are appropriately chosen. The wheel is seen below. The rain, of course, represents the solar radiation, the wheel represents the convection cell turned upside down and the amount of water represents the temperature.



What are the fixed points of Lorenz's equations? We simply solve $\dot{X} = \dot{Y} = \dot{Z} = 0$ and find three solutions:

(a) The single point $\mathcal{Y}$ = (0,0,0), representing no motion of the cell or wheel,

(b) Two points $\mathcal{A}$ and $\mathcal{B}$ given by $X = Y = \pm\sqrt{b(r-1)}$, $Z = r-1$ representing steady clockwise and counter-clockwise convection or rotation.

Next, recall what happened with the full phase plane plot of the pendulum studied in Chapter 7. We had 2 fixed points: the stable fixed point $\mathcal{S}$ where the pendulum points down and is motionless and the unstable fixed point $\mathcal{U}$ where the pendulum points directly up and is also motionless. (The pendulum should be imagined as suspended at the end of a rigid rod.) Of course the slightest nudge and the upward pointing pendulum falls, slowly at first, then faster to the left or right. In the phase plane, the motion of falling left or right was seen as the two sides of a curve through the point $\mathcal{U}$ representing this unstable equilibrium: this curve is sometimes referred to as Out($\mathcal{U}$), the set of outward flowing directions. On the other hand, one might imagine placing the pendulum near the vertical and giving just the right upward push so that it would move up and come to a stop exactly at $\mathcal{U}$! One can do this from both sides and these trajectories of the system give a second curve in the phase-plane plot, called In($\mathcal{U}$) and representing the flows of the system that point inward to $\mathcal{U}$.

We can do exactly this in the 3D Lorenz system. We take X,Y,Z as coordinates in 3-space, so that any solution $X(t)$, $Y(t)$, $Z(t)$ of the Lorenz equations represents a curve in 3-space: this is the 3D phase portrait of the Lorenz equation. Moreover, the equations define a vector at each point of X,Y,Z space, a so-called *vector field*. If we could, we would make a quiver plot showing all these arrows but this gets pretty messy in 3-space. In this 3 space, the equilibrium points $\mathcal{Y},\mathcal{A},\mathcal{B}$ have zero vectors attached to them. But none of these points are stable and they all have some solutions pointing out from them, others pointing in and approaching them in the limit. The picture of how this works out is shown on the next page.

The three semistabile equilibria

Let's look at these three fixed points and think through what happens if we state of the system is very near but not exactly equal to one of them. We can start with $\mathcal{Y}$, the state where the system is stopped and there is no L/R asymmetry in the water, and only the top/down asymmetry caused by the extra rain at the top, stabilized by the drip. If we increase or decrease the top/down asymmetry without any rotation or L/R asymmetry, the system will return to the equilibrium due to more leakage or filling back up from the rain. This is an IN path. But if the perturbed state has a little rotation, it will cause L/R asymmetry of the water and the rotation will accelerate and the system will not return. This is an OUT path. Finally, one can simultaneously introduce a small clockwise rotation and a little more water on the left than on the right: these effects oppose each other and, as the water leaks and friction slows the wheel, it will return to the equilibrium. This is another IN path. Altogether, there is a 2-dimensional slice giving IN($\mathcal{Y}$) and a 1-dimensional curve giving OUT($\mathcal{Y}$). Thus $\mathcal{Y}$ is neither a stable nor an unstable equilibrium: it is a bit of both. This is diagrammed in the cartoon below, from the wonderful cartoon book of illustrated dynamical systems of Abraham and Shaw. (Note that our point $\mathcal{Y}$ is denoted Y in that figure.)

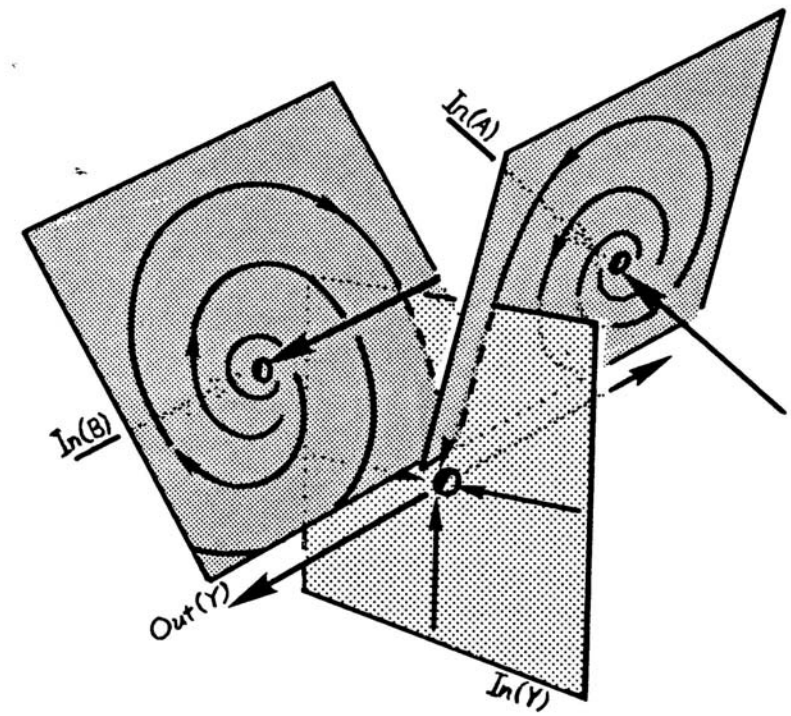
One can also work this out by algebra. One considers the 3 equations near (0,0,0) and, because (X,Y,Z) are all small, one drops the quadratic terms, leaving simply:

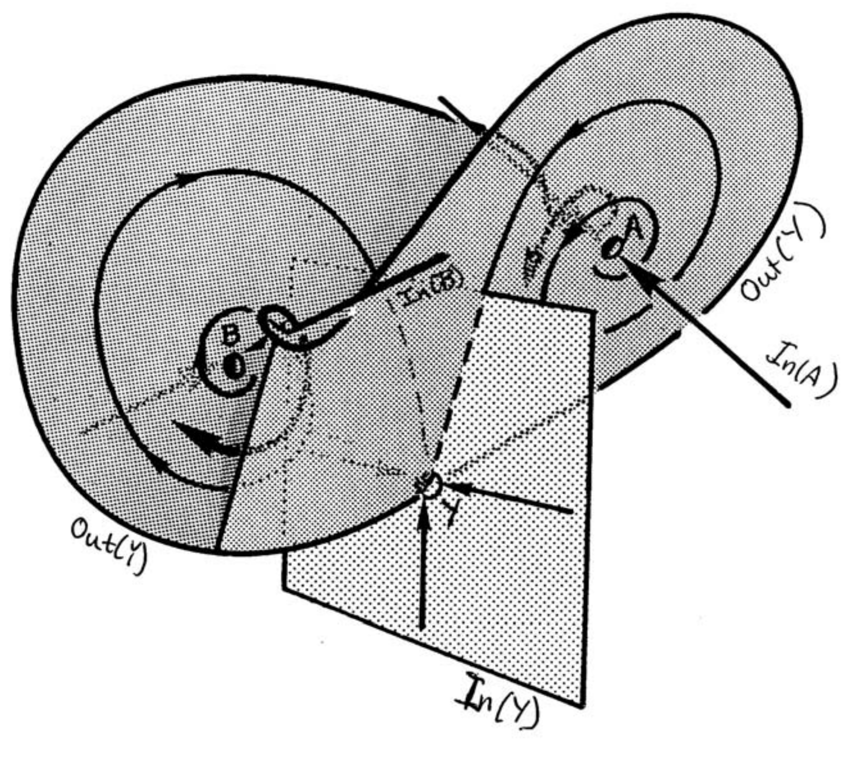
$$\frac{dX}{dt} = -\sigma(X-Y)$$
$$\frac{dY}{dt} = -Y + rX$$
$$\frac{dZ}{dt} = -bZ$$

These are *linear* equations and a small calculation, called *eigenvalue* analysis will show what we claimed by 'hand waving'. (*Eigen* is a German word meaning *proper* which makes this calculation sounds exotic, so it has stuck in English usage too.) We'll stick with the hand waving.

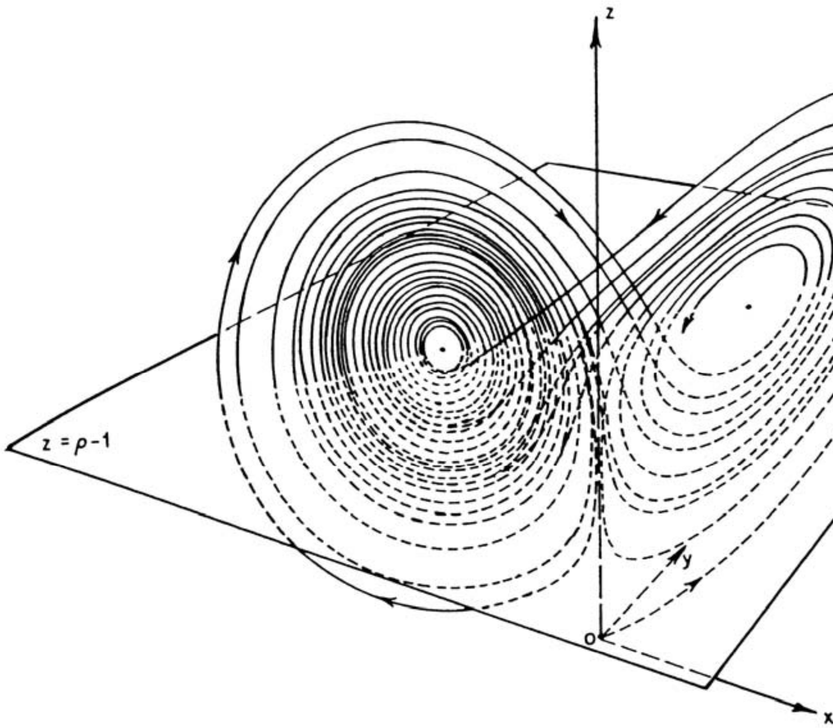
Now suppose we are at the equilibrium point $\mathcal{A}$. What is happening here is that the wheel is rotating clockwise, there is more water on the right and more water on the top than on the bottom. The extra water on the right is working to accelerate the rotation while friction is decelerating it and these two are canceling out. The rotation speed is just right so that as the wheel turns the extra water being carried to the left is cancelled by the drip. And by the extra water on the top being moved to the right by the turning of the wheel. This is achieved by the exact values $X = Y = \sqrt{b(r-1)}, \quad Z = r-1$. Now imagine a small perturbation in which the speed of rotation is increased. This is the most interesting: the extra speed causes the extra water on the right to move partly to the left; this causes the speed of rotation to decrease, in fact beyond the equilibrium value; then the slower speed allows the rain to accumulate on the top; the rotation carries this to the right; this re-accelerates the wheel; etc.etc. The wheel is now oscillating, first faster, then slower, then faster, etc. This is an oscillating instability, a combination of simple harmonic motion and growing strength. It makes OUT($\mathcal{A}$) 2-dimensional.

There is one direction in which $\mathcal{A}$ can be perturbed and relax back to the equilibrium $\mathcal{A}$. Suppose we increase the speed a tiny bit and decrease the extra water on the right and on the top. If we do it just right, friction will slow down the wheel, turning will decrease the extra top water by taking it to the right and decrease the extra right water by taking it to the left. In other words there is a 1-dimensional IN(a). All this can be checked by the magic *eigenvalue* toolkit. The figure below shows all of this.





It's truly hard sometimes to visualize a complicated 3-dimensional diagram. Here is a computer rendition. The origin is still the origin and the other two equilibrium points are shown with dots.



Clockwise or Counter-clockwise: can we predict it?

What we really want to know is how the system behaves over extended periods of time. The analysis above of its behavior near the equilibrium points must be glued together to provide a global picture. The effect is this: suppose the system starts somewhere near $\mathcal{A}$. rotating clockwise. Then it spins outwards, more or less in OUT($\mathcal{A}$) in greater and greater oscillations. In these oscillations, the wheel goes alternately faster and slower. At some point it slows to a stop because too much water has been carried over to the left, the wrong side: and then it starts to rotate in the other direction – counter-clockwise. These rotations at first accelerate as the wheel nears the other equilibrium $\mathcal{B}$. But the opposite rotation is very unlikely to be stable either and it oscillates again in increasing fast and slow cycles of counter-clockwise rotation. At some point, this too slows so far that it stops and then it goes back to clockwise rotation.

It's much like the Democrats and the Republicans in power: one gets elected with great enthusiasm. Then it has to make many decisions and each one irritates more and more people. At some point which varies a lot, depending on what seem like accidents, it gets so unpopular that the other party gets elected. And so the cycle repeats, with a seemingly unpredictable number of turns on each side before flipping to the other.

Lorenz recalled what happened when he began to explore this system of differential equations. At one stage during the computation he decided to take a closer look at a particular solution. For this, he restarted the integration using some intermediate value printed out by the computer as a new initial condition. To his surprise, the new calculation diverged gradually from the first one, yielding totally different results in about four "weather days". Lorenz considered the possibility of hardware failure before he understood what was going on. To speed things up, he had instructed the computer program to print only three decimal digits, although the calculations were carried out to six digits. So the new initial condition entered into the program didn't quite match the value generated in the first integration. The small initial difference was augmented at each integration step, causing the two solutions to look completely different after a while.

The consequences were far-reaching: assuming the weather does behave like these models, then long-range weather prediction is impossible: the unavoidable errors in determining the present state are amplified as time goes by, rendering the values obtained by numerical integration meaningless within a fairly short period of time.

Interestingly, the great mathematical genius Henri Poincaré had anticipated this result in 1909:

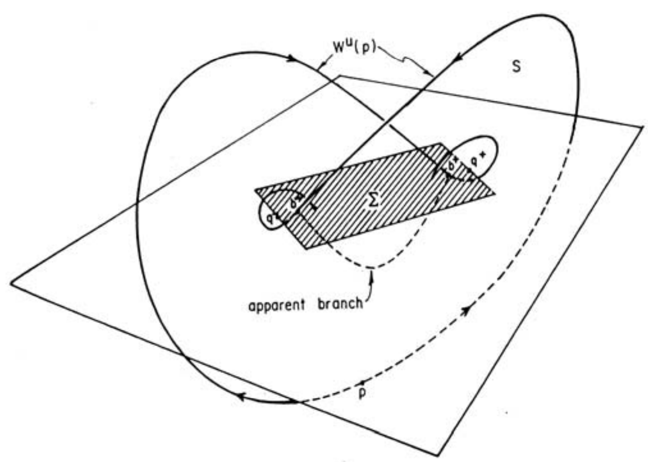
> *Why have meteorologists such difficulty in predicting the weather with any certainty? Why is it that showers and even storms seem to come by chance, so that many think it is quite natural to pray for them, though they would consider it ridiculous to ask for an eclipse by prayer? ....* [If the temperature is varied by] *a tenth of a degree more or less at any given point, the cyclone will burst here and not there, and extend its ravages over districts it would otherwise have spared. If they had been aware of this tenth of a degree, they could have known it beforehand, but the observations were neither sufficiently comprehensive nor sufficiently precise, and that is the reason why it all seems due to the intervention of chance.*

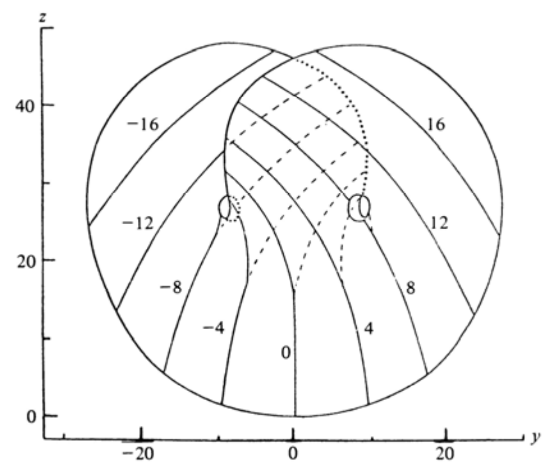
Lorenz put it more poetically when he entitled a lecture given in 1972 to the AAAS: *"Does the flap of a butterflies wings in Brazil set off a tornado in Texas?"*

The return map

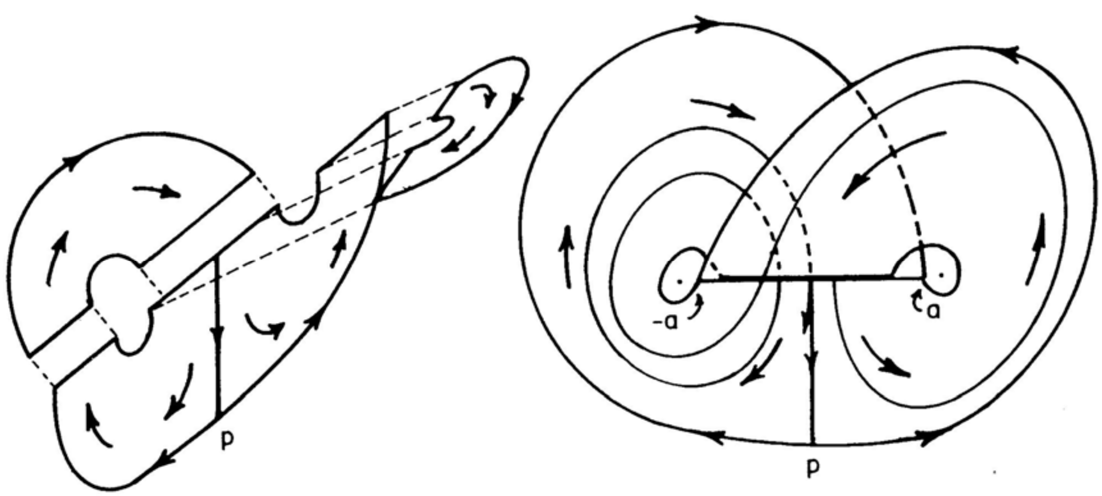
What can you do in the face of such unpredictability? You could give up, throw up your hands and say: math stops here! But Lorenz didn't do this. He looked closely at the 3-dimensional picture and found a way to understand it better. It all hinges on something called the return map, shown here in two renditions.

The key point is to pick out the point on each fast/slow cycle when the wheel slows the most and keep track of these states. It turns out that these states, more or less, all lie on a line between the two fixed points $\mathcal{A}$ and $\mathcal{B}$. The rotation speed X is a coordinate on this line segment, with $X=\sqrt{b(r-1)}$ being one fixed point and $X=-\sqrt{b(r-1)}$ being the other. It's convenient to use $x=(X+\sqrt{b(r-1)})/2\sqrt{b(r-1)}$ as coordinate on the line instead, so that x=0 at one fixed point, x=1 at the other. Then starting at any point on this line, we follow the equations and if $x<.5$, it will rotate around the left wing of the butterfly, while if $x>.5$, it goes around the right. When it comes back to the shaded region in the middle, it's near this line again: ignore the small displacement and push it back to the line. The result is that for x other than 0.5, we get a new value of x , call it $f(x)$. For x=0.5, the orbit falls into the origin (0,0,0). Two renditions of the result are shown below.





A simplified version of how the dynamic system behaves is shown in a third version:



A complete mathematical model: symbolic dynamics

What is the map f which arises from this 'return' map? Experimentally, the graph of f is shown in the figure on the next page. This looks as though it might be complex but.Lorenz conjectured – and after 40 years, ?? has finally proven – that this 'return' mapping f behaves like the very simple map $f(x) = \begin{cases} 2x, & \text{if } 0 < x < 0.5 \\ 2x-1, & \text{if } 0.5 < x < 1 \end{cases}$

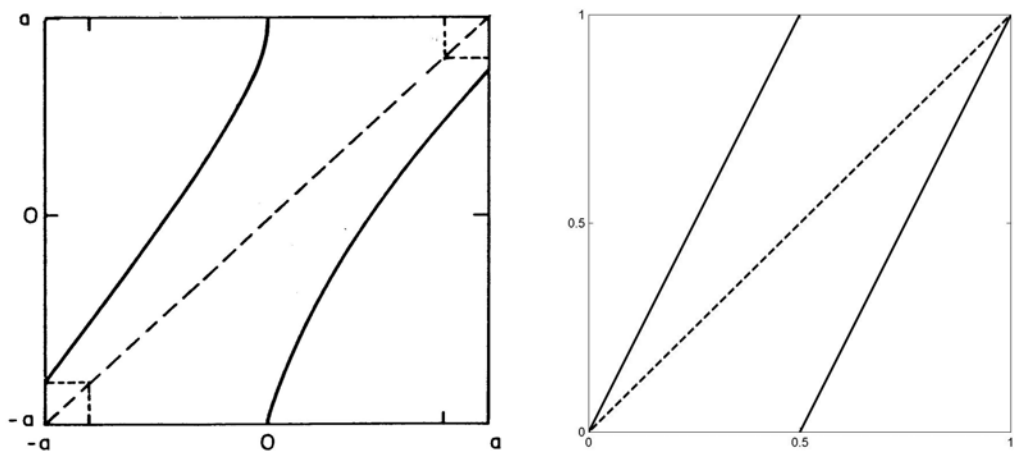
What do we mean by 'behaves like'? It is simply this: if we had a suitably warped ruler to measure where points were between the endpoints $\mathcal{A}$ and $\mathcal{B}$, then using the numbers x read off from this ruler, the return map would be given *exactly* by the above formula.

This shows immediately why the system is so unpredictable: expand x as a binary 'decimal", like x=.1100101001.... (which means

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{32} + \frac{1}{128} + \frac{1}{1024} + \ldots)$$

Then the effect of f is just to strip off one binary place at a time, i.e. for the above x, $f(x)$=.100101001.... Thus the n^{th} binary place is 0 or 1 if and only if the n^{th} loop in the future of x goes around the left or right wing of the butterfly. Thus the n^{th} cycle in the future is buried in the finer and finer approximations to the number x.

The graph of the true Lorenz 'return' map and its symbolic version



What does this leave us with? We cannot predict the future but we have a good understanding of why not. And we can deduce good *statistical* predictions of the future. And if this is all part of a larger more comprehensive model, we may be able to predict some features of the future even though others are beyond our reach. Thus meteorologists may be able to predict the intensity of the hurricane season even though the precise days and locations of hurricanes A through Z may elude us.