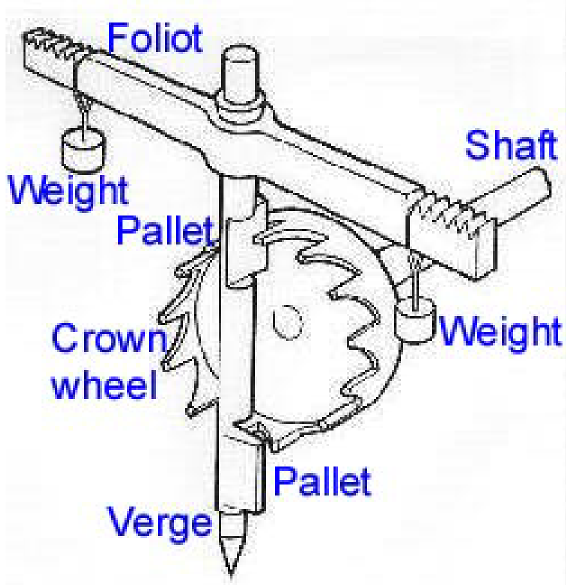
Chapter Three: Oresme and the Invention of Graphing

Alfred Crosby's book "The Measure of Reality" quotes the Arab scholar Ibn Khurradadhbeh as describing Western Europe in the mid-9th century as a source of *"eunuchs, slave girls and boys, brocade, beaver skins, glue, sables and swords"* and not much else. Indeed, the Arab Middle East and China were the intellectual leaders of the world at that time, cultivating science, poetry and art and investing heavily in large libraries and scholarship. In particular, algebra was greatly advanced by al-Khwarizmi's book *Hisab al-jabr w'al-muqabala* (from which the word 'algebra' came into use). Europe was at the margins of civilization. Crosby's book deals with an analysis of what happened next, what sparked the resurgence of Europe and the Renaissance of Western culture. His hypothesis is that the turning point was in the 14th century, when clocks were invented, better sailing charts were devised and better accounting was worked out.

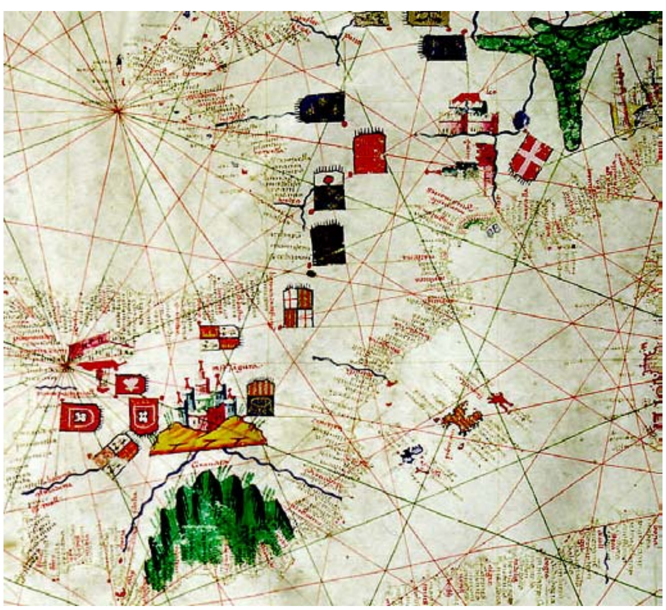
The accepted view of the world in the middle ages had been very finite and very non-empirical. Time, for example, had, on the basis of the bible, been calculated by various scholars to have started at dates like 3952 BCE or 5194 BCE. And Christ's life was supposed to be the midpoint of time, so the last judgment was expected at 4000-5000 AD. This means all of time included only 300 or so generations of mankind. Likewise, the day was divided crudely by times of prayer and the tolling of church bells, *matins* at sunrise, *prime* at breakfast time, *tierce* in the morning, *sext* around noon, *none* in the afternoon, *vespers* at sunset and *compline* in the evening – with intervals growing and shrinking according to the seasons. Space was similarly small: the earth was surrounded by 9 concentric crystal spheres and these had shrunk remarkably since the Greek calculations. Now Roger Bacon had the moon at a distance of only 100,000 miles (a distance which could be *walked* in a pilgrimage of about 15 years!) and Gossoin of Metz placed the outermost crystal sphere at 6,500,000 miles, a 700 years walk. Much later, even Columbus estimated the earth as 25% smaller than it was, hence his naïve optimism at expecting to reach the Spice Islands of the Far East.

So what changed? Crosby's thesis is that people discovered the power and usefulness of measuring things *accurately* and fell in love with this new modern way of doing things. It infected everyday life, from the speculations of the scholastics to the accounting of business to music and art. It was a rediscovery of the true power of mathematics and science. Crosby cites the wave of construction of great town clocks which swept Europe in 1270-1330 as the first sign of this sea change. Never before and nowhere in the world had anything like this been invented. They cost an arm and a leg: townspeople might put a full year's tax to build one. Here is the mechanism of a typical 15th century town clock.



1392 Wells Cathedral Clock

The picture does not show the heavy weight attached to a rope which is wound around the shaft. As we are looking at the crown wheel, this weight seeks to turn the crown wheel counter-clockwise. But it is held back by one of the 2 pallets. The sequence of events is this: the heavy foliot swings counter-clockwise until the pallet ceases to block the crown wheel. The crown wheel then advances until it hits the second pallet. Its pressure on the second pallet stops the motion of the foliot and then swings it clockwise until this pallet twists out of the way of the crown wheel. Then the wheel turns until blocked by the first pallet. This repeats itself indefinitely, the foliot swinging back and forth and the crown wheel advancing once for every swing. It worked although technicians were needed to tune it up and repair it daily and it was never more accurate than 15 minutes a day. But now the town could regulate its life.



A detail of the 1466 Roselli portolan chart of the Mediterranean

A second advance was accurate measurement of the earth itself. 'Portolan' charts, using multiple polar coordinates, were invented. Here is a part of such a chart of the Mediterranean, showing the part of the coastline of Spain and France:

A third advance was the invention of double-entry bookkeeping, which enabled much more complex business enterprises to be tracked and reduced to clear unambiguous numbers. Started in the 14^{th} century, this method was brought to essentially its modern form by the mathematician Luca Pacioli in he renaissance.

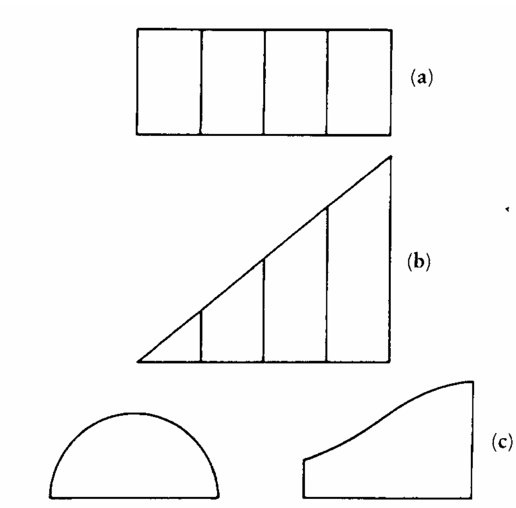
But I want to focus on what seems to me the biggest mathematical step that was taken at that time: the invention of graphing. The Greeks had been concerned with geometry of the plane for its own sake, the complex way in which triangles and circles interact and structure 2-dimensional space. This is very different from using the plane for the sake of *visualization*, which is what graphs are all about. In particular, the idea of tracking some measurable quantity like position, velocity, temperature or brightness over an interval of time or for all the points of an object and making a geometric representation of all these numbers – of this *function* of time and space – was brand new. The idea is due to Nicole Oresme.

Oresme (pronounced 'Orem') was a scholastic philosopher who lived from 1323 to 1382, taught at the then recently founded University of Paris and eventually rose to become a bishop. Among other works, he wrote, in 1353, *Tractatus de configuationibus qualitatum et motuum* (Treatise on the configurations of qualities and motions) which seems to me the first really big European book on mathematics: it made explicit the idea of functions, x- and y-coordinates and, above all, of the graphs of functions. Of course, he didn't use these terms, in Latin or English. I think it's very instructive to see what he actually wrote: you get some insight in the way he viewed the world. Here's how his book begins, in Clagett's English translation together with my gloss:

Every measurable thing except numbers is imagined in the manner of continuous quantity. Therefore, for the mensuration of such a thing, it is necessary that points, lines and surfaces, or their properties be imagined. For in them, as the Philosopher has it, measure or ratio is initially found, while in other things it is recognized by similarity as they are being referred to by the intellect to the geometrical entities. Although indivisible points, or lines, are non-existent, still it is necessary to feign them mathematically for the measures of things and for the understanding of their ratios. Therefore, every intensity which can be acquired successively ought to be imagined by a straight line perpendicularly erected on some point of the space or subject of the intensible thing, e.g. a quality. For whatever ratio is found to exist between intensity and intensity of the same kind, a similar ratio is found to exist between line and line, and vice versa. … Therefore, the measure of intensities can be fittingly imagined as the measure of lines.	1. Roughly, he's saying that all measurable things in the world are either discrete things like whole numbers, or vary continuously. 2. Given 2 such measurements, they always have a ratio, one to the other; and the most basic case of this sort of measurement is the length of line segments or the area of surfaces, because 2 lengths or 2 areas have a definite ratio, one to the other. 'The Philosopher" is Aristotle. 3. Points are infinitely small and lines infinitely thin, so they are idealizations. 4. 'Successive' means a quantity that varies in time $f(t)$. 5. The 'subject' is the set of points on which the function f is defined, its domain. 6. His graph is given by imagining perpendicular lines erected on the domain, like a bar graph.

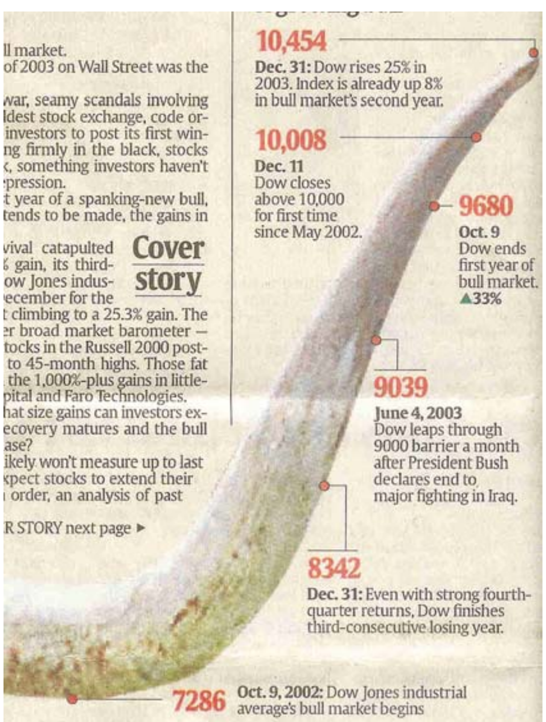
In the next few chapters, he calls 'longitude' the axis of the independent variable (which can be space or time and which varies over a set of values he calls the 'subject' or the 'extension'); and 'latitude' the perpendicular axis plotting values of the 'quality' or dependent variable, these values being called the 'intension' or 'intensity'. A few pages later (I.iv), he states the idea of graphing quite clearly:

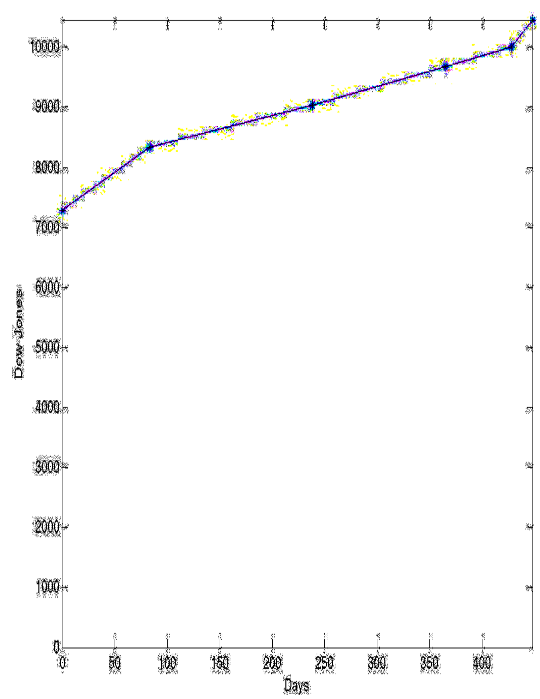
The quantity of any linear quality is to be imagined by a surface whose length or base is a line protracted in a subject of this kind and whose breadth or altitude is designated by a line erected perpendicularly on the aforesaid base. And I understand by “linear quality” the quality of some line in the subject informed with a quality. That the quantity of such a linear quality can be imagined by a surface of this sort is obvious, since one can give a surface equal to the quality in length or extension and which would have an altitude similar to the intensity of the quality. But it is apparent that we ought to imagine a quality in this way in order to recognize its disposition more easily, for its uniformity and its difformity are examined more quickly, more easily and more clearly when something similar to it is described in a sensible figure. … Thus it seems quite difficult for some people to understand the nature of a quality which is uniformly difform. But what is easier to understand than that the altitude of a right triangle is is uniformly difform. … Now, just as the quality of a point is imagined as a line and the quality of a line by a surface, so the quality of a surface is imagined as a body whose base is the surface informed with the quality…. Moreover, since in any kind of a body there is an infinite number of equivalent surfaces and the quality of any one of them is imagined as a body, it is not unfitting but necessary that one body be imagined to be at the same time in the place where another body is imagined to be. We can think of this taking place by penetration or mathematical superposition. … It does not happen that a fourth dimension exists or is imagined, still a corporeal quality is imagined to have a double corporeality: a true one with respect to the extension of the subject in every dimension and another one that is only imagined from the intensity of this quality taken an infinite number of times and dependent upon the multitude of surfaces of the subject.	1. In modern terminology, ‘linear quality’ means a dependent variable y depending on 1 independent variable, i.e. $y=f(x)$. 2. ‘Subject’ means the domain of x, in this case a line segment I. 3. The surface referred to is the plane figure $0\leq y\leq f(x)$, x in I. The “quantity” of the quality means the area of this surface or the integral of f over I. Note that the x and y axes are required to be perpendicular. 4. Next, he says that any such quality can be graphed like this. *Note that his qualities are always positive.* 5. Some such qualities are “uniform”, meaning f is constant, and others “difform”, meaning f is non-constant and he notes that one sees such things much better by making a graph, because it is then “sensible”, i.e. visible to the eye. 6. Finally “uniformly difform” means the rate of change of y is constant, or equivalently the graph is a straight line and so it is part of the hypotenuse of a right triangle erected in the x axis. 7. If a function is defined only at a point, it is represented by a single line segment; if it is defined on a line, its graph is a surface; if it is defined on a surface, its graph is a solid object. Note this graph is the set $0\leq z\leq f(x,y)$, (x,y) in I. 8. Now he says, can we graph a function of three variables? We’d like to use a fourth dimension but this doesn’t exist, so he says we have to imagine the graph as having ‘double corporeality’, consisting of superimposed objects.



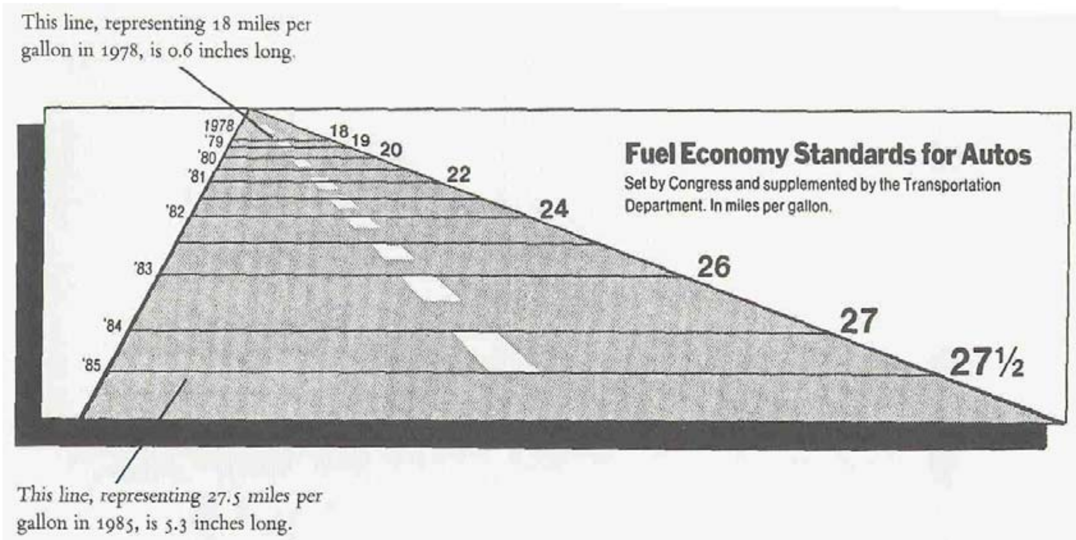
Here are some of the examples of such graphs from his book. (a) is uniform, (b) is uniformly difform and the two examples in (c) are “difformly difform”. They are pretty basic: he didn’t engage in much real life data gathering but only in the theory of this.

One thing he was very clear about is that the key thing about a graph is that its shape should depict accurately the *ratios* of the quality being measured against the true distances in the *subject*, an interval of space or time. Suppose you are graphing some quality like heat. You can choose any scale you want in the vertical axis by assigning some distance to a unit of heat; but regardless of what unit is chosen, if one measurement is twice another, the plot should be twice as high on one as on the other. If this is simple principle is violated, the graph is quite misleading. In recent times, Edward Tufte wrote a wonderful and quite famous book entitled “The Visual Display of Quantitative Information” which, in particular, inveighs against the absurdity of graphs that flaunt this basic principle. Here is a recent example from USA Today, where the ‘bull’ market is forced to look like a bull’s horn in spite of the fact that the same data, plotted correctly climbs relatively a much more modest amount and it has nearly the opposite ‘S’ shape, (unlike the horn, it starts off steeply, then climbs less rapidly and ends with a steep increase).





Here's another example from Tufte's book:

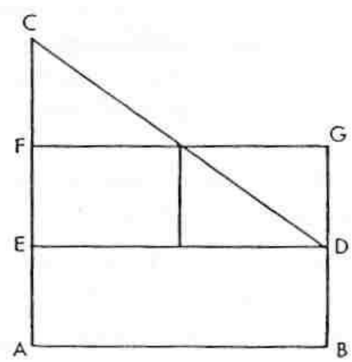


Note that the extreme fuel economy standards have a ratio 27.5/18, about 1.5, while the corresponding line lengths have a ratio 5.3/0.6, about 8.8, violating Oresme's precept to accurately represent ratios by nearly 6. Tufte calls this a 'Lie Factor' of 15 (because a 50% increase is shown as a 780% increase). Allow me to inveigh too: I think that if the educated public was used to reading accurately drawn graphs and to using this geometric presentation of an often complex situation to understand what the data shows, our ability to make sensible economic and political judgments would be vastly improved.

Back to Oresme's book. Oresme goes on to talk about many 'qualities' which he believes ought in principle to be graphed – he discusses temperature, pain and grace. The only requirement is that two instances of such a quality should have a ratio. It is a bit astonishing to see him propose that one person has been given twice as much grace as another, but he was a scholastic philosopher after all. Although his qualities are always positive, from a modern point of view, you can see that he realizes something like negative numbers are needed when he talks about temperature as made up of the opposites hotness and coldness. This leads him to the idea of complementary graphs, where one, placed on the top of the other makes a constant total, i.e. $f(x)$ and $C-f(x)$. Although he never talks of oscillating qualities (which play such a key role for Galileo, 2 centuries later), he does talk of 'rough and difform' qualities to describe a soul '*occupied by many thoughts and affected by many passions*'. Part II of his book is concerned specifically with functions of time and functions of both time and space. He illustrates the significance of graphs having specific non-linear or 'difformly difform' shapes by the example of the optimal force for throwing a javelin: if the force is the right function of time, one can make a better throw than if not. It is also interesting that he makes very little distinction between physical qualities that we know how to measure today and psychological qualities that still defy measurement. Here is a rather interesting passage about the measurement and graphing of pain in which you see that he understands integration (II.xxxix). Pain is conceived of as a function *p(t)* of time. When he says that two pains $p_1(t)$ and $p_2(t)$ are '*simply equal*', he is saying that they have the same integral!

I suppose, therefore, that pain or sorrow is a certain quality of the soul which is extended in time and is intensifiable by degrees. Hence it is possible for two such qualities to be simply equal and yet for one to be more shunned and worse than another. This can happen in two ways: in one way as the result of an inequality in intensity, and in the other way as a result of a diversity in the configuration of their difformity. As an example of the first, let A and B be two pains, with A being twice as intensive as B and half as extensive. Then they will be equal simply ... although pain A is worse than and more to be shunned than pain B. For it is more tolerable to be in less pain for two days than in great pain for one day. But these two equal and uniform pains when mutually compared are differently figured ... so that if pain A is assimilated to a square, then pain B will be assimilated to a rectangle whose longer side will denote the extension and the rectangle and the square will be equal.

Finally, in Part III, he basically defines the Riemann integral and evaluates the integral of several functions including some improper integrals where the graph goes to infinity or the domain, the *subject,* is infinitely long. Integration, you see, has much older roots than Newton and Leibniz. Perhaps the most important fact about integration that he discovered is that area under a linear graph is the product of the length of the base times the height of the graph at the midpoint of the base. The figure below is from one of the manuscripts of his book. In his words (III.vii):

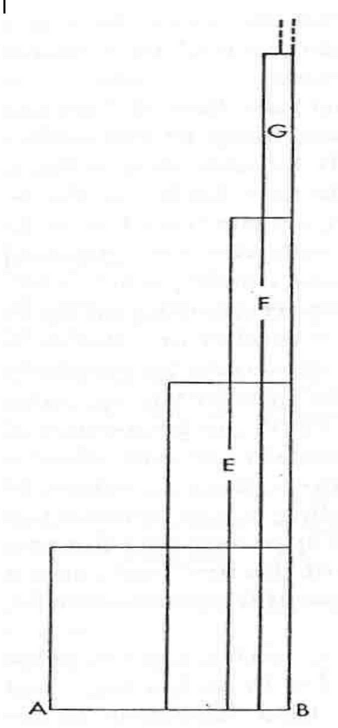


Oresme's assertion:
area(ABDC) =
area(ABGF)

"Every quality, if it is uniformly difform, is of the same quantity as would be the quality of the same or equal subject that is uniform according to the degree of the middle point of the same subject"

In modern terminology:

$$\int_a^b (Cx+D)dx = \left(C\left(\frac{a+b}{2}\right)+D\right).(b-a)$$



Here is a final example to show how far Oresme went with his primitive mathematics. He considers a 'quality' that has value *n* between points 2^{-n} and $2^{-(n+1)}$, so that it 'blows up' when *x*=0. The graph from his book appears on the left. By rearranging the blocks as shown in the figure to make one rectangle, he showed that the area is just twice the length *AB* times the height of the graph over *A*. Again, in modern terminology, he is approximating the evaluation of an improper integral:

$$\int_0^1 \log(1/x)dx = 1$$

Although Oresme did not discover the fundamental theorem of Calculus, he was very aware of one special case of this: that if you plot the *velocity* of an object as a function of time, then the area under this graph equals the distance travelled.

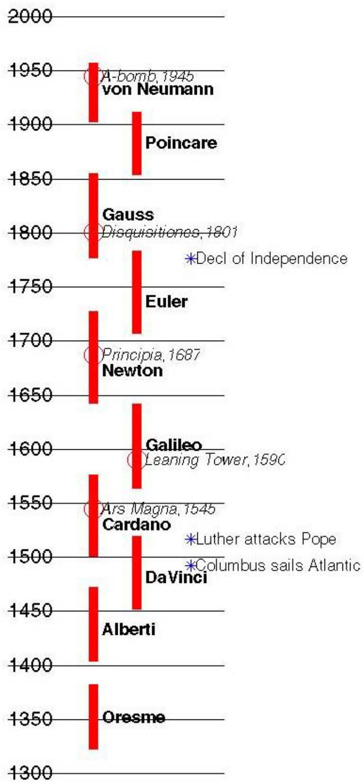
Problems:

(a) Suppose Marco Polo, on his return from China in 1295, had brought with him a marvelous oriental invention in which a Tibetan prayer wheel, a cylinder with many slots was spun around a brilliant light of burning phosphorus. (I'm making this up!) The Chinese had been delighted with its rapidly flickering light and the marvelous illusions it made when dancers were seen by its light. We moderns think of this device as an early precursor to a high frequency strobe light! IF Oresme had seen such a device, he might have related this to his theory of graphing. Write a paragraph in his style doing just this.

(b) Either (i) find an example of an outrageous graph in some newspaper or magazine that violates Oresme's and Tufte's principles and hand it in together with a proper version of the same data or (ii) find some data of real interest to you and make an informative graph of it, describing what its 'configuration' tells you.

Chapter Four: Galileo and the leaning tower of Pisa

With Galileo Galilei (1564-1642), we are entering the period of history that is much closer to us, that we can understand much more easily. Below is a time line of major Western mathematicians (mostly those who will come up in this course), and a few events to help one keep one's bearings.



Galileo lived just after the Renaissance, that wonderful storm of creativity, imagination and exploration. But before he was born, Luther had nailed his theses attacking corruption in the Catholic Church on the door of the Wittenburg Cathedral (1517) and Copernicus has published *De Revolutionibus* (1543) attacking the idea that the earth was the center of the universe. The Church was on the defensive, the counter-reformation and the inquisition had begun and scientists had to wary as Galileo found out all too well. In the middle of his life, Galileo became convinced that Copernicus was right that the earth turned about the sun. But the Church had decided this contradicted scripture, for example:

> *And the sun stood still, and the moon stayed, while the nation took vengeance on its foes* (Joshua 10:12)

and had allowed Copernicus's book to be published only as a scheme for more efficient calculations, not as presenting objective truth. They burned Giordano Bruno when he refused to comply. So Galileo adopted the strategy of writing a dialogue between three characters, Salviati, Sagredo and Simplicio. Salviati is clearly a stand-in for Galileo himself, Simplicio states the Aristotelian and Church positions (though not very well) and Sagredo is meant to be the intelligent man-in-the-street whom Galileo is seeking to convince of his ideas. His book was called *'Dialogues concerning Two Chief World Systems'*. and pits the Ptolemaic model with the sun revolving around the earth against the Copernican one. Ironically, one of his main arguments that the earth turned was based on the tides and was quite wrong. Moreover his ruse to get around the papal prohibition did not work very well and Galileo was hailed before the inquisition. There he was forced to renounce the Copernican model, reportedly, however, muttering under his breath as he left *'Eppur si muove'* (But still it moves). He lived out the rest of his life under house arrest near Pisa.

But what got Galileo into the 'Natural Philosophy' as a young man was not the traditional problems of the heavens but things that were much closer to home, like the motion of falling bodies and the trajectories of cannon balls. He realized that the laws of motions of such bodies on the earth's surface should also be studied and measured and that when you actually checked, things didn't happen at all the way Aristotle had written. Galileo's forte was experimental science and his two major interests were mechanics and later astronomy (he discovered the moons of Jupiter, sunspots and the rings of Saturn). His mathematical models were fairly rudimentary, more or less on the level of Oresme but his discoveries were far reaching. Not until Newton published *Principia Philosophiae* (1687) did the right mathematical models (and a clear general statement of the physical laws) for his ideas emerge.

Galileo's most famous experiment was certainly that of dropping two balls off the leaning tower of Pisa to see if they hit the ground at the same time. It is believed he did this in 1569 at age 25 or soon after. As in studying Oresme, it is much more interesting to hear Galileo tell the tale himself than simply to describe his ideas. He writes about this in the book *'Dialogues concerning Two New Sciences'* published in 1638 when he was an old man under house arrest.

Below and on the next page, we give an excerpt from *Two New Sciences* concerning the speed at which bodies fall. It begins in the middle of a speech by Salviati claiming that Aristotle is wrong on several points and *probably never tested whether heavier and lighter bodies fall at the same rate*. When Simplicio begins to quote Aristotle saying the heavier body falls faster, Sagredo interrupts and says that he has *actually tried the test* and they fell at the same rate. Then Salviati goes on a long 'thought experiment' about dropping two bodies of different weights, what happens when you tie them together or merely let the heavier stone fall while being on top of the lighter. Simplicio now is beginning to doubt. You see the idea: some experiments are referred to with no actual data but supplemented with some general argument from common sense. The dialog continues and after this passage, Salviati does acknowledge that the resistance of the air affects the speed of falling but claims that for heavy objects this is a secondary effect. This is also one of Galileo's strengths: he realized that in many situations one should describe the main effect and ignore small discrepancies as being due to other causes.

And as to the first, I greatly doubt that Aristotle ever tested by experiment whether it be true that two stones, one weighing ten times as much as the other, if allowed to fall, at the same instant, from a height of, say, 100 cubits, would so differ in speed that when the heavier had reached the ground, the other would not have fallen more than 10 cubits.

SIMP. His language would seem to indicate that he had tried the experiment, because he says: *We see the heavier;* now the word *see* shows that he had made the experiment.

SAGR. But I, Simplicio, who have made the test can assure

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you that a cannon ball weighing one or two hundred pounds, or even more, will not reach the ground by as much as a span ahead of a musket ball weighing only half a pound, provided both are dropped from a height of 200 cubits.

SALV. But, even without further experiment, it is possible to prove clearly, by means of a short and conclusive argument, that a heavier body does not move more rapidly than a lighter one provided both bodies are of the same material and in short such as those mentioned by Aristotle. But tell me, Simplicio, whether you admit that each falling body acquires a definite

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speed fixed by nature, a velocity which cannot be increased or diminished except by the use of force [*violenza*] or resistance.

SIMP. There can be no doubt but that one and the same body moving in a single medium has a fixed velocity which is determined by nature and which cannot be increased except by the addition of momentum [*impeto*] or diminished except by some resistance which retards it.

SALV. If then we take two bodies whose natural speeds are different, it is clear that on uniting the two, the more rapid one will be partly retarded by the slower, and the slower will be somewhat hastened by the swifter. Do you not agree with me in this opinion?

SIMP. You are unquestionably right.

SALV. But if this is true, and if a large stone moves with a speed of, say, eight while a smaller moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. Thus you see

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how, from your assumption that the heavier body moves more rapidly than the lighter one, I infer that the heavier body moves more slowly.

SIMP. I am all at sea because it appears to me that the smaller stone when added to the larger increases its weight and by adding weight I do not see how it can fail to increase its speed or, at least, not to diminish it.

SALV. Here again you are in error, Simplicio, because it is not true that the smaller stone adds weight to the larger.

SIMP. This is, indeed, quite beyond my comprehension.

SALV. It will not be beyond you when I have once shown you the mistake under which you are laboring. Note that it is necessary to distinguish between heavy bodies in motion and the same bodies at rest. A large stone placed in a balance not only acquires additional weight by having another stone placed upon it, but even by the addition of a handful of hemp its weight is

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augmented six to ten ounces according to the quantity of hemp. But if you tie the hemp to the stone and allow them to fall freely from some height, do you believe that the hemp will press down upon the stone and thus accelerate its motion or do you think the motion will be retarded by a partial upward pressure? One always feels the pressure upon his shoulders when he prevents the motion of a load resting upon him; but if one descends just as rapidly as the load would fall how can it gravitate or press upon him? Do you not see that this would be the same as trying to strike a man with a lance when he is running away from you with a speed which is equal to, or even greater, than that with which you are following him? You must therefore conclude that, during free and natural fall, the small stone does not press upon the larger and consequently does not increase its weight as it does when at rest.

SIMP. But what if we should place the larger stone upon the smaller?

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SALV. Its weight would be increased if the larger stone moved more rapidly; but we have already concluded that when the small stone moves more slowly it retards to some extent the speed of the larger, so that the combination of the two, which is a heavier body than the larger of the two stones, would move less rapidly, a conclusion which is contrary to your hypothesis. We infer therefore that large and small bodies move with the same speed provided they are of the same specific gravity.

SIMP. Your discussion is really admirable; yet I do not find it easy to believe that a bird-shot falls as swiftly as a cannon ball.

SALV. Why not say a grain of sand as rapidly as a grindstone? But, Simplicio, I trust you will not follow the example of many others who divert the discussion from its main intent and fasten upon some statement of mine which lacks a hair's-breadth of the truth and, under this hair, hide the fault of another which is as big as a ship's cable. Aristotle says that "an iron ball of one hundred pounds falling from a height of one hundred cubits reaches the ground before a one-pound ball has fallen a single cubit." I say that they arrive at the same time. You find, on

Galileo was very concerned with finding the actual law obeyed by projectiles fired off at some angle, including ones simply dropped and by bodies falling down inclined planes. His *piece de resistance* was his deduction of the law

$$x = at$$

$$y = bt - \frac{g}{2}t^2$$

for a projectile with initial speed (a,b) where g denotes the acceleration downwards due to gravity and where we ignore the resistance of air. Three key things should be noted: (i) the *mass* of the body does not appear (hence two falling bodies or projectiles of different mass move in lock step), (ii) except for gravity, the velocity would be constant and (iii) if you solve the first equation for t: $t=x/a$ and substitute this into the second equation, you find that the path of the projectile is a parabola, specifically:

$$y - b^2/2g = -\frac{g}{2a^2}\left(x - ab/g\right)^2$$

Note that when $a=b=0$, we have the case of a falling body starting at rest and the formula is simply $y = -gt^2/2$. Now Galileo did not use equations to work this out! Algebra had been around since the Babylonians, continuing through Diophantus in Alexandria, al-Khwarizmi in Baghdad and Cardano in renaissance Italy, but it was more an esoteric art, not a general purpose language for use in any numerical study. Galileo preferred to use the tried and true methods of Euclid. Here is how he derived this law:

THEOREM I, PROPOSITION I

The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before acceleration began.

Let us represent by the line AB the time in which the space CD is traversed by a body which starts from rest at C and is uniformly accelerated; let the final and highest value of the speed gained during the interval AB be represented by the line EB drawn at right angles to AB; draw the line AE, then all lines drawn from equidistant points on AB and parallel to BE will represent the increasing values of the speed, beginning with the instant A. Let the point F bisect the line EB; draw FG parallel to BA, and GA parallel to FB, thus forming a parallelogram AGFB which will be equal in area to the triangle AEB, since the side GF bisects the side AE at the point I; for if the parallel lines in the triangle AEB are extended to GI, then the sum of all the parallels contained in the quadrilateral is equal to the sum of those contained in the triangle AEB; for those in the triangle IEF are equal to those contained in the triangle GIA, while those included in the trapezium AIFB are common. Since each and every instant of time in the time-interval AB has its corresponding point on the line AB, from which points parallels drawn in and limited by the triangle AEB represent the increasing values of the growing velocity, and since parallels contained within the rectangle represent the values of a speed which is not increasing, but constant, it appears, in like manner, that the momenta [*momenta*] assumed by the moving body may also be represented, in the case of the accelerated motion, by the increasing parallels of the triangle

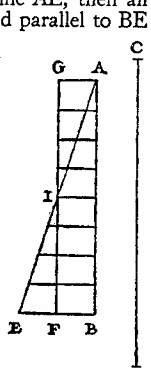

Fig. 47

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AEB, and, in the case of the uniform motion, by the parallels of the rectangle GB. For, what the momenta may lack in the first part of the accelerated motion (the deficiency of the momenta being represented by the parallels of the triangle AGI) is made up by the momenta represented by the parallels of the triangle IEF.

Hence it is clear that equal spaces will be traversed in equal times by two bodies, one of which, starting from rest, moves with a uniform acceleration, while the momentum of the other, moving with uniform speed, is one-half its maximum momentum under accelerated motion. Q. E. D.

THEOREM II, PROPOSITION II

The spaces described by a body falling from rest with a uniformly accelerated motion are to each other as the squares of the time-intervals employed in traversing these distances.

Let the time beginning with any instant A be represented by the straight line AB in which are taken any two time-intervals AD and AE. Let HI represent the distance through which the body, starting from rest at H, falls with uniform acceleration. If HL represents the space traversed during the time-interval AD, and HM that covered during the interval AE, then the space MH stands to the space LH in a ratio which is the square of the ratio of the time AE to the time AD; or we may say simply that the distances HM and HL are related as the squares of AE and AD.

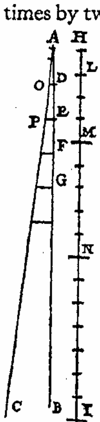

Fig. 48

Draw the line AC making any angle whatever with the line AB; and from the points D and E, draw the parallel lines DO and EP; of these two lines, DO represents the greatest velocity attained during the interval AD, while EP represents the maximum velocity acquired during the interval AE. But it has just been proved that so far as distances traversed are con-

cerned it is precisely the same whether a body falls from rest with a uniform acceleration or whether it falls during an equal time-interval with a constant speed which is one-half the maximum speed attained during the accelerated motion. It follows therefore that the distances HM and HL are the same as would be traversed, during the time-intervals AE and AD, by uniform velocities equal to one-half those represented by DO and EP respectively. If, therefore, one can show that the distances HM and HL are in the same ratio as the squares of the time-intervals AE and AD, our proposition will be proven.

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But in the fourth proposition of the first book [p. 157 above] it has been shown that the spaces traversed by two particles in uniform motion bear to one another a ratio which is equal to the product of the ratio of the velocities by the ratio of the times. But in this case the ratio of the velocities is the same as the ratio of the time-intervals (for the ratio of AE to AD is the same as that of ½ EP to ½ DO or of EP to DO). Hence the ratio of the spaces traversed is the same as the squared ratio of the time-intervals. Q. E. D.

Evidently then the ratio of the distances is the square of the ratio of the final velocities, that is, of the lines EP and DO, since these are to each other as AE to AD.

COROLLARY I

Hence it is clear that if we take any equal intervals of time whatever, counting from the beginning of the motion, such as AD, DE, EF, FG, in which the spaces HL, LM, MN, NI are traversed, these spaces will bear to one another the same ratio as the series of odd numbers, 1, 3, 5, 7; for this is the ratio of the differences of the squares of the lines [which represent time], differences which exceed one another by equal amounts, this excess being equal to the smallest line [viz. the one representing a single time-interval]: or we may say [that this is the ratio] of the differences of the squares of the natural numbers beginning with unity.

While, therefore, during equal intervals of time the velocities increase as the natural numbers, the increments in the distances traversed during these equal time-intervals are to one another as the odd numbers beginning with unity.

SAGR. Please suspend the discussion for a moment since there just occurs to me an idea which I want to illustrate by means of a diagram in order that it may be clearer both to you and to me.

Let the line AI represent the lapse of time measured from the initial instant A; through A draw the straight line AF making any angle whatever; join the terminal points I and F; divide the time AI in half at C; draw CB parallel to IF. Let us consider CB as the maximum value of the velocity which increases from zero at the beginning, in simple proportionality to the intercepts on the triangle ABC of lines drawn parallel to BC; or what is the same thing, let us suppose the velocity to increase in proportion to the time; then I admit without question, in view of the preceding argument, that the space described by a body falling in the aforesaid manner will be equal to the space traversed by the same body during the same length of time travelling with a uniform speed equal to EC, the half of BC. Further let us imagine that the

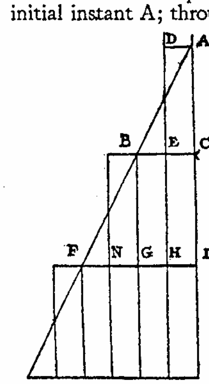

Fig. 49

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body has fallen with accelerated motion so that, at the instant C, it has the velocity BC. It is clear that if the body continued to descend with the same speed BC, without acceleration, it would in the next time-interval CI traverse double the distance covered during the interval AC, with the uniform speed EC which is half of BC; but since the falling body acquires equal increments of speed during equal increments of time, it follows that the velocity BC, during the next time-interval CI will be increased by an amount represented by the parallels of the triangle BFG which is equal to the triangle ABC. If, then, one adds to the velocity GI half of the velocity FG, the highest speed acquired by the accelerated motion and determined by the parallels of the triangle BFG, he will have the uniform velocity with which the same space would have been described in the time CI; and since this speed IN is three times as great as EC it follows that the space described during the interval CI is three times as great as that described during the interval AC. Let us imagine the motion extended over another equal time-interval IO, and the triangle extended to APO; it is then evident that if the motion continues during the interval IO, at the constant rate IF acquired by acceleration during the time AI, the space traversed during the interval IO will be four times that traversed during the first interval AC, because the speed IF is four times the speed EC. But if we enlarge our triangle so as to include FPQ which is equal to ABC, still assuming the acceleration to be constant, we shall add to the uniform speed an increment RQ, equal to EC; then the value of the equivalent uniform speed during the time-interval IO will be five times that during the first time-interval AC; therefore the space traversed will be quintuple that during the first interval AC. It is thus evident by simple computation that a moving body starting from rest and acquiring velocity at a rate proportional to the time, will, during equal intervals of time, traverse distances which are related to each other as the odd numbers beginning with unity, 1, 3, 5;* or considering the total space traversed, that covered

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in double time will be quadruple that covered during unit time; in triple time, the space is nine times as great as in unit time.

And in general the spaces traversed are in the duplicate ratio of the times, i. e., in the ratio of the squares of the times.

SIMP. In truth, I find more pleasure in this simple and clear argument of Sagredo than in the Author's demonstration which to me appears rather obscure; so that I am convinced that matters are as described, once having accepted the definition of uniformly accelerated motion. But as to whether this acceleration is that which one meets in nature in the case of falling bodies, I am still doubtful; and it seems to me, not only for my own sake but also for all those who think as I do, that this would be the proper moment to introduce one of those experiments—and there are many of them, I understand—which illustrate in several ways the conclusions reached.

SALV. The request which you, as a man of science, make, is a very reasonable one; for this is the custom—and properly so—in those sciences where mathematical demonstrations are applied to natural phenomena, as is seen in the case of perspective, astronomy, mechanics, music, and others where the principles, once established by well-chosen experiments, become the foundations of the entire superstructure. I hope therefore it will not appear to be a waste of time if we discuss at considerable length this first and most fundamental question upon which hinge numerous consequences of which we have in this book only a small number, placed there by the Author, who has done so much to open a pathway hitherto closed to minds of speculative turn. So far as experiments go they have not been neglected by the Author; and often, in his company, I have attempted in the following manner to assure myself that the acceleration actually experienced by falling bodies is that above described.

A piece of wooden moulding or scantling, about 12 cubits long, half a cubit wide, and three finger-breadths thick, was taken; on its edge was cut a channel a little more than one finger in breadth; having made this groove very straight, smooth, and polished, and having lined it with parchment, also as smooth and polished as possible, we rolled along it a hard, smooth, and very round bronze ball. Having placed this

board in a sloping position, by lifting one end some one or two cubits above the other, we rolled the ball, as I was just saying, along the channel, noting, in a manner presently to be described, the time required to make the descent. We repeated this experiment more than once in order to measure the time with an accuracy such that the deviation between two observations never exceeded one-tenth of a pulse-beat. Having performed this operation and having assured ourselves of its reliability, we now rolled the ball only one-quarter the length of the channel; and having measured the time of its descent, we found it precisely one-half of the former. Next we tried other distances, comparing the time for the whole length with that for the half, or with that for two-thirds, or three-fourths, or indeed for any fraction; in such experiments, repeated a full hundred times, we always found that the spaces traversed were to each other as the squares of the times, and this was true for all inclinations of the plane, i. e., of the channel, along which we rolled the ball. We also observed that the times of descent, for various inclinations of the plane, bore to one another precisely that ratio which, as we shall see later, the Author had predicted and demonstrated for them.

For the measurement of time, we employed a large vessel of water placed in an elevated position; to the bottom of this vessel was soldered a pipe of small diameter giving a thin jet of water, which we collected in a small glass during the time of each descent, whether for the whole length of the channel or for a part of its length; the water thus collected was weighed, after each descent, on a very accurate balance; the differences and ratios of these weights gave us the differences and ratios of the times, and this with such accuracy that although the operation was repeated many, many times, there was no appreciable discrepancy in the results.

SIMP. I would like to have been present at these experiments; but feeling confidence in the care with which you performed them, and in the fidelity with which you relate them, I am satisfied and accept them as true and valid

SALV. Then we can proceed without discussion.

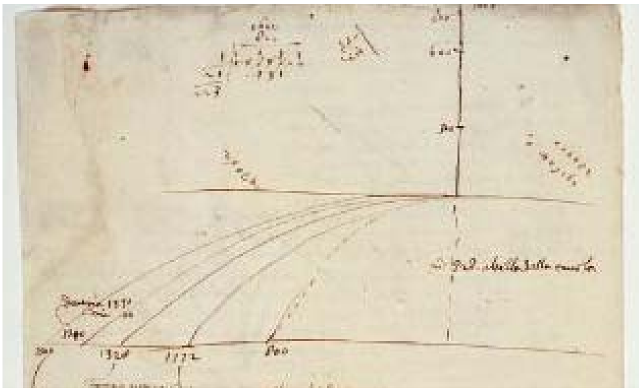
Let's go through this passage in some detail. It starts with a Theorem stating the same result that we saw in Oresme: if the velocity of a body increases at a constant rate over some time interval, then the total movement is the same as if its velocity had a constant value equal to what it was the exact middle of the time interval. It is interesting to speculate whether or not Galileo had read Oresme's work. He never cited Oresme and in the absence of great libraries (and the internet), it was a matter of accident whether he had seen a copy of this manuscript. However this rule was also known to the scholastic Swineshead at Merton College, Oxford, in the 14^{th} century and many others in the intervening 2 centuries and it seems probable that these ideas came down somehow to Galileo, directly or indirectly. Note that his proof of this fact is entirely in the spirit of Euclid.

In the previous several pages, he had been discussing the basic equation for a body moving at constant speed:
distance = (*velocity*) x (*time*)
But instead of writing this algebraically, he states three separate theorems, namely if you fix the velocity, then the distance moved are proportional to the time elapsed; if you fix the time interval, then the distance moved is proportional to the velocities; and if you fix the distance travelled, then the velocities are inversely proportional to the time elapsed. After that come three more theorems in which two of the qauntites are varied and the change in the thrid is described as a 'compound ratio'! All this seems very strange to us, as we are accustomed to the simplicity of putting everything together in the simple formula above. But the huge advantages of using algebra became obvious and universally used only a century later.

The next Theorem draws the conclusion that the distance fallen is proportional to the square of time. But the idea of the expressing this as a formula is again thought very abstract and Sagredo pitches in with what he views as a better way to see it. This is that the distance a body falls in successive equal time intervals will be like the sequence of odd numbers 1,3,5,7,… (because $3 = 2^2\text{-}1^2$, $5 = 3^2\text{-}2^2$, $7 = 4^2\text{-}3^2$, etc.) Which way of expressing these facts do you find most elegant and memorable? Not to belabor a point, but it turns out that Oresme had also stated both these results in another work of his: *Questions on the Geometry of Euclid*.

Then comes one of the most interesting parts: Simplicio asks for empirical evidence that this actually happens. And finally, Galileo goes into some detail on what we nowadays call the 'Methods Section' of a science paper. He describes how 'scantlings' are given a groove, polsihed, etc. and balls rolled down them. Note that he uses balls rolling down inclined planes instead of freely falling objects. He argues elsewhere that they obey the same sort of rule and, as the speed is much less, he can measure them better. This passage is a real window into 16^{th} century technology.

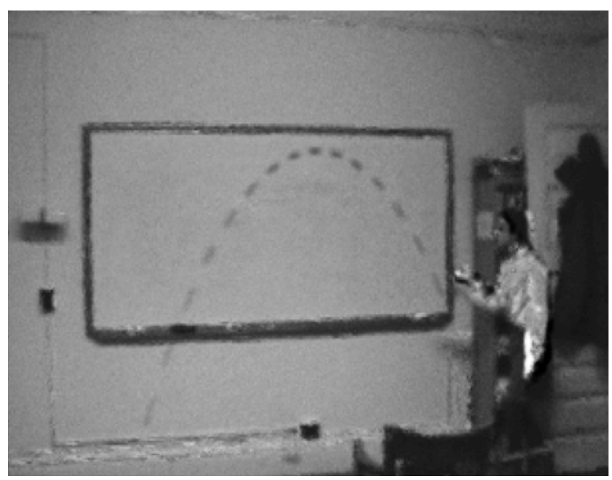
Later in the book he comes to projectiles and the fact that their orbits are parabolas (neglecting air resistance). You've probably read enough Galileo and we won't reproduce this as well. He actually computes the first 'ballistic tables', indicating altitudes and ranges of missiles fired at various angles. But I can't resist including a reproduction of one of his folios on which he apparently was actually testing his parabolic trajectory law:



We will come back to Galileo twice, as his interests touched on many important things that were more fully developed later – especially his interests in the pendulum and in music. But to make our story a bit more coherent, it's easier to go on next to Newton.

Problems:
(I) Suppose Galileo had at his disposal a primitive strobe light, as described in the last Chapter. Write some dialog between Salvatio, Sagredo and Simplicio in which this strobe is cited for its relevance to some of Galileo's (Salvatio's) theories.

(II) Here is a composite photograph of a squash ball being thrown up in front of my blackboard (created from a digital camera set in movie mode).



a) First either print out this photo to fill a piece of paper or, much better, bring it up in a program that allows you to track the cursor position (e.g. Microsoft photo editor). Then you can measure the 15 positions (x_i, y_i) of the ball in its arc. Note that because of the exposure time, each shot shows a blurry streak rather than a ball. I suggest you measure both the initial and final position of the ball as accurately as possible. This way, you get a good measure of the velocity as well as the position at 15 times. Also, measure the outer corners of the backboard, which, in the world is 4 feet by 8 feet: this way, you can convert the measurements of the ball position into feet (I know there is perspective distortion: ignore this). The frame rate for the movie was 15 frames per second, so we have good time measurements too.

Secondly we want to analyze how well these conform to the equations of motion given by Galileo.

b) Put these numbers in columns in Excel (e.g. the time in column A, starting at 0 and adding 1/15 sec for each observation, and then your measurements in the next columns) or in vectors in MatLab (one vector for each set of 15 measurements).

c) Next look at velocities. Compute the horizontal and the vertical velocities as a function of time. You can either use differences between the first and last point of each streak made by the squash ball or the difference between the first points of consecutive streaks. Plot the both the horizontal and vertical velocity -- call them u and v -- against time and print this out. Fit a straight line to each: is the horizontal velocity roughly constant as predicted or not? Don't expect the law to be followed exactly: there is air resistance as well as projective distortion (the camera viewing geometry). Find the time t_0 when the vertical velocity is zero by the zero crossing of the fitted line (this need not be exactly at any single observation but may be between them). Measuring the slope of the line fitted to vertical velocity, estimate the vertical acceleration downwards (which we call g).

d) Then look at positions. Fit Galileo's law this way. For horizontal motion, take the average $\bar{u}$ of the horizontal velocities found above and fit the horizontal position with $x \approx \bar{u}t + x_0$. Then graph the predicted x as well as the measured x and see how close they come to each other. Print out this graph. For vertical positions, in part c, we have estimated the high point t_0 of the trajectory as well as the acceleration downwards g. So the predicted vertical position should be $g(t-t_0)^2/2 + y_0$ where y_0 is the position of the ball at the apex of its trajectory. Estimate y_0, plot the predicted y and the measured y against time and print this out. Finally make a plot of the predicted x and y (no time shown) and the measured x and y from the photo and see whether the shape of the trajectory has come out right!