APPLIED MATHEMATICS 18

Modeling the world with mathematics

Take away number in all things and all things perish. Take calculation from the world and all is enveloped in dark ignorance, nor can he who does not know the way to reckon be distinguished from the rest of animals.
St. Isadore of Seville, 7^{th} century CE

Introduction

Mathematical ideas underpin virtually all of the technology that keeps our civilization going. Fortunately, most of these ideas are, in their essentials, not very complex and there is no reason why the average citizen-in-the-street should not know about them and use this knowledge to have a better understanding of what are the potentials and limitations of this technology. To have a professional mastery of these mathematical ideas requires a lengthy apprenticeship. But the basic ideas are easily illustrated by simple and intuitive examples. In addition, some mathematical formulas are needed in order to give a precise expression to each idea and to make a bridge so one can convert the intuitive idea into an arithmetic calculation of how it will work in the world.

In this course, all these ideas will be presented in a dual way, using, on the one hand, readings from the primary historical sources where they were first discovered and, on the other hand, numerical calculations carried out with the simple yet powerful tool of a spreadsheet (e.g. Excel). We are blessed (and cursed) with computers, which can make manifest in a split second what a formula hides within itself. Many of the mathematicians of earlier centuries loved to calculate and would have been delighted to have such a tool, now available in the cheapest PC.

What we will *not* do is to present the formalism of mathematics for its own sake. This formalism is what you need to master to become a professional in one of the 'exact' sciences, notably pure or applied mathematics, physics, chemistry, engineering, economics or computer science. Many are in love with the elegance and precision of abstract definitions, with deceptively simple but truly deep questions about ordinary integers or the arcane skills employed in the manipulations of analysis: if so, take standard math courses and not this one.

Math is usually seen as divided into 3 areas: algebra, geometry and analysis. This course deals mostly with analysis, which is the area of mathematics which has grown almost exclusively from its applications to the real world and which has in turn fed the technologies with which we master the world. Algebra and geometry, on the other hand, have developed to a large extent by pursuing their own internal logic, with occasional prods from reality. Algebra has had many applications in computer science and has hit the headlines recently due to its use in cryptography and secure internet transactions; and geometry has had major links with physics throughout the 20th century (esp. in the theory of gravity and in particle physics). But analysis applies to virtually every aspect of modeling the world.

Most of what we say here will relate directly to the 'real' world. We will need some half a dozen deeper math-facts, which are not at first obvious. But we will try to explain why they are natural and inevitable and give you a chance to flex your computational muscles with them to see that really are true. We will assume some acquaintance with calculus – e.g. one semester. But we will also review the basic definitions of calculus as we go along and, in fact, see them in their original form in the hands of the old masters Newton, Leibniz and Euler.

So exactly *what* are we going to study? The first topic is what started the Western World on the path of its technological success, whose fruits we enjoy today. I want to look briefly at Babylonian, Greek and Arabic mathematics, the cultures on which modern science rests. The Babylonians were apparently the first inventors of arithmetic, including fractions and decimals and converting between them – except that they used base 60 (sexagesimal) instead of base10 – and they reveled in calculating with these numbers. The Greeks, on the other hand, had less appetite for numbers but loved logic and discovered the striking fact that geometry could be studied on the basis of axioms without any measurements at all. In their study of astronomy, they put geometry and numbers together and came up with a stunningly accurate estimate of the diameter of the earth. This is easy to mimic, as we will do in assignment #1. The Arabs did some great science (e.g. they understood the nature of light correctly) and put zero back in the number system, which had been foreshadowed with small place markers in Babylonian cuneiform but not properly appreciated. The Western World first stirred in the work of Nicole Oresme who introduced the idea of graphing a quantity which changes in time. He introduced x- and y-coordinates and used them for making plots of space, time, velocity, even spiritual quantities like grace. In many ways, he was preaching the contemporary 'hot' topic of visualization and sounded much like its best-known apostle today, Edward Tufte. More or less at the same time, Europe fell in love with clocks and maps and began to appreciate all that can be done if you measure time and space accurately. The take-home message of the 14th century was that arithmetic could be used to measure and better understand the structure of events taking place around us.

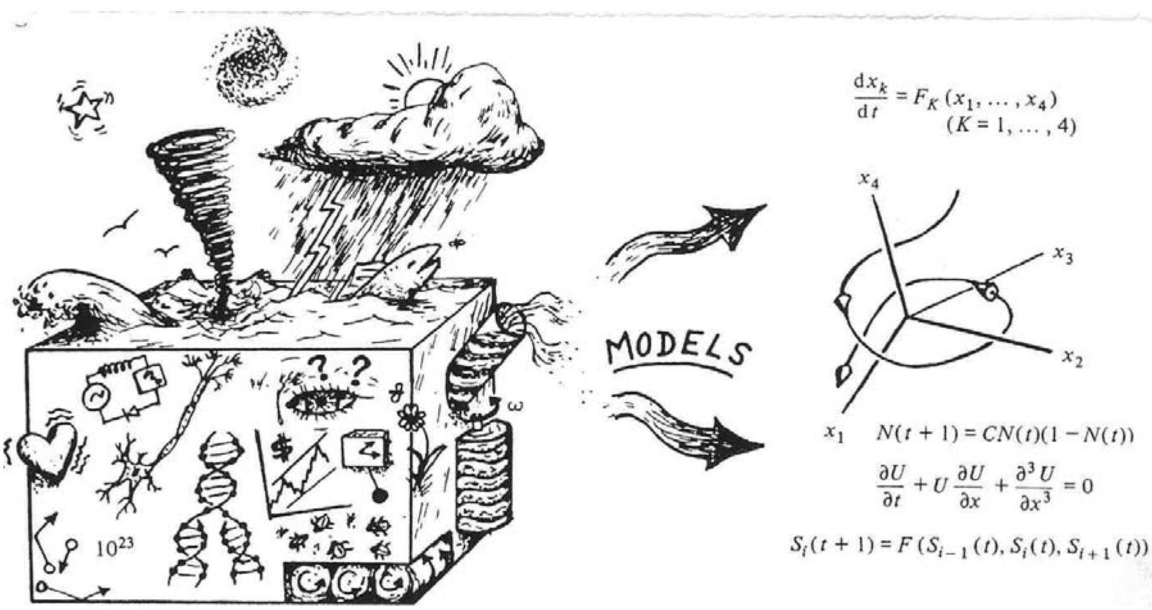
The second topic is the work of Galileo, Newton and his immediate successors. It is usual to say that the key step was the discovery of calculus but I would describe the biggest impact of their work differently. They uncovered the fact that, for a large number of events in the world, if you measure at one point in time (i) where everything *is* and (ii) how it is *moving or changing*, then the laws of nature, once you know them, tell you (iii) how every object is *accelerating* and (iv) that knowing that this law will continue to hold, the *entire future is determined*. This is a stunning simplification of the ancient challenge of prophecy, a job that had pre-occupied every previous culture but which had met with quite limited success. Put mathematically, this discovery is that many aspects of the world are described by differential equations (to be technical, 2nd order ordinary and hyperbolic partial differential equations) and much of the next 3 centuries has been spent extending this model to ever-larger categories of events. I like to put it this way: God's laws for the universe are written in the language of differential equations. So it behooves everyone to have an inkling of what these are. We will read Galileo and Newton and try to see inside their minds a bit. Folklore has it that it all started with Galileo's watching the swing of the censors and oil lamps in the cathedrals of Pisa. With the help of a spreadsheet and some digital movies of a swinging pendulum, we can readily check part of what he and Newton claimed. Ironically, science is quite circular here: an accurate clock needed to check this mathematical model and the mathematical model is needed to prove that a pendulum ticks like a first class clock.

The third topic will be waves. The first insight into wave phenomena was the analysis of musical chords by Pythagoras and his followers, noticing the strange fact that two strings plucked at he same time sound better together if the ratio of their lengths is a simple fraction like 3/2. We will look at some length at the vibrating string and the waves it produces, which will illustrate again how differential equations are the basis of all mechanical systems. With the help of strobe lights and digital speech recordings, we can easily 'see' both simple and complex waves. We will look at a human voice singing the major scale and check what underlies Pythagoras's beliefs and we will use a spreadsheet to solve the vibrating string equation. Finally, these two examples lead us to a very important piece of mathematics: Fourier Analysis. Rather like the familiar idea that a real number can be approximated by an infinite decimal to greater and greater accuracy, Fourier declared that every function could be approximated by an infinite superposition of simple waves of different frequencies. It is just the mathematical expression of a musical score. A quite unexpected twist to the story here is the amazing usefulness of a mathematical artifice of the Renaissance – introducing a new 'number' having nothing to do with lengths, the square root of –1. We will look in passing at waves in earth and water, and most importantly, electro-magnetic waves such as radio, TV and light. How can you tune one antenna or cable box to pick up so many different stations?: it's all Fourier Analysis.

The fourth topic will be chaos! The word has many meanings but the most important one for us is the idea, as Ed Lorenz put it, that the flapping of a butterflies' wings in Rio can set off a tornado in Oklahoma. This means certain phenomena of nature are fundamentally unpredictable. Understanding what is and what is not predictable has been one of the main problems driving applied mathematics in the last 50 years. Whereas the superposition of waves had been the key tool that unlocked the behavior of linear differential equations, the key challenge recently has been to understand all the complex things that happen with non-linear equations. Our focus will be Lorenz's equation for convection rolls in the atmosphere (the instability caused by the sun heating the earth hence the lowest layers of the atmosphere and this then rising). A new idea here will be that to visualize this and other equations, one makes plots in 'phase space', an idea that Oresme would have liked. We will draw for ourselves the 'strange attractor' that explains Lorenz's equation and this will lead us to the idea of fractals, shapes with convolutions on finer and finer scales, which model many of the more complex structures in nature. In the dialectic of mathematical models, the Newtonian thesis of predictable planets in smooth elliptical orbits has been replaced with the antithesis that chaos and fractals are ubiquitous.

Finally, the last topic will be chance. Historically the most important observation of chance in Science was the discovery by Brown of the seemingly random motion of tiny particles in solution. As it turns out, this was the first empirical evidence of the atomic nature of matter! Individual molecules, moving randomly due their heat, collide with the microscopically visible particle and drive it this way and that. The most important thing to know about chance is that the size of its fluctuations is predictable. There is a formula here – another for your toolkit – and it applies to elections, medical trials and the reliability of fingerprints. Another fascinating aspect is how chance can be used to better compute things governed by immensely complex equations. The H-bomb was designed using so-called 'Monte Carlo' algorithms and we will see how it is easy to estimate the critical mass of uranium needed for a bomb, again with a spreadsheet!

Here's a cartoonist's version of what we it means to model the world with mathematics and what we will be studying this term:



Tentative Syllabus

Part I: *Discovering the tools for measuring and modeling the world*

Jan.25: Outline of the course, begin Babylonian Mathematics, Sexagesimals
Jan.27: Pythagorus's theorem and the Square Root of 2.
Jan.30: The Greeks and the Measurement of Astronomical Distances
Feb.1: Archimedes and integration
Feb.3: Oresme and the Invention of Graphing
Feb.6: Galileo and the Leaning Tower of Pisa

Part II: *Differential Equations, or how best to predict the future*

Feb.8: Newton, Fluxions and Forces
Feb.10: Newton's general method and some simple Differential Equations
Feb.13: Simple harmonic motion
Feb.15: The pendulum and Phase-Plane Plots
Feb.17: Gravity
Feb.22: The motion of the planets

Part III: *Music, Sound and a Multiplicity of Waves*

Feb.24: Music, Chords and Harmony
Feb.27: The Vibrating String and PDE's
March 1: Fourier Series, Spectrograms and Modulation
March 3: Fourier series (continued)

March 6 Review
March 8 HOUR EXAM

March 10: A strange twist: 'imaginary' numbers
March 13: Trigonometry and vibrations via imaginary numbers
March 15: Traveling Waves on a string
March 17: Waves in Air, Water and the Ether
March 20: Waves (continued)

Part IV: *The non-linear world of chaos: the real limits of predictability and computability*

March 22: Epidemics and competition: non-linear models
March 24: Weather, convection cells and the Lorenz equation

++++++++++++++++++++ Spring break +++++++++++++++++++++++

Apr.3: Lorenz's eqn is unpredictable
Apr.5: The strange attractor at its heart
Apr.7: Boom and bust with the logistic equation
Apr.10 Ferns and self-similar fractals in nature
Apr.12 Measuring 2½ dimensions (PROJECT PROPOSALS DUE)

Part V: *Harnessing chance: how throwing dice is not so unpredictable and can be useful*

Apr.14 Brown's observations of microscopic motion
Apr.17 Random walks
Apr.19 Elections and standard deviation
Apr.21 Making the atomic bomb at Los Alamos by coin flipping
Apr.24 Fingerprints and ROC curves
Apr.26 Misuse of statistics: IQ's and heavy tails

Apr.28+ READING PERIOD, STUDENT REPORTS ON PROJECTS

Chapter One: Babylonian Mathematics, Sexagesimals and the Square Root of 2

Everybody knows that the Greeks loved mathematics and developed it in major ways, in particular, inventing the idea of proof, of a rigorous and logical proof broken up into small precise steps. But it is not nearly so well-known that a millennium and a half earlier, during the period of Hammurabi, the Babylonians had also fallen in love with mathematics and had developed it in quite different ways. Perhaps other civilizations had also had great mathematicians but the Babylonians were fortunate to have had a handy supply of clay along the banks of the Tigris and Euphrates and to have thought of using this clay to record things they might forget. Fires burn down libraries and the paper books in them but they bake the clay hard and so we now have tens of thousands of tablets from the millions they made and kept in their cities.

As with every civilization, one of the most important things for them was keeping track of their material possessions and transactions and so they wrote on these tablets their accounts, stuff like: "I, …, have so many sheep and hides, so much grain, dates and figs, on such and such a day etc." In addition, they kept track of time and paramount to their time keeping were the phases of the moon: especially the first appearance of the new moon which started a new month but which could happen either 29 or 30 days after its first appearance in the previous month. Predicting astronomical events became an obsession. Between accurate accounting and astronomical modeling, they were led to the invention of arithmetic including fractions and 'decimals' and effective methods of carrying out the 4 arithmetic operations on them. All this really basic stuff now taught in K-8 was also taught at scribal schools in Babylon. Historians often argue that it is misleading to assume they saw things the same way as we do but I don't see much difference between their arithmetic and ours.

They did one thing differently from us: instead of decimals, they used a 'sexagesimal' system. This starts with assigning a single symbol (made up of multiple impressions of the stylus) for every number from 1 to 59:

The Babylonian numbers 1,2,3,4,5,6,7,8,9,10,11,12 and 13 in cuneiform

The Babylonian numbers 10,20,30,40,50

(Note that the symbols used for 1 to 59 are structured in multiples of 10, so they must have started with numbers based on 10 fingers, but apparently, they also wanted to be able to divide exactly by 3 and 4 as easily as by 2 and 5, so they went to sexagesimal.) Numbers bigger than 59 were written just the way we do it. Take a number like 4415. For us, the writing '4415' is shorthand for:

$4\times1000 + 4\times100 + 1\times10 + 5\times1$

For them, you start by expanding:

$4415 = 1\times3600 + 13\times60 + 35\times1$

so then you write the number by placing next to each other their symbol for 1, then their symbol for 13 and finally their symbol for 35, giving something on the clay tablet like:

<1>,<13>,<35>.

Here I have written <35> for their cuneiform symbol for the number 35. In addition to whole numbers, they wanted to measure lots of quantities (lengths, areas, fractional interests in some goods) that were not exactly equal to a whole number so they needed the equivalent of decimals to the right of decimal point. Again their system was exactly like ours. For instance, where we write

271.25 which is decimal shorthand for

$2\times100 + 7\times10 + 1\times1 + 2\times(1/10) + 5\times(1/100)$

they write the same number as

<4>,<31>,<15> meaning for them $4\times60 + 31\times1 + 15\times (1/60)$.

The 'decimal point', the point in the expansion separating the whole numbers from those that are fractions, was left to the reader to deduce from the context! One drawback of the sexagesimal system is that it's a lot harder to learn your 'multiplication tables'. Our children must master 100 products like 7×8 but they had to memorize 3600 products like <23>×<41> in order to do arithmetic fast.

How about the number zero? People always say zero was devised by the Indians and the Arabs nearly 3000 years later but the Babylonians had a reasonable equivalent. If they wanted to distinguish 3601 – which is $1\times(60^2) + 0\times60 + 1\times1$ or <1>,<0>,<1> in sexagesimal – from 61 – which is $1\times60 + 1\times1$ or <1>,<1> in sexagesimal – they either simply left a bigger space between the 2 one's or later used their 'end-of-sentence' symbol (i.e. their 'period' which actually looked like a Greek epsilon) one or more times to indicate the zeros. Thus 3601 was written something like <1>, . , <1>.

a. YBC 7289.

Now here's the amazing thing about the Babylonians which shows they were up to the level of High School math: they discovered **Pythagoras's theorem**, that the square of the hypotenuse is the sum of the squares of the two short sides of a right triangle. It seems unlikely they proved it as nothing resembling proofs appears on their tablets but many many tablets are concerned with related constructions involving geometric squares, the formula $(a+b)^2=a^2+2ab+b^2$, solving for two numbers with given sum and product, arithmetical exercises involving 'Pythagorean' triples, i.e. 3 whole numbers n,m and p such that $n^2+m^2=p^2$.

Most convincingly, on the tablet shown above we find the diagonal of the square labeled with the sexagesimal number **<1>,<24>,<51>,<10>**!!! As just mentioned, the 'unit' column is never specified by the Babylonians, but if we put it after the first <1>, this number is

$$1\times 1 + 24\times(1/60) + 51\times(1/60^2) + 10\times(1/60^3)$$

and pulling out your pocket calculator, you re-express this in our familiar decimals as:

1.41421296…

The length of the diagonal of the unit square is, by Pythagoras's theorem, equal to $\sqrt{2}$, whose exact value in decimals starts as

1.41421356…

which differs from the Babylonian number by less than a millionth. In fact the sexagesimal number written on the tablet is the *correct* sexagesimal expansion of square root of 2 to that number of places. One can make tantalizing speculations about how they worked this out. But several things are clear: (i) they must have known that the diagonal of a unit square has length $\sqrt{2}$ and (ii) they were truly applied mathematicians, not pure mathematicians. That is, they liked good approximations and never bothered with proofs! These approximations were exactly what they needed in their astronomy, which was based on combining successive effects with different periods in quite reasonable approximate ways (we will return to this when we look at Fourier's theory of periodic functions). [ADD LATER BABYLONIAN PERIODIC FCNS]

Why is Pythagorus's theorem so extremely important? At first glance, it seems like an idle fact about right triangles. Not at all: it is the key to systematically measuring the world, the basic link between algebra and geometry and the fact that enables us to reason about distances in the world using arithmetic. It was needed to resurvey fields inundated by floods, to construct buildings and measure altars for example. I discovered it is used by carpenters in Maine when I was laying out a porch: they checked the right angles on the corners by measuring the diagonal of the room!

The space we live in allows us to make measuring sticks, tape measures, etc. and to move these around so as to measure distances between any 2 accessible points. But, if you have more than 2 points, the set of all these measured distances is connected by rules, so that some of them determine others. The basic rule is Pythagoras's theorem. Once this is known, you are led to lay out perpendicular coordinate grids giving us *x* and *y* coordinates and then we can compute the distance between any two points *P=(a,b)* and *Q=(c,d)* by means of the Pythagorean formula [NEED FIGURE HERE]

$$\text{distance } P \text{ to } Q = \sqrt{(a-c)^2 + (b-d)^2}$$

In other words, Pythagoras's theorem plus coordinates tells us what all distances in the world are. Not until Einstein came along did we find we needed to refine our world measurements and go beyond this formula.

Pythagorus's theorem has a remarkable international history. In addition to its discovery by the Babylonians in the 2nd millennium BC, it appears to have been discovered by the

Indians and the Chinese, as well, of course, by the Greeks, in the 1^{st} millennium BC. Thus the fact that the diagonal of a square has length √2 appears in the *Baudhayana Sulbasutra* (c. 800 BC):

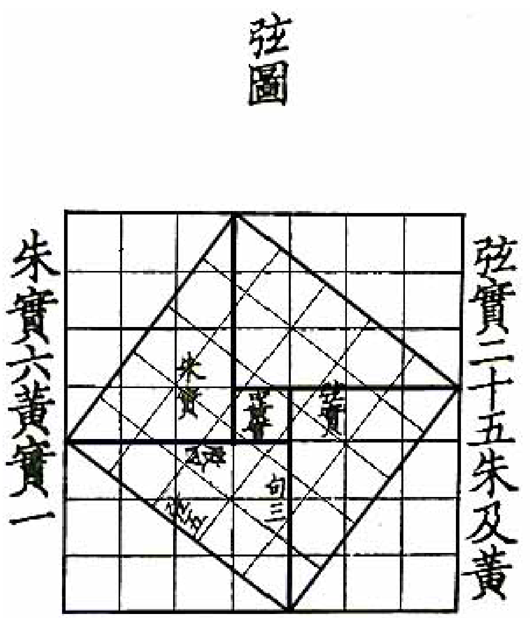
"The rope which is stretched across the diagonal of a square produces an area double the size of the original square."

and the general form appears in the *Katyayana Sublasutra* (c.200 BC):

"The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together."

Whether the Indians had any form of proof is not clear. Lacking the technology of clay tablets, they passed on their work by encoding it in cryptic sanskrit verses which they committed to memory and passed down orally. It was left to the Greek and the Chinese to leave us the first *proofs* of Pythagoras's theorem.

Here is the Chinese proof, as it appears in the *Zhoubi suanjing* ("The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven"), whose date is unknown but has been estimated to be in the 6^{th} century BCE.



I'm not quite sure how the Chinese argued but if the triangle in the corner has short sides a and b and hypotenuse c, then the big square has area $(a+b)^2$ and it breaks up into the tilted square with area c^2 and four triangles in the corners each with area ½ab. Equating these two gives immediately:

$$c^2+4\left(\frac{ab}{2}\right)=(a+b)^2, \text{ or}$$

$$c^2=a^2+b^2$$

The breakup of the tilted square into 4 inner triangles and the inner square gilds the lily, giving this time:

$$c^2=4\left(\frac{ab}{2}\right)+(a-b)^2, \text{ or}$$

$$c^2=a^2+b^2 \text{ again.}$$

The most important mathematical discovery of the Greeks, dating from the very beginning of their golden period, 500-100 BCE, was that √2 was irrational, that is not exactly equal to any whole number fraction. This is so easy to see that we have to include it, even though it is against the philosophy of this book to prove nothing! The argument is this: suppose $\sqrt{2}=n/m$. Note that we can suppose that either n or m are odd because if both were even, we would cancel out a 2 in the fraction n/m and make a simpler expression for

$\sqrt{2}$. Starting with $\sqrt{2}=n/m$, if we square and clear denominators, this becomes $2m^2=n^2$. This is an equality between two whole numbers and obviously the number on the left is an even number. So n^2 is even too. The squares of odd numbers are odd, while squares of even numbers are even, so n must be even too: $n=2k$. Substituting this into our equation, we get $2m^2=n^2=4k^2$. Then $m^2=2k^2$, hence m^2 is even too, hence m is even. This contradicts what we said in the beginning and the only way out is that $\sqrt{2}$ has NO expression as a fraction!

You get the impression that Babylonians felt the same way about clay tablets that we do about computers today. All of a sudden, they could keep track of property, contracts, celestial events, remedies for diseases, etc. better than ever before and so they recorded *everything*. They started scribal schools in which the three R's were taught. They were swamped with tablets and were forced to create mankind's first libraries, complete with 'index tablets' listing the individual tablets. They apparently loved playing with arithmetic now that clay tablets had made it possible, even working out on tablets frivolous problems such as one involving wages in which the sum and product of the number of workmen and the number of days worked on a certain project were given, the challenge being to find the number of workmen and of days.

Problems: Using a pocket calculator,

(a) find the next 2 terms of the sexagesimal expansion of $\sqrt{2}$.

(b) Here's one idea of how the Babylonians computed $\sqrt{2}$. Consider any number a between 1 and 2. Then $2/a$ is also between 1 and 2 and $a=2/a$ only if $a=\sqrt{2}$. In fact, when $a < \sqrt{2}$, then $2/a > \sqrt{2}$ and vice versa: in other words a and $2/a$ always lie on opposite sides of $\sqrt{2}$. So if we average a and $2/a$, it stands to reason that this is much closer to $\sqrt{2}$ than a is. In fact this works really fast. Start with a=1 and replace a by $\frac{1}{2}\left(a+\frac{2}{a}\right)$ three times. Keep track of both a and how different it is from $\sqrt{2}$ each time.

(c) Expand π is 3 sexagesimal places.

Chapter Two: The Greeks and the Measurement of Astronomical Distances

The Greeks were very different from the Babylonians. They seem to have been in love with debating and that led them to discover the rules of logic. They went pretty far along this road, even preferring abstract reasoning to empirical evidence. An extreme example of this can be found in Plato's Republic, where Socrates rejects doing astronomy by observing the stars:

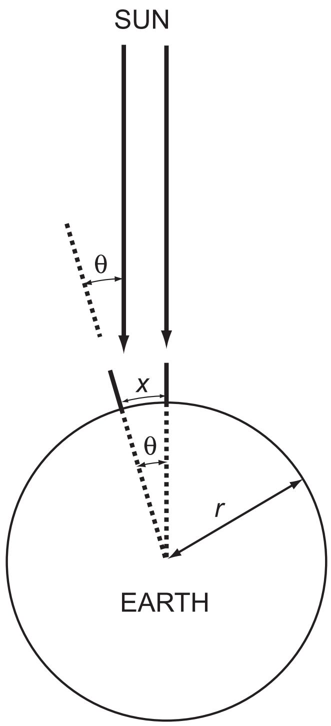
The sparks that paint the sky, since they are decorations on a visible surface, we must regard, to be sure, as the fairest and most exact of material things; but we must recognize that they fall far short of the truth .... both in relation to one another and as vehicles of the things they carry and contain. These can be apprehended only by reason and thought, but not by sight. .... It is by means of problems, then, as in the study of geometry, that we will pursue astronomy too, and we will let be the things in the heavens, if we are to have a part in the true science of astronomy.

As one would expect, by ignoring empirical data they made some amazing goofs. For example, Aristotle decided thought went on in the heart and not the brain because he did not take into account the evidence coming from patients with head trauma (as Galen did a few centuries later).

The Greeks were also hampered by using a terrible type of arithmetic in their commercial transactions, the Egyptian method of 'unit fractions' instead of decimal or sexagesimal expansions. In this system, a number like 4 3/5 was expressed as 4+1/2+1/10, i.e. every fraction had to be written as a sum of unit fractions $1/n$. This made addition of fractions a real nuisance and little books called 'Ready Reckoners' were used to remember rules like (1/5+1/5) = (1/3+1/15). Greek mathematicians who liked to calculate (like Archimedes) and Greek Astronomers (like Ptolemy) realized the superiority of the Babylonian sexagesimal expansions and employed them. Whether the Greeks learned the statement of Pythagoras's theorem from the Babylonians or discovered it on their own is hotly debated.

As Plato is saying in the passage above, the development of geometry meshed beautifully with the study of astronomy. Aristarchus and Eratosthenes, a generation or so after Euclid, fortunately were willing to combine some excellent observations of the 'fair skies' with the use of geometric reasoning and came up with one of the high points of Greek applied mathematics: excellent estimates of the size of the earth and the moon and the distance of the earth to the moon and some not so good estimates of the size of the sun and its distance from the earth. Their ideas are a wonderful example of how a small number of easy observations, combined with the right mathematics, go a long way. It's a bit surprising at first, how easy it is to measure such a huge object as the earth, but in the problem below, you will do this (by a slightly different method).

Eratosthenes' work was based on the height of the sun at noon on the same day in 2 different cities a known distance due north/south of each other. Aristarchus's work was based on (a) the size of the moon and the sun in the sky, in degrees, (b) the shape of the earth's shadow cast on the moon during lunar eclipses and (c) the exact time at which the first quarter moon occurs, that it is closer to new moon than to full moon. Their ideas can be seen in a few simple geometric pictures.



Eratosthenes Measurement: He found two cities, Syene near the modern Aswan and Alexandria, a certain distance *x* (approximately 980 miles) apart approximately on a due north-south line. At a certain day in the year, at noon, the sun shone directly down on the first city (checked by looking down into a deep well). At the other city, he measured the size of the shadow cast by a vertical pole and thus knew the angle θ between the sun and the zenith at that city at the same time. Assuming θ is measured in degrees, then the same angle is $\theta^* = \theta\pi/180$ in radians, but in radians, by definition, the angle is x/r. Thus

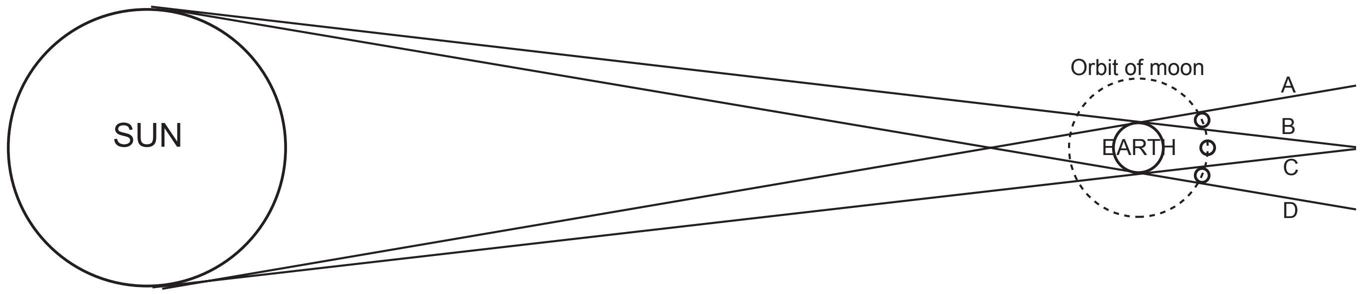
$$r = 180x/(\theta\pi)$$

What is quite stunning is that his result was so accurate: he found a radius of 3890 miles, while the correct value is 3963 miles – people think his errors cancelled, but there is no reason he should not have been correct to 5% anyway.

Eclipse Facts: First of all, seen from the earth's surface, the moon and the sun heave very nearly identical size in the sky, both being close to 0.5° across. This is confirmed by the fact that in eclipses of the sun, the moon nearly exactly covers the sun (sometimes it is a bit smaller, sometimes a bit bigger – because of variation of the distances between the earth and the moon and sun). Also, 0.5° is $\pi/360 \approx 0.009$ radians and thus we have our first relation:

(diameter of moon)/(dist. earth to moon) = (diameter of sun)/(dist. earth to sun) ≈ 0.009

Secondly, what happens during an eclipse of the moon? The moon passes through the shadow of the earth. This shadow has 2 parts: a deep 'umbra' within which the sun is totally behind the earth and a 'penumbra' within which the earth partially covers the sun. During typical lunar eclipses, the moon darkens on one side as it enters the penumbra. After a while, the umbra itself is visible as a curve across the face of the moon on one side of which the moon becomes much darker and is in the umbra proper. The moon may pass entirely into the umbra or just graze it. The eclipse ends as the moon leaves first the umbra and then the penumbra. The key point is that by making sketches of the edge of the umbra as the moon passes through it, one can estimate fairly well relative size of the umbra and of the moon itself. It turns out that the umbra is about 2.7 moon diameters in size. Here's the picture:

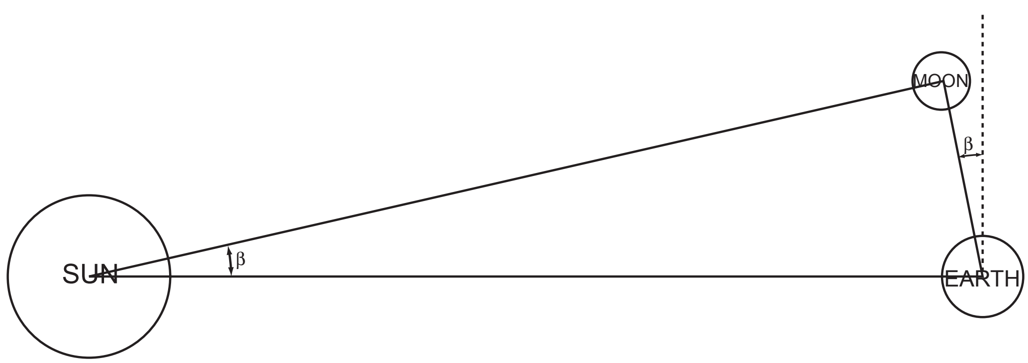


The umbra of the earth is the space between lines B and C, while the penumbra is the space between A and B and between C and D. Because of the relatively large distance to the sun, the lines A and C are very nearly parallel, as are the lines B and D. Therefore the distance between the parallel lines A to C is equal to the diameter of the earth, as is the distance between B and D. Moreover, the angle between A and B is 0.5°, the angular size of the sun as seen from the earth (look at the lines continued to the right). Since this is the same as the angular size of the moon as seen from the earth, we see that, in the moon's orbit, exactly one moon fits between the lines A and B. The same holds for C and D and our drawing has been made to reflect this. The observation was that about 2.7 moons fit between B and C. It follows that one earth diameter is the distance between A and C at the moon's orbit and that this is about 3.7 moon diameters. Thus:

(diameter of earth) ≈ 3.7×(diameter of moon)
hence (as 30 ≈ 1/(3.7×.009)):
(dist. earth to moon) ≈ 30×(diameter of earth)

Aristarchus, using this method, found the distance from the earth to the moon to be 25 earth diameters: again, remarkably close to the correct answer.

<u>And the sun?</u> The Greeks were not nearly so accurate in estimating the distance to the sun. They used a very straightforward idea, shown in the diagram:



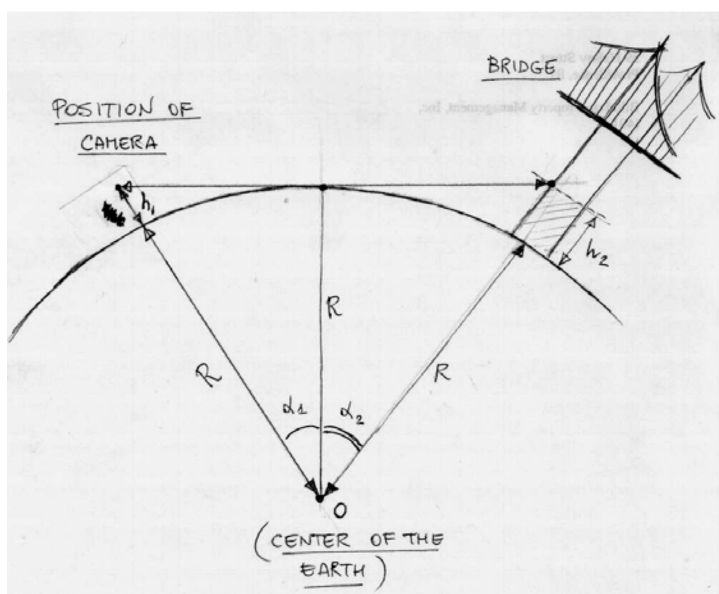
The moon is shown here at 'first quarter' when exactly half of the moon is illuminated by the sun. Note that when this occurs, the triangle formed by the earth, moon and sun has a right angle at the moon. The dotted line, however, which is perpendicular to the line joining the earth and sun, is the position of the moon exactly half way between new moon and full moon: in other words, because of the finite distance of the sun, true first quarter occurs a little after the mid point between new and full moon. Note also that by simple results from Euclidean geometry, the two angles marked β are, in fact, equal.

Using this idea, Aristarchus estimated β at 3°, which is also about 0.05 radians. Since β is also the angle at the sun in the sun-moon-earth triangle, he got:

(*dist. earth to sun*) $\approx 20\times$(*dist.earth to moon*)

which was a nice try. Unfortunately, the correct angle β is about 0.15° and the true ratio is thus about 20 times larger. There is some speculation that Aristarchus knew his estimate of β was too large but that the resulting huge distance to the sun and huge size of the sun seemed nearly incredible and he did not want to put his reputation on the line for this.

Problem: The moral of all this is that it is not all that hard to measure *huge* things. You needn't leave this sort of thing up to experts! Your second assignment is to measure the size of the earth using two photos of the Newport bridge, one taken from 18.5 miles down Narragansett Bay (near Providence) and the other from close up:



You need to know: (a) the tower of the bridge is about 405 feet high and (b) the camera was approximately 25 feet above the water when the first photo was taken. Also note that in this photo, the black is open water while the distant water is frozen (and there is a mirage). So estimate the true horizon by extrapolating the faint line jutting out from the land on the right. You may want to print out these pictures and use a ruler to measure parts of the tower in each shot. The diagram here should help you put all this together.

[Comments on this exercise: Often approximate calculations are just as good and much faster than exact ones. In the above diagram, one can use the *exact* Pythagorean theorem for the right triangles formed by the center of the earth, the camera (or the bridge) and the point where the light between the camera and the bridge grazes the water. But one can also use an *approximation*. If x_1 is the distance from the camera to the point where the light ray grazes the water, we have a right triangle giving us $(h_1+r)^2 = x_1^2 + r^2$. Now use the following: if a is small, $(1+a/2)^2 = 1+a+a^2/4 \approx 1+a$, thus $\sqrt{1+a} \approx 1+a/2$. We apply this to solve for h_1:

$$h_1 = \sqrt{r^2+x_1^2} - r = r\left(\sqrt{1+(x_1/r)^2} - 1\right) \approx \frac{x_1^2}{2r}$$

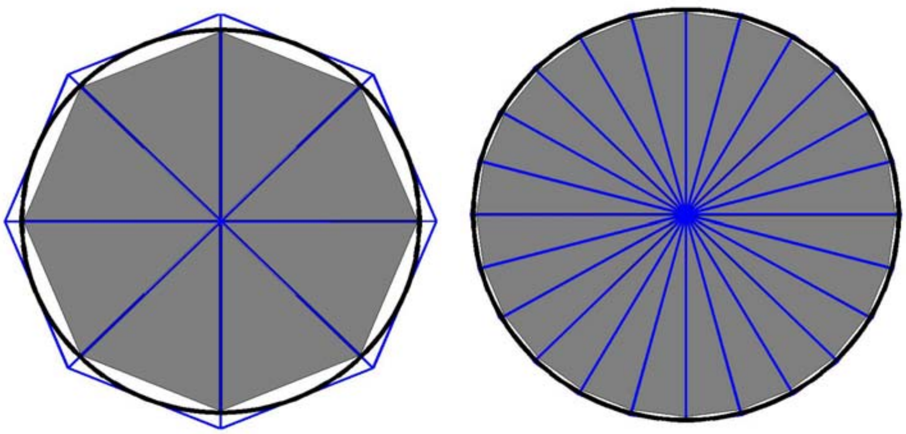
In other words, if you move exactly tangent to the earth a distance x_1, then this straight line will be $x_1^2/2r$ above the surface of the earth at its end. This will also be useful later.]

As the uses of mathematics in technology and especially the use of calculus are the main points of this course, we cannot omit some words about Archimedes. Like all the greatest mathematicians, he excelled in both pure and applied mathematics. In applied mathematics, he discovered hydrostatics, e.g. the force with which water buoys up submersed objects, and he was a master of simple devices involving pulleys, levers and screws. His remarkable machines single-handedly stalled the Roman siege of Syracuse for months and terrified the Roman soldiers. He is said to have devised a machine whereby he was able to draw a three-masted ship up on the beach all by himself – and even if this is apocryphal, he undoubtedly demonstrated some such feat using pulleys, levers, wheels and threaded screws that amazed the King of Syracuse.

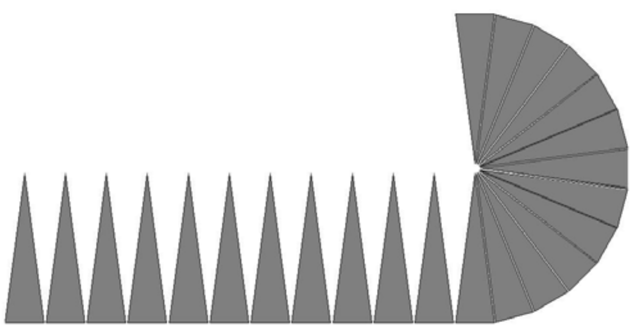
But his greatest triumph mathematically was his discovery of the formulae for the area and volume of a sphere: (in our notation) if the radius is r, then the area is $4\pi r^2$ and the volume is $(4/3)\pi r^3$. He said it this way:

First, the surface of any sphere is four times its greatest circle (i.e. the area of the sphere is four times the area of the circle in the middle of the sphere whose boundary is the equator – namely πr^2)*; ... and further, that any cylinder having its base equal to the greatest circle of those in the sphere and height equal to the diameter of the sphere, is itself half as large again as the sphere* (i.e. the volume of this cylinder, given by the product of the area of the base and its height – namely $(\pi r^2) \times (2r)$, is 3/2 times the volume of the sphere).

In his mathematical work, Archimedes essentially discovered the concept of integration, of breaking up a quantity up into very many very small parts, ultimately infinitely many infinitesimal parts, in order to measure it. As far as I can see, the only thing lacking, when compared to present day ideas, was algebraic notation[1]. I want to illustrate Archimedes' ideas by the simplest example in his papers, which is a proof that the area of a circle is half the product of its circumference and its radius, or $\pi r^2 = (1/2) \times (2\pi r) \times r$. (This is Proposition 1 of his manuscript *Measurement of a Circle,* whose purpose is to estimate π). The proof is illustrated in the figures below.

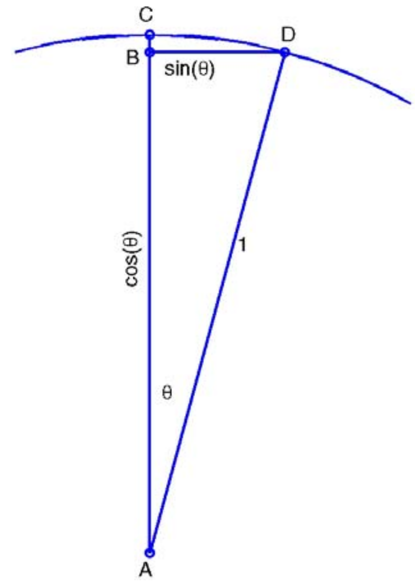


What Archimedes does is to both inscribe and circumscribe a regular polygon with n sides in the given circle. On the left, we have drawn n=8 and n=24. Let A be the area of the circle and let $B^+(n)$ be the area of the circumscribed polygon and $B^-(n)$ the area of the inscribed polygon. Then Archimedes next shows that for any non-zero area, if n is large enough, the total area of the slivers between the circumscribed polygon and the circle, i.e. $B^+(n)–A$ and the area between the circle and the inscribed polygon $A–B^-(n)$ will both be less than this area. To see what $B^+(n)$ and $B^-(n)$ are, we can 'unroll' the two polygons like this:



As the area of any triangle is ½ the product of the length of the base and the altitude, the combined area of all these triangles, making up the polygon, is ½ the product of the combined length of the bases and its altitude. Thus the area of the circumsribed polygon is ½ the product of its perimeter and the radius of the circle; and that of the inscribed polygon is ½ the product of its perimeter and the distance from the center of the circle to the midpoint of one of its edges. If C is the ½ the product of the circumference and the radius of the circle, then this shows that $B^+(n) > C > B^-(n)$. This makes it clear that $A=C$. Nowadays, we like to say 'take the limit as n goes to infinity'. Archimedes used the following more careful argument. He says, suppose $A \neq C$. Then either $A > C$ or $A < C$. Take the first case. Then if n is large enough, we know that the area $A–B^-(n)$ is less than the area $A–C$. Thus $A > B^-(n) > C$. But we just checked that $C > B^-(n)$, so this is impossible. Similarly, using the circumscribed polygon, we check that $A < C$ is impossible. The Greek school didn't have a formal theory of limits: they always argued by contradiction like this.

[1] In addition to using integration to compute areas and volumes, he was inspired by his work in mechanics to study the *center of mass* of planar and solid objects. These are given by integrals too, but now the integral of the coordinate functions *x,y,z* over the object.



We can paraphrase this argument by saying that the circle can be divided up into infinitely many infinitesimal triangles, whose altitude is the radius of the circle and whose bases, taken all together have a length equal to the circumference of the circle. Then the ½ base times altitude formula for a triangle implies that the area of a circle is ½ its radius times its circumfernece.

There are 2 facts from Archimedes' argument that we will want to use later, when we come to use calculus more systematically. These are illustrated in the simple diagram on the right showing part of an inscribed polygon. We want to first introduce the idea of measuring angles in *radians* instead of *degrees*. The idea is simply to measure an angle like that shown in this figure by the length of the arc it subtends on the unit circle. Thus, in the diagram, θ in radians is the length of the arc *CD*. In this system, a 90° angle has measure $\pi/2$, a 45° angle has measure $\pi/4$, a 30° angle has measure $\pi/6$, etc. You just multiply the number of degrees by $\pi/180$ to get the angle in radians. Now back to the figure: the length of the arc *CD* is θ and the length of the line segment *BD* is $\sin(\theta)$. Moreover, as θ gets smaller and smaller, the line segment *BD* and the arc *CD* get closer and closer. Thus the ratio $\sin(\theta)/\theta$ tends to 1 as θ tends to 0. This is just a restatement of Archimedes observation that the circumference of the circle is approximated by the perimeter of the inscribed n-gon as n gets large. The second fact which we see in the diagram is that the length of the line segment *BC* is $1-\cos(\theta)$ and this becomes smaller and smaller compared to *BD* when the angle θ gets smaller. (In fact, working out angles, it is easy to see that the angle *BDC* is $\theta/2$, so $length(BC)/length(BD) = \sin(\theta/2)$.) Thus $(1-\cos(\theta))/\theta$ goes to 0 as θ goes to 0. Summarizing, we have the relations:

$$1 = \overline{AC} = \overline{AD} \approx \overline{AB} >> \mathrm{arc}CD \approx \overline{BD} >> \overline{BC}$$

Problem: Consider first inscribed and circumscribed squares around the unit circle: work out their perimeters and areas, hence get upper and lower bounds for the circumference 2π and the area π of the unit circle. Then go on to do the same for inscribed and circumscribed *octagons*. This requires working out various lengths using Pythagoras' Theorem: draw the inscribed square and octagon and proceed to fill in all lengths you know and you should find there's always another one which you can get from a right triangle with 2 sides known. The circumscribed octagon is just a scaled up version of the inscribed one.