Chapter Thirteen: Traveling Waves and the Wave Equation in Air, Water and the Ether

We have been discussing the waves that are found in the string of a guitar or violin and how they are the physical reality that underlies music. But there are other kinds of waves: there are water waves and non-musical sound waves, such as the sound of hands clapping. These are not confined to a string but propagate over large distances in air and water. Some waves of this kind are easy to create yourself. Take a piece of rope, a jump rope for example and stretch it between 2 people. If one person gives it a shake, you will see the shake in the form of a bump in the rope moving from one person to the other. Slinkies will do the same. These are called traveling waves as are the waves of the ocean or the sound wave of clapping than to music: they can travel large distances without changing their shape very much.

Their mathematical basis is even simpler than the sines and cosines we have worked with above. Their explanation was discovered by the French mathematician Jean Le Rond D'Alembert in 1747 ("*Recherches sur la Courbe que Forme une Corde Tendue mise en Vibration"* – Studies of the Curve Formed by a String set in Vibration – published by Royal Academy of Berlin) .

His quite simple but fundamental observation was that you can start with *any function f(x)* which is to be the shape of the string at time 0, then define the position of the string at all later times by

$$y(x,t) = f(x-t)$$

and this will satisfy the vibrating string equation:

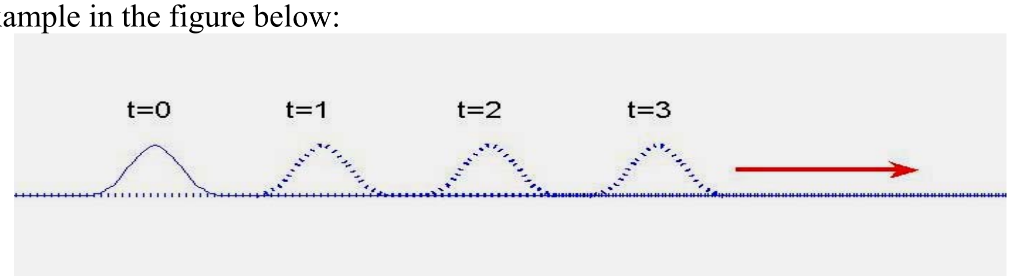
$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

If $\dot{f}$ and $\ddot{f}$ are the first and second derivatives of f, then we can compute all the derivatives of y and check that $y(x,t)$ satisfies the equation like this:

$$\frac{\partial y}{\partial x} = \dot{f}(x-t), \quad \frac{\partial y}{\partial t} = -\dot{f}(x-t), \text{ so}$$

$$\frac{\partial^2 y}{\partial x^2} = \ddot{f}(x-t), \quad \frac{\partial^2 y}{\partial t^2} = -\left(-\ddot{f}(x-t)\right) = \ddot{f}(x-t) = \frac{\partial^2 y}{\partial x^2},$$

It's easy to visualize this solution too. At any fixed $t=a>0$, $y(x,t)$ is the same function f, but *shifted*; this means that the value $f(0)$ occurs not at $x=t=0$, but at $x=t=a$. We can see an example in the figure below:



These are half the traveling wave solutions. After all, why should the wave always travel to the right? The ones which travel to the left are given by

$$y(x,t) = g(x+t)$$

It's easy to check that this solves the vibrating string equation also. In fact, we can add these two waves and have one wave going right, one going left *t the same time*:

$$y(x,t) = f(x-t) + g(x+t)$$

It can look really neat if 2 people shake a rope at the same time, producing short waves, one going right, one going left. They 'collide' in the middle of the rope but, after a short time, they reappear having passed each other and recovering the same shape they had before the collision. This is all a consequence of linearity, that the sum of 2 solutions of the vibrating string equation is also a solution.

> D'Alembert also claimed that *all* solutions of the vibrating string equation could be expressed this way. This result not needed for the our discussion but maybe it's interesting to see what he said. Suppose *y(x,t)* is some function of *x* and *t* that solves the equation. He then looked at the 2 auxiliary functions:
>
> $$p(x,t) = \frac{\partial y}{\partial x} - \frac{\partial y}{\partial t}, \text{ and}$$
>
> $$q(x,t) = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t}$$
>
> He then calculates the derivative with respect to *s* of *p(x+s,t+s)* by using the chain rule:
>
> $$\frac{d}{ds}\big(p(x+s,t+s)\big) = \frac{\partial p}{\partial x}(x+s,t+s) + \frac{\partial p}{\partial t}(x+s,t+s) = \left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial x \partial t}\right) + \left(\frac{\partial^2 y}{\partial t \partial x} - \frac{\partial^2 y}{\partial t^2}\right) = 0$$
>
> Thus $p(x+s,t+s)$ does not change when *s* changes. So it must be equal to some function of *x-t*, i.e. $p(x,t) = h(x-t)$ for some function *h*. Let *f* be the indefinite integral of *h*. Now the same argument shows that $q(x,t) = k(x+t)$. Let *g* be the indefinite integral of *k*. The final step is to check that
>
> $$y(x,t) - \big(f(x-t) + g(x-t)\big)/2 = \text{a constant}$$
>
> We get this by showing that the derivatives of the left hand side with respect to both *x* and *t* are zero, so it must be a constant.

We have ignored one small thing: we are pretending the string or rope is *infinite* so we don't have to think about its 2 ends. For the correct vibrating string, we have to add in one complication: that it never moves at its 2 ends, say at *x*=0 and *x*=*L*. This what D'Alembert looked at next.

Start with the assumptions:

$$y(x,t) = f(x-t) + g(x+t),$$
$$y(0,t) \equiv 0, \;\; y(L,t) \equiv 0$$

Then vanishing at the first end works out to mean:

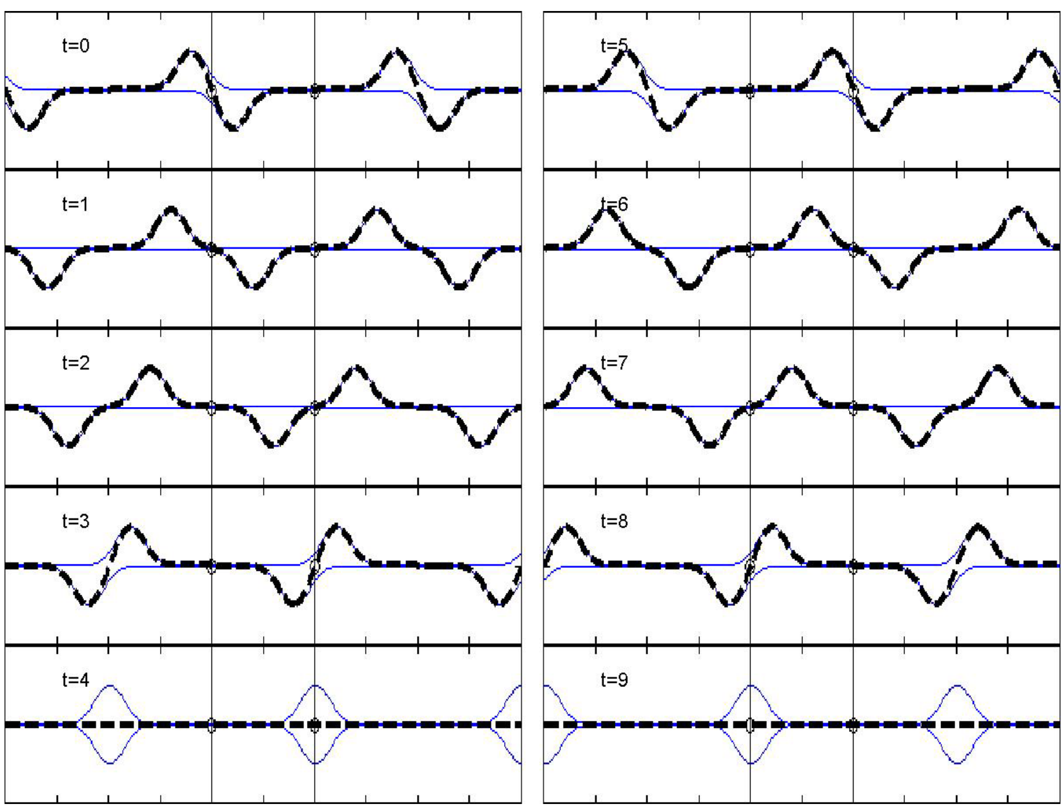
$$0 = y(0,t) = f(-t) + g(t) \text{ implies } f(t) = -g(-t), \text{ or}$$
$$y(x,t) = g(t+x) - g(t-x).$$

This implies that $y(-x,t) = -y(x,t)$ which means that there is always a negative version of wave to the left of 0 of exactly the same shape as the wave to the right of 0. This is what gives us the cancellation at 0. Vanishing at the second end works out like this:

$$0 = y(L,t) = g(t+L) - g(t-L) \text{ or}$$
$$g \text{ is periodic of period } 2L : g(t+2L) \equiv g(t)$$

The conclusion is that the 2 travelling waves manage to cancel each other out at the two endpoints and create a *periodic* wave which repeats when time is advanced by the interval 2*L*.

Here's an example which should help make these formulas clearer. In the example, there is a positive bump, shown in blue, traveling from right to left, and a negative bump, also shown in blue, traveling from left to right. The two vertical lines represent the two ends of the string where the wave will always have value 0. At *t*=0, the negative wave has just separated from the positive one and is moving across [0,1]. The movie proceeds by reading down until the waves meet at *x*=1. Then, for a split second, they cancel, as seen in the bottom panel. On the right, the movie proceeds, now with the positive wave going left from 1 to 0. Note also how we need to have a train of positive and negative waves spaced every 2 units to keep the whole thing going. This is what our equations showed us.



Now we have two ways of producing solutions of the vibrating string equation: by superposition of the sine waves, as in Chapter 10, or by traveling waves. The question is: are they the same? This got the 3 mathematicians D'Alembert, Bernouilli and Euler quite excited. In the polite discourse of the Enlightenment, Bernouilli wrote in the 1753 paper quoted above, that these new solutions were "improper" though strictly speaking correct solutions!

> I. Mr. Taylor was the first to obtain the number of vibrations made in a given time by a string uniformly thick, of given length and given weight, and stretched by a given force. It was not possible to determine this number without knowing in advance the curve taken by the string during the whole time that its vibration lasted; he therefore proved that this curve was always "the companion of an extremely elongated cycloid," for which the ordinates represent the sines of the arcs represented by the abscissas. I think that only in this form can the vibrations become regular, simple, and isochronous despite the inequality of the deviations [*excursions*]. Since I always had this idea I could only be surprised to see in the *Mémoires* [of the Berlin Academy] of the years 1747 and 1748 an infinity of other curves claimed to be endowed with the same property. I really needed the great names of Messrs. D'Alembert and Euler, whom I could not suspect of any carelessness, to make me examine whether there would not be anything in this aggregate of curves that conflicted [*équivoque*] with those of Mr. Taylor, and in what sense they could be admitted. I immediately saw that this multitude of curves could be admitted only in quite an improper sense. I do not the less esteem the calculations of Messrs. D'Alembert and Euler, which certainly contain all that analysis can have at its deepest and most sublime, but which show at the same time that an abstract analysis which is accepted without any synthetic examination of the question under discussion is liable to surprise rather than enlighten us. It seems to me that we have only to pay attention to the nature of the simple vibrations of the strings to foresee without any calculation all that these two great geometers have found by the most thorny and abstract calculations that the analytical mind can perform.

With traveling waves we are able to write down solutions in which the string starts in any position at all. To see this, suppose $f(x)$ is *any* initial position of the string, which we assume tied down at x=0 and x=1. Then first we imagine an *infinite* string which extends the one which is tied down and now goes all the way from -∞ on the left to +∞ on the right. We need to set an initial position of the rest of the string. What we do is place the string in the inverted position on [-1,0]:

$$f(x) = -f(-x), \text{ if } -1 \le x \le 0$$

and then we make the string repeat its position every 2 units of x:

$$f(x+2) \equiv f(x)$$

(See figure.) Then all we need to do is break up the initial position of the string into 2 waves of half the size of $f(x)$, one travelling left and one travelling right:

$$y(x,t) = \frac{1}{2}\big(f(x+t) + f(x-t)\big)$$

This is just the general story behind the example given above.

So we can start the string in *any* position at all, release it and let it vibrate. What bothered Bernouilli and then became a big issue between him, Euler and D'Alembert is whther we get *more* solutions of the vibrating string equation this way. What this means is: can we 'expand' any function $f(x)$ as a sum of sines:

$$f(x) = a_1 \sin(2\pi x) + a_2 \sin(4\pi x) + a_3 \sin(6\pi x) + \cdots$$

in which case the two versions of the solution (D'Alembert's on the left, Bernouilli's on the right) are the same:

$$\frac{1}{2}\big(f(x+t) + f(x-t)\big) = a_1 \sin(2\pi x)\cos(2\pi t) + a_2 \sin(4\pi x)\cos(4\pi t) + a_3 \sin(6\pi x)\cos(6\pi t) + \cdots$$

Can any initial position can be written as a sum of sine waves at the basic frequency and all its harmonics.

[MUST REARRANGE MATERIAL: DISCUSSION OF FOURIER EXPANSION OF ARBITRARY FUNCTIONS SHOULD BE HERE.]

All this has been about the waves which can be produced in a string under tension. But waves are something that occurs in a myriad of other settings. For example the head of a drum vibrates when hit, also in a periodic fashion with the membrane of the drum head going up and down in a definite pattern. The surface of water, in a glass or the ocean, supports many types of waves, some regular, some chaotic. Sound is a wave in the air, caused by alternating compression and rarefaction of the air, oscillating at many possible speeds. And, most important for our technology, radio waves, radar and light are all waves, but remarkably, waves without an underlying medium like air or water. They are waves of rapidly varying electric and magnetic fields.

The Reason we have lingered so long on the vibrating string is because all the basic ideas are present in this simple example. The mathematics appears in its simplest form in this case and we can simulate it most simply on the computer. To go further, we need more than 1 space variable. Instead of describing the wave by $y(x,t)$, with one space variable x (as well as one time variable t and one variable y which 'carries' the wave), we need

[FUTURE MATERIAL: Emphasize superposition and linearity. Computer demos. (Visual illusions: plaids.) Show circularly symmetric solutions of 2D wave equations, as per demo. (Aside on computing Laplacian of 1/sqrt(t^2-|x|^2).)

Sound and pressure waves. Euler's equations with advection term and its linear version. Simplest Example, as in Newton. Non-linear term in equation. Hmm: Eulerian or Lagrangian version?

Maxwell's equations – must not get bogged down in details here – light and EM waves and the speed of light. Maybe doing the vector potential is easiest? Easiest 1D wave with *X_x, H_y* and motion along *z*-axis. Cable equation and story of the trans-Atlantic cable.

The simplest approx to water waves (pics of this). Newton's discussion of water waves and anlogy with pendulum. Modern version with Hilbert transform – easy to do numerically. (Comments on shallow water?, Pego's derivation of KdV)