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Tata Lectures on Theta III

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Preface

The present volume contains the final chapter of this work on theta functions. Like the other chapters, it originated with the senior author's lectures at the Tata Institute of Fundamental Research during November 1978–March 1979. Excellent notes on these lectures were made by M. Nori, but, due to the shortage of time, not all the projected topics were covered. In the next few years, the ideas in this chapter were developed in various directions while the other parts of these lectures were published. However, this final chapter was not completed until a collaboration with Peter Norman beginning in 1988 infused new life into the project. At the same time, the interest of string oriented theoretical physicists in theta functions gave extra impetus to completing these notes. We are pleased that this joint effort has now made it possible to publish this volume.

The idea behind this chapter was to bring together and clarify the interrelations between three ways of viewing theta functions:

- a) as classical holomorphic functions in the vector \vec{z} and/or the period matrix T ,
- b) as matrix coefficients of a representation of the Heisenberg and/or metaplectic groups,
- c) as sections of line bundles on abelian varieties and/or the moduli space of abelian varieties.

Although equivalent on a deep level, superficially these three points of view look totally different and require quite different vocabularies. A more specific motivation was that the purely algebraic theory of theta functions, which comes from (c), has not been very widely understood. This approach originated the senior author's three part paper in *Inventiones Math.*, in 1966–67, *On the Equations defining Abelian Varieties*. This paper is not easy to read, however, and with the exception of a few papers by Kempf, Barsotti, Igusa, Moret-Bailly and Norman, the ideas in it have not been developed very far. For this reason, one goal of these lectures was to give a reasonably simple explicit treatment of the algebraic definition of theta functions, valid over any ground field (or base scheme). In the last few sections many open questions are raised: we hope this will make it clear how little is known beyond the foundations and will stimulate further work in the subject.

Cambridge

February, 1991

1. Heisenberg groups in general

The abstract approach to the theory of theta functions is intimately bound up with a certain class of non-abelian groups, called Heisenberg groups. We begin by developing the representation theory of this class of groups. We consider locally compact groups G which lie in a central extension:

$$1 \longrightarrow \mathbf{C}_1^* \longrightarrow G \longrightarrow K \longrightarrow 0$$

i.e., $\mathbf{C}_1^* = \{z \in \mathbf{C} \mid |z| = 1\}$ is a normal subgroup of G , in the center of G , and G/\mathbf{C}_1^* is an abelian locally compact group K (which we write additively: hence the notation $1 \longrightarrow \dots \longrightarrow 0$ above). We assume that G admits a continuous section over K , so that we can describe G as

$$G = \mathbf{C}_1^* \times K \quad (\text{as a set}).$$

Then the group law on G is given by:

$$(\lambda, x) \cdot (\mu, y) = (\lambda\mu\psi(x, y), x + y)$$

where

$$\psi : K \times K \longrightarrow \mathbf{C}_1^*$$

is a 2-cocycle:

$$\psi(x, y) \cdot \psi(x + y, z) = \psi(x, y + z)\psi(y, z).$$

Next, if we choose any elements $x, y \in K$, let $\tilde{x}, \tilde{y} \in G$ lie over them and form $\tilde{x} \tilde{y} \tilde{x}^{-1} \tilde{y}^{-1}$. This lies in \mathbf{C}_1^* and is independent of the liftings \tilde{x}, \tilde{y} , so we may define:

$$e : K \times K \longrightarrow \mathbf{C}_1^*$$

by

$$e(x, y) = \tilde{x} \tilde{y} \tilde{x}^{-1} \tilde{y}^{-1}.$$

It is easy to verify that

$$e(x + x', y) = e(x, y) \cdot e(x', y)$$

$$e(x, y + y') = e(x, y) \cdot e(x, y')$$

$$e(x, x) = 1, \quad e(x, y) = e(y, x)^{-1}$$

$$e(x, y) = \frac{\psi(x, y)}{\psi(y, x)}.$$

e is a skew-multiplicative pairing. Let \hat{K} be the character group of K (its "Pontrjagin dual"). Define

$$\varphi : K \longrightarrow \hat{K}$$

by

$$\varphi(x)(y) = e(x, y)$$

or

$$ghg^{-1} = \varphi(\bar{g})(\bar{h}) \cdot h, \quad (\forall g, h \in G, \bar{g} = \pi g, \bar{h} = \pi h).$$

DEFINITION 1.1. G is a Heisenberg group if φ is an isomorphism.

Given such a G , we will want to consider closed subgroups $H \subset K$ such that equivalently:

- a) $e_{H \times H} \equiv 1$ and H is maximal with this property.
- b) φ restricts to an isomorphism between $H \subset K$ and $(\widehat{K/H}) \subset \hat{K}$.
- c) $H = H^\perp$, where $H^\perp = \{x \in K \mid e(x, y) = 1, \text{ all } y \in H\}$.

(The equivalence is easy noting that $\varphi^{-1}(\widehat{K/H}) = H^\perp$ and that if $e|_{H \times H} \equiv 1, x \in H^\perp$, then the group H' which is the closure of $H + \mathbb{Z} \cdot x$ in K also satisfies $e|_{H' \times H'} \equiv 1$.) When $e_{H \times H} \equiv 1$, we say H is isotropic; when H is maximal with this property, we say H is maximal isotropic. Note also that the following two properties of a closed subgroup $H \subset K$ are equivalent:

- a) H is isotropic.
- b) $\pi : G \longrightarrow K$ splits over H , i.e., \exists homomorphism $\sigma : H \longrightarrow G$ such that $\pi \circ \sigma = 1_H$.

Here b) \implies a) by the definition of e . To see a) \implies b), let H be isotropic and consider $\pi^{-1}(H)$. Then $\pi^{-1}(H)$ is commutative, and taking duals we have:

$$0 \longleftarrow \mathbb{Z} \longleftarrow \widehat{\pi^{-1}(H)} \longleftarrow \hat{H} \longleftarrow 0.$$

Lift $1 \in \mathbb{Z}$ to an element $\zeta \in \widehat{\pi^{-1}(H)}$. Then ζ is a character of $\pi^{-1}(H)$ such that $\zeta(\lambda) = \lambda$, all $\lambda \in \mathbb{C}_1^*$. Thus π restricts to an isomorphism from $\text{Ker}(\zeta)$ to H and its inverse here is the section σ . In terms of co-cycles, splitting π over H in K amounts to giving

$$\sigma(x) = (\alpha(x), x), \quad \forall x \in H$$

such that

$$\sigma(x + y) = \sigma(x) \cdot \sigma(y), \quad \forall x, y \in H$$

or

$$\frac{\alpha(x + y)}{\alpha(x) \cdot \alpha(y)} = \psi(x, y), \quad \forall x, y \in H.$$

After these preliminaries, we are now ready to state the Main Theorem about the representations of such groups:

THEOREM 1.2. (Stone, Von Neumann, Mackey). Let G be a Heisenberg group. Then

- i) G has a unique irreducible unitary representation

$$U : G \longrightarrow \text{Aut}(\mathcal{H}_0)$$

such that $U_\lambda = \lambda \cdot \text{id.}$, all $\lambda \in \mathbb{C}_1^*$.

- ii) For all maximal isotropic subgroups $H \subset K$ and splittings $\sigma(x) = (\alpha(x), x)$ of π over H , this representation may be realized by

$$\mathcal{H}_0 = \left\{ \begin{array}{l} \text{measurable functions } f : K \rightarrow \mathbb{C} \text{ such that} \\ \text{a) } f(x + h) = \alpha(h)^{-1} \psi(h, x)^{-1} f(x), \quad \forall h \in H \\ \text{b) } \int_{K/H} |f(x)|^2 dx < +\infty \end{array} \right\}$$

$$U_{(\lambda, y)} f(x) = \lambda \cdot \psi(x, y) \cdot f(x + y).$$

We will write this \mathcal{H}_0 as

$$L^2(K//H).$$

iii) All representations (U, \mathcal{H}) such that $U_\lambda = \lambda \cdot \text{id}$, all $\lambda \in \mathbb{C}_1^*$, are isomorphic to $\mathcal{H}_0 \hat{\otimes} \mathcal{H}_1$, G acting trivially on \mathcal{H}_1 .

This theorem was first proven independently by Stone and von Neumann in the case $K = \mathbb{R}^{2n}$. The general case is due to Mackey. We give the proof first for the case where K is finite, where all the steps are completely elementary. Then we indicate the modifications necessary to deal with the general case, but following exactly the same method. We shall use as a reference for the general case the treatment in V.S. Varadarajan, *Geometry of Quantum Mechanics*, vol. II, and simply isolate the steps where analytic lemmas are needed. This does not seem to distort the situation.

Assume then that (U, \mathcal{H}) is any unitary representation of G , that K is finite, that $H \subset K$ is maximal isotropic and that $\sigma(x) = (\alpha(x), x)$ is a splitting of π over H .

Step I. Decompose \mathcal{H} under the action of the abelian group $\sigma(H)$:

$$\mathcal{H} = \bigoplus_{\zeta \in \hat{H}} \mathcal{H}_\zeta$$

where $\mathcal{H}_\zeta = \{a \in \mathcal{H} \mid U_{\sigma(x)} a = \zeta(x) \cdot a, \text{ all } x \in H\}$. Another way of writing this is to classify the ζ 's according to $\dim \mathcal{H}_\zeta$. Let

$$\hat{H}_n = \{\zeta \in \hat{H} \mid \dim \mathcal{H}_\zeta = n\}, \quad (n = 0, 1, 2, \dots; \infty)$$

and let \mathcal{K}_n be a standard n -dimensional Hilbert space. Then

$$\mathcal{H} \cong \bigoplus_{n=0}^{\infty} L^2(\hat{H}_n; \mathcal{K}_n)$$

where $L^2(\hat{H}_n, \mathcal{K}_n) = \{\text{space of } \mathcal{K}_n\text{-valued functions } f_n(x) \text{ on } \hat{H}_n\}$ and the above isomorphism carries $U_{\sigma(x)}$ for $x \in H$ to the map

$$\{\tilde{f}_n\} \longmapsto \{\tilde{g}_n\}, \quad \tilde{g}_n(\zeta) = \zeta(x) \cdot f_n(\zeta).$$

Step II. $\pi^{-1}(H)$, which is $\mathbb{C}_1^* \times H$ via the section σ , is a normal subgroup of G . But if $\pi(g) = y$, $h = (\lambda, x)$, then

$$ghg^{-1} = (\lambda \cdot e(y, x), x) = (\lambda \cdot \varphi(y)(x), x).$$

Therefore if U_h acts on a vector $a \in \mathcal{H}$ by the character $\zeta \in \hat{H}$, then $U_{ghg^{-1}}$ acts on a by the character $\varphi(y) + \zeta$:

$$\begin{aligned} U_{ghg^{-1}} a &= \varphi(y)(x) \cdot U_h(a) \\ &= \varphi(y)(x) \cdot \zeta(x) \cdot a \\ &= (\varphi(y) + \zeta)(x) \cdot a. \end{aligned}$$

Thus

$$\dim \mathcal{H}_\zeta = \dim \mathcal{H}_{\varphi(y) + \zeta}.$$

Since φ is surjective, this proves that $\dim \mathcal{H}_\zeta$ is independent of ζ ; hence

$$\mathcal{H} = L^2(\hat{H}, \mathcal{K}_n)$$

for some fixed n . So for $f \in \mathcal{H}$, $y \in H$, $x \in \hat{H}$,

$$U_{(\alpha(y), y)} f(x) = x(y) f(x).$$

Step III. Now let us write down how the rest of G must act on \mathcal{H} in terms of a 1-cocycle. We use the simple

LEMMA. Automorphisms V of $L^2(\hat{H}, \mathcal{K}_n)$ that commute with the action of H are given by:

$$(Vf)(\zeta) = A(\zeta)(f(\zeta))$$

where

$$A(\zeta) : \mathcal{K}_n \rightarrow \mathcal{K}_n$$

is a set of unitary maps depending on ζ .

To apply the lemma, take $x \in K$ and let ϕ denotes the composition of φ with the restriction from \hat{K} to \hat{H} . Consider the map $f \mapsto Vf$, where

$$(Vf)(\zeta) = (U_{(1,x)}f)(\zeta - \phi(x)).$$

Then for all $y \in H$,

$$\begin{aligned} V U_{(\alpha(y),y)}f(\zeta) &= U_{(1,x)}U_{(\alpha(y),y)}f(\zeta - \phi(x)) \\ &= e(x,y)U_{(\alpha(y),y)}U_{(1,x)}f(\zeta - \phi(x)) \\ &= e(x,y)\alpha(y)(\zeta - \phi(x))(y)U_{(1,x)}f(\zeta - \phi(x)) \\ &= e(x,y)\alpha(y)(\zeta - \phi(x))(y) \cdot Vf(\zeta) \\ &= \zeta(y) \cdot \alpha(y)Vf(\zeta) \\ &= U_{(\alpha(y),y)}Vf(\zeta). \end{aligned}$$

Therefore

$$U_{(1,x)}f(\zeta) = A_x(\zeta)[f(\zeta + \phi(x))]$$

where $A_x(\zeta)$ are unitary isomorphisms of \mathcal{K}_n .

Step IV. Now define a map

$$W : L^2(\hat{H}, \mathcal{K}_n) \longrightarrow L^2(K//H; \mathcal{K}_n)$$

where

$$\begin{aligned} L^2(K//H; \mathcal{K}_n) &= \left\{ \begin{array}{l} \text{space of maps } g : K \rightarrow \mathcal{K}_n \text{ such that} \\ g(x+h) = \alpha(h)^{-1}\psi(h,x)^{-1}g(x), \forall h \in H \end{array} \right\} \\ &= L^2(K//H) \hat{\otimes} \mathcal{K}_n. \end{aligned}$$

Let

$$Wf(x) = U_{(1,x)}f(e) = A_x(e)[f(\phi(x))].$$

(The 2nd and 3rd expressions are equal by Step III). By Step III, we see that

$$\int_{K/H} \|Wf(x)\|^2 dx = \int_{K/H} \|f(\phi(x))\|^2 dx = \int_{\hat{H}} \|f(\zeta)\|^2 d\zeta.$$

So W is a unitary map. Moreover, for $h \in H$

$$\begin{aligned} Wf(x+h) &= U_{(1,x+h)}f(e) \\ &= [U_{(\psi(h,x)^{-1},h)} \cdot U_{(1,x)}f](e) \\ &= \psi(h,x)^{-1} \cdot \alpha(h)^{-1} U_{(\alpha(h),h)}(U_{(1,x)}f)(e) \\ &= \psi(h,x)^{-1} \cdot \alpha(h)^{-1} \cdot e(h) \cdot U_{(1,x)}f(e) \\ &= \psi(h,x)^{-1} \cdot \alpha(h)^{-1} \cdot Wf(x) \end{aligned}$$

hence $Wf \in L^2(K//H; \mathcal{K}_n)$. Thirdly

$$\begin{aligned} WU_{(1,y)}f(x) &= U_{(1,x)}U_{(1,y)}f(e) \\ &= U_{(\psi(x,y),x+y)}f(e) \\ &= \psi(x,y)Wf(x+y) \end{aligned}$$

which is the rule given in the theorem by which G is to act on

$L^2(K//H) \hat{\otimes} \mathcal{K}_n$ (generalized to arbitrary n). Finally, as f can be given arbitrary values in \mathcal{K}_n for each $\zeta \in \hat{H}$, it follows that $A_x(e)f(\phi(x))$ has arbitrary values in \mathcal{K}_n for x ranging over a set of coset representatives in $K \bmod H$. Thus W is surjective. Putting this together, W is a unitary isomorphism of the G -representations \mathcal{H} and $L^2(K//H) \hat{\otimes} \mathcal{K}_n$.

Thus if \mathcal{H} is irreducible, $n = 1$, and every irreducible representation U such that $U_\lambda = \lambda \cdot id$ is isomorphic to $L^2(K//H)$. This proves (i) and (ii). Moreover every non-irreducible one is a direct sum of $L^2(K//H)$ with itself n times for some n .

QED for finite K

We now outline the modifications necessary to deal with general n .

The generalization of Step I is the classification theorem for arbitrary unitary representations of locally compact abelian groups. This is called

the theory of spectral multiplicity, and may be found, e.g., in P. Halmos, "Introduction to Hilbert space and Spectral Multiplicity". The result is that any unitary representation (U, \mathcal{H}) of an abelian H is of the form:

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} L^2(\hat{H}_n, \mu_n; \mathcal{K}_n)$$

where \hat{H} is the disjoint union of Borel sets \hat{H}_n , μ_n is a measure supported on \hat{H}_n and $L^2(\hat{H}_n, \mu_n; \mathcal{K}_n)$ is the Hilbert space of measurable functions $f : \hat{H}_n \rightarrow \mathcal{K}_n$ with norm

$$\|f\|^2 = \int_{\hat{H}_n} \|f(\zeta)\|^2 d\mu_n(\zeta)$$

and H acts by

$$U_h f(\zeta) = \zeta(h) \cdot f(\zeta).$$

In this decomposition, the measure class of μ_n , (i.e., the set of subsets $S \subset \hat{H}$ of measure zero) is uniquely determined.

To generalize Step II, the argument given above shows that for every $\eta \in \hat{H}$, if we modify the representation (U_h, \mathcal{H}) on H by multiplying by the character $\eta(h)$, we get a unitarily equivalent representation. In terms of the decomposition via the measures μ_n , this means that we translate all the measures μ_n on \hat{H} by η . Therefore by Step I, the measures μ_n must have the property:

All translates of μ_n by $\zeta \mapsto \zeta + \eta$, $\eta \in \hat{H}$, are in the same measure class as μ_n .

We now cite the well-known lemma:

LEMMA 1.3. *If H is an abelian locally compact group, there is a unique measure class which is translation invariant, and it contains a unique measure which is also translation-invariant: the Haar measure.*

(See V.S. Varadarajan, op. cit., Lemma 8.12, p. 19). Thus all μ_n may be assumed to be multiples of Haar measure. Since the μ_n 's also have disjoint

support, only one of them can be non-zero, i.e., for some n :

$$\mathcal{H} = L^2(\hat{H}, \mathcal{K}_n).$$

To generalize Step III we need to know that any unitary isomorphism of $L^2(\hat{H}, \mathcal{K}_n)$ commuting with H is given by

$$(Vf)(\zeta) = A(\zeta)(f(\zeta))$$

where

$$A(\zeta) : \mathcal{K}_n \longrightarrow \mathcal{K}_n$$

are unitary maps depending measurably on ζ ; and that if $V_t f$ is a measurable family of such V 's, it is given as above with

$$A_t(\zeta) : \mathcal{K}_n \longrightarrow \mathcal{K}_n$$

depending measurably on t and ζ . This is the content of Lemmas 9.4 and 9.5 in V.S. Varadarajan, op. cit., pp. 63-66.

It is when we reach Step IV that we are in trouble. The definition of W does not make sense because we have to evaluate $U_{(1,x)}f$ or $A_x(\zeta)$ at specific points, and we are dealing with measurable functions, which are only well-defined modulo functions supported on a set of measure zero. This is the key point which Mackey was able to solve. His idea was that $A_x(\zeta)$ is essentially a co-cycle and as such it can be made continuous, mod a coboundary, and then it has good values everywhere. To be precise, let

$$B_x(y) = \frac{A_x(\varphi(y))}{\psi(y, x)}, \quad \forall x, y \in K.$$

Then the fact that for all $x, y \in K$:

$$U_{(1,x)}U_{(1,y)}f = \psi(x, y)U_{(1,x+y)}f$$

tells us that

$$\begin{aligned}
 (B_{x+y}(z)f)(\varphi(z+x+y)) &= \psi(z, x+y)^{-1} (A_{x+y}(\varphi(z)f))(\phi(z+x+y)) \\
 &= \psi(z, x+y)^{-1} U_{(1, x+y)} f(\phi(z)) \\
 &= \psi(z, x+y)^{-1} \psi(x, y)^{-1} U_{(1, x)} U_{(1, y)} f(\phi(z)) \\
 &= \psi(z+x, y)^{-1} \psi(z, x)^{-1} \cdot \\
 &\quad A_x(\varphi(z)) \circ A_y(\phi(z+x)) f(\phi(z+x+y)) \\
 &= (B_x(z) \circ B_y(z+x)) f(\phi(z+x+y))
 \end{aligned}$$

i.e., for all $x, y \in K$, then for all $z \in K$ except in a set of measure zero (depending on x, y):

$$B_{x+y}(z) = B_x(z) \circ B_y(z+x).$$

Then Mackey's fundamental lemma in this quite special case says that there is a measurable function $A : K \rightarrow \text{Aut}(\mathcal{K}_n)$ such that for all $x \in K$, then for all $y \in K$ except in a set of measure zero (depending on x):

$$B_x(y) = A(y)^{-1} \circ A(x+y).$$

We may normalize A by requiring $A(e) = \text{id}$. So $B_x(e) = A(x)$. Our old definition of W can now be rewritten:

$$\begin{aligned}
 (Wf)(x) &= A_x(e)f(\varphi(x)) \\
 &= \psi(e, x) B_x(e) f(\varphi(x)) \\
 &= \psi(e, x) A(x) f(\varphi(x)).
 \end{aligned}$$

This formula does not involve taking a function defined up to a set of measure zero and substituting a particular value for one of its variables. It is now straightforward to verify that this W is a unitary isomorphism of \mathcal{H} with $L^2(K//H) \hat{\otimes} \mathcal{K}_n$, although, since we do not have the formula $U_{(1, x)} f(e)$ for $Wf(x)$, the proofs are more roundabout than in the original Step IV.

We give the proof of Mackey's lemma in order to convey the style of this type of argument. We write " $\forall\forall$ " to mean "for almost all".

a) Let $C(x, y) = B_{y-x}(x) = A(x)^{-1} \circ A(y)$. Then in terms of C the cocycle condition on B gives

$$\forall\forall(x, y, z) \in K^3, C(x, z) = C(x, y) \circ C(y, z).$$

It follows that there is some particular value y_0 of y such that

$$\forall\forall(x, z) \in K^2, C(x, z) = C(x, y_0) \circ C(y_0, z).$$

Write

$$C(y_0, z) = C_1(z), C(x, y_0) = C_2(x),$$

so

$$\forall\forall(x, z) \in K^2, C(x, z) = C_2(x) \circ C_1(z).$$

b) Define $B'_x(y) = C_1(y) \circ B_x(y) \circ C_1(x+y)^{-1}$. Then B' satisfies the same hypothesis as B , and moreover $\forall\forall(x, y) \in K^2$

$$\begin{aligned}
 B'_x(y) &= C_1(y) \circ C(y, x+y) \circ C_1(x+y)^{-1} \\
 &= C_1(y) \circ C_2(y) \circ C_1(x+y) \circ C_1(x+y)^{-1} \\
 &= C_3(y), \text{ say.}
 \end{aligned}$$

c) Use the identities that B satisfies, and it follows that

$$\begin{aligned}
 \forall\forall(x, y, z) \in K^3, C_3(z) &= B'_{x+y}(z) \\
 &= B'_x(z) \circ B'_y(z+x) \\
 &= C_3(z) \circ C_3(z+x)
 \end{aligned}$$

i.e.,

$$\forall\forall x \in K, C_3(x) = \text{id};$$

hence

$$\forall\forall(x, y) \in K^2, B'_x(y) = \text{id}.$$

d) Finally fix any $x \in K$. Then $\forall\forall(y, z) \in K^2$, the 3 formulae:

$$\begin{aligned}
 B'_x(z) &= B'_{x+y}(z) \circ B'_y(z+x)^{-1} \\
 B'_{x+y}(z) &= \text{id.} \\
 B'_y(x+z) &= \text{id.}
 \end{aligned}$$

all hold. Thus $\forall\forall z \in K, B'_x(z) = \text{id.}$, hence $\forall\forall z \in K$

$$B_x(z) = C_1(z)^{-1} \circ C_1(x+z)$$

as asserted.

We finish this section by making three remarks to amplify the idea of the Heisenberg representation.

PROPOSITION 1.4. *Given a Heisenberg group*

$$1 \rightarrow \mathbf{C}_1^* \rightarrow G \rightarrow K \rightarrow 0,$$

let $H \subset K$ be a compact isotropic subgroup of K , and $\sigma(h) = (\alpha(h), h)$ a splitting of G over H . Then

$$1 \rightarrow \mathbf{C}_1^* \rightarrow \pi^{-1}(H^\perp)/\sigma H \rightarrow H^\perp/H \rightarrow 0$$

is a Heisenberg group, and if \mathcal{H}_0 is the Heisenberg representation of G , $(\mathcal{H}_0)^{\sigma H}$ is the Heisenberg representation of $G_H = \pi^{-1}(H^\perp)/\sigma H$.

PROOF: It is easy to see that G_H is Heisenberg (without assuming compactness of H). To check the assertion about representations, let $H \subset H_1 \subset H^\perp$ be a maximal isotropic subgroup of K and extend σ to $\sigma_1 : H_1 \rightarrow G$. Then

$$L^2(K//H_1) = \text{the Heisenberg repres. of } G$$

$$L^2(H^\perp//H_1) = \text{the Heisenberg repres. of } G_H.$$

Via e , H and K/H^\perp are dual abelian locally compact groups; since H is compact, K/H^\perp is discrete, i.e., H^\perp is open in K . Therefore we may identify $L^2(H^\perp//H_1)$ with the subspace of $L^2(K//H_1)$ of functions supported on the open and closed subspace H^\perp . Writing out the quasi-periodicity formula with respect to H_1 , it is easy to see that $f \in L^2(K//H_1)$ is $\sigma(H)$ -invariant if and only if it is supported on H^\perp . Q.E.D.

PROPOSITION 1.5. *Given 2 Heisenberg groups:*

$$1 \rightarrow \mathbf{C}_1^* \rightarrow G_i \rightarrow K_i \rightarrow 0, \quad i = 1, 2$$

then

$$1 \rightarrow \mathbf{C}_1^* \rightarrow G_1 \times G_2 / \{(\lambda, \lambda^{-1}) \mid \lambda \in \mathbf{C}_1^*\} \rightarrow K_1 \times K_2 \rightarrow 0$$

is a Heisenberg group, and its Heisenberg representation is

$$\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2.$$

The proof is straightforward.

The third remark is this. If G is any Heisenberg group, we get a "universal" representation (U, \mathcal{H}_U) where $U_\lambda = \lambda \cdot \text{id}$. by taking the space of all functions

$$f : G \rightarrow \mathbf{C}$$

such that

$$f(\lambda g) = \lambda \cdot f(g)$$

$$\int_{G/\mathbf{C}_1^*} |f|^2 dg < \infty.$$

Call this $L^2_{(1)}(G)$. It is in fact a left and a right G -module via

$$(U_{g_1, g_2} f)(g') = f(g_1^{-1} g' g_2).$$

Equivalently, via the set-theoretic section σ of G over K , this space is just $L^2(K)$ with $G \times G$ acting by:

$$(U_{(\lambda, x); (\mu, y)} f)(z) = \lambda^{-1} \mu \frac{\psi(z - x, y)}{\psi(x, z - x)} f(z - x + y).$$

I claim:

PROPOSITION 1.6. *There is a $G \times G$ -equivariant unitary isomorphism $L^2(K) \cong \mathcal{H}_G^* \hat{\otimes} \mathcal{H}_G$.*

PROOF: Let $\tilde{G} = G \times G / \{(\lambda, \lambda) \mid \lambda \in \mathbf{C}_1^*\}$. This group acts on $L^2(K)$. But it is also a Heisenberg group in a sequence

$$1 \rightarrow \mathbf{C}_1^* \rightarrow \tilde{G} \rightarrow K \times K \rightarrow 0,$$

and K , embedded diagonally in $K \times K$, is a maximal isotropic subgroup. Working out $L^2(K \times K//K)$, we find that the representation of $G \times G$ on

$L^2(K)$ is just the Heisenberg representation of \tilde{G} . On the other hand, if G^0 is G with its center identified with \mathbf{C}_1^* in the opposite way, then

$$\tilde{G} \cong G^0 \times G / \{(\lambda, \lambda^{-1}) \mid \lambda \in \mathbf{C}_1^*\}$$

so by Proposition 1.5, this representation is isomorphic to

$$\mathcal{H}_{G^0} \hat{\otimes} \mathcal{H}_G.$$

But $\mathcal{H}_{G^0} \cong \mathcal{H}_G^*$, hence Prop. 1.6 follows.

QED

2. The real Heisenberg groups

We want to specialize the theory of §1 to the case of Heisenberg groups

$$1 \rightarrow \mathbf{C}_1^* \rightarrow \text{Heis}(V) \rightarrow V \rightarrow 0,$$

V a real vector space. We construct explicit realizations for this Heisenberg representation and use these, given an element T of the Siegel upper-half space, to get special elements $f_T, e_{\mathbf{z}}$ of the representation space. The theta function appears as a matrix coefficient for the representation using f_T and $e_{\mathbf{z}}$.

The commutator $e : V \times V \rightarrow \mathbf{C}_1^*$ can be written in the form

$$e(x, y) = e^{2\pi i A(x, y)}$$

where $A : V \times V \rightarrow \mathbf{R}$ is a non-degenerate, \mathbf{R} -bilinear skew-symmetric form on V . From now on we will write $e(z)$ for $e^{2\pi i z}$. We shall choose V to be \mathbf{R}^{2g} , and A to be:

$$A(x, y) = {}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and x_i and y_i are g -rowed column vectors. Choosing a suitable splitting, the group $\text{Heis}(V)$ may be described as:

$$\text{Heis}(2g, \mathbf{R}) = \left\{ \begin{array}{l} \text{set of pairs } (\lambda, x), \lambda \in \mathbf{C}_1^*, x \in \mathbf{R}^{2g}, \text{ with group law} \\ (\lambda, x) \cdot (\mu, y) = (\lambda\mu e(A(x, y)/2), x + y). \end{array} \right.$$

In the notation of §1, $\psi(x, y) = e(A(x, y)/2) = e(\frac{x}{2}, y)$. The most obvious kind of maximal isotropic subspace in V is a real sub-vector space $W \subset V$ which is a maximal isotropic for A . We shall use the notation

$$W_1 = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \right\} \subset \mathbf{R}^{2g}$$

$$W_2 = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right\} \subset \mathbf{R}^{2g}.$$

Taking W_2 as the group H in Theorem 1.2, we obtain

$$\mathcal{H}_0 = \text{functions } f : \mathbf{R}^{2g} \rightarrow \mathbf{C} \text{ such that}$$

$$f\left(x + \begin{pmatrix} 0 \\ y_2 \end{pmatrix}\right) = e({}^t y_2 \cdot x_1/2) f(x), \quad \int_{W_1} |f\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}\right)|^2 dx_1 < \infty$$

$$U_{(\lambda, y)} f(x) = \lambda e({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1/2) f(x + y).$$

These functions f are obviously determined by their restrictions to W_1 , so setting $\varphi(x_1) = f\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}\right)$, we find:

Realization I:

$$\mathcal{H}_1 = \text{functions } \varphi : \mathbf{R}^g \rightarrow \mathbf{C} \text{ such that } \int |\varphi(x_1)|^2 dx_1 < \infty$$

$$U_{(1, y_1)} \varphi(x_1) = \varphi(x_1 + y_1)$$

$$U_{(1, y_2)} \varphi(x_1) = e({}^t x_1 \cdot y_2/2) f\left(\begin{pmatrix} x_1 \\ y_2 \end{pmatrix}\right)$$

$$= e({}^t x_1 \cdot y_2) \varphi(x_1)$$

hence

$$U_{(\lambda, y_1, y_2)} \varphi(x_1) = \lambda e({}^t x_1 \cdot y_2 + {}^t y_1 \cdot y_2/2) \varphi(x_1 + y_1).$$

Here we have the well-known Heisenberg representation, i.e., the irreducible group of unitary maps of $L^2(\mathbf{R}^g)$ consisting of translations and multiplication by characters.

Another maximal isotropic subgroup of V is a lattice L such that A is integral on $L \times L$, and L is maximal with this property, i.e., if you express A on $L \times L$ by a matrix, then $\det A = 1$. Take L to be \mathbf{Z}^{2g} , the standard lattice in \mathbf{R}^{2g} , and split $\text{Heis}(2g, \mathbf{R})$ over L by

$$\sigma(n) = (e_*(n/2), n), \quad \forall n \in \mathbf{Z}^{2g},$$

where $e_*\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = e(2 \cdot ({}^t x_1 \cdot x_2))$. Then

$$e_*(n/2) \in \{\pm 1\} \quad \text{if} \quad n \in \mathbf{Z}^{2g},$$

and

$$\sigma(n) \cdot \sigma(m) = (e_*(n/2) e_*(m/2) \psi(n, m), n + m)$$

$$= (e({}^t n_1 \cdot n_2 + {}^t m_1 \cdot m_2)/2 + ({}^t n_1 \cdot m_2 - {}^t n_2 \cdot m_1)/2, n + m)$$

$$= (e({}^t (n_1 + m_1)(n_2 + m_2)/2) \cdot e(-{}^t n_2 \cdot m_1), n + m)$$

$$= (e_*(\frac{n+m}{2}), n + m) = \sigma(n + m)$$

if $n, m \in \mathbf{Z}^{2g}$. Notice that e_* is a quadratic form on $\frac{1}{2}\mathbf{Z}^{2g}/\mathbf{Z}^{2g}$ with values in $\{\pm 1\}$ and σ is a section because

$$e_*\left(\frac{x+y}{2}\right) e_*\left(\frac{x}{2}\right)^{-1} e_*\left(\frac{y}{2}\right)^{-1} = e\left(\frac{x}{2}, y\right).$$

The main theorem in the previous section gives

Realization II:

$$\mathcal{H}_2 = \left\{ \begin{array}{l} \text{functions } f : \mathbf{R}^{2g} \rightarrow \mathbf{C} \text{ such that} \\ f(x + n) = e_*\left(\frac{n}{2}\right) e\left(\frac{n}{2}, x\right)^{-1} f(x), \quad \forall n \in \mathbf{Z}^{2g} \\ \int_{\mathbf{R}^{2g}/\mathbf{Z}^{2g}} |f(x)|^2 < \infty \end{array} \right\}$$

$$U_{(\lambda, y)} f(x) = \lambda \cdot e\left(\frac{x}{2}, y\right) f(x + y).$$

The space \mathcal{H}_2 will be called $L^2(\mathbf{R}^{2g}/\mathbf{Z}^{2g})$.

The $\text{Heis}(2g, \mathbf{R})$ -equivariant unitary maps between these 2 realizations are easily written down:

$$f \in L^2(\mathbf{R}^g) \quad \text{corresponds to} \quad f^* \in L^2(\mathbf{R}^{2g}/\mathbf{Z}^{2g})$$

iff

$$f^*\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \sum_{n \in \mathbf{Z}^g} f(x_1 + n) e({}^t n \cdot x_2 + {}^t x_1 \cdot x_2/2)$$

$$f(x_1) = \int_{\mathbf{R}^g/\mathbf{Z}^g} f^*\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) e(-{}^t x_1 \cdot x_2/2) dx_2$$

We leave this to the reader to check.

There is a 3rd realization which is very important and which is based, not on a maximal isotropic subgroup of K , but on a maximal isotropic subalgebra of $\text{Lie}(K)_{\mathbf{C}}$. Before explaining this, it is convenient to make explicit a little linear algebra underlying complexifying the vector space V .

PROPOSITION 2.1. Given a real vector space V of dimension $2g$ and a non-degenerate alternating form $A : V \times V \rightarrow \mathbf{R}$, then the following data on V are all equivalent:

- 1) A complex structure J on V such that $A(Jx, Jy) = A(x, y)$, all $x, y \in V$ and $A(Jx, x) > 0$, all $x \in V, x \neq 0$,
- 2) A complex structure J on V and a positive definite Hermitian form H such that $\text{Im } H = A$,
- 3) A g -dimensional complex subspace $P \subset V_{\mathbf{C}}$ such that $A_{\mathbf{C}}(x, y) = 0$, all $x, y \in P$, $iA_{\mathbf{C}}(x, \bar{x}) < 0$, all $x \in P, x \neq 0$.

The set of all these we call the Siegel space \mathfrak{H}_V associated to V (and A). If, furthermore, we choose a basis $e_i^{(1)}, e_i^{(2)}$ of V in which $A(e_i^{(1)}, e_j^{(1)}) = A(e_i^{(2)}, e_j^{(2)}) = 0$, $A(e_i^{(1)}, e_j^{(2)}) = \delta_{ij}$ and let $x_i^{(1)}, x_i^{(2)}$ be corresponding coordinates, then a point of \mathfrak{H}_V is given by a $g \times g$ complex symmetric matrix T , with $\text{Im } T$ positive definite, i.e., a point $T \in \mathfrak{H}_g$, the Siegel upper half space.

The connections between these data are:

a) Given J ,

$$H(x, y) = A(Jx, y) + iA(x, y),$$

$$P = \text{locus of points } ix - Jx.$$

b) Given P , the complex structure J comes from the isomorphism

$$V \hookrightarrow V_{\mathbf{C}} \twoheadrightarrow V_{\mathbf{C}}/P$$

and T is defined by the property: $e_i^{(1)} - \Sigma T_{ij} e_j^{(2)} \in P$.

c) Given T , H is defined by $H(e_i^{(2)}, e_j^{(2)}) = (\text{Im } T)_{ij}^{-1}$, P is the span of $e_i^{(1)} - \Sigma T_{ij} e_j^{(2)}$, and the complex structure J comes from requiring that $\underline{x}_i = \Sigma T_{ij} x_j^{(1)} + x_i^{(2)}$ are complex coordinates.

Finally, the symplectic group $Sp(V, A)$ acts on the space \mathfrak{H}_V by

$$\gamma(J) = \gamma J \gamma^{-1}, \quad \gamma \in Sp(V, A).$$

This action is transitive, and the stabilizer of J is the unitary group $U(V, H)$, hence

$$\mathfrak{H}_V \cong Sp(V, A)/U(V, H).$$

In terms of a basis, the action is given by

$$T \rightarrow (DT - C)(-BT + A)^{-1}, \quad \text{if } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

(The usual formula, except for the automorphism

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}$$

of $Sp(2g, \mathbf{R})$.) (For more details see Tata Lectures on Theta, I.)

We now chose H, J , and T as in the Proposition, so that we have a complex structure on \mathbf{R}^{2g} , and even a definite isomorphism

$$\mathbf{C}^g \cong \mathbf{R}^{2g}$$

with the complex coordinates

$$\underline{x} = Tx_1 + x_2.$$

From Chapter 2, we recall that the formulae

$$\begin{aligned} \vartheta(\underline{x}, T) &= \vartheta(\underline{x} + n_2, T) & n_2 \in \mathbf{Z}^g \\ \vartheta(\underline{x}, T) &= e^{(n_1 \cdot \underline{x} + n_1 \cdot T \cdot n_1/2)} \vartheta(\underline{x} + Tn_1, T) & n_1 \in \mathbf{Z}^g \end{aligned}$$

describe the invariance of $\vartheta(\underline{x}, T)$, and that these extend to actions of \mathbf{R}^{2g} on the space of *holomorphic* functions on \mathbf{C}^g :

$$U_{(1, y_2)} f(\underline{x}) = f(\underline{x} + y_2)$$

$$U_{(1, y_1)} f(\underline{x}) = e^{(y_1 \cdot \underline{x} + y_1 \cdot T \cdot y_1/2)} f(\underline{x} + T y_1).$$

Combining these, we find that we have a representation of the Heisenberg group $Heis(2g, \mathbf{R})$, provided we set $U_\lambda = \lambda^{-1} \cdot Id.$, hence

$$U_{(\lambda, y)} f(\underline{x}) = \lambda^{-1} e^{(y_1 \cdot \underline{x} + y_1 \cdot y/2)} f(\underline{x} + \underline{y}).$$

This called the *Fock representation* of $Heis(2g, \mathbf{R})$. This λ^{-1} looks like an accident but it is actually rather basic as we shall see. We make this into a unitary representation by setting

$$\|f\|^2 = \int_{\mathbf{C}^g} |f(\underline{x})|^2 e^{-2\pi' \underline{x}_1 \cdot Im T \cdot \underline{x}_1} \cdot d\underline{x}_1 d\underline{x}_2.$$

It is easily checked that the $U_{(\lambda, y)}$'s are unitary on the Hilbert space $\mathcal{H}_\phi^2(\mathbf{C}^g, T)$ of holomorphic f 's for which $\|f\| < +\infty$.

We shall show that the Fock representation is irreducible. It then follows from Theorem 1.2 that it is canonically the *dual* of the other two representations, and also, because it is a Hilbert space, hence is conjugate-linear to its dual, there is a $Heis(2g, \mathbf{R})$ -equivariant conjugate-linear isomorphism of the Fock with the other representations. To prove this irreducibility, a modified version of the Fock representation is more convenient: namely one in which the action of W_1 and W_2 are symmetrical. In previous chapters, we have used the trick of modifying the periodicity of ϑ by multiplying by $e^{Q(\underline{x})}$, Q quadratic. We do the same here: define

$\mathcal{H}_\phi^2(\mathbf{C}^g, T)$ = space of holomorphic functions $f(\underline{x})$ on \mathbf{C}^g such that

$$\|f\|^2 = \int_{\mathbf{C}^g} |f(\underline{x})|^2 e^{-\pi H(\underline{x}, \underline{x})} d\underline{x} < +\infty,$$

where $H(\underline{x}, \underline{x}) = {}^t \underline{x} \cdot (Im T)^{-1} \cdot \underline{x}$ as in Prop. 2.1.

Define a unitary isomorphism

$$\mathcal{H}_\phi^2(\mathbf{C}^g, T) \xrightarrow{\sim} \mathcal{H}_\phi^2(\mathbf{C}^g, T)$$

by

$$f(\underline{x}) \longmapsto e^{\frac{\pi}{2} {}^t \underline{x} \cdot Im T^{-1} \cdot \underline{x}} f(\underline{x}).$$

Then it is an elementary, though tedious, calculation that the group action on \mathcal{H}_ϕ takes the form:

$$(U_{(\lambda, y)} f)(\underline{x}) = \lambda^{-1} e^{-\pi H(\underline{x}, \underline{y}) - \frac{\pi}{2} H(\underline{y}, \underline{y})} f(\underline{x} + \underline{y}).$$

The deeper analysis of real Heisenberg groups depends on the fact that they are Lie groups, hence have Lie algebras. We use the Lie algebra structure to construct for each choice of $T \in \mathfrak{H}$ a unique element f_T of \mathcal{H}_ϕ . This element is used to express ϑ as a matrix coefficient. The uniqueness of f_T will imply that \mathcal{H}_ϕ is irreducible. In general, when a Lie group G acts on a Hilbert space \mathcal{H} , then for all $X \in Lie G$, we let $\exp(tX) \in G$ be the 1-parameter group X generates, and consider for all $x \in \mathcal{H}$:

$$\delta U_X(x) = \lim_{t \rightarrow 0} \frac{(U_{\exp(tX)}x) - x}{t}.$$

This limit will exist for a dense set S of x 's in \mathcal{H} and $\delta U_X : S \rightarrow \mathcal{H}$ will be a skew-adjoint, but not bounded operator. One defines $\mathcal{H}_\infty \subset \mathcal{H}$ to be the set of all $x \in \mathcal{H}$ such that

$$\delta U_{X_1} \circ \cdots \circ \delta U_{X_n}(x)$$

is defined for all $X_1, \dots, X_n \in Lie G$. It is a theorem that \mathcal{H}_∞ is dense in \mathcal{H} . We do not want to discuss the general theory here, but only want to illustrate what it says in our example. Here $Lie G$ has a basis:

$$A_1, \dots, A_g, B_1, \dots, B_g, C$$

such that

$$\exp(\sum x_i^{(1)} A_i) = (1, \begin{pmatrix} x^{(1)} \\ 0 \end{pmatrix})$$

$$\exp(\sum x_i^{(2)} B_i) = (1, \begin{pmatrix} 0 \\ x^{(2)} \end{pmatrix})$$

$$\exp(t C) = (e^{2\pi i t}, 0).$$

Then

$$[A_i, A_j] = [B_i, B_j] = [C, A_i] = [C, B_i] = 0$$

$$[A_i, B_j] = \delta_{ij} C.$$

In realization I:

$$\delta U_{A_i}(f)(x) = \lim_{t \rightarrow 0} \frac{f(x + t e_i) - f(x)}{t} = \frac{\partial f}{\partial x_i}(x)$$

$$\delta U_{B_i}(f)(x) = \lim_{t \rightarrow 0} \frac{e^{2\pi i t x_i} f(x) - f(x)}{t} = 2\pi i x_i \cdot f(x)$$

$$\delta U_C(f)(x) = \lim_{t \rightarrow 0} \frac{e^{2\pi i t} f(x) - f(x)}{t} = 2\pi i f(x).$$

δU_{A_i} are called the momentum operators, δU_{B_i} the position operators. \mathcal{H}_∞ is the set of functions with " L^2 -derivatives $\frac{\partial^\alpha f}{\partial x^\alpha}$ " of all orders with $\|x^\beta \cdot \frac{\partial^\alpha f}{\partial x^\alpha}\| < +\infty$. By Sobolev's lemma, \mathcal{H}_∞ is the set of C^∞ -functions $f(x)$ with

$$\left| \frac{\partial^\alpha f}{\partial x^\alpha} \right| \in O(\|x\|^{-N}), \quad \text{all } N, \alpha,$$

also known as the Schwartz space $\mathcal{S}(\mathbf{R}^g)$. We do not work out δU for realization II, but note that, by our formula for the isomorphism between these, the space \mathcal{H}_∞ will be the space of C^∞ quasi-periodic functions. In the Fock representation, in its symmetrical form, abbreviating $Im T^{-1}$ by S , we find:

$$\begin{aligned} \delta U_{A_i} f(\underline{x}) &= \lim_{t \rightarrow 0} \frac{1}{t} [e^{-\pi' \underline{x} S \cdot \overline{T}(te_i) - (\pi/2)'(te_i) T S \overline{T}(te_i)} f(\underline{x} + t T e_i) - f(\underline{x})] \\ &= -\pi \sum_{j,k} \overline{T}_{ij} \cdot S_{jk} \underline{x}_k \cdot f(x) + \sum_j T_{ij} \frac{\partial f}{\partial \underline{x}_j} \\ \delta U_{B_i} f(\underline{x}) &= \lim_{t \rightarrow 0} \frac{1}{t} [e^{-\pi' \underline{x} S \cdot te_i - \pi/2'(te_i) S \cdot (te_i)} f(\underline{x} + t e_i) - f(\underline{x})] \\ &= -\pi \sum_k S_{ik} \underline{x}_k \cdot f(x) + \frac{\partial f}{\partial \underline{x}_i}. \end{aligned}$$

The complex span of these operators is the same as that of the operators:

$$f \mapsto \partial f / \partial \underline{x}_i, \quad \text{called the annihilation operators}$$

and

$$f \mapsto \underline{x}_i f, \quad \text{called the creation operators.}$$

In fact:

$$W_T = \{\text{Span of } \delta U_{A_i} - \sum_j T_{ij} \cdot \delta U_{B_j} \text{ all } i\} = \{\text{Span of } f \mapsto \underline{x}_i f\}$$

$$W_{\overline{T}} = \{\text{Span of } \delta U_{A_i} - \sum_j \overline{T}_{ij} \cdot \delta U_{B_j} \text{ all } i\} = \{\text{Span of } f \mapsto \frac{\partial f}{\partial \underline{x}_i}\}$$

and these are conjugate abelian complex subalgebras of $Lie(G) \otimes \mathbb{C}$. These subalgebras have a remarkable property:

THEOREM 2.2. Fix $T \in \mathfrak{H}_g$.

- a) In the Heisenberg representation \mathcal{H} of $Heis(2g, \mathbf{R})$, there is a element f_T , unique up to a scalar, such that $\delta U_X(f_T)$ is defined and equal to 0, all $X \in W_T$.
- b) In $L^2(\mathbf{R}^g)$, $f_T = e^{\pi i' x \cdot T x}$.
- c) In $L^2(\mathbf{R}^{2g} // \mathbf{Z}^{2g})$, $f_T = e^{\pi i' x_1 \cdot \underline{x} \vartheta(\underline{x}, T)}$.
- d) In $\mathcal{H}_\phi^2(\mathbf{C}^g, T)$, there is a unique f_T killed by $W_{\overline{T}}$ and it is $f_T = 1$, the "vacuum state". Hence \mathcal{H}_ϕ^2 is irreducible and in the conjugate linear isomorphism with $L^2(\mathbf{R}^g)$, 1 corresponds to $e^{\pi i' x T x}$.

PROOF: In \mathcal{H}_1

$$\begin{aligned} \delta U_{A_i} f &= \frac{\partial f}{\partial x_i}, \\ \delta U_{B_i} f &= 2\pi i x_i f. \end{aligned}$$

So a function f annihilated by W_T is one satisfying

$$\frac{\partial f}{\partial x_i} = 2\pi i \left(\sum_{j=1}^g T_{ij} x_j \right) \cdot f.$$

The only solution to these equations is $f(x) = e^{\pi i' x \cdot T x}$. Going over to $L^2(\mathbf{R}^{2g} // \mathbf{Z}^{2g})$ by the formula given above, the function becomes

$$\begin{aligned} f^* \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \sum_n e^{\pi i' (x_1 + n) T (x_1 + n) + 2\pi i' n \cdot x_2 + \pi i' x_1 \cdot x_2} \\ &= e^{\pi i' (x_1 T x_1 + x_1 \cdot x_2)} \sum_n e^{\pi i' n T n + 2\pi i' n (T x_1 + x_2)} \\ &= e^{\pi i' x_1 \cdot \underline{x} \vartheta(\underline{x}, T)}. \end{aligned}$$

Finally, in \mathcal{H}_ϕ^2 , f is killed by $W_{\overline{T}}$ if and only if $\partial f / \partial \underline{x}_i = 0$, all i , i.e., f is a constant. Q.E.D.

This theorem says that in the big Hilbert space \mathcal{H}_U we have a canonical way of singling out a finite-dimensional submanifold of the "most elementary vectors", i.e., the f_T 's. And in the quasi-periodic representation, we get exactly the theta functions, multiplied by a simple exponential factor

which puts their periodicity in the simplest form. The Fock representation suggests a whole filtration of \mathcal{H} defined by

$$\mathcal{H}_\phi(\mathbb{C}^g, T) \supset V_n = (\text{vector space of polynomials in } \underline{x} \text{ of degree } \leq n).$$

Then

$$V_n = \text{Span of } \{(\delta U_{x_1} \circ \cdots \delta U_{x_n})f_T \mid x_i \in \text{Lie}(\text{Heis}(2g, \mathbb{R}))\}.$$

In physics, V_n is called the space where there are $\leq n$ particles present.

In order to complete our construction of theta as a matrix coefficient we now single out a second element $e_{\mathbf{Z}}$. Following the idea of distributions, we can enlarge \mathcal{H} canonically by defining

$$\mathcal{H}_{-\infty} = \{\text{Space of conjugate linear continuous maps } \ell : \mathcal{H}_{\infty} \rightarrow \mathbb{C}\}$$

where

$$\mathcal{H} \subset \mathcal{H}_{-\infty}$$

by mapping $x \in \mathcal{H}$ to $\ell_x : \mathcal{H}_{\infty} \rightarrow \mathbb{C}$,

$$\ell_x(y) = \langle x, y \rangle.$$

(The inner product $\langle x, y \rangle$ is taken to be linear in x , conjugate-linear in y . Here continuous means that there is a finite set of conditions $\|\delta U_{x_1} \circ \cdots \circ \delta U_{x_n} x\| < \delta$ on $x \in \mathcal{H}_{\infty}$ implying $|\ell(x)| < 1$.) In our case, it can be shown that:

In realization I, $\mathcal{H}_{-\infty} = \text{Space of tempered distributions on } \mathbb{R}^g$.

In realization II, $\mathcal{H}_{-\infty} = \text{Space of all distributions } f \text{ on } \mathbb{R}^{2g} \text{ such that}$

$$f(x+n) = e_{\star}(\frac{n}{2})e(x, \frac{n}{2})f(x) \quad \text{all } x \in \mathbb{Z}^{2g}.$$

In the Fock representation \mathcal{H}_{ϕ}^2 :

$\mathcal{H}_{\infty} = \text{Space of holomorphic functions } f(\underline{x}) \text{ such that}$

$$f(\underline{x}) = O(e^{\frac{\gamma}{2}H(\underline{x}, \underline{x})}/\|\underline{x}\|^n), \quad \text{all } n$$

$\mathcal{H}_{-\infty} = \text{Space of holomorphic functions } f(\underline{x}) \text{ such that}$

$$f(\underline{x}) = O(\|\underline{x}\|^n \cdot e^{\frac{\gamma}{2}H(\underline{x}, \underline{x})}), \quad \text{some } n.$$

We will not use these facts though. Our reason for introducing $\mathcal{H}_{-\infty}$ is to prove the following theorem.

THEOREM 2.3. *Let $\sigma(\mathbb{Z}^{2g}) \subset \text{Heis}(2g, \mathbb{Z})$ be the set of elements:*

$$\{(e_{\star}(\frac{n}{2}), n),$$

$n \in \mathbb{Z}^{2g}\}$.

a) *In the distribution completion $\mathcal{H}_{-\infty}$ of \mathcal{H} , there is a element $e_{\mathbf{Z}}$, unique up to scalars, which is invariant under U_g for all $g \in \sigma(\mathbb{Z}^{2g})$.*

b) *In $L^2(\mathbb{R}^g)$, $e_{\mathbf{Z}} = \sum_{n \in \mathbb{Z}^g} \delta_n$, (δ_a = the delta function at a).*

c) *In $L^2(\mathbb{R}^{2g} // \mathbb{Z}^{2g})$, $e_{\mathbf{Z}} = \sum_{n \in \mathbb{Z}^{2g}} e_{\star}(\frac{n}{2})\delta_n$.*

d) *In $\mathcal{H}_{\phi}^2(\mathbb{C}^g, T)$, $e_{\mathbf{Z}} = \vartheta(\underline{x}, T)$.*

PROOF: Consider an arbitrary (no continuity assumed) conjugate-linear map

$$\ell : \mathcal{S}(\mathbb{R}^{2g} // \mathbb{Z}^{2g}) = \left\{ \begin{array}{l} C^{\infty} - \text{functions } f \text{ on } \mathbb{R}^{2g} \text{ s.t.} \\ f(x+n) = e_{\star}(\frac{n}{2})e(x, \frac{n}{2})f(x) \end{array} \right\} \longrightarrow \mathbb{C}$$

such that $\ell(U_{(e_{\star}(\frac{n}{2}), m)}f) = \ell f$, all f . I claim

$$\ell(f) = \alpha \cdot \overline{f(0)}, \quad \text{some } \alpha.$$

In fact,

$$\begin{aligned} U_{(e_{\star}(\frac{m}{2}), m)}f(x) &= e_{\star}(\frac{m}{2})e(\frac{x}{2}, m)f(x+m) \\ &= e(x, m)f(x) \end{aligned}$$

so

$$\ell([1 - e(x, m)]f(x)) = 0, \quad \text{all } m \in \mathbb{Z}^{2g}.$$

Let $\epsilon_1, \dots, \epsilon_{2g}$ be the unit vectors in \mathbb{Z}^{2g} and for some δ cover \mathbb{R}^{2g} by

$$U_0 = B_{\delta} + \mathbb{Z}^{2g}, \quad B_{\delta} = \{x \mid \|x\| < \delta\}$$

$$U_i = \{x \mid e(x, \epsilon_i) \neq 1\}, \quad 1 \leq i \leq 2g.$$

Let $\{\rho_i\}$ be a partition of unity by C^∞ functions on $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ such that $\text{supp}(\rho_i) \subset \text{image } U_i$. Take any $f \in \mathcal{S}(\mathbb{R}^{2g}/\mathbb{Z}^{2g})$ such that $f(0) = 0$. If δ is small enough, then by Taylor's theorem, we can write

$$f = \sum x_i f_i \text{ in } B_\delta.$$

Then $\rho_0 f = \sum g_i$, where $\text{supp } g_i \subset B_\delta$, and $x_i = 0 \implies g_i(x) = 0$. Extend g_i to a quasi-periodic function in $\mathcal{S}(\mathbb{R}^{2g}/\mathbb{Z}^{2g})$. Then

$$\begin{aligned} f &= \rho_0 f + \sum_{i=1}^{2g} \rho_i f \\ &= \sum_{i=1}^{2g} g_i + \sum_{i=1}^{2g} (1 - e(x, \epsilon_i)) \frac{\rho_i f}{1 - e(x, \epsilon_i)} \\ &= \sum (1 - e(x, \epsilon_i)) \cdot \left[\frac{g_i + \rho_i f}{1 - e(x, \epsilon_i)} \right]. \end{aligned}$$

Then $h_i = \frac{g_i + \rho_i f}{1 - e(x, \epsilon_i)} \in \mathcal{S}(\mathbb{R}^{2g}/\mathbb{Z}^{2g})$, so

$$\ell(f) = \sum \ell((1 - e(x, \epsilon_i)) h_i) = 0.$$

Thus ℓ is of the form $g \mapsto \overline{\alpha g(0)}$. This particular linear functional is continuous by Sobolev's lemma again. We may write $g \mapsto \overline{g(0)}$ as an inner product between quasi-periodic functions:

$$g(0) = \int_{\mathbb{R}^{2g}/\mathbb{Z}^{2g}} \overline{g(x)} \cdot \sum_{n \in \mathbb{Z}^{2g}} e_*(\frac{n}{2}) \delta_n(x) dx$$

and this proves (c) and (a) together. (b) follows from the transformation formula between Realizations I and II: In fact, for all $f(x_1) \in L^2(\mathbb{R}^g)$, f corresponds to

$$f^*(x_1, x_2) = \sum_{n \in \mathbb{Z}^g} f(x_1 + n) e^{2\pi i n x_2 + \pi i x_1 \cdot x_2}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^{2g}/\mathbb{Z}^{2g}} \overline{f^*(x_1, x_2)} \left\{ \sum_{m_1, m_2 \in \mathbb{Z}^g} e^{\pi i^t m_1 \cdot m_2} \cdot \delta_{m_1, m_2}(x) \right\} dx \\ &= \int_F \overline{f^*(x_1, x_2)} \cdot e^{\pi i^t m_1 \cdot m_2} \delta_{m_1, m_2}(x) dx, \end{aligned}$$

(if $F \subset \mathbb{R}^{2g}$ is a fundamental domain containing the lattice point m_1, m_2)

$$\begin{aligned} &= \overline{f^*(m_1, m_2)} \cdot e^{\pi i^t m_1 \cdot m_2} \\ &= \sum_{n \in \mathbb{Z}^g} \overline{f(m_1 + n)} \cdot e^{-2\pi i^t n \cdot m_2 - \pi i^t m_1 \cdot m_2} \cdot e^{\pi i^t m_1 \cdot m_2} \\ &= \sum_{n \in \mathbb{Z}^g} \overline{f(n)} \\ &= \int_{\mathbb{R}^g} \overline{f(x)} \cdot \left(\sum_{n \in \mathbb{Z}^g} \delta_n(x) \right) dx. \end{aligned}$$

As for (d), the transformation formula for ϑ says that ϑ is invariant under the required U_g 's. Q.E.D.

With Theorems 2.2 and 2.3, everything can be fitted together very elegantly. First, the essential relation between the theta function and the Heisenberg representation is given by:

COROLLARY 2.4. Fix $T \in \mathfrak{H}_g$, let \mathcal{H} be the Heisenberg representation of $\text{Heis}(2g, \mathbb{R})$, let $f_T \in \mathcal{H}_\infty$ be killed by W_T , and let $e_{\mathbf{z}} \in \mathcal{H}_{-\infty}$ be fixed by $\sigma(\mathbb{Z}^{2g})$. Then if $\underline{x} = T x_1 + x_2$, we have:

$$\langle U_{(1, \underline{x})} f_T, e_{\mathbf{z}} \rangle = c \cdot e^{\pi i^t x_1 \cdot \underline{x} \vartheta(\underline{x}, T)}$$

for some $c \in \mathbb{C}^*$.

PROOF: We compute in $L^2(\mathbb{R}^{2g}/\mathbb{Z}^{2g})$. Then $e_{\mathbf{z}} = c_1 \sum_{n \in \mathbb{Z}^{2g}} e_*(\frac{n}{2}) \delta_n$,

$f_T = c_2 e^{\pi i^t x_1 \cdot \underline{x} \vartheta(\underline{x}, T)}$, so

$$\begin{aligned} \langle U_{(1, \underline{x})} f_T, e_{\mathbf{z}} \rangle &= \bar{c}_1 U_{(1, \underline{x})} f_T(0) \\ &= \bar{c}_1 f_T(x) \\ &= \bar{c}_1 c_2 e^{\pi i^t x_1 \cdot \underline{x} \vartheta(\underline{x}, T)}. \end{aligned}$$

Q.E.D.

Recall that in Ch. 2, §5 we defined

$$\vartheta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (T) = e^{\pi i^t x_1 \cdot (Tx_1 + x_2)} \cdot \vartheta(Tx_1 + x_2, T).$$

In terms of theta-functions with characteristic,

$$\vartheta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (T) = e^{-\pi i^t x_1 x_2} \vartheta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (0; T).$$

We will see that in many ways ϑ^α is the most natural and most important variant of ϑ . In particular, Cor. 2.4 says that ϑ^α is just a matrix coefficient of the representation \mathcal{H} ! Moreover, by looking at matrix coefficients, we construct the quasi-periodic and Fock realizations of \mathcal{H} directly, i.e.,

COROLLARY 2.5.

$$\left\{ \begin{array}{l} \text{The space of functions on } \mathbf{R}^{2g} \\ x \mapsto \phi_f(x) = \langle U_{(1,x)} f, e_{\mathbf{Z}} \rangle, \\ \text{some } f \in \mathcal{H}_\infty \end{array} \right\} = \left\{ \begin{array}{l} \text{The space of } C^\infty \text{ - functions } g(x), x \in \mathbf{R}^{2g}, \\ \text{such that} \\ g(x+n) = e_*(\frac{n}{2}) e(x, \frac{n}{2}) g(x), \text{ all } n \in \mathbf{Z}^{2g} \end{array} \right\}.$$

Hence the space on the right is contained in $L^2(\mathbf{R}^{2g} // \mathbf{Z}^{2g})$. Let $L^2(\mathbf{R}^{2g} // \mathbf{Z}^{2g})$ be acted on as in Realization II; then the map

$$\mathcal{H}_\infty \ni f \mapsto \phi_f \in L^2(\mathbf{R}^{2g} // \mathbf{Z}^{2g})$$

is equivariant:

$$\phi_{U_g(f)} = U_g(\phi_f).$$

PROOF: Calculating the space on the left in the realization $L^2(\mathbf{R}^{2g} // \mathbf{Z}^{2g})$, we see that

$$\langle U_{(1,x)} f, e_{\mathbf{Z}} \rangle = f(x)$$

so the space on the left is $L^2(\mathbf{R}^{2g} // \mathbf{Z}^{2g})_\infty$, i.e., the C^∞ functions in $L^2(\mathbf{R}^{2g} // \mathbf{Z}^{2g})$. Now we check the assertion concerning the group actions:

$$\begin{aligned} \phi_{U_{(\lambda,y)} f}(x) &= \langle U_{(1,x)} U_{(\lambda,y)} f, e_{\mathbf{Z}} \rangle \\ &= \langle U_{(\lambda e(\frac{x}{2}, y), x+y)} f, e_{\mathbf{Z}} \rangle \\ &= \lambda \cdot e(\frac{x}{2}, y) \phi_f(x+y) \\ &= U_{(\lambda,y)} \phi_f(x). \end{aligned}$$

QED

Likewise the Fock representation¹ is just the action of $Heis$ on a space of matrix coefficients.

COROLLARY 2.6.

$$\left\{ \begin{array}{l} \text{The space of functions on } \mathbf{C}^g \\ x \mapsto \psi_f(x) = \langle U_{(1,x)} f_T, f \rangle \\ \text{some } f \in \mathcal{H}_{-\infty} \end{array} \right\} = \left\{ \begin{array}{l} \text{The space of holomorphic} \\ \text{functions } g(\underline{x}), \underline{x} \in \mathbf{C}^g \text{ such that} \\ g(\underline{x}) \in O(\|\underline{x}\|^n e^{\frac{n}{2} H(\underline{x}, \underline{x})}), \text{ some } n \end{array} \right\} \cdot e^{-\frac{n}{2} H(\underline{x}, \underline{x})}.$$

The action of $Heis(2g, \mathbf{R})$ on the space on the right given in the Fock representation is just

$$U_{(\lambda, -y)} [e^{\frac{n}{2} H} \psi_f] = e^{\frac{n}{2} H} \cdot \psi_{U_{(\lambda,y)} f}.$$

PROOF: We calculate the space on the left in the Fock representation \mathcal{H}_ϕ^2 to which \mathcal{H} is conjugate linear isomorphic. Here f_T corresponds to the function 1, so if f corresponds to a holomorphic function $\tilde{f}(\underline{x})$

¹We state this assuming the identification of $\mathcal{H}_{-\infty}$ in the Fock representation asserted above. If this is not assumed, one can limit the corollary to the subspaces where $f \in \mathcal{H}$ on the left and $g \in \mathcal{H}_\phi^2$ on the right.

$$\begin{aligned}
\psi_f(x) &= \langle U_{(1,x)} f_T, f \rangle = \overline{\langle U_{(1,x)} f_T, \tilde{f} \rangle_{H^2}} \\
&= \int \tilde{f}(\underline{y}) \cdot \overline{U_{(1,x)} f_T(\underline{y})} e^{-\pi H(\underline{y}, \underline{y})} d\underline{y} \\
&= \int \tilde{f}(\underline{y}) e^{-\pi H(\underline{y}, \underline{x}) - \frac{\pi}{2} H(\underline{x}, \underline{x})} e^{-\pi H(\underline{y}, \underline{y})} d\underline{y} \\
&= \int \tilde{f}(\underline{y}) e^{-\pi H(\underline{x}, \underline{y}) - \frac{\pi}{2} H(\underline{x}, \underline{x}) - \pi H(\underline{y}, \underline{y})} d\underline{y} \\
&= e^{-\frac{\pi}{2} H(\underline{x}, \underline{x})} \int \tilde{f}(\underline{y}' - \underline{x}) e^{\pi H(\underline{y}', \underline{x})} \cdot e^{-\pi H(\underline{y}', \underline{y}')} d\underline{y}'
\end{aligned}$$

if $y' = y + x$. But by the Mean Value Theorem, there is a constant c such that

$$\int_{\mathbf{C}^g} g(\underline{z}) e^{-\pi H(\underline{z}, \underline{z})} d\underline{z} = c \cdot g(0)$$

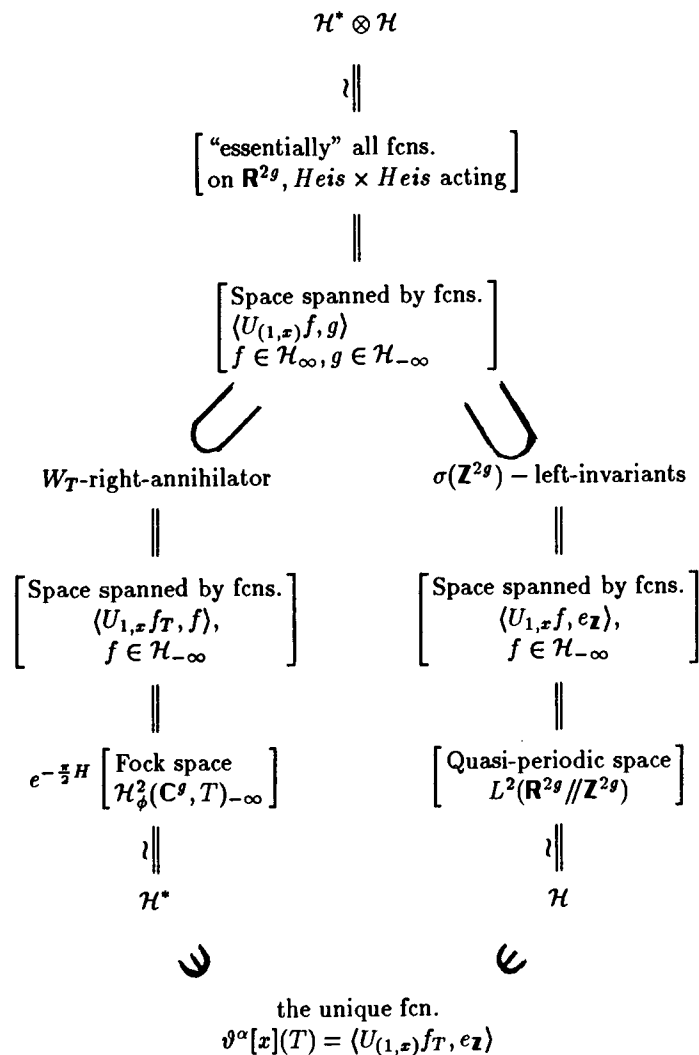
for all holomorphic functions g . Therefore

$$\psi_f(x) = c \cdot e^{-\frac{\pi}{2} H(\underline{x}, \underline{x})} \cdot \tilde{f}(-\underline{x}).$$

We calculate easily:

$$\begin{aligned}
e^{\frac{\pi}{2} H(\underline{x}, \underline{x})} \psi_{U_{(\lambda, y)} f}(x) &= e^{\frac{\pi}{2} H(\underline{x}, \underline{x})} \langle U_{(1,x)} f_T, U_{(\lambda, y)} f \rangle \\
&= e^{\frac{\pi}{2} H(\underline{x}, \underline{x})} \langle U_{(\lambda, y)}^{-1} U_{(1,x)} f_T, f \rangle \\
&= e^{\frac{\pi}{2} H(\underline{x}, \underline{x})} \cdot \lambda^{-1} e(\frac{x}{2}, y) \langle U_{(1, x-y)} f_T, f \rangle \\
&= e^{\frac{\pi}{2} H(\underline{x}, \underline{x})} \cdot \lambda^{-1} \cdot e^{\pi i \operatorname{Im} H(\underline{x}, \underline{y})} \cdot e^{-\frac{\pi}{2} H(\underline{x} - \underline{y}, \underline{x} - \underline{y})} \tilde{f}(\underline{x} - \underline{y}) \\
&= \lambda^{-1} e^{\pi H(\underline{x}, \underline{y}) - \frac{\pi}{2} H(\underline{y}, \underline{y})} \tilde{f}(\underline{x} - \underline{y}) \\
&= U_{(\lambda, -y)} \tilde{f}(x) \\
&= U_{(\lambda, -y)} (e^{\frac{\pi}{2} H} \cdot \psi_f)(x). \quad \text{QED}
\end{aligned}$$

We may summarize the situation in the important diagram:



To explain this, we define the left and right actions of $Heis(2g, \mathbf{R})$ on the space of functions on \mathbf{R}^{2g} by

$$U_{(\lambda, y)}^{\text{left}} f(x) = \lambda^{-1} e(\frac{x}{2}, y) f(x - y)$$

$$U_{(\lambda, y)}^{\text{right}} f(x) = \lambda e(\frac{x}{2}, y) f(x + y).$$

We urge the reader to check that these are commuting actions of $Heis(2g, \mathbf{R})$ and that

$$U_{(\lambda, y)}^{\text{left}} [(U_{(1, x)} f, g)] = (U_{(1, x)} f, U_{(\lambda, y)} g)$$

$$U_{(\lambda, y)}^{\text{right}} [(U_{(1, x)} f, g)] = (U_{(1, x)} \cdot U_{(\lambda, y)} f, g).$$

But as we saw in Proposition 2.3, $L^2(\mathbf{R}^{2g})$ is the Heisenberg representation of $Heis \times Heis / \{(\lambda, \lambda) | \lambda \in \mathbf{C}_1^*\}$, and is isomorphic to $\mathcal{H}^* \otimes \mathcal{H}$. Thus, if we replaced the space spanned by the matrix coefficients

$$(U_{(1, x)} f, g), f \in \mathcal{H}_\infty, g \in \mathcal{H}_{-\infty},$$

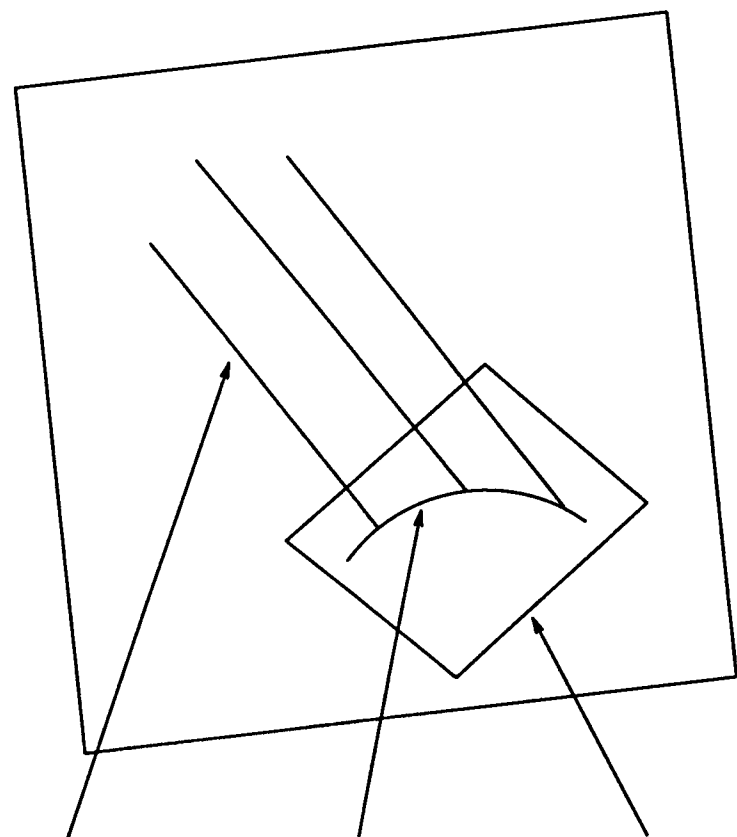
by the space of L^2 -convergent combinations

$$\sum a_{ij} (U_{(1, x)} f_i, f_j), \quad \sum |a_{ij}|^2 < \infty, \quad f_i \in \mathcal{H} \text{ an orthonormal basis,}$$

we would get $L^2(\mathbf{R}^{2g})$. As it stands, the space in the upper right is essentially all functions on \mathbf{R}^{2g} , in a function space that we do not make precise.

What the above diagram does, in essence, is to give a representation theoretic proof that ϑ^α is the only function on \mathbf{R}^{2g} which is a) quasi-periodic for \mathbf{Z}^{2g} , and b) multiplied by a simple factor, holomorphic.

We can describe the situation by a picture:



The Fock spaces
 $\mathcal{H}_\phi^2(\mathbf{C}^g, T)_{-\infty}$
for various T

Mysterious subset
of the functions
 $\vartheta^\alpha[x](T)$
for some T

$L^2(\mathbf{R}^{2g} // \mathbf{Z}^{2g})$

3. Finite Heisenberg groups and sections of line bundles on abelian varieties

We have seen in §2 that the theta function occurs as a matrix coefficient of the real Heisenberg representation. In Ch.2, §1, we used a finite set of translates of the theta function to embed various complex tori $X_{T,L}$ in projective space. We now want to relate these two aspects of the theory of theta functions and show how a finite version of the Heisenberg representation theorem occurs naturally in each of these projective embeddings.

We first recall some fundamental definitions. A complex line bundle on a complex analytic space consists of

- (A) a complex analytic space \mathbf{L} and a morphism $p : \mathbf{L} \rightarrow X$ such that, for each point $x \in X$, $p^{-1}(x)$ is endowed with the structure of a one-dimensional complex vector space, and
- (B) each point of X has a neighbourhood U and an isomorphism $\varphi_U : \mathbf{C} \times U \rightarrow p^{-1}(U)$ of analytic spaces such that $p(\varphi_U(z, x)) = x$ and for each $x \in U$, the map from \mathbf{C} to $p^{-1}(x)$ given by $z \mapsto \varphi_U(z, x)$ is a linear transformation.

The trivial line bundle on X is simply $\mathbf{L} = \mathbf{C} \times X$, and $p : \mathbf{L} \rightarrow X$ is the projection: $p(z, x) = x$ for $z \in \mathbf{C}, x \in X$.

A section of a line bundle $p : \mathbf{L} \rightarrow X$ is a holomorphic map $s : X \rightarrow \mathbf{L}$ such that $ps(x) = x$ for all $x \in X$. The collection of all sections of \mathbf{L} forms a vector space denoted by $\Gamma(X, \mathbf{L})$.

Note that a section of the trivial line bundle is necessarily of the type $s(x) = (h(x), x)$ where h is a holomorphic function on X . Thus, in this case, $\Gamma(X, \mathbf{L})$ is just the vector space of all holomorphic functions on X .

An automorphism of a line bundle \mathbf{L} on X is a pair (ϕ, ψ) where ϕ and

ψ are analytic isomorphisms of X and \mathbf{L} respectively such that the diagram

$$\begin{array}{ccc} \mathbf{L} & \xrightarrow{\psi} & \mathbf{L} \\ p \downarrow & & \downarrow p \\ X & \xrightarrow{\phi} & X \end{array}$$

commutes, and in addition, the restriction of ψ from $p^{-1}(x)$ to $p^{-1}(\phi(x))$ is complex linear.

The collection of automorphisms of \mathbf{L} , to be denoted by $\text{Aut } \mathbf{L}$, forms a group under the obvious composition law $(\phi_1, \psi_1) \circ (\phi_2, \psi_2) = (\phi_1 \circ \phi_2, \psi_1 \circ \psi_2)$. The group $\text{Aut } \mathbf{L}$ acts on the vector space $\Gamma(X, \mathbf{L})$ by $U_{(\phi, \psi)}s = \psi \circ s \circ \phi^{-1}$.

We shall now define an action of $\text{Heis}(2g, \mathbf{R})$ on the trivial line bundle on \mathbf{C}^g (equivalently, a homomorphism from $\text{Heis}(2g, \mathbf{R})$ to the group of automorphisms of the trivial line bundle on \mathbf{C}^g) such that the corresponding action of $\text{Heis}(2g, \mathbf{R})$ on the sections, that is, on entire functions on \mathbf{C}^g , is identical to the Fock representation $\mathcal{H}_g^2(\mathbf{C}^g, T)$ if one ignores the square-integrability hypothesis. Fix $T \in \mathfrak{H}_g$ and hence fix an isomorphism $\mathbf{R}^{2g} \xrightarrow{\sim} \mathbf{C}^g$ via $\underline{y} = Ty_1 + y_2$ as in §2. For any $h = (\lambda, y) \in \text{Heis}(2g, \mathbf{R})$ where $y = (y_1, y_2)$ put

$$\phi_h(z) = z - \underline{y} \quad \text{for all } z \in \mathbf{C}^g, \text{ and}$$

$$\psi_h(\alpha, z) = (\alpha \lambda^{-1} \exp \pi i^t y_1 (2z - \underline{y}), z - \underline{y}), \quad \text{for all } \alpha \in \mathbf{C}, z \in \mathbf{C}^g.$$

Then $h \mapsto (\phi_h, \psi_h)$ is the action we are looking for.

If $s(z) = (f(z), z)$ is a section of the trivial line bundle on \mathbf{C}^g , then

$$\begin{aligned} \psi_h s \phi_h^{-1}(z) &= \psi_h s(z + \underline{y}) \\ &= \psi_h(f(z + \underline{y}), z + \underline{y}) \\ &= (f(z + \underline{y}) \lambda^{-1} \exp \pi i^t y_1 (2z + 2\underline{y} - \underline{y}), z) \\ &= (f(z + \underline{y}) \lambda^{-1} \exp \pi i^t y_1 (2z + \underline{y}), z), \end{aligned}$$

which is the Fock representation $\mathcal{H}_g^2(\mathbf{C}^g, T)$.

Now for any lattice $L \subset \mathbb{Z}^{2g}$, we defined, in §1, a complex torus $X_{T,L} = \mathbb{C}^g/L$. We can now define a basic line bundle \mathbf{L} on each of these complex tori. Recall that there is a section $\sigma : \mathbb{Z}^{2g} \rightarrow \text{Heis}(2g, \mathbb{R})$ defined by

$$\sigma(n) = (e_*(n/2), n)$$

where $e_*(n/2) = e(\frac{1}{2}(n_1 \cdot n_2))$ if $n = (n_1, n_2)$. Via σ , \mathbb{Z}^{2g} acts on the trivial line bundle on \mathbb{C}^g : $n \in \mathbb{Z}^{2g}$ acts by the pair $(\phi_{\sigma(n)}, \psi_{\sigma(n)})$, and of course, $\phi_{\sigma(n)}$ is simply translation by $-\underline{n}$. The action of \mathbb{Z}^{2g} , and therefore of any sub-lattice L , on $\mathbb{C} \times \mathbb{C}^g$ and \mathbb{C}^g is free and discontinuous. Thus the quotient of $\mathbb{C} \times \mathbb{C}^g$ by L is a complex manifold and we get a commutative diagram

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}^g & \longrightarrow & \mathbf{L} = \mathbb{C} \times \mathbb{C}^g/L \\ \downarrow & & \downarrow \\ \mathbb{C}^g & \longrightarrow & X_{T,L} = \mathbb{C}^g/L \end{array}$$

where the horizontal arrows are universal covering maps and the vertical maps give line bundles on \mathbb{C}^g and $X_{T,L}$.

DEFINITION. The basic line bundle on $X_{T,L}$ is the line bundle $\mathbf{L} = \mathbb{C} \times \mathbb{C}^g/\sigma L$ constructed above.

DEFINITION. For a line bundle \mathbf{L} on a complex torus X ,

$$\mathcal{G}(\mathbf{L}) = \{(\phi, \psi) \in \text{Aut } \mathbf{L} \mid \phi \text{ is a translation}\},$$

$$K(\mathbf{L}) = \{a \in X \mid \text{if } \phi(x) = x + a, \text{ then there exists a pair } (\phi, \psi) \in \mathcal{G}(\mathbf{L})\}.$$

Thus $\mathcal{G}(\mathbf{L})$ comes equipped with the exact sequence:

$$1 \longrightarrow \mathbf{C}^* \longrightarrow \mathcal{G}(\mathbf{L}) \xrightarrow{\pi} K(\mathbf{L}) \longrightarrow 0.$$

In fact the kernel of π is identified with the group of all holomorphic nowhere zero functions defined on all of X , which by compactness of X and the maximum modulus principle is just \mathbf{C}^* .

Let $N(\sigma L) = \{\text{normalizer of } \sigma L \in \text{Heis}(2g, \mathbb{R})\}$. There is an action of $N(\sigma L)$ on \mathbf{L} : Let $\psi \in N(\sigma L)$. Under the action of $\sigma(L)$, x is identified with $\sigma(\ell)(x)$ for all $\ell \in L$. Is $\psi_n(x)$ identified with $\psi_n(\sigma(\ell)(x))$? Yes, since $\psi_n \sigma(\ell)(x) = \sigma(m) \psi_n(x)$ for some $m \in L$ by normality. This shows that ψ_n descends to a map $\bar{\psi}_n : \mathbf{L} \rightarrow \mathbf{L}$. Also, if $n \in L$, the induced action of $\sigma(n)$ on \mathbf{L} is clearly the identity by the very definition of \mathbf{L} . Let $\pi : \text{Heis}(2g, \mathbb{R}) \rightarrow \mathbb{R}^{2g}$ be the usual projection. Since $\pi^{-1}(L) \cap \sigma(L)$ has exactly one element, any element which under conjugation normalizes $\sigma(L)$ actually centralizes it. Thus if $x \in N(\sigma(L))$, then $\pi(x) \in L^\perp$. This gives a homomorphism between exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{C}_1^* & \longrightarrow & N(\sigma L)/\sigma L & \longrightarrow & L^\perp/L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{C}^* & \longrightarrow & \mathcal{G}(\mathbf{L}) & \longrightarrow & K(\mathbf{L}) \longrightarrow 0. \end{array}$$

Because of our conventions, the reader will see that the vertical arrows here $\mathbf{C}_1^* \rightarrow \mathbf{C}^*$ and $L^\perp/L \rightarrow K(\mathbf{L})$ (between subgroups of $X_{T,L}$) are both the reverse of the identity.

From the general results of §1, we see that

$$1 \longrightarrow \mathbf{C}_1^* \longrightarrow N(\sigma L)/\sigma L \longrightarrow L^\perp/L \longrightarrow 0$$

is a Heisenberg group built from the finite abelian group L^\perp/L . Let us define, for any field k , an algebraic Heisenberg group over k as a group G plus an exact sequence:

$$1 \longrightarrow k^* \longrightarrow G \xrightarrow{\pi} K \longrightarrow 0$$

with K finite abelian and $k^* = \text{center of } G$. Then we have:

PROPOSITION 3.1. $L^\perp/L \cong K(\mathbf{L})$, hence $N(\sigma L)/\sigma L \cong \mathcal{G}(\mathbf{L})$ and $\mathcal{G}(\mathbf{L})$ is an algebraic Heisenberg group over \mathbb{C} .

PROOF: The homomorphism from $N(\sigma L)/\sigma L$ to $\mathcal{G}(\mathbf{L})$ already showed that L^\perp/L is contained in $K(\mathbf{L})$. If $a \in \mathbb{C}^g$ is such that its image in $X_{T,L}$ is in

$K(\mathbf{L})$, then by applying an elementary lifting argument there is an automorphism (T_a, ψ) of the trivial line bundle on \mathbf{C}^g (where T_a = translation by a) which normalizes the σL -action, and therefore centralizes it. Now putting $\psi(\alpha, z) = (\alpha h(z), z + a)$ where h is nowhere zero, the above condition can be read off as

$$h(z - a)^{-1} h(z - a - \underline{y}) = \exp 2\pi i {}^t y_1 a$$

for all $y \in L$ and $z \in \mathbf{C}^g$. Putting $h(z) = \exp 2\pi i f(z)$, one finds that $f(z - \underline{y}) - f(z)$ is a constant function for all $y \in L$. It follows that all the partial derivatives of f are invariant under translation by L and are therefore constant because $X_{T,L} = \mathbf{C}^g/L$ is compact, implying that $f(z) = A + g(z)$ where $A \in \mathbf{C}$ and g is a linear form on \mathbf{C}^g . Substituting for h in terms of g in the last equation now gives

$$g(\underline{y}) \equiv {}^t y_1 a \pmod{\mathbf{Z}}$$

for all $y \in L$. If $a = \underline{b} = T b_1 + b_2$, then

$$\begin{aligned} g(\underline{y}) - {}^t \underline{y} b_1 &\equiv {}^t y_1 a - {}^t \underline{y} b_1 \pmod{\mathbf{Z}} \\ &= {}^t y_1 (T b_1 + b_2) - {}^t (T y_1 + y_2) b_1 \\ &= {}^t y_1 b_2 - {}^t y_2 b_1 \\ &\equiv A(y, b) \pmod{\mathbf{Z}}. \end{aligned}$$

Now L generates \mathbf{C}^g as a real vector space, which implies that the complex linear form $g(\underline{y}) - {}^t \underline{y} b_1$ takes only real values and is therefore identically zero. This implies that $A(L, b) \subset \mathbf{Z}$, or equivalently, that $b \in L^\perp$. Therefore $K(\mathbf{L}) = L^\perp/L$. QED

Next we consider the sections $\Gamma(X_{T,L}, \mathbf{L})$ of the basic line bundle \mathbf{L} over $X_{T,L}$. We easily see that

$$\begin{aligned} \Gamma(X_{T,L}, \mathbf{L}) &\cong \left\{ \begin{array}{l} \text{Space of } \sigma(L) - \text{invariant sections of the} \\ \text{trivial bundle } \mathbf{C} \times \mathbf{C}^g \text{ on } \mathbf{C}^g \end{array} \right\} \\ &\cong \left\{ \begin{array}{l} \text{Space of entire functions } f(z) \text{ on } \mathbf{C}^g \text{ invariant by } \\ \sigma(L) \text{ in the Fock representation } \mathcal{H}_\theta(\mathbf{C}^g, T) \end{array} \right\}. \end{aligned}$$

In fact these invariant functions must belong to the space $\mathcal{H}_\theta^2(\mathbf{C}^g, T)_{-\infty}$, as one checks as follows:

a) Define the norm

$$N : \mathbf{C} \times \mathbf{C}^g \longrightarrow \mathbf{R}^+$$

by

$$N(\alpha, \underline{z}) = \alpha \cdot e^{-\pi {}^t z_1 \cdot \text{Im } T \cdot z_1}.$$

b) a small calculation shows that

$$N(\alpha, \underline{z}) = N(\psi_h(\alpha, \underline{z})), \quad \text{all } h \in \text{Heis}(2g, \mathbf{R}),$$

where

$$\psi_h(\alpha, z) = (\alpha \lambda^{-1} \exp \pi i {}^t y_1 (2z - \underline{y}), z - \underline{y}).$$

c) hence if $\{(f(\underline{z}), z)\}$ is a $\sigma(L)$ -invariant section, $N(f(\underline{z}), z)$ is a function of $\underline{z} \in \mathbf{C}^g/L$, hence is bounded,

d) hence

$$f(\underline{z}) \in O(e^{\pi {}^t z_1 \cdot \text{Im } T \cdot z_1}).$$

Using the characterization of $\mathcal{H}_{\phi, -\infty}$ before Theorem 2.3 and the map from \mathcal{H}_θ to \mathcal{H}_ϕ we see that

$$\Gamma(X_{T,L}, \mathbf{L}) \cong (\mathcal{H}_\theta^2(\mathbf{C}^g, T)_{-\infty})^{\sigma L},$$

where $\mathbf{L} = \mathbf{C} \times \mathbf{C}^g/L$ is as constructed above.

In fact, $\Gamma(X_{T,L}, \mathbf{L})$ is the irreducible Heisenberg representation of $\mathcal{G}(\mathbf{L})$. To prove this, we next define a whole set of elements $e[\frac{a}{b}]$ in the distributional completion $\mathcal{H}_{-\infty}$ of the real Heisenberg representation space \mathcal{H} . For all $a, b \in \mathbf{Q}^g$, set

$$\begin{aligned} e[\frac{a}{b}] &= \sum_{n \in \mathbf{Z}^g} e^{2\pi i {}^t (n-a) \cdot b} \delta_{n-a}(x) \quad \text{in realization I} \\ &= \sum_{n, m \in \mathbf{Z}^g} e^{\pi i {}^t (n-a)(m+b)} \cdot \delta_{n-a, m-b}(x) \quad \text{in realization II} \end{aligned}$$

These are easily seen to correspond by the proof used in Theorem 2.3. Or else, their equality follows by noting that $e_{\mathbf{z}} = e\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is the unique $\sigma(\mathbf{Z}^{2g})$ -invariant vector of that Theorem and that

$$e\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = U_{(e^{-\pi i^t a \cdot b}, a, b)} e\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right].$$

Moreover, in the conjugate-linear isomorphism with \mathcal{H}_{θ}^2 , $e\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ corresponds to the unique $\sigma(\mathbf{Z}^{2g})$ -invariant element, namely $\vartheta(\underline{x})$, hence $e\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ corresponds to

$$\begin{aligned} U_{(e^{-\pi i^t a \cdot b}, a, b)} \vartheta(\underline{x}) &= e^{\pi i^t a \cdot b} \cdot e^{\pi i^t a \cdot (Ta + b + 2\underline{x})} \vartheta(\underline{x} + Ta + b) \\ &= \sum e^{\pi i^t a \cdot b + i^t a \cdot (Ta + b + 2\underline{x}) + i^t n \cdot T \cdot n + 2^t n \cdot (\underline{x} + Ta + b)} \\ &= \sum e^{\pi i^t [(n+a)T(n+a) + 2^t(n+a)(\underline{x}+b)]} \\ &= \vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](\underline{x}) \end{aligned}$$

the theta function with characteristic of Ch. 2, §1.

We now prove

PROPOSITION 3.2. *Let (a_i, b_i) be coset representatives for $L^\perp / \mathbf{Z}^{2g}$. Then identifying $\Gamma(X_{T,L}, \mathbf{L})$ with its image in $\mathcal{H}_{\theta}(\mathbf{C}^g, T)_{-\infty}$, the $\vartheta\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right]$ form a basis of $\Gamma(X_{T,L}, \mathbf{L})$, and $\Gamma(X_{T,L}, \mathbf{L})$ is the irreducible Heisenberg representation of $\mathcal{G}(\mathbf{L})$.*

PROOF: We show that $\Gamma(X_{T,L}, \mathbf{L})$ is irreducible under the action of the group $N(\sigma L) / \sigma L$. Now any $\lambda \in \mathbf{C}_1^* \subset N(\sigma L) / \sigma L$ acts on this vector space by multiplication by λ^{-1} , and by §1 it follows that $\Gamma(X_{T,L}, \mathbf{L}) = \mathcal{K} \oplus \mathcal{K} \dots \oplus \mathcal{K}$ where \mathcal{K} is the unique irreducible representation such that $\lambda \in \mathbf{C}_1^*$ acts by λ^{-1} . To show that $\Gamma(X_{T,L}, \mathbf{L}) = \mathcal{K}$ it suffices to prove that the subspace of $\Gamma(X_{T,L}, \mathbf{L})$ fixed by $\sigma \mathbf{Z}^{2g} / \sigma L$ has dimension one. Now this subspace of $\Gamma(X_{T,L}, \mathbf{L})$ is identified canonically with the subspace of $\Gamma(\mathbf{C} \times \mathbf{C}^g, \mathbf{C}^g)$ fixed by $\sigma \mathbf{Z}^{2g}$. For $n_1 \in \mathbf{Z}^g \times \{0\}$ and $n_2 \in \{0\} \times \mathbf{Z}^g$ we have by the formulas in §2.

$$U_{\sigma(n_1)} f(z) = f(z + Tn_1) \exp(\pi i^t n_1 T n_1 + 2\pi i n_1^t z)$$

$$U_{\sigma(n_2)} f(z) = f(z + n_2).$$

And any holomorphic function on \mathbf{C}^g invariant under these two sets of operations is a scalar times $\vartheta(z, T)$.

If $c_i = (a_i, b_i)$ form a system of representatives for $L^\perp / \mathbf{Z}^{2g}$, then

$$N(\sigma L) = \bigcup_i U_{(1, c_i)} \cdot \sigma(\mathbf{Z}^{2g}) \cdot \mathbf{C}_1^*$$

and therefore the functions $U_{(1, c_i)} \vartheta(z, T)$ span a $N(\sigma L) / \sigma L$ -invariant subspace of $\Gamma(X_{T,L}, \mathbf{L})$. Because $\Gamma(X_{T,L}, \mathbf{L})$ is irreducible and its dimension is $[L^\perp : \mathbf{Z}^{2g}]$, it follows that

$$U_{(1, c_i)} \vartheta(z, T) = e^{-\pi i^t a_i \cdot b_i} \vartheta\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right](z, T)$$

form a basis of $\Gamma(X_{T,L}, \mathbf{L})$.

QED

Thus we have arrived again at the situation of Ch. 2, §1: to embed $X_{T,L}$ in projective space, we take the basis $\vartheta\left[\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right]$ for the global sections of the basic line bundle. Now we use:

LEMMA 3.3. *If $L \subseteq nL^\perp$, then there is a line bundle \mathbf{M} on $X_{T,L}$ such that $\mathbf{M}^{\otimes n} \cong \mathbf{L}$.*

PROOF: Note that if $e^{(i)}$ is a basis of L , then \mathbf{L} is $\mathbf{C} \times \mathbf{C}^g$ modulo the automorphisms generated by

$$(\alpha, z) \mapsto (\alpha \cdot e^{\pi i^t e_1^{(i)} \cdot (2z - T \cdot e_1^{(i)})}, z - \underline{e}^{(i)}), \quad e^{(i)} \in L.$$

Define \mathbf{M} to be $\mathbf{C} \times \mathbf{C}^g$ modulo the automorphisms generated by

$$(\alpha, z) \mapsto (\alpha e^{\pi i^t e_1^{(i)} \cdot (2z - T \cdot e_1^{(i)})/n}, z - \underline{e}^{(i)}), \quad e^{(i)} \in L.$$

Using the fact that $\frac{1}{n} e^{(i)} \in L^\perp$, i.e., $A(e^{(i)}, e^{(j)}) \in n\mathbf{Z}$, one checks easily that these maps commute, hence define the needed \mathbf{M} . QED

Now a theorem of Lefschetz (cf. D. Mumford, *Abelian Varieties*, p. 29) says that if $n \geq 2$, the sections of $\mathbf{M}^{\otimes n}$ define a holomorphic map

$$\phi_{T,L} : X_{T,L} \longrightarrow \mathbf{P}^{\nu-1}, \quad \nu = [\mathbf{Z}^{2g} : L]$$

and if $n \geq 3$, $\phi_{T,L}$ is an embedding. This gives us a projective version of the Heisenberg representation: the action of $\mathcal{G}(\mathbf{L})$ on $\Gamma(X_{T,L}, \mathbf{L})$ induces an action of the finite abelian group $K(\mathbf{L})$ on $\mathbf{P}^{\nu-1}$ which is irreducible, (i.e., there is no linear subspace $\mathbf{P}^{\mu-1} \subset \mathbf{P}^{\nu-1}$ which is mapped to itself by $K(L)$) and which makes $\phi_{T,L}$ $K(L)$ -equivariant, i.e., if $a \in K(L)$ induces $p_a : \mathbf{P}^{\nu-1} \rightarrow \mathbf{P}^{\nu-1}$, then

$$\phi_{T,L}(x+a) = p_a(\phi_{T,L}(x)).$$

This group action leads in low dimensional cases to very beautiful explicit descriptions of $\text{Im } \phi_{T,L}$. In chapter I, we studied the case $g = 1, L = 2 \cdot \mathbf{Z}^2$. The cases $g = 1, L = 2\mathbf{Z} + \mathbf{Z}$ and $g = 1, L = 3\mathbf{Z} + \mathbf{Z}$ are the well-known representation of elliptic curves as double covers of \mathbf{P}^1 ramified in 4 points $\pm a, \pm a^{-1}$ and as cubic curves $X_0^3 + X_1^3 + X_2^3 + \lambda X_0 X_1 X_2 = 0$, respectively. The case $g = 2, L = 2\mathbf{Z}^2 + \mathbf{Z}^2$ is the representation of principally polarized a 2-dimensional abelian surface as a double cover of a "Kummer" quartic surface with 16 nodes. The case $g = 2, L = 4\mathbf{Z} + \mathbf{Z}^3$ leads to a beautiful class of octic surfaces in \mathbf{P}^3 ; the case $g = 2, L = 5\mathbf{Z} + \mathbf{Z}^3$ leads to an interesting story in \mathbf{P}^4 (cf. Horrocks & Mumford, *Topology*, vol. 12, 1973).

Much of the above theory concerns only the algebraic varieties obtained when a complex torus is embedded in projective space. This part of the theory is really a branch of algebraic geometry and has nothing to do with analysis. We want next to sketch this variant of the Heisenberg circle of ideas. We begin with some basic definitions:

DEFINITION. Let k be a field. An abelian variety defined over k is a projective variety X defined over k with a morphism $f : X \times X \rightarrow X$ and a k -rational point $0 \in X$ which makes X into a group. For every k -rational point a , let $T_a : X \rightarrow X$ be given by $T_a(x) = f(x, a)$.

Facts about abelian varieties that we shall freely use are:

LEMMA 3.4. (A) X is a commutative group, whose inverse is a morphism $-1 : X \rightarrow X$.

(B) The set of all points x of X defined over the algebraic closure \bar{k} of k such that $nx = 0$ for some $n \geq 1$ is dense in X , and

(C) $X_n = \{x \in X(\bar{k}) | nx = 0\} \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$, if n is not divisible by $\text{char } k$.

When $X = X_{T,L}$ these facts are obvious. For the general case, see D. Mumford, *Abelian Varieties*, §§4,6. The fundamental definitions related to line bundles in the algebraic case are:

DEFINITION. A line bundle \mathbf{L} on a variety X defined over k is a morphism $p : \mathbf{L} \rightarrow X$ and isomorphisms $\varphi_\alpha : U_\alpha \times \mathbf{A}^1 \xrightarrow{\sim} p^{-1}(U_\alpha)$ on an open cover U_α of X such that

$$(\varphi_\beta|_{U_\alpha \cap U_\beta \times \mathbf{A}^1})^{-1} \circ (\varphi_\alpha|_{U_\alpha \cap U_\beta \times \mathbf{A}^1}) : U_\alpha \cap U_\beta \times \mathbf{A}^1 \rightarrow U_\alpha \cap U_\beta \times \mathbf{A}^1$$

is given by $(x, t) \mapsto (x, \psi_{\alpha\beta}(x)t)$ where $\psi_{\alpha\beta}$ and $\psi_{\alpha\beta}^{-1}$ are regular functions on the open set $U_\alpha \cap U_\beta$.

A section of \mathbf{L} on an open subset U of X is a morphism $s : U \rightarrow \mathbf{L}$ such that $ps = i_U$ where i_U is the inclusion morphism of U in X . The collection of sections of \mathbf{L} defined on U is a k -vector space denoted by $\Gamma(U, \mathbf{L})$. The sheaf of sections of \mathbf{L} is the sheaf on X given by attaching to every open set U in X the abelian group $\Gamma(U, \mathbf{L})$. This sheaf is a locally free sheaf of rank one and this sets up a one-to-one correspondence between line bundles \mathbf{L} on X and locally free sheaves of rank one on X .

DEFINITION. The tensor product $\mathbf{L}_1 \otimes \mathbf{L}_2$ of two line bundles \mathbf{L}_1 and \mathbf{L}_2 on X is a line bundle on X such that its sheaf of sections is the tensor product of the sheaves of sections of \mathbf{L}_1 and \mathbf{L}_2 . Equivalently there is a morphism $\mathbf{L}_1 \times_X \mathbf{L}_2 \rightarrow \mathbf{L}_1 \otimes \mathbf{L}_2$ making the fibre of $\mathbf{L}_1 \otimes \mathbf{L}_2$ over each k -valued point of X into the tensor product over k of the fibres of \mathbf{L}_1 and \mathbf{L}_2 . The line bundles $\mathbf{L} \otimes \mathbf{L}, \mathbf{L} \otimes \mathbf{L} \otimes \mathbf{L}, \dots$ are denoted by $\mathbf{L}^{\otimes 2}, \mathbf{L}^{\otimes 3}, \dots$. The line bundle \mathbf{L} is ample if for some $n \geq 1$, $\mathbf{L}^{\otimes n}$ is generated by its sections, and these sections define an embedding of X in \mathbf{P}^N .

DEFINITION. For a line bundle \mathbf{L} on an abelian variety X defined over an

algebraically closed field k ,

$$\mathcal{G}(\mathbf{L}) = \{(\phi, \psi) \in \text{Aut } \mathbf{L} \mid \phi = T_a \text{ for some } a \in X\}$$

$$K(\mathbf{L}) = \{a \in X \mid T_a^*(\mathbf{L}) \cong \mathbf{L}\}.$$

Clearly there is an exact sequence

$$1 \longrightarrow k^* \longrightarrow \mathcal{G}(\mathbf{L}) \longrightarrow K(\mathbf{L}) \longrightarrow 0.$$

The main results of the algebraic theory parallel those for locally compact Heisenberg groups and assert the existence of a canonical Heisenberg representation:

PROPOSITION 3.5. *Consider all k -vector spaces (finite-dimensional or not) equipped with an action of an algebraic Heisenberg group G such that $\lambda \in k^* \subset G$ acts by multiplication by λ . There is one among these, denoted by \mathcal{H} , with $\dim \mathcal{H} = \sqrt{\#K}$, which is irreducible, and furthermore, any such representation is a direct sum of copies of \mathcal{H} . \mathcal{H} will be called the Heisenberg representation.*

PROPOSITION 3.6. *If \mathbf{L} is an ample line bundle on an abelian variety X defined over an algebraically closed field k , with first chern class $c_1(\mathbf{L})$, then*

$$(A) \dim \Gamma(X, \mathbf{L}) = \frac{1}{g!} (c_1(\mathbf{L})^g) > 0, \text{ and this integer is called } \deg \mathbf{L}.$$

If the characteristic of k does not divide $\deg \mathbf{L}$, then

$$(B) \mathcal{G}(\mathbf{L}) \text{ is an algebraic Heisenberg group; } K(\mathbf{L}) \text{ has } (\deg \mathbf{L})^2 \text{ elements, and}$$

$$(C) \Gamma(X, \mathbf{L}) \text{ is the Heisenberg representation of } \mathcal{G}(\mathbf{L}).$$

There is still another variant of the Heisenberg theory to cover the case

$\text{char}(k) \nmid \deg \mathbf{L}$. Using the language of schemes, we define group *schemes*

$\underline{\mathcal{G}}(\mathbf{L}) =$ group scheme whose R -valued points are the pairs

$$\begin{array}{ccc} \mathbf{L} \times_{\text{Spec } k} \text{Spec } R & \xrightarrow{\psi} & \mathbf{L} \times_{\text{Spec } k} \text{Spec } R \\ \downarrow & & \downarrow \\ X \times_{\text{Spec } k} \text{Spec } R & \xrightarrow{\phi} & X \times_{\text{Spec } k} \text{Spec } R \end{array}$$

where ϕ is translation by an R -valued point of X and

ψ is an isomorphism of line bundles; the bijection

between points of $\underline{\mathcal{G}}(\mathbf{L})$ and pairs (ϕ, ψ) is functorial in R .

$\underline{K}(\mathbf{L}) =$ the sub-group scheme of X whose R -valued points a

(for any local ring R) are those such that if ϕ is translation by a ,

then

$$\phi^*(\mathbf{L} \times_{\text{Spec } k} \text{Spec } R) \cong \mathbf{L} \times_{\text{Spec } k} \text{Spec } R.$$

If $\text{char } k \nmid \deg \mathbf{L}$ and k is algebraically closed, then $\underline{K}(\mathbf{L})$ will be discrete and finite, and the k -valued points of $\underline{\mathcal{G}}(\mathbf{L})$, $\underline{K}(\mathbf{L})$ will be the ordinary groups $\mathcal{G}(\mathbf{L})$, $K(\mathbf{L})$ defined above. If $\text{char } k \nmid \deg \mathbf{L}$, but k is not algebraically closed, the "Scheme" structure just amounts to an action of $\text{Gal}(\bar{k}/k)$ on $\mathcal{G}(\mathbf{L} \times_k \bar{k})$ and $K(\mathbf{L} \times_k \bar{k})$. But if $\text{char } k \mid \deg \mathbf{L}$, $\underline{K}(\mathbf{L})$ may have nilpotent elements in its structure sheaf. Next, we define a Heisenberg group scheme to be a group scheme $\underline{\mathcal{G}}$ plus an exact sequence

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \underline{\mathcal{G}} \longrightarrow \underline{K} \longrightarrow 0$$

where \underline{K} is a finite abelian group scheme and \mathbf{G}_m is the center of $\underline{\mathcal{G}}$. The main results generalize to:

PROPOSITION 3.7. *Consider all representations $\rho : \underline{\mathcal{G}} \longrightarrow GL(n)$ such that the center \mathbf{G}_m acts by $\rho(\lambda) = \lambda \cdot I_n$. There is one among these, denoted*

by \mathcal{H} , with $\dim \mathcal{H} = \sqrt{\text{order } \underline{K}}$ (here $\text{order } \underline{K} = \dim_k \Gamma(\mathcal{O}_{\underline{K}})$), which is irreducible. And furthermore any such representation is a direct sum of copies of \mathcal{H} . \mathcal{H} will be called the Heisenberg representation of \underline{G} .

PROPOSITION 3.8. *If \underline{L} is an ample line bundle on an abelian variety X defined over k , then $\underline{\mathcal{G}}(\underline{L})$ is a Heisenberg group scheme, $\underline{K}(\underline{L})$ has $(\deg \underline{L})^2$ elements and $\Gamma(X, \underline{L})$ is the Heisenberg representation of $\underline{\mathcal{G}}(\underline{L})$.*

For proofs, see D. Mumford, *Abelian Varieties* §16, §23 and T. Sekiguchi, *J. Math. Soc. Japan*, 29 (1977), p. 709.

4. Adelic Heisenberg groups and towers of abelian varieties

The algebraic theory of Heisenberg groups of §3 seems like merely a faint residue of the real Heisenberg representation in the context of projective varieties and might suggest that only a small part of the theta function could be reconstructed from abelian varieties. This is not correct, however, and by using adelic methods, we will see that a purely algebraic theory of adelic Heisenberg groups and adelic theta functions can be given which is quite parallel to the analytic theory. We will explain how this goes in §5. In this section, we will merely define the adelic Heisenberg group $\widehat{\mathcal{G}}(\underline{L})$ associated to any line bundle \underline{L} on any abelian variety X and give a systematic exposition of its basic properties.

We begin with some basic definitions:

DEFINITION 4.1. *Let A be an abelian group. The groups $V_p(A)$, $V(A)$, $V^p(A)$ and $T_p(A)$, $T(A)$, $T^p(A)$ are:*

i) $V_p(A)$ = the group of all sequences (a_0, a_1, a_2, \dots) where $a_i \in A$ such that:

$$pa_{i+1} = a_i \text{ and } p^N a_0 = 0 \text{ for some } N \geq 0$$

ii) $V(A)$ = the group of all sequences (a_1, a_2, a_3, \dots) where $a_i \in A$ such that:

$$ma_{m+1} = a_m \text{ and } Na_1 = 0 \text{ for some } N \geq 1$$

iii) $V^p(A)$ is the same as $V(A)$ except that the a_i are defined only for i not divisible by p and there is an N , not divisible by p , such that $Na_1 = 0$.

iv) $T_p(A)$, $T(A)$ and $T^p(A)$ are the subgroups of $V_p(A)$, $V(A)$ and $V^p(A)$ given by $a_0 = 0$, $a_1 = 0$ and $a_1 = 0$, respectively.

Note that $V_p(\mathbb{Q}/\mathbb{Z})$ is the field of p -adic numbers \mathbb{Q}_p ; $T_p(\mathbb{Q}/\mathbb{Z})$ is the subring of p -adic integers \mathbb{Z}_p ; $V(\mathbb{Q}/\mathbb{Z})$ is the ring of finite adeles \mathbb{A}_f ; $T(\mathbb{Q}/\mathbb{Z})$ is the subring of integral finite adeles, $\widehat{\mathbb{Z}}$, a completion of \mathbb{Z} ; $V^p(\mathbb{Q}/\mathbb{Z})$ is the ring of finite adeles without the p -factor, which we write $\mathbb{A}_f^{(p)}$; and $T^p(\mathbb{Q}/\mathbb{Z})$

is the subring of integral finite adeles without the p -factor, which we write $\widehat{\mathbf{Z}}^{(p)}$.

LEMMA 4.2. $T(A) = \prod_{\ell} T_{\ell}(A)$ and $T^p(A) = \prod_{\ell \neq p} T_{\ell}(A)$ where the ℓ range through all primes, and $V(A)$ and $V^p(A)$ are the "restricted" direct products:

$$V(A) = \left\{ (\alpha_{\ell}) \in \prod_{\ell} V_{\ell}(A) \mid \text{all but finitely many } \alpha_{\ell} \text{ belong to } T_{\ell}(A) \right\}$$

$$V^p(A) = \left\{ (\alpha_{\ell}) \in \prod_{\ell \neq p} V_{\ell}(A) \mid \text{all but finitely many } \alpha_{\ell} \text{ belong to } T_{\ell}(A) \right\}.$$

There are exact sequences,

$$0 \rightarrow T(A) \rightarrow V(A) \rightarrow A_{\text{tor}} \rightarrow 0$$

$$0 \rightarrow T_p(A) \rightarrow V_p(A) \rightarrow A(p^{\infty}) \rightarrow 0$$

$$0 \rightarrow T^p(A) \rightarrow V^p(A) \rightarrow A_{\text{tor}}^p \rightarrow 0$$

where A_{tor} is the torsion subgroup of A , $A(p^{\infty})$ is the subgroup of A of points of order p^N for some N , A_{tor}^p is the prime-to- p torsion subgroup, and the homomorphism from $V(A)$ to A_{tor} is given by $(a_1, a_2, a_3, \dots) \rightarrow a_1$.

We omit the proof which depends only on the fact that any element a of finite order in any abelian group A has a unique decomposition $a = \sum_{\ell} a_{\ell}$ where each a_{ℓ} is annihilated by some power of ℓ .

DEFINITION 4.3. Let G be any commutative group scheme defined over a field k . In practice we shall put G = an abelian variety or $G = \mathbf{G}_m$ (the multiplicative group scheme of the field k) only. Let $G(\bar{k})$ = all points of G defined over the algebraic closure \bar{k} of k . We put

$$V_p(G) = V_p(G(\bar{k})), \quad \text{if } p \neq \text{char. } k,$$

$$T_p(G) = T_p(G(\bar{k})), \quad \text{if } p \neq \text{char. } k,$$

$$V(G) = V(G(\bar{k})) \quad \text{and} \quad T(G) = T(G(\bar{k})) \quad \text{if char. } k = 0,$$

and finally

$$V(G) = V^p(G(\bar{k})) \quad \text{and} \quad T(G) = T^p(G(\bar{k})) \quad \text{if char. } k = p.$$

Actually, these groups remain unchanged if \bar{k} is replaced by k_{sep} the separable closure of k , and all of them are acted upon naturally by the galois group $\text{Gal}(k_{sep}/k)$.

LEMMA 4.4. For an abelian variety X of dimension g defined over k of characteristic $\neq p$, the pair $(V_p(X), T_p(X))$ is isomorphic to $(\mathbf{Q}_p^{2g}, \mathbf{Z}_p^{2g})$ and the pair $(V_p(\mathbf{G}_m), T_p(\mathbf{G}_m))$ to $(\mathbf{Q}_p, \mathbf{Z}_p)$. Also the pairs $(V(X), T(X))$ and $(V(\mathbf{G}_m), T(\mathbf{G}_m))$ are isomorphic to $(\mathbf{A}_f^{2g}, \widehat{\mathbf{Z}}^{2g})$ and $(\mathbf{A}_f, \widehat{\mathbf{Z}})$ when the characteristic of k is 0 and to $((\mathbf{A}_f^{(p)})^{2g}, (\widehat{\mathbf{Z}}^{(p)})^{2g})$ and $(\mathbf{A}_f^{(p)}, \widehat{\mathbf{Z}}^{(p)})$ when the characteristic is p .

PROOF: The second statement follows from the first after an application of Lemma 4.2.

Let $X(p^{\infty})$ = all points of X defined over \bar{k} annihilated by some power of p . By definition $V_p(X)$ is the inverse limit of the chain of arrows:

$$\dots \rightarrow X(p^{\infty}) \xrightarrow{p} X(p^{\infty}) \xrightarrow{p} X(p^{\infty}).$$

There is an isomorphism $X(p^{\infty}) \cong (\mathbf{Q}_p/\mathbf{Z}_p)^{2g}$. Once this isomorphism has been fixed, $V_p(X)$ is the inverse limit of:

$$\begin{array}{ccccccc} \dots & \rightarrow & (\mathbf{Q}_p/\mathbf{Z}_p)^{2g} & \xrightarrow{p} & (\mathbf{Q}_p/\mathbf{Z}_p)^{2g} & \xrightarrow{p} & (\mathbf{Q}_p/\mathbf{Z}_p)^{2g} \\ & & p^2 \downarrow \cong & & p \downarrow \cong & & 1 \downarrow \cong \\ \mathbf{Q}_p^{2g} & \dots \rightarrow & (\mathbf{Q}_p/p^2\mathbf{Z}_p)^{2g} & \rightarrow & (\mathbf{Q}_p/p\mathbf{Z}_p)^{2g} & \rightarrow & (\mathbf{Q}_p/\mathbf{Z}_p)^{2g} \end{array}$$

which by the commutative diagram above and the completeness of \mathbf{Q}_p^{2g} in the p -adic topology is just \mathbf{Q}_p^{2g} . It is clear that $T_p(X)$, given by the first member of the chain = 0, gets identified to \mathbf{Z}_p^{2g} . The Lemma holds for \mathbf{G}_m because $\mathbf{G}_m(p^{\infty}) \cong \mathbf{Q}_p/\mathbf{Z}_p$ holds too. Q.E.D.

On various occasions below we will fix an isomorphism

$$\mathbf{e} : \mathbf{A}_f/\widehat{\mathbf{Z}} \xrightarrow{\sim} \bar{k}_{\text{tor}}^*$$

(or $\mathbf{A}_f^{(p)}/\widehat{\mathbf{Z}}^{(p)} \xrightarrow{\sim} \bar{k}_{\text{tor}}^*$ in char. p) and we will always use bold-face \mathbf{e} for such a map.

DEFINITION 4.5. If E is a line bundle on X and $f : Y \rightarrow X$ a morphism, then the line bundle f^*E on Y is the fibre product:

$$\begin{array}{ccc} f^*E = Y \times_X E & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

If $g = (\phi, \psi) \in \text{Aut } E$ and $\phi' : Y \rightarrow Y$ is an isomorphism so that $\phi \circ f = f \circ \phi'$ then $(\phi', \psi') \in \text{Aut } f^*E$ is the lift of g covering ϕ' where $\psi'(y, \ell) = (\phi'(y), \psi(\ell))$ for all $(y, \ell) \in Y \times_X E$.

The most important group associated to a line bundle L on an abelian variety is:

DEFINITION 4.6. Assume first that characteristic $k = 0$. Then $\hat{G}(L) =$ all sequences of pairs (x_n, ϕ_n) where:

- A. The $x_n \in X(\bar{k})$ define a member of $V(X)$, i.e., $mx_{mn} = x_n$ and $Nx_1 = 0$ for some $N \geq 1$.
- B. The ϕ_n are defined for all n such that $x_n \in K(n^*L)$.
- C. $(T_{x_{mn}}, \phi_{mn})$ is the lift of (T_{x_n}, ϕ_n) covering $T_{x_{mn}}$ as in Definition 4.5. with $E = nm_X^*L$, $Y = X$, $f = m_X$.
- D. The group law in $\hat{G}(L)$ is given by

$$(x_n, \phi_n) \circ (y_n, \psi_n) = (x_n + y_n, \phi_n \circ \psi_n)$$

In $\hat{G}(L)$, if char. $k \neq 0$, then (x_n, ϕ_n) are to be defined only for n not divisible by p , and $\exists N$ such that $p \nmid N$ and $Nx_1 = 0$.

The power of $\hat{G}(L)$ is due to this: For any element $x = (x_1, \dots) \in V(X)$, there is an N such that for all n divisible by N , $x_n \in K(n^*L)$. Indeed we have

PROPOSITION 4.7. The group $\hat{G}(L)$ has an exact sequence:

$$1 \longrightarrow \bar{k}^* \longrightarrow \hat{G}(L) \xrightarrow{\pi} V(X) \longrightarrow 0$$

where π takes (x_n, ϕ_n) to the sequence (x_n) defining an element of $V(X)$ and the inclusion of \bar{k}^* in $\hat{G}(L)$ is given by $\alpha \mapsto (x_n, \phi_n)$ where $x_n = 0$ and $\phi_n = \alpha(n^*1_L)$ for all n . Further, \bar{k}^* is contained in the centre of $\hat{G}(L)$.

PROOF: To see that π is surjective, let $\{x_n\} \in V(X)$ and let N be the smallest integer so that $Nx_0 \in K(L)$, so $T_{Nx_0}L \otimes L^{-1}$ is trivial. Now

$$\begin{aligned} T_{x_N}^* N^* L \otimes (N^* L)^{-1} &= N^* (T_{Nx_N}^* L \otimes L^{-1}) \\ &\cong T_{N^2 x_N} L \otimes L^{-1} \quad (\text{by the theorem of the square}) \\ &\cong (\text{trivial bundle}). \end{aligned}$$

QED

COROLLARY 4.8. If $\varphi : L_1 \xrightarrow{\sim} L_2$ is an isomorphism of line bundles on X , then the induced isomorphism of $\hat{G}(L_1)$ and $\hat{G}(L_2)$ is independent of φ .

PROOF: To check the Corollary, it suffices to show that any isomorphism $\varphi : L \cong L$ induces the identity isomorphism from $\hat{G}(L)$ to itself. And this is so because any such φ is α_L where $\alpha \in \bar{k}^*$ and therefore the corresponding automorphism of $\hat{G}(L)$ is conjugation by $\alpha \in \bar{k}^* \subset \hat{G}(L)$ which is the identity. QED

The next proposition describes the functorial nature of $\hat{G}(L)$.

PROPOSITION 4.9. If $f : Y \rightarrow X$ is a homomorphism of abelian varieties and L is a line bundle on X , there is a commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \bar{k}^* & \longrightarrow & \hat{G}(f^*L) & \xrightarrow{\pi} & V(Y) \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow j(f, L) & & \downarrow V(f) \\ 1 & \longrightarrow & \bar{k}^* & \longrightarrow & \hat{G}(L) & \xrightarrow{\pi} & V(X) \longrightarrow 0 \end{array}$$

If L_1 and L_2 are line bundles on X and

$$H = \{g_1, g_2\} \in \hat{G}(L_1) \times \hat{G}(L_2) \mid \pi(g_1) = \pi(g_2)\} \quad \text{and}$$

$$A = \{(\alpha_1, \alpha_2) \in \bar{k}^* \times \bar{k}^* \subset H \mid \alpha_1 \alpha_2 = 1\},$$

then there is a diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \bar{k}^* & \longrightarrow & H/A & \longrightarrow & V(X) \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow h & & \downarrow 1 \\ 1 & \longrightarrow & \bar{k}^* & \longrightarrow & \hat{G}(L_1 \otimes L_2) & \longrightarrow & V(X) \longrightarrow 0 \end{array}$$

PROOF: Let (y_n, ϕ_n) be a sequence in $\widehat{\mathcal{G}}(f^*\mathbb{L})$. Choose an N such that $N^2 y_N = 0$; this implies that ϕ_N is defined. Then $f(y_N) \in K(N_X^*\mathbb{L})$ allowing one to choose a $(T_{f(y_N)}, \beta) \in \text{Aut } N_X^*\mathbb{L}$. By modifying β by a unique element of \bar{k}^* one may assume that the lift of $(T_{f(y_N)}, \beta)$ to $\text{Aut } f^*N_X^*\mathbb{L}$ is (T_{y_N}, ϕ_N) . Let $(f(y_n), \phi_n)$ be the unique sequence in $\widehat{\mathcal{G}}(\mathbb{L})$ so that $\phi_N = \beta$. Put $j(f, \mathbb{L})(y_n, \phi_n) = (f(y_n), \phi_n)$. This gives $j(f, \mathbb{L})$ in a well-defined manner, and gives $j(f, \mathbb{L})(\alpha) = \alpha$ for $\alpha \in \bar{k}^*$; that $\pi \circ j(f, \mathbb{L}) = V(f) \circ \pi$ is immediate.

We shall content ourselves with defining $h' : H \rightarrow \widehat{\mathcal{G}}(\mathbb{L}_1 \otimes \mathbb{L}_2)$ such that $h'(A) = 0$. Let $u = (x_n, \phi_n) \in \widehat{\mathcal{G}}(\mathbb{L}_1)$ and $v = (x_n, \psi_n) \in \widehat{\mathcal{G}}(\mathbb{L}_2)$. Then (T_{x_n}, ϕ_n) and (T_{x_n}, ψ_n) are in $\text{Aut } n_X^*\mathbb{L}_1$ and $\text{Aut } n_X^*\mathbb{L}_2$ respectively; they induce $(T_{x_n}, \phi_n \otimes \psi_n)$ an automorphism of $\mathbb{L}_1 \otimes \mathbb{L}_2$. Now put $h'(u, v) = (T_{x_n}, \phi_n \otimes \psi_n)$. This satisfies $h'(A) = 0$ and gives a factoring of h' :

$$h' : H \rightarrow H/A \xrightarrow{h} \widehat{\mathcal{G}}(\mathbb{L}_1 \otimes \mathbb{L}_2)$$

QED

The homomorphism $\pi : \widehat{\mathcal{G}}(\mathbb{L}) \rightarrow V(X)$ has a section $\sigma^{\mathbb{L}}$ over the subgroup $T(X) \subset V(X)$:

DEFINITION 4.10. If $x \in X_n$, let $(T_x, \sigma_n(x)) \in \text{Aut } n_X^*\mathbb{L}$ be the lift of $(1_X, 1_{\mathbb{L}}) \in \text{Aut } \mathbb{L}$ covering T_x . For $x \in T(X)$, where $x = (x_1, x_2, x_3, \dots)$, let $\sigma^{\mathbb{L}}(x) \in \widehat{\mathcal{G}}(\mathbb{L})$ be the sequence of pairs $(x_n, \sigma_n(x_n))$. Alternately, $\sigma^{\mathbb{L}}(x)$ is the unique element of $\widehat{\mathcal{G}}(\mathbb{L})$ with $\pi(\sigma^{\mathbb{L}}(x)) = x$ and $\phi_1 = 1_{\mathbb{L}}$ (where $\sigma^{\mathbb{L}}(x) = (x_n, \phi_n)$).

LEMMA 4.11. The map σ_n defines an action of X_n on $n_X^*\mathbb{L}$ so that $p(\sigma_n(x)\ell) = p(\ell) + x$. The quotient of $n_X^*\mathbb{L}$ by this action is precisely the line bundle \mathbb{L} on X .

The functoriality of $\sigma^{\mathbb{L}}$ is given by:

LEMMA 4.12. With notation as in Proposition 4.9, we have

$$j(f, \mathbb{L})\sigma^{f^*\mathbb{L}}(x) = \sigma^{\mathbb{L}}(V(f)(x))$$

for all $x \in T(Y)$ and $h(\sigma^{\mathbb{L}_1}(x), \sigma^{\mathbb{L}_2}(x)) = \sigma^{\mathbb{L}_1 \otimes \mathbb{L}_2}(x)$ for all $x \in T(X)$.

Lemmas 4.11 and 4.12 follow by just plain checking.

PROPOSITION 4.13. The three subgroups of $\widehat{\mathcal{G}}(\mathbb{L})$ defined below are identical:

- (A) the normaliser of $\sigma^{\mathbb{L}}T(X)$,
- (B) the subgroup of (x_n, ϕ_n) with $x_1 \in K(\mathbb{L})$, and
- (C) the elements g of $\widehat{\mathcal{G}}(\mathbb{L})$ which have a representative of the type (x_n, ϕ_n) with ϕ_n defined for all n .

Consequently, this subgroup, modulo $\sigma^{\mathbb{L}}T(X)$, is isomorphic to the subgroup of elements g of $\mathcal{G}(\mathbb{L})$ with $\pi(g)$ a torsion element of $K(\mathbb{L})$.

PROOF: Step I: We prove $A \subseteq B$, i.e., if $g \in N(\sigma^{\mathbb{L}}T(X))$, then $x_1 \in K(\mathbb{L})$ where $g = (x_n, \phi_n)$. If $u \in \sigma^{\mathbb{L}}T(X)$, then $gug^{-1}u^{-1} \in k^* \cap \sigma^{\mathbb{L}}T(X) = 1$ showing g commutes with u . Assume that ϕ_m is defined. Reading off the m^{th} element of the sequence for $gug^{-1}u^{-1}$ one sees that $\phi_m \sigma_m(y) = \sigma_m(y) \phi_m$ for all $y \in X_m$. By Lemma 4.11, because $\phi_m : m_X^*\mathbb{L} \rightarrow m_X^*\mathbb{L}$ commutes with the σ_m -action of X_m , there exists $\psi : \mathbb{L} \rightarrow \mathbb{L}$ and a commutative diagram

$$\begin{array}{ccc} m_X^*\mathbb{L} & \xrightarrow{\phi_m} & m_X^*\mathbb{L} \\ \downarrow & & \downarrow \\ \mathbb{L} & \xrightarrow{\psi} & \mathbb{L} \end{array}$$

Clearly $(T_{mx_m}, \psi) = (T_{x_1}, \psi) \in \text{Aut } \mathbb{L}$ implying that $x_1 \in K(\mathbb{L})$.

Step II: Next prove $B \subseteq C$. If $g = (x_n, \phi_n)$ has $x_1 \in K(\mathbb{L})$, then ϕ_1 is defined and so $x_n \in K(n^*\mathbb{L})$ for all n .

Step III: If $g = (x_m, \phi_m) \in \widehat{\mathcal{G}}(\mathbb{L})$ has ϕ_1 defined, then g commutes with $\sigma^{\mathbb{L}}(y)$ for all $y \in T(X)$. This is so because both $g\sigma^{\mathbb{L}}(y)$ and $\sigma^{\mathbb{L}}(y)g$ have their m^{th} coordinate as the lift of (T_{x_1}, ϕ_1) covering $T_{x_m+y_m}$.

This shows that the three groups are equal. Now define

$$h : N(\sigma^{\mathbf{L}}T(X)) \rightarrow \mathcal{G}(\mathbf{L})$$

$$h((x_1, \phi_1), (x_2, \phi_2), \dots) = (T_{x_1}, \phi_1) \in \mathcal{G}(\mathbf{L}).$$

By definition the kernel is just $\sigma^{\mathbf{L}}T(X)$ and the image is all $(T_{x_1}, \phi_1) \in \mathcal{G}(\mathbf{L})$ with x_1 a torsion point of $K(\mathbf{L})$. In particular, if \mathbf{L} is ample,

$$\mathcal{G}(\mathbf{L}) \cong N(\sigma^{\mathbf{L}}T(X))/\sigma^{\mathbf{L}}(T(X))$$

QED

PROPOSITION 4.14. Let X be an abelian variety and \mathbf{L} a line bundle on X . For $x, y \in V(X)$ choose $\bar{x}, \bar{y} \in \hat{\mathcal{G}}(\mathbf{L})$ with $\pi(\bar{x}) = x$ and $\pi(\bar{y}) = y$. Put $e^{\mathbf{L}}(x, y) = \bar{x} \bar{y} \bar{x}^{-1} \bar{y}^{-1}$, then

- A) $e^{\mathbf{L}}(x, y)$ is independent of the choices of \bar{x} and \bar{y} and is an alternating form with values in the group of all the roots of unity in \bar{k}^* .
- B) $e^{\mathbf{L}}(x, y) = 1$ for $x, y \in T(X)$.
- C) Let $A^{\mathbf{L}}(x, y) = (e^{\mathbf{L}}(x, y), e^{\mathbf{L}}(\frac{x}{2}, y), e^{\mathbf{L}}(\frac{x}{3}, y), \dots)$ in $V(\mathbf{G}_m)$. Then

$$A^{\mathbf{L}} : V(X) \times V(X) \longrightarrow V(\mathbf{G}_m)$$

is an alternating \mathbf{A}_f -bilinear form on $V(X)$, and if

$$0 \rightarrow T(\mathbf{G}_m) \rightarrow V(\mathbf{G}_m) \xrightarrow{\bar{e}} \bar{k}_{tor}^* \rightarrow 0$$

is the exact sequence of Lemma 4.2, then

$$e(A^{\mathbf{L}}(x, y)) = e^{\mathbf{L}}(x, y) \quad \text{for all } x, y \in V(X).$$

- D) If \mathbf{L} is an ample line bundle on X , then define

$$\begin{aligned} T(X)^{\perp} &= \{y \in V(X) | e^{\mathbf{L}}(x, y) = 1 \text{ for all } x \in T(X)\} \\ &= \{y \in V(X) | A^{\mathbf{L}}(x, y) \in \hat{\mathbf{Z}} \text{ for all } x \in T(X)\}. \end{aligned}$$

Then $[T(X)^{\perp} : T(X)] = (\deg \mathbf{L})^2$ in characteristic 0, or the prime-to- p factor of $(\deg \mathbf{L})^2$ in characteristic p . (See Proposition 3.4 for definition of $\deg \mathbf{L}$.)

E) In particular, if $\deg \mathbf{L} = 1$, then $T(X) = T(X)^{\perp}$.

F) If $f : Y \rightarrow X$ is a homomorphism of abelian varieties, then

$$A^{\mathbf{L}}(V(f)x, V(f)y) = A^{f^*\mathbf{L}}(x, y)$$

and

$$e^{\mathbf{L}}(V(f)x, V(f)y) = e^{f^*\mathbf{L}}(x, y)$$

and finally,

G)

$$A^{\mathbf{L}_1 \otimes \mathbf{L}_2}(x, y) = A^{\mathbf{L}_1}(x, y) + A^{\mathbf{L}_2}(x, y)$$

and

$$e^{\mathbf{L}_1 \otimes \mathbf{L}_2}(x, y) = e^{\mathbf{L}_1}(x, y) e^{\mathbf{L}_2}(x, y)$$

The group-law on $V(\mathbf{G}_m)$ is written $+$ rather than \cdot because it is an \mathbf{A}_f -module. We give only some of the proofs.

A) $e^{\mathbf{L}}(x, y)^{m^2} = e^{\mathbf{L}}(mx, my) = 1$ if m is chosen so that mx and my both belong to $T(X)$.

B) The existence of the group homomorphism $\sigma^{\mathbf{L}} : T(X) \rightarrow \hat{\mathcal{G}}(\mathbf{L})$ shows that

$$e(x, y) = [\text{the commutator of } \sigma^{\mathbf{L}}(x) \text{ and } \sigma^{\mathbf{L}}(y)] = 1.$$

C) $A^{\mathbf{L}}(x, y)$ is clearly additive in y . Also $e^{\mathbf{L}}(\frac{x}{m}, y) = e^{\mathbf{L}}(\frac{x}{m}, \frac{y}{m})^m = e^{\mathbf{L}}(x, \frac{y}{m})$ showing that

$$A^{\mathbf{L}}(x, y) = (e^{\mathbf{L}}(x, y), e^{\mathbf{L}}(x, \frac{y}{2}), e^{\mathbf{L}}(x, \frac{y}{3}), \dots)$$

from which $A^{\mathbf{L}}(x, y)$ is additive in x . Also $e^{\mathbf{L}}(x, \frac{x}{m}) = e^{\mathbf{L}}(\frac{x}{m}, \frac{x}{m})^m = 0$ showing $A^{\mathbf{L}}(x, x) = 0$. We omit the rest of C).

D) Letting X_{tor}^p be the prime-to- p torsion subgroup, the sequence

$$0 \rightarrow T(X) \rightarrow V(X) \rightarrow X_{tor}^p \rightarrow 0$$

from Lemma 4.2 allows us to identify $T(X)^\perp/T(X)$ with a subgroup of X_{tor}^p . It follows from Proposition 4.13 that this subgroup is precisely $X_{tor}^p \cap K(\mathbf{L})$. By Proposition 4.4, $K(\mathbf{L})$ has $(\deg \mathbf{L})^2$ elements and is therefore contained in X_{tor} .

F) This follows from Proposition 4.9. Put $j = j(f, \mathbf{L})$. Choose $\bar{x}, \bar{y} \in \widehat{G}(f^*\mathbf{L})$ so that $\pi(\bar{x}) = x$ and $\pi(\bar{y}) = y$. Then:

$$\begin{aligned} e^{f^*\mathbf{L}}(x, y) &= \bar{x} \bar{y} \bar{x}^{-1} \bar{y}^{-1} \\ &= j(\bar{x} \bar{y} \bar{x}^{-1} \bar{y}^{-1}) \\ &= j(\bar{x}) j(\bar{y}) j(\bar{x})^{-1} j(\bar{y})^{-1} \\ &= e^{\mathbf{L}}(\pi j(\bar{x}), \pi j(\bar{y})) \\ &= e^{\mathbf{L}}(V(f)x, V(f)y) \end{aligned}$$

Now $A^{\mathbf{L}}$ is defined using $e^{\mathbf{L}}$ so the same formula holds for it, and G follows from Proposition 4.9 too in a similar manner. QED

LEMMA 4.15. Let K be the reduced connected component of $K(\mathbf{L})$, where \mathbf{L} is a line bundle on an abelian variety X .

A) The null-space of $e^{\mathbf{L}}$ is $V(K)$, and $e^{\mathbf{L}}$ gives a non-degenerate pairing on $V(X)/V(K)$.

B) Also, for any prime $p \neq \text{char. } k$, the null-space of $e^{\mathbf{L}}$ on $V_p(X)$ is $V_p(K)$.

C) If $K(\mathbf{L})$ is finite and an isomorphism $\mathbf{A}_f \cong V(\mathbf{G}_m)$ is fixed, then there is an isomorphism $V(X) \cong \mathbf{A}_f^{2g}$ such that $A^{\mathbf{L}}$ becomes the form ${}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1$ for all $(x_1, x_2), (y_1, y_2) \in \mathbf{A}_f^{2g}$, and $T(X)$ becomes a subgroup of finite index in $\widehat{\mathbf{Z}}^{2g}$ (replace \mathbf{A}_f by $\mathbf{A}_f^{(p)}$ in characteristic $p > 0$).

PROOF A: The pairing $e^{\mathbf{L}} : K(\mathbf{L}) \times K(\mathbf{L}) \rightarrow \mathbf{G}_m$ is trivial when restricted to $K \times K(\mathbf{L})$ because K is complete, connected and reduced and $e^{\mathbf{L}}(0, x) = 1$. The same argument shows that $e^{n^*\mathbf{L}}$ is trivial on $K \times K(n^*\mathbf{L}) \supset K \times X_n$

for all $n \geq 1$ because $K \subset K(n^*\mathbf{L})$, and consequently, $e^{\mathbf{L}}(x, y) = 1$ for all $x \in V(K), y \in V(X)$. Denoting the induced pairing on $V(X)/V(K)$ by $\bar{e}^{\mathbf{L}}$ and the image of $T(X)$ in $V(X)/V(K)$ by T , we have $T^\perp/T = K(\mathbf{L})/K$ by 4.13, where

$$T^\perp = \{x \in V(X)/V(K) \mid \bar{e}^{\mathbf{L}}(x, y) = 1 \text{ for all } y \in T\}.$$

Thus $[T^\perp : T]$ is finite, showing that $e^{\mathbf{L}}$ is non-degenerate.

B) follows from A) by looking only at $T_p(X), V_p(X)$.

To prove B) implies C), we cite the normal form for non-degenerate skew-symmetric forms on each $V_p(X)$, defining $V_p(X) \xrightarrow{\sim} \mathbf{Q}_p^{2g}$ separately for each p such that $T_p(X)$ goes over to \mathbf{Z}_p^{2g} except for finitely many p . All these isomorphisms together give us the isomorphism of $V(X)$ with \mathbf{A}_f^{2g} . We leave the details to the reader. QED

We add the last bit of structure to $\widehat{\mathcal{G}}(\mathbf{L})$. Assume \mathbf{L} is isomorphic to $i^*\mathbf{L}$ where i is the inverse map. This induces an automorphism of $\widehat{\mathcal{G}}(\mathbf{L})$ which allows us to define a section τ to the map $\pi : \widehat{\mathcal{G}}(\mathbf{L}) \rightarrow V(X)$. The section τ is the key to an algebraic definition of ϑ .

PROPOSITION 4.16. Let $\text{Aut}^{\mathbf{L}} X$ be the group of all automorphisms f of X such that $f^*\mathbf{L} \cong \mathbf{L}$. Then there is an action of the group $\text{Aut}^{\mathbf{L}} X$ on $\widehat{\mathcal{G}}(\mathbf{L})$.

PROOF: Any $f \in \text{Aut} X$ induces a $j(f, \mathbf{L})$ as in Proposition 4.9:

$$\begin{array}{ccc} \widehat{\mathcal{G}}(f^*\mathbf{L}) & \xrightarrow{\pi} & V(X) \\ \downarrow j(f, \mathbf{L}) & & \downarrow f \\ \widehat{\mathcal{G}}(\mathbf{L}) & \xrightarrow{\pi} & V(X). \end{array}$$

By functoriality $j(f, \mathbf{L}) \circ j(g, f^*\mathbf{L}) = j(f \circ g, \mathbf{L})$. Now $f \in \text{Aut}^{\mathbf{L}}(X)$, so there exists $\lambda : f^*\mathbf{L} \cong \mathbf{L}$; further the isomorphism induced by $\lambda : \widehat{\mathcal{G}}(f^*\mathbf{L}) \rightarrow \widehat{\mathcal{G}}(\mathbf{L})$ is independent of the choice of λ (Cor. 4.8). Thus $j(f, \mathbf{L})$ can be regarded as an automorphism of $\widehat{\mathcal{G}}(\mathbf{L})$ when $f \in \text{Aut}^{\mathbf{L}} X$; furthermore if $f, g \in \text{Aut}^{\mathbf{L}} X$,

then $j(f, \mathbf{L})j(g, \mathbf{L}) = j(f \circ g, \mathbf{L})$. By Proposition 4.9, $j(f, \mathbf{L})$ acts on the exact sequence for $\widehat{\mathcal{G}}(\mathbf{L})$ as follows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \bar{k}^* & \longrightarrow & \widehat{\mathcal{G}}(\mathbf{L}) & \xrightarrow{\pi} & V(X) \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow j(f, \mathbf{L}) & & \downarrow V(f) \\ 1 & \longrightarrow & \bar{k}^* & \longrightarrow & \widehat{\mathcal{G}}(\mathbf{L}) & \xrightarrow{\pi} & V(X) \longrightarrow 0. \end{array}$$

QED

DEFINITION 4.17. A line bundle \mathbf{L} on X is symmetric if $(-1_X)^*\mathbf{L} \cong \mathbf{L}$, or equivalently if $-1_X \in \text{Aut}^{\mathbf{L}} X$. The involution $j(-1_X, \mathbf{L})$ on $\widehat{\mathcal{G}}(\mathbf{L})$ will be denoted by $i^{\mathbf{L}}$, so that there is an exact sequence:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \bar{k}^x & \longrightarrow & \widehat{\mathcal{G}}(\mathbf{L}) & \xrightarrow{\pi} & V(X) \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow i^{\mathbf{L}} & & \downarrow -1 \\ 1 & \longrightarrow & \bar{k}^x & \longrightarrow & \widehat{\mathcal{G}}(\mathbf{L}) & \xrightarrow{\pi} & V(X) \longrightarrow 0. \end{array}$$

This allows us to make the important

DEFINITION. Let $x \in V(X)$ and choose $y \in \widehat{\mathcal{G}}(\mathbf{L})$ such that $2\pi(y) = x$. Define $\tau(x) \in \widehat{\mathcal{G}}(\mathbf{L})$ to be $y i^{\mathbf{L}}(y)^{-1}$.

REMARK 1: $\pi(\tau(x)) = x$.

REMARK 2: $\tau(x)$ is independent of the choice of y . To show this note y can only be replaced by $\alpha \cdot y$ with $\alpha \in k^*$ and $\alpha y(i(\alpha y))^{-1} = y(iy)^{-1}$ because $i(\alpha y) = \alpha i(y)$.

We shall abbreviate $\sigma^{\mathbf{L}}, i^{\mathbf{L}}$, etc., to σ, i , etc. when there is no possibility of confusion.

PROPOSITION 4.18. Let \mathbf{L} be a symmetric line bundle on an abelian variety X .

A. $i \circ \sigma(x) = \sigma(-x)$ for $x \in T(X)$.

B. Any element of $\widehat{\mathcal{G}}(\mathbf{L})$ can be written uniquely as $\lambda \cdot \tau(x)$ with $\lambda \in \bar{k}^*$ and $x \in V(X)$; the multiplication table of $\widehat{\mathcal{G}}(\mathbf{L})$ is given in this set-up by

$$\lambda \tau(x) \cdot \mu \tau(y) = \lambda \mu e^{\mathbf{L}}\left(\frac{x}{2}, y\right) \tau(x+y) = \lambda \mu e\left(\frac{1}{2} A^{\mathbf{L}}(x, y)\right) \tau(x+y).$$

The involution i is given by $i(\lambda \tau(x)) = \lambda \tau(-x)$.

C. If we define $e_*^{\mathbf{L}}$ by $\sigma^{\mathbf{L}}(x) = e_*^{\mathbf{L}}(\frac{x}{2})\tau(x)$, for all $x \in T(X)$, then $e_*^{\mathbf{L}}$ is a quadratic form on $\frac{1}{2}T(X)$ with values in the group ± 1 ; $e^{\mathbf{L}}$ satisfies:

$$e_*^{\mathbf{L}}(x+y)e_*^{\mathbf{L}}(x)^{-1}e_*^{\mathbf{L}}(y)^{-1} = e^{\mathbf{L}}(x, y)^2.$$

PROOF: A. Lemma 4.12 says $j(f, \mathbf{L})\sigma^{f^*\mathbf{L}}(x) = \sigma^{\mathbf{L}}Vf(x)$ which gives for $f = i$

$$j(-1_X, \mathbf{L})\sigma^{-1_X^*\mathbf{L}} = \sigma^{\mathbf{L}}V(-1_X)x = \sigma^{\mathbf{L}}(-x),$$

which in our notation reads as $i^{\mathbf{L}}\sigma^{\mathbf{L}}(x) = \sigma^{\mathbf{L}}(-x)$.

B. We verify the multiplication formula: assume $2\pi(p) = x$ and $2\pi(q) = y$, then

$$\begin{aligned} \lambda \tau(x) \mu \tau(y) &= \lambda \mu p i(p)^{-1} q i(q)^{-1} \\ &= \lambda \mu e^{\mathbf{L}}(\pi(p i(p)^{-1}), \pi(q)) q p i(p)^{-1} i(q)^{-1} \\ &= \lambda \mu e^{\mathbf{L}}\left(x, \frac{y}{2}\right) q p \cdot i(q p)^{-1} \\ &= \lambda \mu e^{\mathbf{L}}\left(\frac{x}{2}, y\right) \tau(x+y). \end{aligned}$$

Also

$$i(\lambda \tau(x)) = i(\lambda p i(p)^{-1}) = \lambda i(p) p^{-1} = \lambda p^{-1} i(p) = \lambda \tau(-x).$$

C. $\sigma^{\mathbf{L}}(2x) = \sigma^{\mathbf{L}}(x)^2 = e_*^{\mathbf{L}}(\frac{x}{2})^2 \tau(x)^2 = e_*^{\mathbf{L}}(\frac{x}{2})^2 \tau(2x)$ by B. But $\tau(2x) = \sigma^{\mathbf{L}}(2x)$ for all $x \in T(X)$ because $\sigma^{\mathbf{L}}(x)$ is a lift of x so that by definition

$$\tau(2x) = \sigma^{\mathbf{L}}(x) i \sigma^{\mathbf{L}}(x)^{-1} = \sigma^{\mathbf{L}}(x) \sigma^{\mathbf{L}}(x) = \sigma^{\mathbf{L}}(x)^2.$$

This shows $e_*^{\mathbf{L}}(\frac{x}{2})^2 = 1$ for all $x \in T(X)$. It only remains to check that

$$e_*^{\mathbf{L}}(x+y)e_*^{\mathbf{L}}(x)^{-1}e_*^{\mathbf{L}}(y)^{-1} = e^{\mathbf{L}}(x, y)^2.$$

We have

$$\begin{aligned} \sigma^{\mathbf{L}}(x+y) &= \sigma^{\mathbf{L}}(x) \sigma^{\mathbf{L}}(y) \\ &= e_*^{\mathbf{L}}\left(\frac{x}{2}\right) e_*^{\mathbf{L}}\left(\frac{y}{2}\right) \tau(x) \tau(y) \\ &= e_*^{\mathbf{L}}\left(\frac{x}{2}\right) e_*^{\mathbf{L}}\left(\frac{y}{2}\right) e^{\mathbf{L}}\left(\frac{x}{2}, y\right) \tau(x+y) \end{aligned}$$

by the multiplication formula in B. On the other hand, by the definition of e_*^L we have

$$\sigma^L(x+y) = e_*^L\left(\frac{x+y}{2}\right)\tau(x+y).$$

QED

LEMMA 4.19. Any non-degenerate quadratic form in $2g$ variables over $\mathbf{Z}/2\mathbf{Z}$ is equivalent to either $\sum_{i=1}^g a_i b_i$ or $a_1^2 + a_1 b_1 + b_1^2 + \sum_{i=2}^g a_i b_i$.

PROOF: Let q be the quadratic form and $A(x, y) = q(x+y) - q(x) - q(y)$ the associated alternating form. If $g > 1$ and $x \neq 0$ there is certainly a non-zero y different from x such that $A(x, y) = 0$ because $\{y | A(x, y) = 0\}$ has 2^{2g-1} elements. For such a pair at least one of the values $q(x), q(y), q(x+y)$ is zero. So let $v_1 \neq 0$ be such that $q(v_1) = 0$ and choose v_2 such that $q(v_2) = 0$ and $A(v_1, v_2) = 1$, and take the orthogonal complement and proceed by induction until one has chosen a basis so that

$$q(a, b) = \sum_{i=1}^{g-1} a_i b_i + \epsilon_1 a_g^2 + \epsilon_2 a_g b_g + \epsilon_3 b_g^2.$$

The only possibilities for the last bit in a_g and b_g are

$$1) a_g^2 + a_g b_g + b_g^2 \quad 2) a_g b_g \quad 3) a_g(a_g + b_g) \quad 4) (a_g + b_g)b_g,$$

and 2), 3) and 4) are all equivalent. It only remains to show that the two quadratic forms in the statement of the lemma are inequivalent. Simply count the number of zeroes. The first has $2^{2g-1} + 2^{g-1}$ and the second has $2^{2g-1} - 2^{g-1}$ zeros. QED

From this, it is now easy to show

PROPOSITION 4.20. Let L be a symmetric ample line bundle of degree one on an abelian variety X . Fix an isomorphism, once and for all, of $(V(\mathbf{G}_m), T(\mathbf{G}_m))$ with $(\mathbf{A}_f, \hat{\mathbf{Z}})$. Then there is an isomorphism of $(\mathbf{A}_f^{2g}, \hat{\mathbf{Z}}^{2g})$ with $(V(X), T(X))$ so that A^L and e_*^L are now given by $A^L(u_i, v_j) = \delta_{ij}$

and $A^L(u_i, u_j) = A^L(v_i, v_j) = 0$, and

$$\begin{aligned} e_*^L\left(\frac{1}{2}\left(\sum_i x_i u_i + \sum_i y_i v_i\right)\right) &= (-1)^{\sum_i x_i y_i} \quad \text{or} \\ &= (-1)^{x_1^2 + x_1 y_1 + y_1^2 + \sum_{i \geq 2} x_i y_i}, \end{aligned}$$

where $(u_1, u_2, \dots, u_g, v_1, v_2, \dots, v_g)$ is a basis. Of course this means that the u_i and v_i form a $\hat{\mathbf{Z}}$ -basis for $T(X)$ and an \mathbf{A}_f -basis for $V(X)$. The L 's for which the first normal form holds are called "even symmetric" and those for which the second holds are called "odd symmetric".

One uses Proposition 4.14E, Proposition 4.18C, Lemma 4.19 and works a little to get to the proof of this fact. If the characteristic is p , replace \mathbf{A}_f and \mathbf{Z} by R_p, S_p .

We remark here that this problem of the two choices for e_* occurred earlier in §2. We had L , a maximal isotropic lattice in V , and found that the collection $(\pm 1, \ell)$ with $\ell \in L$ formed a subgroup of $\text{Heis}(2g, \mathbf{R})$. Calling this subgroup G for the moment, the exact sequence

$$0 \rightarrow \{\pm 1\} \rightarrow G \rightarrow L \rightarrow 1$$

has its splittings in one-to-one correspondence with quadratic forms: $\ell \mapsto ((-1)^{f(\ell)}, \ell)$ is a splitting if and only if $f(\ell_1 + \ell_2) - f(\ell_1) - f(\ell_2) \equiv A(\ell_1, \ell_2) \pmod{2}$. And under the symplectic group of the lattice L these quadratic forms break up into two orbits (Lemma 4.18). Only in the case $g = 1$ is there a canonical choice: $a^2 + ab + b^2$ has the property that any quadratic form is equivalent to this one. This corresponds to the fact that there is a canonical choice in an algebraic equivalence class of line bundles of degree 1 on an elliptic curve, but not so when $g > 1$.

The subgroups of $V(X)$ can be used to classify isogenies.

DEFINITION 4.21. A "good" isogeny $f: X \rightarrow Y$ of abelian varieties is a homomorphism which is surjective and has finite kernel whose order is prime to char. k . A rational "good" isogeny or a \mathbf{Q} -isogeny from X to Y is a triple

(Z, f_1, f_2) where Z is an abelian variety and $f_1 : Z \rightarrow X$ and $f_2 : Z \rightarrow Y$ are good isogenies. In what follows, we shall omit the adjective good, but always make the "prime-to- p " assumption. Two \mathbb{Q} -isogenies (Z, f_1, f_2) and (W, g_1, g_2) are equivalent if there is an abelian variety R and isogenies $a : R \rightarrow Z$ and $b : R \rightarrow W$ so that $f_i \circ a = g_i \circ b$ for $i = 1$ and 2 . A line bundle \mathbf{L} on X and a line bundle \mathbf{L}' on X' are related by a \mathbb{Q} -isogeny if there is a \mathbb{Q} -isogeny (Z, f_1, f_2) from X to X' such that $f_1^* \mathbf{L} = f_2^* \mathbf{L}'$.

Many of the formal properties of isogenies carry over to \mathbb{Q} -isogenies. Since an isogeny $f : X \rightarrow Y$ induces an isomorphism $V(f) : V(X) \rightarrow V(Y)$, a \mathbb{Q} -isogeny $\alpha = (Z, f_1, f_2)$ from X to Y induces an isomorphism $V(\alpha) : V(X) \rightarrow V(Y)$ defined by $V(\alpha) = V(f_2) \circ V(f_1)^{-1}$. Equivalent \mathbb{Q} -isogenies induce the same map.

LEMMA 4.22. *Equivalence classes of \mathbb{Q} -isogenies from X to Y for a fixed X are in 1-1 correspondence with compact open subgroups of $V(X)$, henceforth referred to as lattices.*

PROOF: We can describe the correspondence between isogenies and finite subgroups of X in a slightly different way. An isogeny induces:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(X) & \longrightarrow & V(X) & \longrightarrow & X_{\text{tor}} \longrightarrow 0 \\ & & \downarrow & & \downarrow V(f) & & \downarrow f \\ 0 & \longrightarrow & T(Y) & \longrightarrow & V(Y) & \longrightarrow & Y_{\text{tor}} \longrightarrow 0 \end{array}$$

as in Lemma 4.2. It follows that $\ker f$ can now be identified with the group $V(f)^{-1}T(Y)/T(X)$. Associate to a \mathbb{Q} -isogeny α from X to Y the lattice $L(\alpha) = V(\alpha)^{-1}T(Y)$. Note that $L(\alpha)$ depends only on the equivalence class of α .

Conversely, given a lattice $L \subset V(X)$ there is some $m > 1$ for which $mT(X) \subset L$. Put $Z = X$ and $f_1 = m_X$. Then $V(f_1)^{-1}L = M \subset V(Z)$ contains $T(Z)$ and $M/T(Z)$ gets identified to a finite subgroup H of Z . Now let $f_2 : Z \rightarrow Y$ be the quotient map of Z by H . It follows that $V(f_2)^{-1}T(Y) = M$ and therefore $L = V(f_1)M = V(f_1)V(f_2)^{-1}T(Y) = V(\alpha)^{-1}T(Y)$. QED

Let $f : X \rightarrow Y$ be an isogeny and let \mathbf{L}, \mathbf{M} be ample line bundles on X and Y respectively so that $f^*(\mathbf{M}) \cong \mathbf{L}$. We can recover \mathbf{M} from $\ker(f)$ plus a splitting of the natural map $\mathcal{G}(\mathbf{L}) \rightarrow K(\mathbf{L})$ over the subgroup $\ker(f) \subset K(\mathbf{L})$:

$$\begin{array}{ccc} \mathcal{G}(\mathbf{L}) & \longrightarrow & K(\mathbf{L}) \\ & \nwarrow & \uparrow \\ & & \ker(f). \end{array}$$

Equivalently, to give (Y, \mathbf{M}) we need to give a lattice L with $T(X) \subseteq L \subseteq V(X)$ and an extension of $\sigma : T(X) \rightarrow \hat{\mathcal{G}}(\mathbf{L})$ to L . This procedure generalizes to rational isogenies.

Let $\alpha = (Z, f_1, f_2)$ be a rational isogeny from X_1 to X_2 , and let \mathbf{L}_1 and \mathbf{L}_2 be line bundles related by α , so, $f_1^* \mathbf{L}_1 = f_2^* \mathbf{L}_2$. We write $\alpha^* \mathbf{L}_2 = \mathbf{L}_1$.

$$\begin{array}{ccc} & Z & \\ f_1 \swarrow & & \searrow f_2 \\ X_1 & \xrightarrow[\alpha]{} & X_2 \end{array}$$

Since $j(f_i, \mathbf{L}_i) : \hat{\mathcal{G}}(f_i^* \mathbf{L}_i) \rightarrow \hat{\mathcal{G}}(\mathbf{L}_i)$ are both isomorphisms we can define an isomorphism $j(\alpha) : \hat{\mathcal{G}}(\mathbf{L}_1) \rightarrow \hat{\mathcal{G}}(\mathbf{L}_2)$ by $j(\alpha) = j(f_2, \mathbf{L}_2) \circ j(f_1, \mathbf{L}_1)^{-1}$. Let $L(\alpha) = V(\alpha)^{-1}T(X_2)$ and define $\sigma(\alpha) : L(\alpha) \rightarrow \hat{\mathcal{G}}(\mathbf{L}_1)$ by $\sigma(\alpha)x = j(\alpha)^{-1}(\sigma^{\mathbf{L}_2}(V(\alpha)x))$ for all $x \in L(\alpha)$. Conversely if L is a lattice in $V(X_1)$ and $\sigma_L : L \rightarrow \hat{\mathcal{G}}(\mathbf{L}_1)$ agrees with $\sigma^{\mathbf{L}_1}$ on $L \cap T(X_1)$, then we can construct a rational isogeny $\alpha : X_1 \rightarrow X_2$ and a line bundle, \mathbf{L}_2 , on X_2 so that $\alpha^*(\mathbf{L}_2) \cong \mathbf{L}_1$.

PROPOSITION 4.23. *A. If \mathbf{L}_1 and \mathbf{L}_2 are both symmetric line bundles related by α , then $i^{\mathbf{L}_2} \cdot j(\alpha) = j(\alpha) \cdot i^{\mathbf{L}_1}$ and $i^{\mathbf{L}_1} \sigma(\alpha)x = \sigma(\alpha)(-x)$ for all $x \in L(\alpha)$.*

B. If \mathbf{L}_1 is assumed to be symmetric, then \mathbf{L}_2 is symmetric if and only if $i^{\mathbf{L}_1} \sigma_L(x) = \sigma_L(-x), \forall x \in L$.

C. Choose $a \in X_n$ and $\xi = (\xi_1, \xi_2, \dots) \in V(X)$ with $\xi_1 = a$. Let $\mathbf{E} = T_a^ \mathbf{L}$. Then $n_X^* \mathbf{L} \cong n_X^* \mathbf{E}$, i.e., $\alpha = (X, n_X, n_X)$ is a \mathbb{Q} -isogeny from X*

to itself by which \mathbf{L} and \mathbf{E} are related. The pair $(L(\alpha), \sigma(\alpha))$ associated to the triple (X, \mathbf{E}, α) , is then: $L(\alpha) = T(X)$ and $\sigma(\alpha) : T(X) \rightarrow \widehat{\mathcal{G}}(\mathbf{L})$ is given by $\sigma(\alpha)(x) = \bar{\xi} \sigma^{\mathbf{L}}(x) \bar{\xi}^{-1} = e^{\mathbf{L}}(\xi, x) \sigma^{\mathbf{L}}(x)$ where $\bar{\xi}$ is some lift of ξ to $\widehat{\mathcal{G}}(\mathbf{L})$.

D. In the notation of C if \mathbf{L} is symmetric and $2a = 0$, then \mathbf{E} is symmetric and $e_*^{\mathbf{E}}(x) = e_*^{\mathbf{L}}(x + \xi) e_*^{\mathbf{L}}(\xi)$.

PROOF: A. follows quickly from the definitions. For example, showing $i^{\mathbf{L}^2} j(\alpha) = j(\alpha) i^{\mathbf{L}}$ reduces to the case of a plain isogeny. B. is left to the reader.

C. For $g \in X_n$, let $\phi_g : n^* \mathbf{L} \cong T_g^* n^* \mathbf{L}$ be the descent data for \mathbf{L} . Let $b \in X$ with $nb = a$; let $\psi : n^* \mathbf{L} \cong T_b^* n^* \mathbf{L}$. The descent data for $T_a^* \mathbf{L}$ is just $\{T_b^* \phi_g\}$. The diagram

$$n^* \mathbf{L} \xrightarrow{\psi} T_b^* n^* \mathbf{L} \xrightarrow{T_b^* \phi_g} T_g^* T_b^* n^* \mathbf{L} \xrightarrow{T_g^* \psi^{-1}} T_g^* n^* \mathbf{L}$$

shows that

$$(\psi, b)^{-1} \cdot (\phi, g) \cdot (\psi, b)$$

is the descent data needed to construct $T_a^* \mathbf{L}$ from $n^* \mathbf{L} \cong T_b^* n^* \mathbf{L}$. To finish it is only necessary to formulate this in terms of the big group $\widehat{\mathcal{G}}(\mathbf{L})$.

D. If $2a = 0$, then the line bundle $\mathbf{E} = T_a^* \mathbf{L}$ is symmetric because

$$i^* \mathbf{E} = i^* T_a^* \mathbf{L} = T_{-a}^* i^* \mathbf{L} \cong T_a^* \mathbf{L}.$$

Define $j : \widehat{\mathcal{G}}(\mathbf{L}) \rightarrow \widehat{\mathcal{G}}(\mathbf{E})$ by fitting together the various

$$j_m : \mathcal{G}(m^* \mathbf{L}) \xrightarrow{\cong} \mathcal{G}(m^* \mathbf{E})$$

for m divisible by 2. For $\varphi \in \widehat{\mathcal{G}}(\mathbf{L})$, by the construction of j , $j(\varphi) \in \widehat{\mathcal{G}}(\mathbf{E})$ over the fiber $x \in X$ "is" $\varphi(x)$. Consequently $j \circ i^{\mathbf{L}}(\varphi)$ is $\varphi(-x)$ on the fiber over $x \in X$, but by the same kind of reasoning $i^{\mathbf{E}} \circ j(\varphi)$ "is" $\varphi(-x)$ over

$x \in X$. Hence $i^{\mathbf{E}} \circ j = j \circ i^{\mathbf{L}}$. This implies $\tau^{\mathbf{E}} = j \circ \tau^{\mathbf{L}}$: Let $2y = x \in V(X)$, and $\bar{y} \in \widehat{\mathcal{G}}(\mathbf{L})$ such that $\pi(\bar{y}) = y$. Then

$$\begin{aligned} \tau^{\mathbf{E}}(x) &= j(\bar{y})(i^{\mathbf{E}} \circ j(\bar{y}))^{-1} = j(\bar{y})(j \circ i^{\mathbf{L}}(\bar{y}))^{-1} \\ &= j(\bar{y} \circ i^{\mathbf{L}}(\bar{y})^{-1}) = j \circ \tau^{\mathbf{L}}(x). \end{aligned}$$

Apply j^{-1} to the equation $\sigma^{\mathbf{E}}(x) = \tau^{\mathbf{E}}(x) e_*^{\mathbf{E}}(x/2)$ and we get

$$\sigma(\alpha)(x) = \tau^{\mathbf{L}}(x) e_*^{\mathbf{E}}\left(\frac{x}{2}\right).$$

On the other hand, by 4.23C

$$\begin{aligned} \sigma(\alpha)(x) &= \sigma^{\mathbf{L}}(x) e^{\mathbf{L}}(\xi, x), \\ &= \tau^{\mathbf{L}}(x) e_*^{\mathbf{L}}\left(\frac{x}{2}\right) e^{\mathbf{L}}(\xi, x), \\ &= \tau^{\mathbf{L}}(x) e_*^{\mathbf{L}}\left(\frac{x}{2} + \xi\right) e_*^{\mathbf{L}}(\xi) \quad (\text{by 4.18C}). \end{aligned}$$

Now by comparing the RHS of the first and last equations we get D. QED

COROLLARY 4.24. If \mathbf{L} is a symmetric line bundle of degree one on X , there is a point $a \in X_2$ such that $T_a^* \mathbf{L}$ is even symmetric.

PROOF: By 4.23D, it suffices to prove that if q is a non-degenerate quadratic form in $2g$ variables with $\mathbf{Z}/2$ -coefficients, then there is a $y \in (\mathbf{Z}/2)^{2g}$ such that q_y defined by $q_y(x) = q(x + y) - q(y)$ and q are non-isomorphic quadratic forms. But if q has s zeros and q_y has t zeros, it is clear that $s = t$ if $q(y) = 0$ and $s + t = 2^{2g}$ if $q(y) = 1$. But $s \neq \frac{1}{2} \cdot 2^{2g}$, by Lemma 4.19, showing that any y with $q(y) = 1$ will do. QED

The following will be of use in §6.

LEMMA 4.25. If $e^{\mathbf{L}_1} = e^{\mathbf{L}_2}$ and $e_*^{\mathbf{L}_1} = e_*^{\mathbf{L}_2}$ for symmetric line bundles \mathbf{L}_1 and \mathbf{L}_2 on X , then $\mathbf{L}_1 \cong \mathbf{L}_2$ (assuming $\text{char } k \neq 2$).

PROOF: If $\mathbf{L} = \mathbf{L}_1 \otimes \mathbf{L}_2^{-1}$, then $e^{\mathbf{L}} = 1$ and $e_*^{\mathbf{L}} = 1$. By 4.15 we have $K(\mathbf{L}) = X$ and the group-scheme $\mathcal{G}(\mathbf{L})$ with involution i sits in the exact sequence: $1 \rightarrow \mathbf{G}_m \rightarrow \mathcal{G}(\mathbf{L}) \xrightarrow{\pi} X \rightarrow 0$. Mimicking the construction of τ , define $h : \mathcal{G}(\mathbf{L}) \rightarrow \mathcal{G}(\mathbf{L})$ by $h(x) = x \cdot i(x)^{-1}$. Then $h(\lambda x) = h(x)$ for $\lambda \in \mathbf{G}_m$

and $x \in \mathcal{G}(\mathbf{L})$ showing that $h = \phi \cdot \pi$ and $\pi \cdot \phi = 2_X$ where $\phi : X \rightarrow \mathcal{G}(\mathbf{L})$ is a morphism.

Now ϕ is actually a homomorphism: $\psi(x, y) = \phi(x + y)\phi(x)^{-1}\phi(y)^{-1}$ defines a morphism from $X \times X$ to \mathbf{G}_m such that $\psi(0, 0) = 1$, and therefore $\psi = 1$ (because $X \times X$ is projective and \mathbf{G}_m is affine). Also for any $x \in X_2$, $\phi(x) = e_*^{\mathbf{L}}(\xi)$ where $\xi = (x, \dots) \in \frac{1}{2}T(X)$. This shows that ϕ is trivial on X_2 so that $\phi = 2\tau$ where $\tau : X \rightarrow \mathcal{G}(\mathbf{L})$ is a homomorphism so that $\pi\tau = 1_X$. The morphism from $\mathbf{L}(0) \times X$ to \mathbf{L} given by $(\alpha, x) \rightarrow \tau(x)\alpha$ is an isomorphism and this finishes the proof that \mathbf{L} is the trivial line bundle. QED

A construction which makes the definition of $\mathcal{G}(\mathbf{L})$ and the theorems that followed more transparent is:

DEFINITION 4.26. Consider the so called "tower" of all isogenies $f : Y \rightarrow X$ (degrees prime-to- p if characteristic $k = p$):

$$\begin{array}{ccc} & \swarrow & \searrow \\ & Y_1 & \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

More precisely, we have a partially ordered set whose elements are isogenies $f : Y \rightarrow X$ and $f \succ g$ means there exists a homomorphism h with $f = goh$. Cofinal in this tower are the isogenies $n_X : X \rightarrow X$ under the ordering $n_X \succ m_X$ if $m|n$.

DEFINITION 4.27. Let \tilde{X} be the inverse limit of the tower on X so there are maps $f_n : \tilde{X} \rightarrow X$ so that $n_X f_{mn} = f_m$.

$$\tilde{X} = \varprojlim_{f:Y \rightarrow X} Y.$$

Put $f_1 = f$. The limit \tilde{X} has the structure of a proper k -scheme. Put $\mathbf{E} = f^*\mathbf{L}$.

A. Then $\hat{\mathcal{G}}(\mathbf{L})$ is the subgroup of $\mathcal{G}(\mathbf{E})$ which sits above $V(X) \subset K(\mathbf{E}) \subset$ geometric points of \tilde{X} .

B. X is the quotient of \tilde{X} by $T(X)$ and \mathbf{L} is the quotient of \mathbf{E} by $\sigma(T(X))$.

It may seem that \tilde{X} is a rather abstract and "unreal" sort of object. This is not so: to make it explicit, let $\{U_\alpha\}$ be an open affine cover of X , and consider the inverse and direct systems:

$$\begin{array}{ccccc} \Gamma(nm_X^{-1}U_\alpha, \mathcal{O}_X) & & (nm_X^{-1}(U_\alpha)) \subset & X & \\ \uparrow m_X^* & & & \downarrow m_X & \\ \Gamma(n_X^{-1}U_\alpha, \mathcal{O}_X) & & n_X^{-1}(U_\alpha) \subset & X & \\ \uparrow n_X^* & & & \downarrow n_X & \\ \Gamma(U_\alpha, \mathcal{O}_X) & & U_\alpha \subset & X. & \end{array}$$

Let $R_\alpha = \varprojlim_n \Gamma(n_X^{-1}U_\alpha, \mathcal{O}_X)$. Then $\text{Spec } R_\alpha$ is a scheme over U_α , and \tilde{X} is formed as the union of the affine opens $\text{Spec } R_\alpha$. In characteristic p , a more comprehensive theory will be obtained if we replace X by the inverse limit of all coverings

$$n_X : X \rightarrow X$$

including $n = p^k$. If $p|n$, n_X is inseparable, and $\ker(n_X)$ must be considered as a group scheme. We can construct $V(X)$ as a formal scheme, viz. the direct limit of schemes $f^{-1}(X_n) \subset \tilde{X}$ for all n . Likewise, we get a formal group scheme $\hat{\mathcal{G}}(\mathbf{L})$ extending $V(X)$. An immediate advantage is that for \mathbf{L} ample, $e^{\mathbf{L}}$ is non-degenerate on p -torsion too, and that in 4.22 we can treat all isogenies. Our primary interest is in characteristic 0 however and we will not discuss this further here.

We now tie together the algebraic approach of this section with the analytic approach of §3.

Now we shall put $k = \mathbf{C}$, $X = X_{T,L}$ as in §3 where $T \in \mathfrak{H}_g$ and $L \subset \mathbf{Z}^{2g}$, and \mathbf{L} the basic line bundle defined on X . For $x \in \mathbf{Q}^{2g}$, $\underline{x} = Tx_1 + x_2$, the sequence $\frac{1}{n}\underline{x}$ (or rather its image in $X_{T,L}$) gives an element of $V(X)$. Thus, there is a homomorphism from \mathbf{Q}^{2g} to $V(X)$, inducing an isomorphism $\mathbf{Q}^{2g} \otimes \mathbf{A}_f = \mathbf{A}_f^{2g} \cong V(X)$ so that $T(X)$ is the closure of the image of L .

Recall that \mathbf{L} was defined, using a particular action of $Heis(2g, \mathbf{R})$ on the trivial line bundle on \mathbf{C}^g in which $(\lambda, y) = (\lambda, y_1, y_2)$ acts on $\mathbf{C} \times \mathbf{C}^g$ by

$$(\alpha, z) \mapsto (\alpha \lambda^{-1} \varphi_y(z), z - \underline{y}) = (\alpha \lambda^{-1} \exp(\pi i^t(y_1) \cdot (2z - \underline{y})), z - \underline{y}).$$

This is the Fock action. The bundle \mathbf{L} is the quotient of $\mathbf{C} \times \mathbf{C}^g$ by the maximal isotropic subgroup of $Heis(2g, \mathbf{R})$ consisting of

$$\{(e_*(\frac{n}{2}), n) | n \in \mathbf{Z}^{2g}\}.$$

To connect the algebraic to the analytic constructions first note that we can replace the tower of coverings of X

$$\begin{array}{ccc} & \searrow & \swarrow \\ & X & \\ & \swarrow & \searrow \\ & X & \end{array} \quad \begin{array}{ccc} & \searrow & \swarrow \\ & X & \\ & \swarrow & \searrow \\ & X & \end{array}$$

with a single universal covering space \mathbf{C}^g . Then $a \in \mathbf{Q}^{2g}$ corresponds to the sequence $(\underline{a}, \dots, \frac{a}{n}, \dots)$ in $V(X)$. Secondly we relate $\widehat{\mathcal{G}}(\mathbf{L})$ to $Heis(2g, \mathbf{R})$. Let $b \in \mathbf{Q}^{2g}$, so b corresponds to a sequence $(\dots, \frac{b}{n}, \dots) \in V(X)$. Assume $Nb \in \mathbf{Z}^{2g}$, and $N|n$. Consider $n^*\mathbf{L}$ as the quotient of $\mathbf{C} \times \mathbf{C}^g$ by the pullback of multiplication by n of the relations defining \mathbf{L} . Then $(\lambda, \frac{b}{n}) \in Heis(2g, \mathbf{R})$ descends to an action on $n^*\mathbf{L}$, hence $(\lambda, \frac{b}{n}) \in K(n^*\mathbf{L})$. This defines a map

$$Heis(2g, \mathbf{R}) \supseteq Heis(2g, \mathbf{Q}) \rightarrow \widehat{\mathcal{G}}(\mathbf{L})$$

$$(\lambda, b) \mapsto (\dots, (\lambda, b/n), \dots),$$

and gives the diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{C}_1^* & \longrightarrow & Heis(2g, \mathbf{R}) & \xrightarrow{\pi} & \mathbf{R}^{2g} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbf{C}_1^* & \longrightarrow & Heis(2g, \mathbf{Q}) & \longrightarrow & \mathbf{Q}_{2g} \longrightarrow 0 \\ & & \downarrow \text{inverse} & & \downarrow & & \downarrow \text{inverse} \\ 1 & \longrightarrow & \mathbf{C}_1^* & \longrightarrow & \widehat{\mathcal{G}}(\mathbf{L}) & \longrightarrow & V(X) = \mathbf{A}_f^{2g} \longrightarrow 0. \end{array}$$

REMARK: This diagram of groups respects the various actions on $\mathbf{C} \times \mathbf{C}^g$ and $n^*\mathbf{L}$. Indeed part A of the proposition below follows immediately.

PROPOSITION 4.28. A. For $x, y \in V(X) = \mathbf{A}_f^{2g}$, we have

$$e^{\mathbf{L}}(x, y) = e^{(x_1 y_2 - x_2 y_1)^{-1}}.$$

B. The line bundle \mathbf{L} is symmetric and the subgroup $Heis(2g, \mathbf{Q})$ of $\widehat{\mathcal{G}}(\mathbf{L})$ is stable under the involution i . The element

$$(\lambda, z) \mapsto (\lambda \exp(\pi i^t y_1 (2z - \underline{y})), z - \underline{y}) = (\lambda \varphi_y(z), z - \underline{y})$$

of $Heis(2g, \mathbf{Q})$ has its image in $\widehat{\mathcal{G}}(\mathbf{L})$ equal to $\tau(y)$ whenever $y \in \mathbf{Q}^{2g}$.

PROOF: For $T \in \mathfrak{H}$, let \mathbf{L} be the basic line bundle on $X = \mathbf{C}^g / (T\mathbf{Z}^g \oplus \mathbf{Z}^g)$; the bundle $i^*\mathbf{L}$ is constructed as the quotient of $\mathbf{C} \times \mathbf{C}^g$ by the pullback via i of the action defining \mathbf{L} :

$$\begin{array}{ccc} (\alpha, z) & \longrightarrow & (\alpha \varphi_{-n}(-z) e_*(-\frac{n}{2})^{-1}, z - n) \\ \downarrow i & & \downarrow i \\ (\alpha, -z) & \longrightarrow & (\alpha \varphi_{-n}(-z) e_*(-\frac{n}{2})^{-1}, -z + n). \end{array}$$

Since $\varphi_{-n}(-z) e_*(-\frac{n}{2})^{-1} = \varphi_n(z) e_*(\frac{n}{2})$, the bundles $i^*\mathbf{L}$ and \mathbf{L} are the same.

We compute the map $\tau : \mathbf{Q}^{2g} \rightarrow Heis(2g, \mathbf{Q})$. Let $y \in \mathbf{Q}^{2g}$, and let g be an element of $Heis(2g, \mathbf{Q})$ that projects to $y/2$; then, by definition, $\tau(y) = g \cdot i^{\mathbf{L}}(g)^{-1}$. First we compute $i^{\mathbf{L}}(g)$. By definition $i^{\mathbf{L}}$ is the composition

$$\widehat{\mathcal{G}}(\mathbf{L}) \xrightarrow{\approx} \widehat{\mathcal{G}}(i^*\mathbf{L}) \xrightarrow{j(i, \mathbf{L})} \widehat{\mathcal{G}}(\mathbf{L}).$$

Since $i^*\mathbf{L} = \mathbf{L}$, the first map is the identity. If $\varphi \in \widehat{\mathcal{G}}(\mathbf{L})$, then $j(i, \mathbf{L})(\psi) = \varphi$ means $i^*\varphi = \psi$. Since $i^* \circ i^*$ is the identity, $j(i, \mathbf{L})$ is just i^* . If $g = (\lambda, y/2) \in Heis(2g, \mathbf{Q})$, then $i^{\mathbf{L}}(g)$ is the map

$$\mathbf{C} \times \mathbf{C}^g \longrightarrow \mathbf{C} \times \mathbf{C}^g$$

$$(\alpha, z) \mapsto (\alpha \lambda^{-1} \varphi_{y/2}(-z), z + y/2);$$

in terms of $Heis(2g, \mathbf{Q})$,

$$i^{\mathbf{L}}(g) = (\lambda \varphi_{y/2}(-z)^{-1} \varphi_{-y/2}(z), -y/2).$$

Plugging in the explicit formula for φ gives $i^{\mathbf{L}}(g) = (\lambda, -y/2)$. From this we see that

$$\tau(y) = g \cdot i^{\mathbf{L}}(g)^{-1} = (1, y).$$

QED

5. Algebraic theta functions

Our aim is to define a ϑ -function algebraically. Let \mathbf{L} be an ample symmetric line bundle on an abelian variety X , and let $s \in \Gamma(\mathbf{L})$ (if the characteristic is p , we assume $p \nmid \deg \mathbf{L}$). We will associate to every s a function ϑ_s on $V(X)$. We begin by studying the basic representation of $\hat{\mathcal{G}}(\mathbf{L})$:

DEFINITION 5.1. $\hat{\Gamma}(\mathbf{L}) = \varinjlim \Gamma(X, n_x^* \mathbf{L})$, where $\hat{\mathcal{G}}(\mathbf{L})$ acts on $\hat{\Gamma}(\mathbf{L})$ as follows: for $g = (x_n, \phi_n) \in \hat{\mathcal{G}}(\mathbf{L})$, $U_g(s) = \phi_n \circ s \circ T_{-x_n}$, which is independent of the n chosen. (In the above limit, the n 's are assumed prime to $\text{char}(k)$ if $\text{char}(k) > 0$.)

According to 4.15 and 4.18, $\hat{\mathcal{G}}(\mathbf{L})$ has the normal form $\bar{k}^* \times \mathbf{A}_f^{2g}$ or $\bar{k}^* \times R_p^{2g}$ ($\text{char} = 0$ or $\text{char} > 0$) with group law

$$(\lambda, x) \cdot (\mu, y) = (\lambda \mu e(\frac{1}{2}({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1)), x + y).$$

PROPOSITION 5.2. Consider \bar{k} -vector spaces W with an action U of the group $\text{Heis}(2g, \mathbf{A}_f)$ such that $U_\lambda = \lambda \cdot 1_W$ for $\lambda \in \bar{k}^*$, and for each $w \in W$ there is an m such that $U_{\sigma(mx)} w = w$ for all $x \in \hat{\mathbf{Z}}^{2g}$. We call these continuous representations.

Among all such representations there is a unique irreducible one which we call the Heisenberg representation \mathcal{H} , and any such representation is a direct sum of copies of \mathcal{H} . \mathcal{H} has the following models:

A. \mathcal{H} can be realized as the space of all locally constant \bar{k} -valued functions g on \mathbf{A}_f^g with compact support and the action of $\text{Heis}(2g, \mathbf{A}_f)$ is given by:

$$U_{(\lambda, y_1, y_2)} g(x) = \lambda e({}^t y_2 \cdot x + \frac{1}{2} {}^t y_1 \cdot y_2) g(x + y_1).$$

B. \mathcal{H} can be realized as the space of \bar{k} -valued functions f on \mathbf{A}_f^{2g} with compact support and quasi-periodicity:

$$f(x_1 + n_1, x_2 + n_2) = (-1)^{n_1 \cdot n_2} e(\frac{1}{2}({}^t x_1 \cdot n_2 - {}^t x_2 \cdot n_1)) \cdot f(x_1, x_2),$$

for all (n_1, n_2) in \mathbb{Z}^{2g} , and the action of $\text{Heis}(2g, \mathbf{A}_f)$ is given by

$$U_{(\lambda, y_1, y_2)} f(x_1, x_2) = \lambda e\left(\frac{1}{2}({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1)\right) f(x_1 + y_1, x_2 + y_2).$$

The map between these 2 realizations is given by

$$f(x_1, x_2) = N^{-g} \sum_{n \in (\widehat{\mathbb{Z}}^g / N\widehat{\mathbb{Z}}^g)} g(x_1 + n) e({}^t n \cdot x_2 + \frac{1}{2} {}^t x_1 \cdot x_2),$$

(N large enough so that g is constant on cosets of $N\widehat{\mathbb{Z}}^g$ and $x_2 \in \frac{1}{N}\widehat{\mathbb{Z}}^g$)

$$g(x_1) = \sum_{x_2 \in \mathbf{A}_f^g / \widehat{\mathbb{Z}}^g} f(x_1, x_2) e(-\frac{1}{2} {}^t x_1 \cdot x_2).$$

PROOF: Let $\sigma : \widehat{\mathbb{Z}}^{2g} \rightarrow \text{Heis}(2g, \mathbf{A}_f)$ be the section

$$\sigma(x_1, x_2) = ((-1)^{{}^t x_1 \cdot x_2}, x_1, x_2)$$

Let $G_m = N(\sigma(m\widehat{\mathbb{Z}}^{2g})) / \sigma(m\widehat{\mathbb{Z}}^{2g})$. This is a finite Heisenberg group.

For the representations V of $\text{Heis}(2g, \mathbf{A}_f)$ under consideration, it is true that $V = \cup_{m \geq 1} V_m$ where $V_m = \{v \in V \mid U_{\sigma(mx)} v = v, x \in \widehat{\mathbb{Z}}^{2g}\}$. Given a representation V of G_m , by the Stone-Von Neumann theorem it is isomorphic to $V^{\sigma(\widehat{\mathbb{Z}}^{2g})} \otimes \mathcal{H}_m$ where \mathcal{H}_m is the unique irreducible representation of G_m . Each V_m is a representation of G_m and hence $V_m \cong \mathcal{H}_m \otimes V_1$ canonically where \mathcal{H}_m is the irreducible representation of G_m with the following model: all functions f on $N(\sigma(m\widehat{\mathbb{Z}}^{2g}))$ such that $f(\lambda\sigma(my) \cdot x) = \lambda f(x)$ for all $x \in N(\sigma(m\widehat{\mathbb{Z}}^{2g}))$, $y \in \widehat{\mathbb{Z}}^{2g}$, $\lambda \in \bar{k}^*$. Extending these functions by zero outside $N(\sigma(m\widehat{\mathbb{Z}}^{2g}))$, \mathcal{H}_m is realized as a space of \bar{k} -valued functions on $\text{Heis}(2g, \mathbf{A}_f)$ and $\mathcal{H}_m \subset \mathcal{H}_{mn}$ for all m and n . Call \mathcal{H} the increasing union of the \mathcal{H}_m . The canonical isomorphisms $V_m \cong \mathcal{H}_m \otimes V_1$ for all m lead to an isomorphism $V \cong \mathcal{H} \otimes V_1$: equivalently, V is a direct sum of copies of \mathcal{H} . That \mathcal{H} is irreducible follows from the fact that $\mathcal{H}_1 = \{f \in \mathcal{H} \mid U_{\sigma(x)} f = f \text{ for all } x \in \widehat{\mathbb{Z}}^{2g}\}$ is one-dimensional. Thus if $0 \neq V \subset \mathcal{H}$, then $V^{\sigma(\widehat{\mathbb{Z}}^{2g})} \neq 0$ and consequently $V^{\sigma(\widehat{\mathbb{Z}}^{2g})} = \mathcal{H}_1$ which generates \mathcal{H} as a $\text{Heis}(2g, \mathbf{A}_f)$ -module. The rest of the proof we leave to the reader. QED

To apply this theorem, we prove:

PROPOSITION 5.3. $\widehat{\Gamma}(\mathbf{L})$ is the irreducible Heisenberg representation of $\widehat{\mathcal{G}}(\mathbf{L})$.

PROOF: When $\deg \mathbf{L} = 1$, this follows from observing that the subspace of $\widehat{\Gamma}(X, \mathbf{L})$ fixed by $\sigma T(X)$ is just $\Gamma(X, \mathbf{L})$ which is one-dimensional.

If $\deg \mathbf{L} > 1$ and H is a maximal isotropic subgroup of $K(\mathbf{L})$ then there is a line bundle \mathbf{L}' on $X/H = Y$ such that the pull-back of \mathbf{L}' is isomorphic to \mathbf{L} . As we have seen in the last section $\widehat{\mathcal{G}}(\mathbf{L}) \cong \widehat{\mathcal{G}}(\mathbf{L}')$, and clearly $\widehat{\Gamma}(X, \mathbf{L}) = \widehat{\Gamma}(Y, \mathbf{L}')$ so that we are reduced to the previous case. QED

Next the realizations constructed for the Heisenberg representation in Proposition 5.2 can be adapted to irreducible representation of $\widehat{\mathcal{G}}(\mathbf{L})$ as follows:

(B) becomes $C_k^c(V(X)//T(X))$, the space of \bar{k} -valued functions f on $V(X)$, with compact support that are quasi-periodic for $T(X)$:

$$f(x+t) = e_*\left(\frac{t}{2}\right) e\left(x, \frac{t}{2}\right) f(x) \quad \forall t \in T$$

with $\widehat{\mathcal{G}}(\mathbf{L})$ acting by

$$U_{\lambda \cdot \tau(y)}^{\text{right}} f(x) = \lambda \cdot e\left(\frac{y}{2}, x\right) f(x-y).$$

The superscript o denotes compact support.

To give a version of A, we fix $V(X) = V_1 \oplus V_2$, a decomposition into subspaces V_1, V_2 maximal isotropic for e . Then the analogue of (A) is $C_k^c(V_1)$, the space of \bar{k} -valued locally constant functions f on V_1 with compact support, with $\widehat{\mathcal{G}}(\mathbf{L})$ acting by

$$U_\lambda f = \lambda f,$$

$$U_{\tau(y)} f(x) = f(x+y) \quad \text{for } y \in V_1$$

$$U_{\tau(y)} f(x) = e(y, x) f(x) \quad \text{for } y \in V_2.$$

COROLLARY 5.4. *There are $\hat{\mathcal{G}}(\mathbf{L})$ -linear isomorphisms, unique up to a constant:*

$$\begin{aligned}\hat{\Gamma}(\mathbf{L}) &\cong C_k^0(V(X)//T(X)) \\ &\cong C_k^0(V_1).\end{aligned}$$

Next we are going to interpret members of $\hat{\Gamma}(X, \mathbf{L})$ as functions on $V(X)$. The following is the main definition of this book:

DEFINITION 5.5. *Fix an isomorphism $\epsilon : \mathbf{L}(0) \rightarrow k$. Let $x \in V(X)$ and assume $\tau(x) = (x_n, \phi_n)$. For each $s \in \hat{\Gamma}(X, \mathbf{L})$, define a k -valued function on $V(X)$, ϑ_s , by*

$$\vartheta_s(x) = \epsilon(\phi_n^{-1}s(x_n)).$$

This is defined of all sufficiently divisible n and it is independent of the n chosen.

Thus $\vartheta_s(x)$ is defined by the chain of maps

$$s(x_n) \in (n^*\mathbf{L})(x_n) = (T_{x_n}^* n^*\mathbf{L})(0) \xrightarrow{\phi_n(0)} n^*\mathbf{L}(0) \xrightarrow{\epsilon} k$$

There are 2 ways to interpret this definition, both quite important. One is that via the section τ of $\hat{\mathcal{G}}(\mathbf{L})$ over $V(X)$, we can trivialize the pull-back of the bundle \mathbf{L} to $V(X)$. To be precise, consider

$$\begin{array}{ccc} p^*\mathbf{L} & \longrightarrow & \mathbf{L} \\ \pi \downarrow & & \downarrow \pi \\ V(X) & \xrightarrow{p} & X. \end{array}$$

Then $\hat{\mathcal{G}}(\mathbf{L})$ acts as a group of automorphisms of $p^*\mathbf{L}$ and we define an isomorphism

$$A^1 \times V(X) \xrightarrow{\Phi} p^*\mathbf{L}$$

by requiring firstly that for all $x \in V(X)$, $\tau(x)$ should be identity map between the fibres over 0 and over x , i.e.,

$$\tau(x)[\Phi(\lambda, 0)] = \Phi(\lambda, x),$$

and requiring secondly that over 0, Φ is ϵ^{-1} . By this trivialization, each section of $p^*\mathbf{L}$ over $V(X)$ is just a function $f_s(x)$ via

$$(f_s(x), x) = \Phi^{-1}s(x).$$

It then follows that

$$\begin{aligned}s(x) &= \Phi((f_s(x), x)) \\ &= \tau(x)\Phi((f_s(x), 0)) \\ &= \tau(x)\epsilon^{-1}(f_s(x))\end{aligned}$$

or

$$\begin{aligned}f_s(x) &= \epsilon(\tau(x)^{-1}(s(x))) \\ &= \vartheta_s(x).\end{aligned}$$

Thus ϑ_s is the function corresponding to the section s .

This definition can be beefed-up to show that when X is a family of abelian varieties parametrized by a scheme S (an abelian scheme over S), then $V(X)$ over S becomes a direct limit of schemes $T_n(X)$ over S and ϑ_s becomes a family of morphisms

$$\begin{array}{ccc} T_n(X) & & \\ \downarrow & \searrow \vartheta_s^n & \\ T_{nm}(X) & \xrightarrow{\vartheta_s^{n+1}} & A_S^1 \\ \downarrow & \dots\dots & \\ \vdots & & \end{array}$$

We shall work this out in an appendix. For the present, we note only one important consequence which follows without any fancy apparatus:

PROPOSITION 5.6. *If X, \mathbf{L} and $s \in \Gamma(X, \mathbf{L})$ are all defined over the field k , then*

$$\vartheta_s(\sigma x) = \sigma \vartheta_s(x)$$

for all $\sigma \in \text{Gal}(\bar{k}/k)$, $x \in V(X)$.

PROOF: Note that every $\sigma \in \text{Gal}(\bar{k}/k)$ acts as automorphism of $\hat{\mathcal{G}}(\mathbf{L})$ and that $\sigma(\tau x) = \tau(\sigma x)$. The formula is then immediate from the definition of ϑ_s . QED

The second way of interpreting ϑ_s is as a matrix coefficient of the irreducible Heisenberg representation $\hat{\Gamma}(\mathbf{L})$ of $\hat{\mathcal{G}}(\mathbf{L})$. In fact, *evaluation at 0* defines linear functionals

$$\ell_0 : \Gamma(X, n_X^* \mathbf{L}) \longrightarrow \mathbf{L}(0) \xrightarrow{\sim} k,$$

and passing to the limit, a linear functional

$$\ell_0 : \hat{\Gamma}(\mathbf{L}) \longrightarrow k.$$

Then for all $s \in \hat{\Gamma}(\mathbf{L})$,

$$x \longmapsto \ell_0(U_{\tau(x)} s)$$

is a matrix coefficient of this representation. I claim this is essentially $\vartheta_s(x)$.

In fact, say $\tau(x)$ is represented by (ϕ_n, x_n) and s by $s_n \in \Gamma(X, n_x^* \mathbf{L})$. Then

$$(U_{\tau(x)} s)(y) = \phi_n(s(y - x_n)),$$

hence

$$\ell_0(U_{\tau(x)} s) = \epsilon(\phi_n(s(-x_n))).$$

Since $\tau(-x)$ is represented by $(\phi_n^{-1}, -s_n)$, it follows that

$$\vartheta_s(x) = \ell_0(U_{\tau(-x)} s).$$

This interpretation makes it clear that $\vartheta_{U_{g(s)}}$ can be computed from ϑ_s for all $g \in \hat{\mathcal{G}}(\mathbf{L})$. In fact:

$$\text{LEMMA 5.7. } \vartheta_{U_{\lambda \cdot \tau(y)}(s)}(x) = \lambda \cdot e(y, \frac{x}{2}) \vartheta_s(x - y).$$

PROOF:

$$\begin{aligned} \vartheta_{U_{\lambda \cdot \tau(y)}(s)}(x) &= \ell_0(U_{\tau(-x)} U_{\lambda \tau(y)}(s)) \\ &= \ell_0(U_{\lambda e(\frac{-x}{2}, y) \tau(y-x)}(s)) \\ &= \lambda e(y, \frac{x}{2}) \ell_0(U_{\tau(y-x)}(s)) \\ &= \lambda e(y, \frac{x}{2}) \vartheta_s(x - y). \end{aligned}$$

QED

Thus one of the functions ϑ_s determines the rest. We express this as follows: as above let $C_k(V(X))$ be the space of all locally constant \bar{k} -valued function on $V(X)$, and let $\hat{\mathcal{G}}(\mathbf{L})$ act on $C_k(V(X))$ by the right action, so

$$U_{\lambda \cdot \tau(y)}^{\text{right}} f(x) = \lambda e(\frac{y}{2}, x) f(x - y).$$

Then the map

$$s \longmapsto \vartheta_s$$

$$\hat{\Gamma}(\mathbf{L}) \hookrightarrow C_k(V(X))$$

is linear with respect to these actions of $\hat{\mathcal{G}}(\mathbf{L})$.

DEFINITION 5.8. When $\deg \mathbf{L} = 1$, let $0 \neq s \in \Gamma(X, \mathbf{L})$ and put $\vartheta^\alpha(x) = \vartheta_s(x)$, a \bar{k} -valued function on $V(X)$.

We shall see below that if $k = \mathbf{C}$, $X = X_{T, \mathbf{Z}^{2g}}$, then the ϑ^α above coincides with the ϑ^α of §3 on \mathbf{Q}^{2g} .

From Lemma 5.7 and the invariance of $s \in \Gamma(X, \mathbf{L})$ under $\sigma T(X)$, we have

$$\text{LEMMA 5.9. } e(\frac{y}{2}, x) e_*(\frac{y}{2}) \vartheta^\alpha(x) = \vartheta^\alpha(x + y) \text{ for all } x \in V(X), y \in T(X).$$

PROOF: Using that $e_*(\frac{y}{2}) \tau(y) s = s$, we have

$$\begin{aligned} \vartheta^\alpha(x + y) &= \epsilon(\tau(x + y)^{-1} s(x + y)) \\ &= \epsilon(\tau(x)^{-1} \tau(y)^{-1} e(\frac{y}{2}, x) s(x + y)) \\ &= e(\frac{y}{2}, x) e_*(\frac{y}{2}) \epsilon(\tau(x)^{-1} s)(x) \\ &= e(\frac{y}{2}, x) e_*(\frac{y}{2}) \vartheta^\alpha(x) \end{aligned}$$

QED

Let us see what happens now if we investigate the space of all matrix coefficients of the representation of $\hat{\mathcal{G}}(\mathbf{L})$ on $\Gamma(\mathbf{L})$, as we did in §2 for the real Heisenberg representation. We start with some linear algebra revolving around the actions of $\hat{\mathcal{G}}(\mathbf{L})$ on C_k , the space of locally constant k -valued functions on $V(X)$. There are two actions:

$$U_{\lambda \tau(y)}^{\text{left}} f(x) = \lambda^{-1} e(\frac{y}{2}, x) f(x + y)$$

$$U_{\lambda \tau(y)}^{\text{right}} f(x) = \lambda e(\frac{y}{2}, x) f(x - y)$$

One checks that

$$U_{\lambda\tau(y)}^{\text{left}} \circ U_{\lambda\tau(y)}^{\text{right}} = U_{\lambda\tau(y)}^{\text{right}} \circ U_{\lambda\tau(y)}^{\text{left}}.$$

Let $C_k(V//T)$ be the space of functions

$$\begin{aligned} & \{f \in C_k \mid U_{\sigma(y)}^{\text{right}} f = f, \forall y \in T(X)\} \\ & = \{f \in C_k \mid f(x+y) = e_*(\frac{y}{2})e(\frac{y}{2}, x)f(x), \forall y \in T(X)\}, \end{aligned}$$

and recall that $C_k^o(V//T)$ is the space of functions on V that have compact support and satisfy

$$f(x+y) = e_*(\frac{y}{2})e(x, \frac{y}{2})f(x), \quad \forall y \in T(X),$$

and that $C_k^o(V//T)$ is an irreducible right $\widehat{\mathcal{G}}(\mathbf{L})$ module.

LEMMA. There is a perfect pairing

$$C_k(V//T) \times C_k^o(V//T) \longrightarrow k$$

respecting the action of $\widehat{\mathcal{G}}(\mathbf{L})$ up to a change of sign in argument:

$$\langle (U_{\lambda\tau(-y)}^{\text{left}})^{-1} f, g \rangle = \langle f, U_{\lambda\tau(y)}^{\text{right}} g \rangle.$$

PROOF: Define

$$\langle f, g \rangle = \sum_{x \in V(X)/T(X)} f(x)g(x).$$

This is well-defined: if $t \in T(X)$, then

$$\begin{aligned} g(x+t)f(x+t) &= g(x)e_*(\frac{t}{2})e(x, \frac{t}{2})f(x)e_*(\frac{t}{2})e(\frac{t}{2}, x) \\ &= g(x)f(x). \end{aligned}$$

We check the assertion regarding group actions:

$$\begin{aligned} \sum_{x \in V//T} ((U_{\lambda\tau(-y)}^{\text{left}})^{-1} f)(x)g(x) &= \sum_{x \in V//T} (U_{\lambda^{-1}\tau(y)}^{\text{left}} f)(x)g(x) \\ &= \sum_{x \in V//T} \lambda e(\frac{y}{2}, x)f(x+y)g(x). \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{x \in V//T} f(x)(U_{\lambda\tau(y)}^{\text{right}} g)(x) &= \sum_{x \in V//T} f(x)\lambda e(\frac{y}{2}, x)g(x-y) \\ &= \sum f(x+y)\lambda e(\frac{y}{2}, x+y)g(x). \end{aligned}$$

QED

In the case of real Heisenberg representations we found that a choice of $T \in \mathfrak{H}$ gives two subspaces of the space of functions on \mathbf{R}^{2g} , one a right $\text{Heis}(2g, \mathbf{R})$ module, the other a left $\text{Heis}(2g, \mathbf{R})$ module; furthermore the functions in one space enjoy the quasi-periodicity of theta while the functions in the other space are analytic with respect to the complex structure associated with T . We have the adelic analogue of this. We assume $\deg \mathbf{L} = 1$ and $0 \neq s_0 \in \Gamma(X, \mathbf{L})$. Then we get the diagram of spaces:

$$\begin{array}{ccc} \widehat{\Gamma}(\mathbf{L}) = \left\{ \begin{array}{l} \text{space of functions} \\ \vartheta_s(x) = \ell_o(U_{\tau(-x)}s), \\ \text{all } s \in \widehat{\Gamma}(\mathbf{L}) \end{array} \right\} & \subset & \left\{ \begin{array}{l} \text{space of functions} \\ f(x) = \sum \ell_i(U_{\tau(-x)}s_i), \\ \ell_i \in \text{Hom}(\widehat{\Gamma}(\mathbf{L}), k) \\ s_i \in \widehat{\Gamma}(\mathbf{L}) \end{array} \right\} = V \\ \Downarrow & & \\ \left\{ \begin{array}{l} \ell_o(U_{\tau(-x)}s_o) \\ \parallel \\ \{\vartheta^\alpha(x)\} \end{array} \right\} & \in & \left\{ \begin{array}{l} \text{space of functions} \\ f(x) = \ell(U_{\tau(-x)}s_o) \\ \text{all linear functionals} \\ \ell: \widehat{\Gamma}(\mathbf{L}) \rightarrow k \end{array} \right\} = V_1. \\ & & \parallel \\ & & \widehat{\Gamma}(\mathbf{L})^* \end{array}$$

PROPOSITION 5.10.

- A) $V =$ the space $C_k'(V(X))$ of all $f \in C_k(V(X))$ such that for some N we have $U_{\lambda\tau(y)}^{\text{right}} f = f$, all $y \in N \cdot T(X)$.
- B) $V_1 =$ the space $C_k(V(X)//T(X))$ of all $f \in C_k$ such that $U_{\sigma(y)}^{\text{right}} f = f$, all $y \in T(X)$, i.e.,

$$f(x+y) = e_*(\frac{y}{2})e(\frac{y}{2}, x)f(x), \quad \text{all } y \in T(X), x \in V(X).$$

- C) There is an isomorphism of $\widehat{\mathcal{G}}(\mathbf{L}) \times \widehat{\mathcal{G}}(\mathbf{L})$ -modules

$$V \cong \mathcal{H} \otimes_k \mathcal{H}^*$$

such that for some e_1, e_2 :

$$\vartheta^\alpha \text{ corresponds to } e_1 \otimes e_2$$

$$\Gamma(\mathbf{L}) \text{ corresponds to } \mathcal{H} \otimes e_2$$

$$V_1 \text{ corresponds to } e_1 \otimes \mathcal{H}^*.$$

Here the action U^{right} of $\widehat{\mathcal{G}}(\mathbf{L})$ on V restricts to the given action of $\widehat{\mathcal{G}}(\mathbf{L})$ on $\widehat{\Gamma}(\mathbf{L})$ and the action U^{left} of $\widehat{\mathcal{G}}(\mathbf{L})$ restricts on V_1 to the dual action of $\widehat{\mathcal{G}}(\mathbf{L})$ on $\widehat{\Gamma}(\mathbf{L})^*$. So if $f \in C_k$ satisfies the quasi-periodicity condition in B then it corresponds to an element of the form $e_1 \otimes h$ for some $h \in \mathcal{H}^*$.

PROOF: We begin with (B). Since $\widehat{\Gamma}(\mathbf{L})$ is an irreducible representation of $\widehat{\mathcal{G}}(\mathbf{L})$ we know there is an isomorphism

$$\psi : C_k^0(V(X)//T(X)) \xrightarrow{\sim} \widehat{\Gamma}(\mathbf{L}).$$

Note that $e_*(\frac{x}{2})\delta_{T(X)}(x)$ is the unique $\sigma(T(X))$ -invariant function in $C_k^0(V(X)//T(X)$ (here $\delta_*(x)$ is the function with value 1 if $x \in S$, 0 if $x \notin S$). Therefore $\psi(e_*(\frac{x}{2})\delta_{T(X)}(x)) = s_o$. Using the $\widehat{\mathcal{G}}(\mathbf{L})$ -equivariance of ψ , it follows that for all $y \in V(X)$

$$\begin{aligned} \psi(e_*(\frac{x+y}{2})e(\frac{-y}{2}, x)\delta_{T(X)-y}(x)) &= \psi(U_{\tau(-y)}^{\text{right}} e_*(\frac{x}{2})\delta_{T(X)}(x)) \\ &= U_{\tau(-y)} \psi(e_*(\frac{x}{2})\delta_{T(X)}(x)) \\ &= U_{\tau(-y)} s_o. \end{aligned}$$

Now we identify $C_k(V//T)$ with the dual of $C_k^0(V//T)$. Using our description of ψ we calculate the $\widehat{\mathcal{G}}(\mathbf{L})$ isomorphism

$$\psi^* : \widehat{\Gamma}(\mathbf{L})^* \xrightarrow{\sim} C_k(V(X)//T(X)).$$

If $\psi^*(\ell) = g$, then

$$\ell(U_{\tau(-y)} s_o) = (e_*(\frac{x+y}{2})e(\frac{-y}{2}, x)\delta_{T(X)-y}(x), g(x)) = g(-y).$$

Thus the matrix coefficient $\ell(U_{\tau(-y)} s_o)$ (except for reversing the sign in y) is the element g of $C_k(V(X)//T(X)$, and since ψ^* is an isomorphism, this proves (B).

To prove (A) and (C), note that the matrix coefficient map

$$s \otimes \ell \longmapsto \text{the function } f(x) = \ell(U_{\tau(-x)} s)$$

is always a map

$$\phi : \widehat{\Gamma}(\mathbf{L}) \otimes \widehat{\Gamma}(\mathbf{L})^* \longrightarrow C_k(V(X))$$

and with our conventions it will carry the action of $\widehat{\mathcal{G}} \times \widehat{\mathcal{G}}$ on the left hand side to the action of $\widehat{\mathcal{G}} \times \widehat{\mathcal{G}}$ by $U^{\text{right}} \times U^{\text{left}}$ on the right hand side. We verify this:

$$U_{\tau(y)}^{\text{right}}(s \otimes \ell) = ((U_{\tau(y)} s) \otimes \ell).$$

This maps to the function

$$\begin{aligned} [\ell(U_{\tau(y)} s)](x) &= \ell(U_{\tau(-x)\tau(y)} s) \\ &= e(\frac{-x}{2}, y) f(x-y) \\ &= (U_{\tau(y)}^{\text{right}} f)(x). \end{aligned}$$

On the other hand

$$(U_{\tau(y)}^{\text{left}})(s \otimes \ell) = s \otimes (U_{\tau(y)}^{\text{left}} \ell)$$

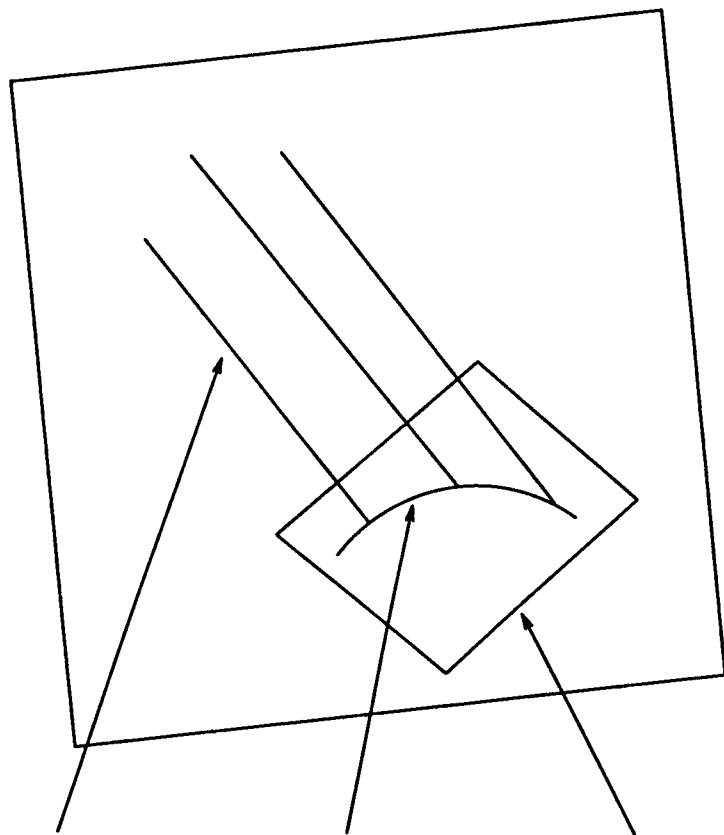
which maps to

$$\begin{aligned} (U_{\tau(y)}^{\text{left}} \ell)(U_{\tau(-x)} s) &= \ell(U_{\tau(-y)\tau(-x)} s) \\ &= \ell(U_{\tau(-x-y)} e(\frac{y}{2}, x) s) \\ &= f(x+y) e(\frac{y}{2}, x) \\ &= (U_{\tau(y)}^{\text{left}} f)(x). \end{aligned}$$

The action of $\widehat{\mathcal{G}}$ on $\Gamma(\mathbf{L})$ is continuous in the sense of 5.2: $\forall w \in \Gamma(\mathbf{L})$, $\exists m$ s.t. $U_{\sigma(mx)} w = w$, all $x \in T(X)$. Thus under U^{right} , the image of ϕ is continuous, i.e., $\text{Im } \phi \subset C'_k(V(X))$. By Proposition 5.2, C'_k is isomorphic

in the U^{right} -action to $\mathcal{H} \otimes [\sigma(T(X))\text{-invariant subspace}]$, i.e., to $\mathcal{H} \otimes C_k(V/T)$. Thus ϕ is an isomorphism if its restriction to the $\sigma(T(X))$ -invariant subspace is an isomorphism. But this restriction is ψ^* . This proves (A) and (C). QED

We can describe the situation by a picture as in §3:



images of $\hat{\Gamma}(\mathbf{L})$
under ϑ for various
abelian varieties
irred. by U^{right}
action of Heis

Mysterious subset
of those fcn.s. which
occur as ϑ^α for some
abelian variety over k
some symplectic isom.
($V(X), T(X) \cong (\mathbf{A}_f^{2g}, \hat{\mathbf{Z}}^{2g})$)

$C_k(\mathbf{A}_f^{2g} // \hat{\mathbf{Z}}^{2g})$
'possible' functions ϑ^α
Dual of irreducible
Heisenberg module

Note that one feature that is missing, compared to §3, is the characterization of the image of $\hat{\Gamma}(\mathbf{L})$ as the holomorphic functions for various abelian varieties. An obvious question is to try to characterize those functions which arise as ϑ^α in some more algebraic way. We will try to do this in the next section and in §11. Another way of phrasing this problem is to use realization A of the Heisenberg representation. Choose $V(X) = V_1 \oplus V_2$. We get

$$\hat{\Gamma}(\mathbf{L}) \cong C_k^2(V(X)//T(X)) \cong C_k^2(V_1)$$

Note that the dual of $C_k^2(V_1)$ is the space $\mathcal{M}_k(V_1)$ of all finitely additive \bar{k} -valued measures μ on the Boolean algebra of all compact open subsets of V_1 , hence

$$\hat{\Gamma}(\mathbf{L})^* \cong C_k(V(X)//T(X)) \cong \mathcal{M}_k(V_1)$$

Thus $\ell_o \in \hat{\Gamma}(\mathbf{L})^*$ (or $\vartheta^\alpha \in C_k(V//T)$) defines, up to scalars, a measure μ . Can we characterize those measures μ which arise from abelian varieties?

Let us go back to the classical case: Let $T \in \mathfrak{H}$, $X = X_{T,L}$ and \mathbf{L} the basic line bundle and let's see what $\hat{\Gamma}, \hat{\mathcal{G}}, \vartheta_s, \vartheta^\alpha$ and μ are. The vector space $\mathbf{L}(0)$ is canonically identified with the fibre of the trivial line bundle $\mathbf{C} \times \mathbf{C}^g$ at 0 and thus there is a natural $\epsilon : \mathbf{L}(0) \cong \mathbf{C}$. We pick up the situation as in Proposition 4.28 and the remarks preceding it, as well as §2, big diagram.

PROPOSITION 5.11. A. $\Gamma(\hat{X}, \mathbf{L})$ is the linear space spanned by $\vartheta_{a,b}(z, T)$ where $(a, b) \in \mathbf{Q}^{2g}$, and in fact the $\vartheta_{a,b}(z, T)$ for $(a, b) \in \mathbf{Q}^{2g}/\mathbf{Z}^{2g}$ form a basis.

B. The natural action of $\text{Heis}(2g, \mathbf{Q})$ on the span of the $\vartheta_{a,b}$'s coincides with the action of $\hat{\mathcal{G}}(\mathbf{L})$ on $\hat{\Gamma}(X, \mathbf{L})$ restricted to $\text{Heis}(2g, \mathbf{Q})$.

C. For any $x \in \mathbf{Q}^{2g}$, $s \in \Gamma(X, \mathbf{L})$ (s is identified with an entire function on \mathbf{C}^g),

$$\vartheta_s(x) = \exp \pi i^t x_1 (Tx_1 + x_2) \cdot s(Tx_1 + x_2).$$

D. In particular, when $L = \mathbf{Z}^{2g}$, $\deg \mathbf{L} = 1$ and $x \in \mathbf{Q}^{2g}$,

$$\vartheta^\alpha(x) = \vartheta(Tx_1 + x_2, T) \exp \pi i^t x_1 (Tx_1 + x_2) \quad (\text{defined as } \vartheta^\alpha[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}](T))$$

whenever $x \in \mathbf{Q}^{2g}$.

E. From Proposition 4.28, $\mathbf{A}_f^g \times 0$ and $0 \times \mathbf{A}_f^g$ are maximal isotropic subspaces of $\mathbf{A}_f^{2g} = V(X)$. Call them V_1 and V_2 respectively. Then the finitely additive measure μ on V_1 (now identified with $\mathbf{A}_f^g \times 0$) corresponding to $\ell_o \in \mathcal{H}^*$ is given (up to scalars) by

$$\mu(U) = \sum_{x \in U \cap \mathbf{Q}^g} \exp \pi i^t x T x \quad \text{for all compact open } U \subset \mathbf{A}_f^g.$$

Thus μ is countably additive when restricted to compact subsets of \mathbf{A}_f^g (i.e., a Radon measure on \mathbf{A}_f^g) and is totally singular, being supported exactly on the countable subset $\mathbf{Q}^g \subset \mathbf{A}_f^g$.

PROOF: A follows from Proposition 3.2.

Parts B, C and D follow without any tedium in our set-up. In particular, B is contained in Proposition 4.27. The specific formulae in C and D rely on Proposition 4.28B. When $x \in \mathbf{Q}^{2g} \subset V(X)$, $\tau(x)$ is the transformation

$$(\lambda, z) \longmapsto (\lambda \exp \pi i^t x_1(-\underline{x} + 2z), z - \underline{x}).$$

Therefore $\tau(x)^{-1}$ takes $(s(\underline{x}), \underline{x})$ to $(s(\underline{x}) \exp \pi i^t x_1 \underline{x}, 0)$ showing that

$$\vartheta_s(x) = s(\underline{x}) \exp \pi i^t x_1 \underline{x}$$

for all $x \in \mathbf{Q}^g$. D follows by putting $s(\underline{x}) = \vartheta(\underline{x}, T)$.

We prove E. Let \mathcal{H} be the space of locally constant functions on \mathbf{A}_f^g and $\psi: \widehat{\Gamma}(\mathbf{L}) \longrightarrow \mathcal{H}$ the unique equivariant map. The evaluation at 0 map: $\widehat{\Gamma}(\mathbf{L}) \longrightarrow k$ induces via ψ a measure $d\mu$ on \mathbf{A}_f^g so that

$$\int \psi(t) d\mu = t(0) \quad t \in \widehat{\Gamma}(\mathbf{L}).$$

Since $\deg \mathbf{L} = 1, 0 \neq s \in \Gamma(\mathbf{L})$ corresponds, under ψ , to δ , the characteristic function of \mathbf{Z}^g . Since $\vartheta_{U_{\lambda, \tau(y)} t}(0) = \lambda \vartheta_t(-y)$ we have

$$\begin{aligned} \vartheta^\alpha[x](T) &= \vartheta_s(x) = \vartheta_{\tau(-x)s}(0) = \int \psi(\tau(-x)s) d\mu \\ &= \int_{U_{(1, -x)}(\delta)} d\mu. \end{aligned}$$

Recall that $\vartheta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(T) = \sum e(\frac{1}{2} {}^t n T n + {}^t n(Tx_1 + x_2) + \frac{1}{2} {}^t x_1(Tx_1 + x_2))$. To complete the proof we must verify that for the measure μ in E that

$$\begin{aligned} \vartheta^\alpha \begin{bmatrix} x_1 \\ 0 \end{bmatrix}(T) &= \int (U_{(1, -x_1, 0)} \delta) d\mu = \int_{\mathbf{Z}^g + x_1} d\mu, \\ \vartheta^\alpha \begin{bmatrix} 0 \\ x_2 \end{bmatrix}(T) &= \int (U_{(1, 0, -x_2)} \delta) d\mu = \int_{\mathbf{Z}^g} e({}^t x_2 \cdot y) d\mu. \end{aligned}$$

This is straightforward. QED

When $k = \mathbf{C}$ we may also relate the real and adelic Heisenberg representations. We do this in an appendix to this section.

Proposition 5.11 plus Proposition 5.4 have the following corollary:

COROLLARY 5.12. Let $T \in \mathfrak{H}_g$ and consider the abelian varieties $X_{T, \mathbf{Z}^{2g}}$, i.e.,

$$X_T = \mathbf{C}^g / (T \cdot \mathbf{Z}^g + \mathbf{Z}^g)$$

and the basic line bundle \mathbf{L} on it. Also, for all $a, b \in \mathbf{Q}^g$ and let

$$x(a, b)_m = \text{Image in } X_T \text{ of } \frac{1}{m}(Ta + b).$$

Then suppose that $(a, b) \in \frac{1}{n} \mathbf{Z}^{2g}$, $k \subset \mathbf{C}$ is a subfield such that X_T and \mathbf{L} can be defined over k and $x(a, b)_{2n}$ is rational over k . Then

$$\vartheta^\alpha \begin{bmatrix} a \\ b \end{bmatrix}(T) / \vartheta^\alpha \begin{bmatrix} 0 \\ 0 \end{bmatrix}(T) \in k.$$

PROOF: The Galois group $\text{Aut}(\mathbf{C}/k)$ acts on X_T , hence on $V(X)$, which is isomorphic to \mathbf{A}_f^{2g} . This representation does not preserve the "lattice" \mathbf{Q}^{2g} in \mathbf{A}_f^{2g} . However, if $x \in V(X)$ has its image $x_0 \in X \bmod T(X)$ rational over k , then $\sigma x - x \in T(X)$, all $\sigma \in \text{Aut}(\mathbf{C}/k)$. Therefore for the point $(a, b) \in \mathbf{Q}^{2g} \subset V(X)$, the hypotheses of the Corollary imply that for all $\sigma \in \text{Aut}(\mathbf{C}/k)$:

$$\sigma(a, b) = (a, b) + \delta$$

some $\delta \in 2nT(X)$.

Now using the equality of the analytic ϑ^α with the algebraic ϑ^α , we get

$$\begin{aligned} \left(\frac{\vartheta^\alpha \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right](T)}{\vartheta^\alpha \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right](T)} \right)^\sigma &= \left(\frac{\vartheta^\alpha((a, b))}{\vartheta^\alpha((0, 0))} \right)^\sigma \\ &= \frac{\vartheta^\alpha(\sigma(a, b))}{\vartheta^\alpha((0, 0))} \\ &= \frac{e_*(\frac{\delta}{2})e(\frac{\delta}{2}, (a, b)) \cdot \vartheta^\alpha(a, b)}{\vartheta^\alpha(0, 0)} \\ &= \frac{\vartheta^\alpha(a, b)}{\vartheta^\alpha(0, 0)} \\ &= \frac{\vartheta^\alpha \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right](T)}{\vartheta^\alpha \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right](T)}. \end{aligned} \quad QED$$

It is this result that distinguishes ϑ^α from the other functions which differ from it by an elementary exponential factor and that motivates the definition of ϑ^α . This is *not* true for $\vartheta(Ta + b, T)$; it is almost true for $\vartheta_{a,b}(0, T)$ since this differs from $\vartheta^\alpha \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right](T)$ only by a root of unity, but still not true. It is unfortunate to add another notation for an almost identical function, but this corollary is the ultimate reason.

Appendix I: ϑ , as a morphism

We shall now define the ϑ , scheme-theoretically. This has three substantial advantages:

- A. Statements like Proposition 5.6 follow trivially.
- B. It proves that when an abelian variety in characteristic 0 has good reduction mod p , the values of its theta function are integers which reduce mod p to the char p theta-function.
- C. There is no need to avoid p -torsion when working over a field of characteristic p .

Let $j : X \rightarrow S$ be an abelian S -scheme (see D. Mumford and J. Fogarty, *Geometric Invariant Theory*, Ch. 6) and let \mathbf{L} be a relatively ample line bundle on X such that $(-1_X)^*\mathbf{L} = \mathbf{L}$ with a fixed isomorphism $\epsilon : 0^*\mathbf{L} \cong \mathcal{O}_S$ where $0 : S \rightarrow X$ is the zero section.

Step I. The construction of $f_1 = f : \tilde{X} \rightarrow X$ and $f_n : \tilde{X} \rightarrow X$ such that $n_X f_{mn} = f_m$ as in Definition 4.26 goes through without a hitch. Define $T_n(X) = f_n^{-1}(0) = f^{-1}(X_n)$ which is a closed subgroup scheme of \tilde{X} . Each $T_n(X)$ is a closed subgroup scheme of $T_{nm}(X)$. For each k and n , multiplication by k from $T_{kn}(X)$ to itself maps $T_{kn}(X)$ isomorphically onto the subscheme $T_n(X)$. In particular we have inverses

$$\frac{1}{k} : T_n \xrightarrow{\sim} T_{nk}.$$

We define $V(X)$ to be the directed system of closed immersions:

$$T(X) = T_1(X) \hookrightarrow T_n(X) \hookrightarrow T_{nm}(X) \hookrightarrow \dots$$

Step II. $\mathcal{G}(n_X^*\mathbf{L})$ makes sense as a group scheme over S with the exact sequence:

$$1 \rightarrow G_m \rightarrow \mathcal{G}(n_X^*\mathbf{L}) \xrightarrow{\pi} K(n_X^*\mathbf{L}) \rightarrow 0.$$

Put $G_n = \pi^{-1}(X_{n^2})$, and $\Gamma_n = j_* n_X^*\mathbf{L}$ a locally free coherent sheaf on S . There is a natural action of G_n on Γ_n (when S is affine, Γ_n is given as a co-module over the co-algebra which is the coordinate ring of G_n).

Step III. Now f_n induces a map from $T_{n^2}(X)$ onto X_{n^2} . Put $\widehat{\mathcal{G}}_n(\mathbf{L}) = G_n \times_{X_{n^2}} T_{n^2}(X)$.

We get the following commutative diagrams of exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & G_m & \longrightarrow & \widehat{\mathcal{G}}_n(\mathbf{L}) & \xrightarrow{\pi} & T_{n^2}(X) \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G_m & \longrightarrow & \widehat{\mathcal{G}}_{nk}(\mathbf{L}) & \xrightarrow{\pi} & T_{(nk)^2}(X) \longrightarrow 0. \end{array}$$

We can then define $\widehat{\mathcal{G}}(\mathbf{L})$ as the directed system of inclusions:

$$\widehat{\mathcal{G}}_1(\mathbf{L}) \hookrightarrow \widehat{\mathcal{G}}_n(\mathbf{L}) \hookrightarrow \widehat{\mathcal{G}}_{mn}(\mathbf{L}) \hookrightarrow \dots$$

Finally, put $\widehat{\Gamma}(X, \mathbf{L}) = \varinjlim_n \Gamma_n = (j \circ f)_* f^* \mathbf{L}$. We have a homomorphism from $\widehat{\mathcal{G}}_n(\mathbf{L})$ to G_n and an action of G_n on Γ_n , hence we get an action of $\widehat{\mathcal{G}}_n(\mathbf{L})$ on $\widehat{\Gamma}(X, \mathbf{L})$.

Step IV. Given a global section of $\Gamma(X, \mathbf{L})$, we shall construct a compatible system of morphisms from $\widehat{\mathcal{G}}_n(\mathbf{L})$ to \mathbf{A}^1 which we shall call a morphism from $\widehat{\mathcal{G}}(\mathbf{L})$ to \mathbf{A}^1 . One only has to check that the old formula $\epsilon(U_{g^{-1}} s)(0)$ for $g \in G_n, s \in \Gamma_n$ makes scheme-theoretic sense.

Step V. The construction of τ . The symmetry of \mathbf{L} gives as usual a compatible system of involutions i on $\widehat{\mathcal{G}}_n(\mathbf{L})$. The morphism $x \mapsto xi(x)^{-1}$ from $\widehat{\mathcal{G}}_n(\mathbf{L})$ to itself clearly factors through a morphism $h : T_n(X) \rightarrow \widehat{\mathcal{G}}_n(\mathbf{L})$ such that πh is multiplication by two. Put $\tau = h \circ \frac{1}{2}$.

Combining this with Step IV, each $s \in \Gamma(S, \widehat{\Gamma}(X, \mathbf{L}))$ gives a sequence of morphisms $\vartheta_s : T_n(X) \rightarrow \mathbf{A}^1$.

Appendix II: Relating all Heisenberg representations

First consider the abstract Heisenberg representations

\mathcal{H} = the irreducible Hilbert space representation of $Heis(2g, \mathbf{R})$ with $U_\lambda = \lambda, \forall \lambda \in \mathbf{C}_1^*$

$\mathcal{H}_\infty \subset \mathcal{H} \subset \mathcal{H}_{-\infty}$, the C^∞ -vectors in \mathcal{H} and its dual,

\mathcal{H}_f = the irreducible continuous representation of $Heis(2g, \mathbf{A}_f)$ with $U_\lambda = \lambda, \forall \lambda \in \mathbf{C}^*$

(cf. 5.2; here $Heis(2g, \mathbf{A}_f)$ is $\mathbf{C}^* \times \mathbf{A}_f^{2g}$ with group law as in 5.1 ff;

$$(\lambda, x) \cdot (\mu, y) = (\lambda\mu e({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1), x + y)$$

$$e(a + b) = e^{2\pi i a}, a \in \mathbf{Q}, b \in \widehat{\mathbf{Z}}.$$

PROPOSITION 5.13. *There is a unique $Heis(2g, \mathbf{Q})$ -linear embedding*

$$\phi : \mathcal{H}_f \longrightarrow \mathcal{H}_{-\infty}$$

the image of which is the linear span of the vectors $e[\frac{a}{b}], a, b \in \mathbf{Q}^g$.

PROOF: The image of such a map must contain a vector fixed by $\sigma(\mathbf{Z}^{2g})$ and $e[\frac{0}{1}]$ is the only such vector in $\mathcal{H}_{-\infty}$. Thus the image of ϕ is the linear span of the $e[\frac{a}{b}]$'s. But $Heis(2g, \mathbf{Q})$ acts continuously on the span of the $e[\frac{a}{b}]$'s with respect to the topology defined by the subgroups $\sigma(m\widehat{\mathbf{Z}}^{2g})$, and $Heis(2g, \mathbf{A}_f)$ is the completion of $Heis(2g, \mathbf{Q})$ for this topology. So $Heis(2g, \mathbf{A}_f)$ acts continuously on this span and by 5.2 it is isomorphic to \mathcal{H}_f . QED

This abstract result allows us to pass between the realization of these representations via the full adeles as follows:

$$\mathcal{A} \xrightarrow{r_1} \mathcal{B} \xrightarrow{r_2} \mathcal{C}$$

where

$$\begin{aligned} \mathcal{A} &= \left\{ \begin{array}{l} \text{span of the fns.} \\ f_{a,b}(x) = (U_{\tau(x)}a, b) \\ a \in \mathcal{H}_\infty, b \in \mathcal{H}_{-\infty} \\ x \in \mathbb{R}^{2g} \end{array} \right\} \\ \mathcal{B} &= \left\{ \begin{array}{l} \text{span of the fns.} \\ f_{a,b}(x) = (U_{\tau(x_\infty)}a, U_{\tau(x_f)}b) \\ a \in \mathcal{H}_\infty, b \in \mathcal{H}_f \\ x \in \mathbb{A}^{2g} \end{array} \right\} \\ \mathcal{C} &= \left\{ \begin{array}{l} \text{span of the fns.} \\ f_{a,b}(x) = (U_{\tau(x)}a, b) \\ a \in \mathcal{H}_\infty^*, b \in \mathcal{H}_f \\ x \in \mathbb{A}_f^{2g} \end{array} \right\} \end{aligned}$$

i) Note that the functions in the middle space are quasi-periodic w.r.t.

$$\mathbb{Q}^{2g} \subset \mathbb{A}^{2g}: \text{ in fact for } y \in \mathbb{Q}^{2g}$$

$$\begin{aligned} f_{a,b}(x+y) &= (U_{\tau(x_\infty+y_\infty)}a, U_{\tau(x_f+y_f)}b) \\ &= (e(\frac{x_\infty}{2}, y_\infty) \cdot U_{\tau(y)}U_{\tau(x_\infty)}a, e(\frac{1}{2}A(x_f, y_f)) \cdot U_{\tau(y)}U_{\tau(x_f)}b) \\ &= e^{\pi i A(x_\infty, y_\infty)} e(-\frac{1}{2}A(x_f, y_f)) (U_{\tau(x_\infty)}a, U_{\tau(x_f)}b) \\ &= e'(\frac{1}{2}A(x, y)) \cdot f_{a,b}(x) \end{aligned}$$

where $e' : \mathbb{A} \rightarrow \mathbb{C}_1^*$ is the map $e'(\lambda_\infty, \lambda_f) = e^{2\pi i \lambda_\infty} e(-\lambda_f)$ and $A(x, y) = {}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1$.

ii) The map r_1 is given by restricting a function on \mathbb{A}^{2g} to the infinite factor, a map which gives an isomorphism

$$\begin{array}{ccc} \left[\begin{array}{l} \text{Fcns. } f \text{ on } \mathbb{A}^{2g}, C^\infty \text{ in real} \\ \text{variable, locally constant} \\ \text{in finite-adele variable, and} \\ \text{quasi-periodic w.r.t. } \mathbb{Q}^{2g} \end{array} \right] & \xrightarrow{r_1} & \left[\begin{array}{l} C^\infty \text{ fncs. } f \text{ on } \mathbb{R}^{2g}, \\ \text{quasi-periodic w.r.t. } N \cdot \mathbb{Z}^{2g} \\ \text{some } N, \text{ i.e.} \\ f(x+k) = e^{\pi i A(x,k)} f(x), \\ \text{for all } k \in N\mathbb{Z}^{2g} \end{array} \right] \\ \parallel & & \parallel \\ \mathcal{B} & & \mathcal{A} \end{array}$$

iii) The map r_2 is given by restricting a function on \mathbb{A}^{2g} to the finite-adele variables and substitutes $-x$ for x , giving an *injective* map:

$$\begin{array}{ccc} \left[\begin{array}{l} \text{Fcns. } f \text{ on } \mathbb{A}^{2g}, C^\infty \text{ in real} \\ \text{variable, locally constant} \\ \text{in finite-adele variable, and} \\ \text{quasi-periodic w.r.t. } \mathbb{Q}^{2g} \end{array} \right] & \xrightarrow{r_2} & \left[\begin{array}{l} \text{Loc. constant fncs. } f \text{ on } \mathbb{A}^{2g}, \\ \text{quasi-periodic w.r.t. } N \cdot \mathbb{Z}^{2g} \\ \text{some } N, \text{ i.e.} \\ f(x+k) = e(\frac{k}{2}, x) f(x), \\ \text{for all } k \in N\mathbb{Z}^{2g} \end{array} \right] \\ \parallel & & \parallel \\ \mathcal{B} & & \mathcal{C} \end{array}$$

iv) The group $\text{Heis}(2g, \mathbb{A})$ of pairs (λ, x) , group law

$$(\lambda, x)(\mu, y) = (\lambda\mu e'(\frac{1}{2}A(x, y)), x+y)$$

acts on the middle space by combining the action of $\text{Heis}(2g, \mathbb{R})$ on a , $\text{Heis}(2g, \mathbb{A}_f)$ on b , and the middle space is its irreducible Heisenberg representation (more precisely, the subspace of "Schwartz functions" in it).

v) The Heisenberg actions match up (with various sign changes) as follows:

left $\text{Heis}(2g, \mathbb{Q})$ on $\mathcal{A} \xleftarrow{r_1} \text{Heis}(2g, \mathbb{Q}) \subset \text{Heis}(2g, \mathbb{A}_f)$ action on \mathcal{B}

right $\text{Heis}(2g, \mathbb{R})$ on $\mathcal{A} \xleftarrow{r_1} \text{Heis}(2g, \mathbb{R})$ action on \mathcal{B}

right $\text{Heis}(2g, \mathbb{A}_f)$ on $\mathcal{C} \xrightarrow{r_2}$ action on \mathcal{B}

left $\text{Heis}(2g, \mathbb{A}_f)$ on $\mathcal{C} \xleftarrow{r_2} \text{Heis}(2g, \mathbb{Q}) \subset \text{Heis}(2g, \mathbb{R})$ action on \mathcal{B}

The situation is summarized in the diagram below:

$$\begin{array}{ccc}
e^{2\pi i x_1 z} \cdot \left[\begin{array}{l} \text{Fock repres.} \\ \text{Holo. fns. on} \\ \mathbb{C}^g, \text{Heis}(2g, \mathbb{R}) \\ \text{acting on left} \end{array} \right] & \subset & \left[\begin{array}{l} \text{Fns. on } \mathbb{R}^{2g} \\ \text{Heis}(2g, \mathbb{R})^2 \text{ acting} \\ U_{(\lambda, y)}^{\text{left}} f(x) = \lambda^{-1} e(\frac{x}{2}, y) f(x-y) \\ U_{(\lambda, y)}^{\text{right}} f(x) = \lambda e(\frac{x}{2}, y) f(x+y) \end{array} \right] \\
\cup \text{dense} & & \cup \text{dense} \\
\left[\begin{array}{l} \text{span of} \\ \vartheta_{a,b}^{\alpha}(z), a, b \in \mathbb{Q}^g \\ \text{Heis}(2g, \mathbb{Q}) \text{ acts on left} \end{array} \right] & \subset & \bigcup_n \left[\begin{array}{l} C^\infty \text{ fns. on } \mathbb{R}^{2g}, \text{ fixed} \\ \text{by left action of} \\ (1, n\mathbb{Z}^{2g}) \\ \text{Heis}(2g, \mathbb{Q}) \text{ acts on left} \\ \text{Heis}(2g, \mathbb{R}) \text{ acts on right} \\ \text{extending to Heis}(2g, \mathbb{A}) \\ \text{action} \end{array} \right] \xrightarrow[\text{res}]{\sim} \\
\cup & & \cup \\
\mathbb{C} \cdot \vartheta^\alpha(z) & \in & \left[\begin{array}{l} \text{Quasi-periodic repres.} \\ C^\infty \text{ fns. on } \mathbb{R}^{2g}, \text{ such that} \\ f(x+a) = e_*(\frac{a}{2}) e(x, \frac{a}{2}) f(x) \\ \text{all } a \in \mathbb{Z}^{2g} \\ \text{Heis}(2g, \mathbb{R}) \\ \text{acting on right} \end{array} \right] \xrightarrow[\text{res}]{\sim}
\end{array}$$

$$\begin{array}{ccc}
\left[\begin{array}{l} \text{Schwartz fns. on } \mathbb{A}^{2g} \\ (C^\infty \text{ in real variable, loc. const.} \\ \text{in } p\text{-adic variable}) \\ \text{left quasi-periodic for } \mathbb{Q}^{2g}: \\ f(x+a) = e'(A(\frac{a}{2}, x)) f(x), a \in \mathbb{Q}^{2g} \\ \text{Heis}(2g, \mathbb{A}) \text{ acts} \\ U_{(\lambda, y)} f(x) = \lambda e'(A(\frac{x}{2}, y)) f(x+y) \end{array} \right] & \xrightarrow[\text{res} \circ (-1)]{\subset} & \bigcup_n \left[\begin{array}{l} \text{Fns. on } \mathbb{A}_f^{2g}, \text{ loc. const.} \\ \text{Heis}(2g, \mathbb{A}_f)^2 \text{ acting} \\ U_{(\lambda, y)}^{\text{left}} f(x) = \lambda^{-1} e(\frac{x}{2}, x) f(x+y) \\ U_{(\lambda, y)}^{\text{right}} f(x) = \lambda e(\frac{x}{2}, x) f(x-y) \\ \text{fixed by } U_{(1, y)}^{\text{right}}, y \in n\mathbb{Z}^{2g} \end{array} \right] \\
\cup & & \cup \\
\left[\begin{array}{l} \text{Schwartz fns. on } \mathbb{A}^{2g} \\ \text{left quasi-periodic wrt } \mathbb{Q}^{2g} \\ \text{right quasi-periodic wrt } \mathbb{Z}^{2g} \\ \text{Heis}(2g, \mathbb{R}) \text{ acts} \end{array} \right] & \xrightarrow[\text{res} \circ (-1)]{\subset} & \left[\begin{array}{l} \text{Fns. on } \mathbb{A}_f^{2g}, \text{ s.t.} \\ f(x+a) = e_*(\frac{a}{2}) e(\frac{a}{2}, x) f(x) \\ \text{all } a \in \mathbb{Z}^{2g} \\ \text{Heis}(2g, \mathbb{R}) \text{ acts on left} \end{array} \right]
\end{array}$$

6. Theta functions with quadratic forms

The purpose of this section is to construct generalizations of the theta functions considered so far, which are associated to an auxiliary positive definite rational form. These functions are important for various reasons. In Chapter I, we discussed the application of theta functions to the problem of the representation of integers by quadratic forms and these general theta functions can be used to study the representation of one quadratic form by another. Our interest, however, is in the *algebra* of theta functions, e.g., the polynomial identities satisfied by them. We have encountered Riemann's theta relation repeatedly in earlier parts of this book, but this is only the simplest in a large class of theta relations. These relations are naturally deduced from the more inclusive algebra of the full family of theta functions associated to quadratic forms.

We begin by constructing theta functions with quadratic forms over \mathbb{C} . In what follows, for any ring R , $R(g, h)$ will denote the $g \times h$ matrices over R .

DEFINITION 6.1. For a rational, symmetric, positive definite $h \times h$ matrix Q and a $T \in \mathfrak{H}_g$, define

$$\vartheta^Q(Z, T) = \sum_{N \in \mathbb{Z}(g, h)} \exp \pi i \operatorname{Tr}({}^t N T N Q + 2 {}^t N Z)$$

where $Z \in \mathbb{C}(g, h)$.

It is easy to see that the following holds:

$$\vartheta^Q(Z + T M Q + N, T) \exp \pi i \operatorname{Tr}({}^t M T M Q + 2 {}^t M Z) = \vartheta^Q(Z, T)$$

for all $M, N \in \mathbb{Z}(g, h)$. This suggests that $\vartheta^Q(Z, T)$ is just Riemann's theta function for the complex torus $\mathbb{C}(g, h)/T\mathbb{Z}(g, h)Q + \mathbb{Z}(g, h)$. To see this, fix the isomorphism $Z = (Z_1, Z_2, \dots, Z_h) \longrightarrow z = \begin{pmatrix} Z_1 \\ Z_2 \\ \dots \\ Z_h \end{pmatrix}$ of $\mathbb{C}(g, h)$ with

$\mathbb{C}(gh, 1) = \mathbb{C}^{gh}$. If $Z, W, F, \dots \in \mathbb{C}(g, h)$ we shall denote the corresponding elements of \mathbb{C}^{gh} by z, w, f, \dots . Now $T^* = T \otimes Q$ can be represented by the $gh \times gh$ block matrix:

$$T^* = \begin{pmatrix} TQ_{11} & TQ_{12} & \cdots & TQ_{1h} \\ TQ_{21} & TQ_{22} & \cdots & TQ_{2h} \\ \cdots & \cdots & \cdots & \cdots \\ TQ_{h1} & TQ_{h2} & \cdots & TQ_{hh} \end{pmatrix}.$$

Once it is checked that

(i) $W = TZQ$ implies $w = T^*z$, and

(ii) $\operatorname{Tr}({}^t W Z) = {}^t w z$,

we can prove our claim:

LEMMA 6.2.

A. $\mathbb{C}(g, h)/(\mathbb{Z}(g, h) + T\mathbb{Z}(g, h)Q) \cong X_{T^*, \mathbb{Z}^{2g, h}}$. This torus will be called X^Q .

B. $\vartheta^Q(Z, T) = \vartheta(z, T^*)$.

C. The complex tori X^Q and $(X_{T, \mathbb{Z}^{2g}})^h$ are isogenous.

PROOF: A is clear. By (i) and (ii) above, we get

$$\operatorname{Tr}({}^t N T N Q + 2 {}^t N Z) = {}^t n T^* n + 2 {}^t n z$$

which shows that

$$\vartheta^Q(Z, T) = \sum_{n \in \mathbb{Z}^{gh}} \exp \pi i ({}^t n T^* n + 2 {}^t n z) = \vartheta(z, T^*).$$

This proves B.

The two complex tori in part C are $\mathbb{C}(g, h)$ modulo the lattices $T\mathbb{Z}(g, h)Q + \mathbb{Z}(g, h)$ and $T\mathbb{Z}(g, h) + \mathbb{Z}(g, h)$ respectively. But both these lattices generate the same rational vector space (because Q is a rational invertible matrix) showing that the two tori are isogenous. QED

More of the underlying geometry that connects X^Q and $X_{T, \mathbb{Z}^{2g}}$ and the basic line bundles they inherit will be explained later. Meanwhile we define the analogues of $\vartheta[\frac{a}{b}](z, T)$ and $\vartheta^a[\frac{a}{b}](T)$:

DEFINITION 6.3. With Q and T as above, $A, B \in \mathbf{Q}(g, h)$ and $X_1, X_2 \in \mathbf{R}(g, h)$,

$$\begin{aligned} & A. \\ \vartheta^Q \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (Z, T) &= \sum_{N \in \mathbf{Z}(g, h)} \exp \pi i \text{Tr}({}^t(N+A)T(N+A)Q + 2{}^t(N+A)(Z+B)) \\ &= \sum_{N \in \mathbf{Q}(g, h)} \chi \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (N) \exp \pi i \text{Tr}({}^tNTNQ + 2{}^tNZ) \\ &= \exp \pi i \text{Tr}({}^tATAQ + 2{}^tA(Z+B)) \cdot \vartheta^Q(Z+TAQ+B, T), \end{aligned}$$

where

$$\begin{aligned} \chi \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (N) &= \exp 2\pi i \text{Tr}{}^tNB, \text{ if } N-A \in \mathbf{Z}(g, h) \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

$$\begin{aligned} & B. \\ \vartheta^{\alpha, Q} \left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix} \right] (T) &= \exp \pi i \text{Tr}({}^tX_1(TX_1Q + X_2)) \cdot \vartheta^Q(TX_1Q + X_2, T) \\ &= \sum_{N \in \mathbf{Z}(g, h)} \exp \pi i \text{Tr}({}^tX_1TX_1Q + {}^tX_1X_2 + {}^tNTNQ + 2{}^tN(TX_1Q + X_2)) \\ &= \exp(-\pi i \text{Tr}({}^tX_1X_2)) \cdot \sum \exp \pi i \text{Tr}({}^tX_1+N)T(X_1+N)Q + 2{}^t(X_1+N)X_2 \\ &= \exp(-\pi i \text{Tr}{}^tX_1 \cdot X_2) \cdot \vartheta^Q \left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix} \right] (0, T). \end{aligned}$$

By (i) and (ii) above, we have

$$A'. \quad \vartheta^Q \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (Z, T) = \vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z, T^*).$$

B'.

$$\begin{aligned} \vartheta^{\alpha, Q} \left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix} \right] (T) &= \exp \pi i {}^t x_1 (T^* x_1 + x_2) \cdot \vartheta^Q(T^* x_1 + x_2, T^*) \\ &= \vartheta^{\alpha} \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \right] (T^*), \end{aligned}$$

the "algebraic" theta function for the torus X^Q .

Following the approach in §2 we define $\text{Heis}(2(g, h); \mathbf{R})$ to be the set of (λ, X_1, X_2) with $\lambda \in \mathbf{C}_1^*$ and $X_1, X_2 \in \mathbf{R}(g, h)$ with multiplication defined as follows:

$$(\lambda, X_1, X_2) \cdot (\mu, Y_1, Y_2) = (\lambda \mu \exp \pi i \text{Tr}({}^tX_1Y_2 - {}^tY_1X_2), X_1 + Y_1, X_2 + Y_2).$$

In fact, $\text{Heis}(2(g, h); \mathbf{R}) \cong \text{Heis}(2gh, \mathbf{R})$, and it acts on the space of all holomorphic functions on $\mathbf{C}(g, h)$ by the formulae:

$$\begin{aligned} U_{(\lambda, 0, 0)} f &= \lambda^{-1} f \\ U_{(1, A, 0)} f(Z) &= \exp(\pi i \text{Tr}({}^tATAQ + 2{}^tAZ)) f(Z + TAQ) \\ U_{(1, 0, B)} f(Z) &= f(Z + B) \\ U_{(\lambda, A, B)} f(Z) &= \lambda^{-1} f(Z + TAQ + B) \exp(\pi i \text{Tr}({}^tA((TAQ + B) + 2Z))). \end{aligned}$$

Then $\vartheta^Q(Z, T)$ considered as a function of Z is, up to a scalar, the only holomorphic function on $\mathbf{C}(g, h)$ invariant under the action of the discrete subgroup

$$(1 \times \mathbf{Z}(g, h) \times 0) \cdot (1 \times 0 \times \mathbf{Z}(g, h))$$

of $\text{Heis}(2(g, h); \mathbf{R})$. We shall call this subgroup $\sigma(\mathbf{Z}(g, h)^2)$. Similarly, $\vartheta^{\alpha, Q} \left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix} \right] (T)$ can be realized as a function invariant under $\sigma(\mathbf{Z}(g, h)^2)$ for the following action of $\text{Heis}(2(g, h); \mathbf{R})$ on the space of all functions defined on $\mathbf{R}(g, h)^2$:

$$U_{(\lambda, A_1, A_2)} f(X_1, X_2) = \lambda^{-1} \exp \pi i \text{Tr}({}^tX_2A_1 - {}^tX_1A_2) f(X_1 + A_1, X_2 + A_2).$$

Denoting the first action by U^1 and the second by U^2 , and by R the linear transformation from the first space to the second given by

$$R : f(X_1, X_2) \longrightarrow \exp(\pi i \text{Tr}{}^tX_1(TX_1Q + X_2)) f(X_1, X_2),$$

then we see that

$$U_g^2 R f = R U_g^1 f \quad \text{for all } g \in \text{Heis}(2(g, h); \mathbf{R}).$$

REMARK: Since

$$R \vartheta^Q(TX_1Q + X_2, T) = \vartheta^{\alpha, Q} \left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix} \right] (T),$$

we can convert ϑ^Q -identities to $\vartheta^{\alpha, Q}$ -identities.

The following results contain the generalization of Riemann's theta relation:

6.4 FIRST FUNDAMENTAL IDENTITY: Let Q_1 and Q_2 be rational symmetric positive definite $(h_1 \times h_1)$ and $(h_2 \times h_2)$ matrices. Let $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$. Then

A .

$$\vartheta^Q(Z_1, Z_2; T) = \vartheta^{Q_1}(Z_1, T) \vartheta^{Q_2}(Z_2, T),$$

and

B .

$$\vartheta^{\alpha, Q} \begin{bmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{bmatrix} (T) = \vartheta^{\alpha, Q_1} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} (T) \vartheta^{\alpha, Q_2} \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} (T),$$

where $Z_i \in \mathbb{C}(g, h_i)$ and $X_i, Y_i \in \mathbb{R}(g, h_i)$.

6.5 SECOND FUNDAMENTAL IDENTITY: Let $A \in GL_h(\mathbb{Q})$ and $Q' = {}^t A Q A$.

Set $P = [\mathbb{Z}^h : A\mathbb{Z}^h \cap \mathbb{Z}^h]^{-g}$. Then

A.

$$\begin{aligned} \vartheta^{Q'}(ZA, T) &= P \sum_R \sum_S U_{(1,R,0)}^1 U_{(1,0,S)}^1 \vartheta^Q(Z, T) \\ &= P \sum_R \sum_S U_{(1,0,S)}^1 U_{(1,R,0)}^1 \vartheta^Q(Z, T) \\ &= P \sum_R \sum_S \vartheta^Q \begin{bmatrix} R \\ S \end{bmatrix} (Z, T). \end{aligned}$$

B.

$$\vartheta^{\alpha, Q'} \begin{bmatrix} X_1 \cdot {}^t A^{-1} \\ X_2 \cdot A \end{bmatrix} (T) = P \sum_R \sum_S U_{(1,R,0)}^2 U_{(1,0,S)}^2 \vartheta^{\alpha, Q} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} (T).$$

C.

$$\vartheta^{Q'}(Z, T) = P \sum_R \sum_S \vartheta^Q \begin{bmatrix} R \\ S \end{bmatrix} (ZA^{-1}, T).$$

D.

$$\begin{aligned} \vartheta^{\alpha, Q'} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} (T) &= \\ P \sum_R \sum_S \exp \pi i \operatorname{Tr}({}^t R X_2 A^{-1} - {}^t S X_1 {}^t A - {}^t S R) \cdot \vartheta^{\alpha, Q} \begin{bmatrix} X_1 {}^t A + R \\ X_2 {}^t A^{-1} + S \end{bmatrix} (T), \end{aligned}$$

where R and S are $(g \times h)$ -matrices whose g rows run through a (finite) system of coset representatives for $\mathbb{Z}(1, h) {}^t A / \mathbb{Z}(1, h) {}^t A \cap \mathbb{Z}(1, h)$ and $\mathbb{Z}(1, h) A^{-1} / \mathbb{Z}(1, h) A^{-1} \cap \mathbb{Z}(1, h)$.

PROOF: Formulae C and D will be used in applications for various choices of Q, A, Q' , and they are immediate consequences of 6.5A and B: replace Z by ZA^{-1} and $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ by $\begin{bmatrix} X_1 {}^t A \\ X_2 A^{-1} \end{bmatrix}$.

PROOF OF 6.5A:

$$\begin{aligned} \vartheta^{Q'}(ZA, T) &= \sum_{N \in \mathbb{Z}(g, h)} \exp \pi i \operatorname{Tr}({}^t N T N {}^t A Q A + 2 {}^t N Z A) \\ &= \sum_{N \in \mathbb{Z}(g, h)} \exp \pi i \operatorname{Tr}({}^t A {}^t N T N {}^t A Q + 2 {}^t A {}^t N Z) \\ &= \sum_{N \in \mathbb{Z}(g, h) {}^t A} \exp \pi i \operatorname{Tr}({}^t N T N Q + 2 {}^t N Z) \\ &= \sum_R \sum_{N \in \mathbb{Z}(g, h) {}^t A \cap \mathbb{Z}(g, h)} \exp \pi i \operatorname{Tr}({}^t (N + R) T (N + R) Q + 2 {}^t (N + R) Z) \\ &= \sum_R \sum_{N \in \mathbb{Z}(g, h) {}^t A \cap \mathbb{Z}(g, h)} \exp \pi i \operatorname{Tr}({}^t N T N Q + 2 {}^t N (Z + T R Q) + {}^t R T R Q + 2 {}^t R Z) \end{aligned}$$

where the R range through a system of coset representatives for

$$\begin{aligned} \mathbb{Z}(g, h) {}^t A / \mathbb{Z}(g, h) {}^t A \cap \mathbb{Z}(g, h), \\ = P \sum_R \sum_{N \in \mathbb{Z}(g, h) {}^t A \cap \mathbb{Z}(g, h)} \sum_{\chi} \chi(N) \exp \pi i \operatorname{Tr}({}^t N T N Q + 2 {}^t N (Z + T R Q) + {}^t R T R Q + 2 {}^t R Z) \end{aligned}$$

where the χ range through the character group of $\mathbb{Z}(g, h) / \mathbb{Z}(g, h) {}^t A \cap \mathbb{Z}(g, h)$

and

$$\begin{aligned} P &= [\mathbb{Z}(g, h) : \mathbb{Z}(g, h) {}^t A \cap \mathbb{Z}(g, h)]^{-1} \\ &= [\mathbb{Z}(1, h) : \mathbb{Z}(1, h) {}^t A \cap \mathbb{Z}(1, h)]^{-g} \\ &= [\mathbb{Z}^h : A\mathbb{Z}^h \cap \mathbb{Z}^h]^{-g}. \end{aligned}$$

Observing that any such χ can be uniquely represented by

$\chi(N) = \exp 2\pi i \operatorname{Tr}({}^t S N)$ where $S \in [\mathbb{Z}(g, h) A^{-1} + \mathbb{Z}(g, h)] / \mathbb{Z}(g, h)$, the above expression boils down to

$$\begin{aligned} P \sum_R \sum_S \sum_{N \in \mathbb{Z}(g, h)} \exp \pi i \operatorname{Tr}({}^t N T N Q + 2 {}^t N (Z + T R Q + S) + {}^t R T R Q + 2 {}^t R Z) \\ = P \sum_R \sum_S \exp \pi i \operatorname{Tr}({}^t R T R Q + 2 {}^t R Z) \cdot \vartheta^Q(Z + T R Q + S, T) \end{aligned}$$

$$= [\mathbf{Z}^h : A\mathbf{Z}^h \cap \mathbf{Z}^h]^{-g} \sum_R \sum_S U_{(1,R,0)}^1 U_{(1,0,S)}^1 \vartheta^Q(Z, T).$$

The S above were chosen as coset representatives for $\mathbf{Z}(g, h)A^{-1} + \mathbf{Z}(g, h)/\mathbf{Z}(g, h)$; they could equally well have been chosen to be coset representatives for $\mathbf{Z}(g, h)A^{-1}/\mathbf{Z}(g, h)A^{-1} \cap \mathbf{Z}(g, h)$. For such R and S , $R = M^t A$ and $S = NA^{-1}$ for some $M, N \in \mathbf{Z}(g, h)$ and therefore $\exp 2\pi i \text{Tr}({}^t RS) = \exp 2\pi i \text{Tr}({}^t MN) = 1$. Consequently, the $U_{(1,R,0)}^1$ and $U_{(1,0,S)}^1$ can be interchanged; also

$$\begin{aligned} & U_{(1,0,S)}^1 U_{(1,T,0)}^1 \vartheta^Q(Z, T) \\ &= \exp \pi i \text{Tr}({}^t RTRQ + 2{}^t R(Z+S)) \vartheta^Q(Z + TRQ + S, T) \\ &= \vartheta^Q \begin{bmatrix} R \\ S \end{bmatrix} (Z, T). \end{aligned}$$

This proves the second and third formulae in 6.6A.

We show that upon applying R to formula (A) we get (B). It is evident that $R\vartheta^Q = \vartheta^{\alpha, Q}$. On the other hand, if $Z = TX_1 Q + X_2$, then

$$\begin{aligned} R\vartheta^{Q'}(ZA, T) &= \exp(\pi i \text{Tr}({}^t X_1(TX_1 Q + X_2)) \vartheta^{Q'}((TX_1 Q + X_2)A, T) \\ &= \exp(\pi i \text{Tr}({}^t(X_1{}^t A^{-1}) \cdot (TX_1{}^t A^{-1} Q' + X_2 A)). \\ &\quad \vartheta^{Q'}(TX_1{}^t A^{-1} Q' + X_2 A, T) \\ &\quad (\text{since } {}^t A^{-1} Q' A^{-1} = Q) \\ &= \vartheta^{\alpha, Q'} \begin{bmatrix} X_1{}^t A^{-1} \\ X_2 A \end{bmatrix} (T). \end{aligned}$$

QED

REMARK 1: Note that the summation over R (resp. S) disappears when A (resp. A^{-1}) is an integral matrix.

REMARK 2: In 6.5D, the $\exp \pi i \text{Tr}({}^t R X_2 A^{-1} - {}^t S X_1{}^t A - {}^t S R)$ are roots of unity when X_1 and X_2 are rational.

REMARK 3: Replacing Z by $Z + TMQ' + N$ in 6.5C and multiplying both sides by $\exp \pi i \text{Tr}({}^t MTMQ' + 2{}^t M(Z+N))$ gives a generalization of 6.5C:

$$\begin{aligned} & \vartheta^{Q'} \begin{bmatrix} M \\ N \end{bmatrix} (Z, T) = \\ & c^{-g} \cdot \sum_{R,S} \exp 2\pi i (-\text{Tr}(M^t AS)) \vartheta^Q \begin{bmatrix} M^t A + R \\ NA^{-1} + S \end{bmatrix} (ZA^{-1}, T), \end{aligned}$$

where $c = [A\mathbf{Z}^h : \mathbf{Z}^h \cap A\mathbf{Z}^h]$

COROLLARY 6.6. If $Q = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & & d_h \end{pmatrix}$ is diagonal, and

$$Z = (z_1, z_2, \dots, z_h), \quad X = (x_1, x_2, \dots, x_h), \quad Y = (y_1, y_2, \dots, y_h),$$

then

$$\vartheta^Q(Z, T) = \prod_{i=1}^h \vartheta(z_i, d_i T)$$

and

$$\vartheta^{\alpha, Q} \begin{bmatrix} X \\ Y \end{bmatrix} (T) = \prod_{i=1}^h \vartheta^{\alpha} \begin{bmatrix} x_i \\ y_i \end{bmatrix} (d_i T)$$

This follows immediately from the first fundamental identity.

COROLLARY 6.7. Let Q be rational positive definite. Choose $B \in GL_h(\mathbb{Q})$ so that $D = {}^t B \cdot Q \cdot B$ is diagonal. Replacing the matrices (Q, Q', A) in 6.5C by (D, Q, B^{-1}) , we get:

$$\vartheta^Q(Z, T) = a^{-g} \sum_{R,S} \prod_{i=1}^h \vartheta \begin{bmatrix} r_i \\ s_i \end{bmatrix} \left(\sum_j b_{ji} z_j, d_i T \right)$$

where a is some natural number, the d_i are the diagonal entries of D , and the r_i, s_i, z_i are the i -th columns of R, S, Z respectively.

This shows that the theta functions with arbitrary quadratic forms are easily calculated in terms of the usual theta functions for the period matrices dT .

Put $Q = 2I_2, Q' = I_2$ and $A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ in 6.5C; so

$A^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is integral, and $\mathbf{Z}(g, 2) \cdot {}^t A / \mathbf{Z}(g, 2) {}^t A \cap \mathbf{Z}(g, 2) = \mathbf{Z}(g, 2) \cdot A / \mathbf{Z}(g, 2)$ has coset representatives (η, η) where $\eta \in \frac{1}{2} \mathbf{Z}^g / \mathbf{Z}^g$. Therefore,

COROLLARY 6.8.

$$\vartheta(z_1, T) \vartheta(z_2, T) = \sum_{\eta \in \frac{1}{2} \mathbf{Z}^g / \mathbf{Z}^g} \vartheta \begin{bmatrix} \eta \\ 0 \end{bmatrix} (z_1 + z_2, 2T) \vartheta \begin{bmatrix} \eta \\ 0 \end{bmatrix} (z_1 - z_2, 2T).$$

This theta relation will play a major role in §7.

Next put $Q = Q' = I_4$ and

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

in 6.5C. Here $A = {}^t A = A^{-1}$ and coset representatives for

$\mathbf{Z}(g, 4)A / [\mathbf{Z}(g, 4)A \cap \mathbf{Z}(g, 4)]$ are (η, η, η, η) where $\eta \in \frac{1}{2}\mathbf{Z}^g / \mathbf{Z}^g$. Thus:

COROLLARY 6.9 (RIEMANN'S THETA RELATION).

$$\begin{aligned} \vartheta(z_1)\vartheta(z_2)\vartheta(z_3)\vartheta(z_4) = \\ 2^{-g} \sum_{\zeta \in \frac{1}{2}\mathbf{Z}^g / \mathbf{Z}^g, \eta \in \frac{1}{2}\mathbf{Z}^g / \mathbf{Z}^g} \vartheta\left[\begin{smallmatrix} \zeta \\ \eta \end{smallmatrix}\right]\left(\frac{1}{2}(z_1+z_2+z_3+z_4)\right) \cdot \vartheta\left[\begin{smallmatrix} \zeta \\ \eta \end{smallmatrix}\right]\left(\frac{1}{2}(z_1+z_2-z_3-z_4)\right) \\ \cdot \vartheta\left[\begin{smallmatrix} \zeta \\ \eta \end{smallmatrix}\right]\left(\frac{1}{2}(z_1-z_2+z_3-z_4)\right) \cdot \vartheta\left[\begin{smallmatrix} \zeta \\ \eta \end{smallmatrix}\right]\left(\frac{1}{2}(z_1-z_2-z_3+z_4)\right). \end{aligned}$$

Finally, choose an orthogonal basis β_1, \dots, β_h of \mathbf{Q}^h starting with $\beta_1 = (1, \dots, 1)$. Let B be the $h \times h$ matrix whose i^{th} column is β_i and let d_i be the square of the length of β_i . Put $A = B^{-1}$, $Q' = I_h$ and $Q = \begin{pmatrix} h & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & & d_h \end{pmatrix}$. Then $Q' = {}^t A \cdot Q A$. Apply Remark 3 following 6.5

and specialize $Z \in \mathbf{C}^{(g,h)}$ to $Z = (z, z, \dots, z)$ with $z \in \mathbf{C}^g$; we get:

COROLLARY 6.10.

$$\prod_{i=1}^h \vartheta \begin{bmatrix} m_i \\ n_i \end{bmatrix} (z, T) = c^{-g} \sum_{R,S} \exp 2\pi i \text{Tr}(M \cdot {}^t AS) \cdot \vartheta \begin{bmatrix} \cdots \\ \cdots \end{bmatrix} (hz, hT) \prod_{j=2}^h \vartheta \begin{bmatrix} \cdots \\ \cdots \end{bmatrix} (0, d_j T).$$

On the left hand side, we are taking products of the $\vartheta \begin{bmatrix} m_i \\ n_i \end{bmatrix} \in \widehat{\Gamma}(X, \mathbf{L})$ where $X = X_{T, \mathbf{Z}^{2g}}$ and \mathbf{L} is the basic line bundle. The R.H.S. consists of linear combinations of $\vartheta \begin{bmatrix} \cdots \\ \cdots \end{bmatrix} (hz, hT) \in \widehat{\Gamma}(X, \mathbf{L}^h)$. Thus the above formula gives an explicit description of the multiplication map $S^h(\widehat{\Gamma}(\mathbf{L})) \rightarrow \widehat{\Gamma}(\mathbf{L}^h)$ in terms of the natural bases that both vector spaces possess.

We now wish to explain how, up to separate scalars for each Q , we can define the functions

$$\vartheta^{\alpha, Q} : V(X)^h \rightarrow \overline{k}$$

purely algebraically and over any ground field. The basic idea is quite straightforward, although its execution is notationally involved. We start with an abelian variety X and an ample degree one symmetric line bundle \mathbf{L} on X , to which we can associate the function ϑ^α on $V(X)$ as in §5. However suppose you consider \mathbf{L}^n and wish to associate to (X, \mathbf{L}^n) a single theta function. Since \mathbf{L}^n has degree bigger than one you need to construct an isogeny

$$f^{(n)} : X \rightarrow Y,$$

and a line bundle \mathbf{M} on Y such that

$$(f^{(n)})^* \mathbf{M} \cong \mathbf{L}^n$$

where \mathbf{M} has degree one. Over \mathbf{C} one obtains in this way $\vartheta(z, \frac{1}{n}T)$ for this bigger lattice, hence it defines the new function. Indeed, if $h = 1$, $Q(x) = \frac{x^2}{n}$, then $\vartheta^Q(Z, T)$ is just $\vartheta(Z, \frac{1}{n}T)$. Algebraically, the operation of passing to a bigger lattice is dividing X by a so-called "Göpel group" $H \subset X_n$

($X_n = n$ -torsion in X): write $V(X) = V_1 \oplus V_2$, V_i isotropic for $e^{\mathbf{L}}$, so that $T(X) = T_1 \oplus T_2$, $T_i = V_i \cap T(X)$. Then $H = \frac{1}{n}T_1/T_1$. We must verify that for such H , if $Y = X/H$, then \mathbf{L}^n is the pull-back of a degree 1 line bundle on Y .

This construction can be extended to products X^h of X . Thus on X^2 , for example, we have three line bundles to play with:

$$p_1^* \mathbf{L}, p_2^* \mathbf{L}, (p_1 + p_2)^* \mathbf{L}$$

where $p_i : X^2 \rightarrow X$ are the projections and $p_1 + p_2 : X^2 \rightarrow X$ is addition. If Q is a 2×2 integral symmetric matrix, we can form the combination

$$\mathbf{L}^{(Q)} = (p_1^* \mathbf{L})^{\otimes Q_{11}} \otimes (p_2^* \mathbf{L})^{\otimes Q_{22}} \otimes ((p_1 + p_2)^* \mathbf{L} \otimes p_1^* \mathbf{L}^{-1} \otimes p_2^* \mathbf{L}^{-1})^{\otimes Q_{12}}$$

which turns out to be ample if Q is positive definite. It is again not hard, via a splitting $V = V_1 \oplus V_2$, to define an isogeny $f : X^2 \rightarrow Y$

and a degree 1 bundle \mathbf{M} on Y which pulls back to $\mathbf{L}^{(Q)}$. This gives us still more theta functions. Over \mathbb{C} , one gets Y by dividing $\mathbb{C}^g \oplus \mathbb{C}^g$ not by $(T\mathbb{Z}^g, 0) + (\mathbb{Z}^g, 0) + (0, T\mathbb{Z}^g) + (0, \mathbb{Z}^g)$ but by the bigger lattice of vectors $(T\underline{n}(Q^{-1})_{11}, T\underline{n}(Q^{-1})_{12})$, and $(T\underline{n}(Q^{-1})_{21}, T\underline{n}(Q^{-1})_{22})$, $\underline{n} \in \mathbb{Z}^g$ plus $(\mathbb{Z}^g, 0) + (0, \mathbb{Z}^g)$. This torus is the torus $X^{Q^{-1}}$, so it follows that the new theta function is just $\vartheta^{Q^{-1}}(z, T)$.

To work this out in detail, we first do the complex case geometrically and describe the relations between $X = X_{T, \mathbb{Z}^{2g}}$ and $X^Q = X_{T^*, \mathbb{Z}^{2g, h}}$ very explicitly. Let \mathbf{L} and \mathbf{L}^Q be the basic line bundles on X and X^Q respectively and $f^Q : X^h \rightarrow X^Q$ the rational isogeny¹ defined by $f^Q(z) = z$ for all $z \in \mathbb{C}^{g, h}$.

We have already seen in §4 that there is a canonical isomorphism $\mathbf{A}_f^g \oplus \mathbf{A}_f^g \cong V(X)$ such that any $(x, y) \in \mathbf{Q}^g \oplus \mathbf{Q}^g$ corresponds to the sequence $\frac{1}{n}(Tx + y)$ in $V(X)$. By this isomorphism, $\widehat{\mathbb{Z}}^g \oplus \widehat{\mathbb{Z}}^g$ goes over to $T(X)$ and $\sigma^{\mathbf{L}} = \tau^{\mathbf{L}}$ on the subgroups $\widehat{\mathbb{Z}}^g \oplus 0$ and $0 \oplus \widehat{\mathbb{Z}}^g$.² We have $e^{\mathbf{L}}(x, y) = e(-A(x, y)) = e(-{}^t x_1 y_2 + {}^t y_1 x_2)$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $e(\alpha + \beta) = \exp 2\pi i \alpha$ where $\alpha \in \mathbf{Q}$ and $\beta \in \widehat{\mathbb{Z}}$. Under the canonical isomorphism of $\mathbf{A}_f(2g, h)$ with $V(X)$, the pairing becomes

$$e^{\mathbf{L}}\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\right) = e(\text{Tr}(-{}^t X_1 Y_2 + {}^t X_2 Y_1)).$$

PROPOSITION 6.11. A. Identify $V(X^h)$ with $\mathbf{A}_f(2g, h)$ as above; similarly

¹This means that f^Q is given by an equivalence class of diagrams

$$\begin{array}{ccc} & Y & \\ p \swarrow & & \searrow q \\ X^h & & X^Q \end{array}$$

where p and q are isogenies and f^Q is the formal combination $q \circ p^{-1}$.

²Equivalently, $e_*^{\mathbf{L}}(x) = 1$ if $x \in \frac{1}{2}\widehat{\mathbb{Z}}^g \oplus 0$ or $x \in 0 \oplus \frac{1}{2}\widehat{\mathbb{Z}}^g$. To determine a symmetric line bundle uniquely, we need to specify both $e^{\mathbf{L}}$ and $e_*^{\mathbf{L}}$, so for all our line bundles we shall keep track of $e^{\mathbf{L}}$ and $e_*^{\mathbf{L}}$.

identify $V(X^Q) = \mathbf{A}_f(2g, h)$ using $T \otimes Q$. In these terms

$$V(f^Q) : V(X^h) \cong \mathbf{A}_f(2g, h) \longrightarrow V(X^Q) \cong \mathbf{A}_f(2g, h) \\ \begin{pmatrix} X \\ Y \end{pmatrix} \longmapsto \begin{pmatrix} XQ^{-1} \\ Y \end{pmatrix}$$

where $X, Y \in \mathbf{A}_f(g, h)$.

B. $V(f^Q)(\widehat{\mathbb{Z}}(g, h)Q \oplus \widehat{\mathbb{Z}}(g, h)) = T(X^Q)$ and $\sigma^{\mathbf{L}^Q} = \tau^{\mathbf{L}^Q}$ on the subgroups $V(f^Q)(\widehat{\mathbb{Z}}(g, h)Q \oplus 0)$ and $V(f^Q)(0 \oplus \widehat{\mathbb{Z}}(g, h))$ of $T(X^Q)$.

C.

$$e^{\mathbf{L}^Q}(V(f^Q)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, V(f^Q)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}) = e^{\mathbf{L}^{(h)}}\left(\begin{pmatrix} X_1 Q^{-1} \\ X_2 Q^{-1} \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\right)$$

where $\mathbf{L}^{(h)}$ is the basic bundle on X^h , i.e., $\mathbf{L}^{(h)} \cong p_1^*(\mathbf{L}) \otimes \dots \otimes p_h^*(\mathbf{L})$, the p_i are the projections from X^h to X , and $X_i, Y_i \in \mathbf{A}_f(g, h)$.

D. If $X_1, X_2 \in \mathbf{Q}(g, h)$ and x_1, x_2 are the corresponding elements of $\mathbf{Q}^{g, h}$, then

$$\vartheta^{\alpha, Q} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} (T) = \vartheta^{\alpha} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (T^*) = \text{const. } \vartheta^{\alpha, \mathbf{L}^Q} \left(V(f^Q) \begin{pmatrix} x_1 Q \\ x_2 \end{pmatrix} \right).$$

Here the first two are analytically defined expressions while the last is algebraically defined (see 6.5).

PROOF: A. The two isomorphisms j_1 and j_2 of $\mathbf{R}(2g, h)$ with $\mathbf{C}(g, h)$ corresponding to the complex tori X^h and X^Q are $j_1 \begin{pmatrix} X \\ Y \end{pmatrix} = TX + Y$ and $j_2 \begin{pmatrix} X \\ Y \end{pmatrix} = TXQ + Y$. This shows that $V(f^Q) \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} XQ^{-1} \\ Y \end{pmatrix}$ for $X, Y \in \mathbf{Q}(g, h)$. Because \mathbf{Q} is dense in \mathbf{A}_f , $V(f^Q) \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} XQ^{-1} \\ Y \end{pmatrix}$ for $X, Y \in \mathbf{A}_f(g, h)$.

B follows from A and the remarks preceding the Proposition. We prove

C. Under the identification of $V(X^Q)$ with $\mathbf{A}(2g, h)$ using $T \otimes Q$,

$$e^{\mathbf{L}^Q} \left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right) = e(\text{Tr}(-{}^t X_1 Y_2 + {}^t X_2 Y_1)).$$

Given this, C is straightforward:

$$\begin{aligned} e^{\mathbf{L}^Q} \left(V(f^Q) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, V(f^Q) \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right) &= e^{\mathbf{L}^Q} \left(\begin{pmatrix} X_1 Q^{-1} \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 Q^{-1} \\ Y_2 \end{pmatrix} \right) \\ &= e(\text{Tr}(-{}^t(X_1 Q^{-1})Y_2 + {}^tX_2 Y_1 Q^{-1})) \\ &= e^{\mathbf{L}^{(h)}} \left(\begin{pmatrix} X_1 Q^{-1} \\ X_2 Q^{-1} \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right). \end{aligned}$$

6.3 B' says that the first two expressions in D are equal and by 5.11D the second is equal to $\text{const. } \vartheta^{\alpha, \mathbf{L}^Q} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \text{const. } \vartheta^{\alpha, \mathbf{L}^Q} \left[V(f^Q) \begin{pmatrix} x_1 Q \\ x_2 \end{pmatrix} \right]$.

QED

We take this Proposition as our cue for defining (f^Q, X^Q, \mathbf{L}^Q) and $\vartheta^{\alpha, Q}$ purely algebraically. For ease of exposition, we shall assume we are dealing with an abelian variety over a field k of characteristic 0. With straightforward modifications and suitable restrictions that various integers are prime-to- p , everything extends to $\text{char}(k) = p \neq 0$.

DEFINITION 6.12. Let k be a field of characteristic zero, X an abelian variety defined over k , and \mathbf{L} an ample, even, symmetric line bundle of degree one on X . A Göpel structure for (X, \mathbf{L}) is a pair (V_1, V_2) of subspaces of $V(X)$ isotropic for the pairing $e^{\mathbf{L}}$ such that $V(X) = V_1 \oplus V_2$ and

- A. $\sigma^{\mathbf{L}} = \tau^{\mathbf{L}}$ on $T_i = V_i \cap T(X)$ for $i = 1, 2$; or equivalently $e_*^{\mathbf{L}}(x) = 1$ for $x \in \frac{1}{2}T_i$, and
- B. $T(X) = T_1 \oplus T_2$.

In what follows, members of $V(X)^h$ will be thought of as row vectors with entries in $V(X)$, and if a Göpel structure is chosen, then members of $V(X^h)$ will be thought of as a $2 \times h$ matrix with top row from V_1 , bottom row from V_2 .

PROPOSITION AND DEFINITION 6.13. Let (V_1, V_2) be a Göpel structure for (X, \mathbf{L}) and let Q be a rational symmetric $h \times h$ positive definite matrix. Then there is a triple (f, Y, \mathbf{M}) where f is a rational isogeny from X^h to

Y and \mathbf{M} is an even symmetric ample line bundle of degree one on Y such that

- A. $e^{\mathbf{M}}(V(f)A, V(f)B) = e^{\mathbf{L}^{(h)}}(AQ^{-1}, B)$ for all $A, B \in V(X)^h$, where $\mathbf{L}^{(h)} = p_1^*(\mathbf{L}) \otimes \dots \otimes p_n^*(\mathbf{L})$,
- B. $V(f)(T_1^h Q \oplus T_2^h) = T(Y)$, and
- C. $(V(f)V_1^h, V(f)V_2^h)$ is a Göpel structure for (Y, \mathbf{M}) .

If (f', Y', \mathbf{M}') is another such triple, then there is an isomorphism $g : Y' \rightarrow Y$ such that $f = g \circ f'$ and $g^*\mathbf{M} \cong \mathbf{M}'$. We shall denote such a triple by (f^Q, X^Q, \mathbf{L}^Q) .

If $Q = Q_1 \oplus Q_2$, then

$$(f^{Q_1} \times f^{Q_2}, X^{Q_1} \times X^{Q_2}, \mathbf{L}^{Q_1} \otimes \mathbf{L}^{Q_2}) = (f^Q, X^Q, \mathbf{L}^Q).$$

PROOF: Choose a natural number n such that $S = n^2 Q^{-1}$ is an integral matrix. Let $\mathbf{L}_i = p_i^* \mathbf{L}$ and $\mathbf{L}_{ij} = (p_i + p_j)^* \mathbf{L} \otimes p_i^* \mathbf{L}^{-1} \otimes p_j^* \mathbf{L}^{-1}$. Define $\mathbf{H} = \bigotimes_i \mathbf{L}_i^{S_{ii}} \bigotimes_{i < j} \mathbf{L}_{ij}^{S_{ij}}$; this is a line bundle on X^h . Our immediate aim is to compute the pairing $e^{\mathbf{H}}$. From 4.14F and 4.14G it follows that

$$\begin{aligned} e^{\mathbf{L}_i}(a, b) &= e^{\mathbf{L}}(a_i, b_i) \quad \text{and} \\ e^{\mathbf{L}_{ij}}(a, b) &= e^{\mathbf{L}}(a_i + a_j, b_i + b_j) e^{\mathbf{L}}(a_i, b_i)^{-1} e^{\mathbf{L}}(a_j, b_j)^{-1} \\ &= e^{\mathbf{L}}(a_i, b_j) e^{\mathbf{L}}(a_j, b_i) \end{aligned}$$

where $a = (a_1, a_2, \dots, a_h)$ and $b = (b_1, b_2, \dots, b_h)$ with the a_i and b_i in $V(X)$. A repeated application of 4.14G now shows that

$$\begin{aligned} e^{\mathbf{H}}(a, b) &= \prod_i e^{\mathbf{L}}(a_i, b_i)^{S_{ii}} \prod_{i < j} e^{\mathbf{L}}(a_i, b_j)^{S_{ji}} e^{\mathbf{L}}(a_j, b_i)^{S_{ij}} \\ &= \prod_{(i,j)} e^{\mathbf{L}}(a_i, b_j)^{S_{ji}} = \prod_j e^{\mathbf{L}}\left(\sum_i a_i S_{ij}, b_j\right) \\ &= e^{\mathbf{L}^{(h)}}(aS, b) = e^{\mathbf{L}^{(h)}}(n^2 a Q^{-1}, b). \end{aligned}$$

Consequently V_1^h and V_2^h are maximal isotropic spaces for this pairing, and in addition, $n^{-1}(T_1^h Q \oplus T_2^h)$ is a maximal isotropic subgroup for this pairing: if $a_1 \in V_1^h$ and $a_2 \in V_2^h$, then

$$e^{\mathbf{H}}(a_1 + a_2, n^{-1}b_1 Q + n^{-1}b_2) = e^{\mathbf{L}^{(h)}}(na_1 Q^{-1}, b_2)^{-1} \cdot e^{\mathbf{L}^{(h)}}(na_2, b_1) = 1$$

for all $b_1 \in T_1^h, b_2 \in T_2^h$ if and only if $na_1Q^{-1} \in T_1^h$ and $na_2 \in T_2^h$, i.e., if and only if $a_1 \in n^{-1}T_1^hQ$ and $a_2 \in n^{-1}T_2^h$.

Now the subgroup $\tau^H(n^{-1}T_1^hQ) \cdot \tau^H(n^{-1}T_2^h)$ of $\widehat{G}(H)$ gives rise to an abelian variety X^Q , a symmetric line bundle L^Q and a rational isogeny $g : X^h \rightarrow X^Q$ by 4.22B so that H and L^Q are related by the Q -isogeny g (so if g is an actual isogeny $g^*L^Q \cong H$). In addition $j(g, L^Q) : \widehat{G}(H) \cong \widehat{G}(L^Q)$ takes $\tau^H(n^{-1}T_1^hQ) \cdot \tau^H(n^{-1}T_2^h)$ isomorphically onto $\sigma^{L^Q}(TX^Q)$. Now put $f^Q = n^{-1}g$, and it is easily seen that the triple (f^Q, X^Q, L^Q) satisfies A, B and C. What is missing, however, is the ampleness of L^Q which depends on the fact that Q is positive definite and not just non-degenerate which is all we have used so far. We postpone the proof of this fact.

Now let (f, Y, M) and (f', Y', M') be two such triples. Because

$$V(f)^{-1}TY = V(f')^{-1}TX = T_1^hQ \oplus T_2^h$$

4.21D implies that there is $g : Y' \cong Y$ such that $f = g \circ f'$. In addition, $e^{L'} = e^{g^*L}$ and $e_*^{L'}(x) = e_*^{g^*L}(x) = 1$ for $x \in \frac{1}{2}V(f')T_1^hQ$ and for $x \in \frac{1}{2}V(f')T_2^h$, showing that $e_*^{L'}(x) = e_*^{g^*L}(x)$ for all $x \in \frac{1}{2}TY'$ by 4.17C. By 4.23 $g^*L \cong L'$ finishing the proof of the Proposition. QED

We head towards a proof of the Fundamental Identities in the algebraic case; along the way we show that L^Q is ample.

When A is an integral $(h \times h)$ matrix, let $\psi_A : X^h \rightarrow X^h$ be the homomorphism defined by $\psi_A(x) = y$ where $y_i = \sum_j A_{ji}x_j$. If $A \in GL_h(Q)$ choose a non-zero $n \in \mathbb{Z}$ such that nA is integral and define

$\psi_A = \psi_{nA} \circ (n_X)^{-1}$ which is a Q -isogeny from X^h to itself (see 4.20).

PROPOSITION 6.14. Let $Q' = {}^tA \cdot Q \cdot A$ where $A \in GL_h(Q)$ and Q is a rational $h \times h$ positive definite symmetric matrix. Choose an $n \geq 1$ so the

Q -isogenies $f^Q, f^{Q'}, \psi_A$ are represented by morphisms in:

$$\begin{array}{ccccc} & & X^h & & \\ & \swarrow & \downarrow n & \searrow & \\ X^Q & \xleftarrow{f^Q} & X^h & \xrightarrow{\psi_A} & X^h & \xrightarrow{f^{Q'}} & X^{Q'} \end{array}$$

Then, for suitable n , the pull-backs of L^Q and $L^{Q'}$ to the top X^h are isomorphic.

PROOF: Let $f^{Q'} \circ \psi_A \circ (f^Q)^{-1}$ be the Q -isogeny ψ'_A . We first show that

$$e^{L^{Q'}}(V(\psi'_A)x, V(\psi'_A)y) = e^{L^Q}(x, y)$$

for all $x, y \in V(X^Q)$; this clearly being necessary for the Proposition to be correct. If $x = V(f^Q)v$ and $y = V(f^Q)w$, then

$$\begin{aligned} e^{L^{Q'}}(V(\psi'_A)x, V(\psi'_A)y) &= e^{L^{Q'}}(V(f^{Q'} \circ \psi_A)v, V(f^{Q'} \circ \psi_A)w) \\ &= e^{L^{Q'}}(V(f^{Q'})(vA), V(f^{Q'})(wA)) = e^{L^{(h)}}(vA \cdot Q'^{-1}, wA) \\ &= e^{L^{(h)}}(v \cdot A \cdot Q'^{-1} \cdot {}^tA, w) = e^{L^{(h)}}(v \cdot Q^{-1}, w) = e^{L^Q}(x, y). \end{aligned}$$

Choose a non-zero m so that $m\psi'_A = g$ is a morphism. The above shows that $e^{m^*L^Q} = e^{g^*L^{Q'}}$. Replacing m by $2m$ if necessary, we may assume that e_* of both bundles vanish identically, so by 4.23, $g^*L^{Q'} \cong m^*L^Q$. QED

We can now prove the ampleness of L^Q . It is clear that if two line bundles are related by a rational isogeny both are ample or neither is ample. By the above Proposition we need to show that L^Q is ample only for diagonal, integral Q (because any quadratic form can be diagonalised), and by the last statement in Proposition 6.13, it suffices to do so when $h = 1$, $Q = d > 0$. In this case, L^Q on X^Q and L^d on X are related by a suitable rational isogeny, showing that L^Q is ample.

Notice that given ϑ^Q and A , with ϑ^Q the unique element fixed by a maximal isotropic subgroup of $\text{Heis}(2(g \times h), A_f)$, we can construct $\vartheta^{Q'}$ in a purely group theoretic way without reference to the underlying abelian varieties.

We now make the algebraic definition:

DEFINITION 6.15. Given any rational symmetric positive definite $h \times h$ matrix Q , abelian variety X , ample, degree 1, even symmetric line bundle \mathbf{L} , and Göpel structure for (X, k) , define:

$$\vartheta^{\alpha, Q} \begin{bmatrix} x \\ y \end{bmatrix} = \vartheta^\alpha [V(f^Q)(xQ + y)]$$

for all $x \in V_1^h, y \in V_2^h$. Here ϑ^α is the algebraic theta function on $V(X^Q)$ associated to \mathbf{L}^Q .

Comparison with 7.9D shows that when $k = \mathbb{C}$, we have the same theta function as the analytically defined $\vartheta^{\alpha, Q}$. We wish to prove the two fundamental identities for these algebraic theta functions:

(6.16). If $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, and $x_1 \in V(X)^{h_1}, x_2 \in V(X)^{h_2}$, then

$$\vartheta^{\alpha, Q_1}[x_1] \vartheta^{\alpha, Q_2}[x_2] = \text{cnst. } \vartheta^{\alpha, Q}[x_1, x_2],$$

(6.17). If $Q' = {}^t A \cdot Q \cdot A$, and $x \in V_1^h, y \in V_2^h$ then

$$\vartheta^{\alpha, Q'} \begin{bmatrix} x \cdot {}^t A^{-1} \\ y \cdot A \end{bmatrix} = \text{cnst. } \sum_R \sum_S U_{\tau(R)} U_{\tau(S)} \vartheta^{\alpha, Q} \begin{bmatrix} x \\ y \end{bmatrix},$$

where R and S run through a system of coset representatives for $T_1^h \cdot {}^t A / (T_1^h \cap T_1^h \cdot {}^t A)$ and $T_2^h \cdot A^{-1} / (T_2^h \cdot A^{-1} \cap T_2^h)$ respectively, and $\tau(R), \tau(S) \in \widehat{\mathcal{G}}(\mathbf{L}^{(h)})$ act on functions on $V(X)^h$ as usual (right action of §5 $U_{\lambda\tau(y)} f(x) = \lambda e(\frac{y}{2}, x) f(x - y)$).

PROOF: We shall consider only (6.17) as (6.16) is quite simple. Note that by (6.14), we have an isomorphism

$$(\psi'_A)^* : \widehat{\Gamma}(\mathbf{L}^{Q'}) \xrightarrow{\sim} \widehat{\Gamma}(\mathbf{L}^Q)$$

equivariant for $j(\psi'_A)^* : \widehat{\mathcal{G}}(\mathbf{L}^Q) \xrightarrow{\sim} \widehat{\mathcal{G}}(\mathbf{L}^{Q'})$. (6.17) states a relation between 2 distinguished elements in these spaces:

$$s \in \Gamma(\mathbf{L}^Q) \subset \widehat{\Gamma}(\mathbf{L}^Q)$$

and

$$t \in \Gamma(\mathbf{L}^{Q'}) \subset \widehat{\Gamma}(\mathbf{L}^{Q'}).$$

The idea of the proof is this: the elements s and t are characterized as the unique elements of an irreducible representation fixed by certain maximal isotropic subgroups. To compare these elements we need to compare explicitly the representation spaces and the maximal isotropic subgroups of s and t . To do this, it is convenient to have the *same* Heisenberg group acting on both spaces.

Define $\beta^Q : \widehat{\mathcal{G}}(\mathbf{L}^{(h)}) \cong \widehat{\mathcal{G}}(\mathbf{L}^Q)$ as follows:

$$\beta^Q(\lambda \tau x \cdot \tau y) = \lambda \tau V(f^Q)(xQ) \cdot \tau V(f^Q)(y) \quad \text{for all } \lambda \in \bar{k}^*,$$

$x \in V_1^h, y \in V_2^h$. To show that β^Q is a homomorphism we must check that

$$e^{\mathbf{L}^Q}(V(f^Q)(xQ), V(f^Q)(y)) = e^{\mathbf{L}^{(h)}}(x, y),$$

but this is Proposition 6.13A; furthermore β^Q commutes with the involution and there is a commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \bar{k}^* & \longrightarrow & \widehat{\mathcal{G}}(\mathbf{L}^{(h)}) & \xrightarrow{\pi} & V(X^h) \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow \beta^Q & & \downarrow \alpha^Q \\ 1 & \longrightarrow & \bar{k}^X & \longrightarrow & \widehat{\mathcal{G}}(\mathbf{L}^{(Q)}) & \xrightarrow{\pi} & V(X^Q) \longrightarrow 0 \end{array}$$

where $\alpha^Q(x + y) = V f^Q(xQ + y)$. In addition,

$$\alpha^Q(TX^h) = TX^Q$$

and

$$\beta^Q \sigma^{\mathbf{L}^{(h)}}(x) = \sigma^{\mathbf{L}^Q} \alpha^Q(x) \quad x \in TX^h.$$

Composing β^Q with the action of $\widehat{\mathcal{G}}(\mathbf{L}^Q)$ on $\widehat{\Gamma}(\mathbf{L}^Q)$ we get the desired action of $\widehat{\mathcal{G}}(\mathbf{L}^{(h)})$ on $\widehat{\Gamma}(\mathbf{L}^Q)$. Moreover, we define

$$\vartheta_s^Q[x] = \vartheta_s[\alpha^Q x], \quad \text{for all } s \in \widehat{\Gamma}(\mathbf{L}^Q), x \in V(X^h).$$

As in §5, we check that $U_g \vartheta_s^Q = \vartheta_{U_g(s)}^Q$, for all $g \in \widehat{G}(\mathbf{L}^{(h)})$, $s \in \widehat{\Gamma}(\mathbf{L}^Q)$ where $\widehat{G}(\mathbf{L}^{(h)})$ acts on the space of functions on $V(X^h)$ as usual.

Let h_A be the composition

$$h_A : \widehat{G}(\mathbf{L}^{(h)}) \xrightarrow{\beta^Q} \widehat{G}(\mathbf{L}^Q) \xrightarrow{j(\psi'_A)} \widehat{G}(\mathbf{L}^{Q'}) \xrightarrow{(\beta^{Q'})^{-1}} \widehat{G}(\mathbf{L}^{(h)}),$$

and define ϕ_A by the diagram

$$\begin{array}{ccc} \widehat{G}(\mathbf{L}^{(h)}) & \xrightarrow{\pi} & V(X^h) \\ \downarrow h_A & & \downarrow \phi_A \\ \widehat{G}(\mathbf{L}^{(h)}) & \xrightarrow{\pi} & V(X^h). \end{array}$$

Putting this together we have the diagram:

$$\begin{array}{ccccc} \mathbf{L}^{(h)} & & & & \mathbf{L}^{(h)} \\ & \searrow & & & \swarrow \\ & V(X^h) & \xrightarrow{\text{dotted}} & V(X^h) & \\ & \downarrow \alpha^Q & & \downarrow \alpha^{Q'} & \\ (f^Q)^*(\mathbf{L}^Q) & & & & (f^Q)^*\mathbf{L}^{Q'} \\ & \searrow & & & \swarrow \\ & V(X^h) & \xrightarrow{V(A)} & V(X^h) & \\ & \downarrow f^Q & & \downarrow f^{Q'} & \\ \mathbf{L}^Q & & & & \mathbf{L}^{Q'} \\ & \searrow & & & \swarrow \\ & V(X^Q) & & & V(X^{Q'}) \end{array}$$

The map ϕ_A is the dotted arrow.

LEMMA 6.18. The map h_A preserves the involution and $\phi_A(x + y) = x^t A^{-1} + yA$ where $x \in V_1^h$ and $y \in V_2^h$.

PROOF: h_A preserves the involution simply because $\beta^{Q'}, \beta^Q$ and $j(\psi'_A, \mathbf{L}^{Q'})$ preserve the involutions in question.

$$\begin{aligned} \phi_A(x_1, x_2) &= (\alpha^{Q'})^{-1} \circ V(A) \circ \alpha^Q(x_1, x_2) \\ &= \alpha^{Q'^{-1}} \circ V(A)(x_1, x_2) \\ &= \alpha^{Q'^{-1}}(x_1 Q A, x_2 A) \\ &= (x_1 Q A (Q')^{-1}, x_2 A) \\ &= (x_1 {}^t A^{-1}, x_2 A). \end{aligned}$$

QED for the lemma.

LEMMA 6.19. If U^Q and $U^{Q'}$ denote the actions of $\widehat{G}(\mathbf{L}^{(h)})$ on $\widehat{\Gamma}(\mathbf{L}^Q)$ and on $\widehat{\Gamma}(\mathbf{L}^{Q'})$ induced by $\beta^Q, \beta^{Q'}$ respectively and $\psi'_A : \widehat{\Gamma}(\mathbf{L}^{Q'}) \rightarrow \widehat{\Gamma}(\mathbf{L}^Q)$ is the isomorphism induced by ψ'_A , then

- A. $\psi'_A U_{h_A g}^{Q'} s = U_g^Q \psi'_A s$ for all $g \in \widehat{G}(\mathbf{L}^{(h)})$, $s \in \widehat{\Gamma}(\mathbf{L}^{Q'})$, and
 B. $\vartheta_{\psi'_A s}^Q = \vartheta_s^{Q'} \circ \phi_A$ for all $s \in \widehat{\Gamma}(\mathbf{L}^{Q'})$.

The proof is left to the reader.

Let $0 \neq s \in \Gamma(\mathbf{L}^Q)$ and $0 \neq t \in \Gamma(\mathbf{L}^{Q'})$. Put $s' = \psi'_A t$. In the classical case, s corresponds to $\vartheta^Q(Z, T)$ and s' to $\vartheta^{Q'}(ZA, T)$, and Proposition 6.6 describes the relation between them.

Note that s is fixed by the subgroup

$$(\beta^Q)^{-1} \sigma^{\mathbf{L}}(TX^Q) = \sigma^{\mathbf{L}^{(h)}}(TX^h) = T$$

while, on the other hand, using Lemma 6.19 we see that s' is fixed by

$h_A^{-1}(T) = \tau(T_1^h \cdot {}^t A) \cdot \tau(T_2^A \cdot A^{-1}) = T'$. The following simple lemma tells us how to get hold of s' given s :

LEMMA 6.20. Let T and T' be subgroups of $\text{Heis}(2g, \mathbf{A}_f)$ such that $T \cap \bar{k}^* = T' \cap \bar{k}^* = 1$ and $\pi(T)$ and $\pi(T')$ are maximal isotropic lattices (i.e., compact open subgroups) of \mathbf{A}_f^{2g} . Let $0 \neq s$ be a T -invariant vector in the Heisenberg representation \mathcal{H} of $\text{Heis}(2g, \mathbf{A}_f)$. Then $s' = \sum_g U_g s$, where g runs through a system of coset representatives of $T'/T \cap T'$, is T' -invariant and is non-zero if and only if $\pi(T) \cap \pi(T') = \pi(T \cap T')$.

PROOF: It is clear that s' is T' -invariant. The important thing is that it is non-zero when $\pi(T) \cap \pi(T') = \pi(T \cap T')$! Let $s'' = \sum_h U_h s'$ where h runs through a system of coset representatives of $T/T \cap T'$. Then

$$\begin{aligned} s'' &= \sum_h \sum_g U_h U_g s = \sum_g \sum_h e(\pi(h), \pi(g)) U_g U_h s \\ &= \sum_g U_g \left(\sum_h e(\pi(h), \pi(g)) \right) s = [T : T \cap T'] s \end{aligned}$$

because if $g \notin T \cap T'$, $\sum_h e(\pi(h), \pi(g)) = 0$. This shows that $s' \neq 0$. We don't need the converse. QED for the lemma.

Applying Lemma 6.20 to the situation $T = \tau(T_1^h) \cdot \tau(T_2^h)$,
 $T' = \tau(T_1^h \cdot {}^t A) \cdot \tau(T_2^h \cdot A^{-1})$ we find:

$$s' = \text{const.} \sum_R \sum_S U_{\tau(R)} U_{\tau(S)} s.$$

Taking ϑ^Q of both sides and using Lemma 6.19B, the fundamental identity

$$\vartheta^{\alpha, Q'} \circ \phi_A = \text{const.} \sum_R \sum_S U_{\tau(R)} U_{\tau(S)} \vartheta^{\alpha, Q},$$

is proven. QED

The algebraic development of this theory goes through if char. $k = p$ and Q and A both belong to $\text{GL}_h(\mathbb{Z}[\frac{1}{p}])$ and $p \nmid a$. If $p > 0$ we may either ignore p -torsion to formulate 6.17 or assume that X is an ordinary abelian variety, in which case a Göpel structure at the prime p can still be put on (X, \mathbf{L}) . If $2 \nmid a$ the assumption that \mathbf{L} is even symmetric can be dropped for 6.1 to be true (assuming that \mathbf{L} is still symmetric).

COROLLARY 6.21. *If \mathbf{L} is a symmetric ample line bundle of degree one on X , then $\vartheta^\alpha(x) = \epsilon \vartheta^\alpha(-x)$ where $\epsilon = 1$ if \mathbf{L} is even symmetric and $\epsilon = -1$ if \mathbf{L} is odd symmetric (char. $k \neq 2$).*

When $k = \mathbb{C}$ and \mathbf{L} is even symmetric, then $X = X_T$ for some $T \in \mathfrak{H}_g$ and \mathbf{L} is then the basic line bundle. Therefore

$$\vartheta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \exp \pi i {}^t x_1 (T x_1 + x_2) \cdot \vartheta(T x_1 + x_2, T)$$

which is obviously an even function of (x_1, x_2) . However there seems to be no straightforward algebraic proof of this fact.

PROOF: Assume that \mathbf{L} is even symmetric. Put $h = 1$ and $Q = Q' = 1$ and $A = -1$ in 6.17. This gives $\vartheta^\alpha[-x] = \epsilon \vartheta^\alpha[x]$ for some constant ϵ , and therefore $\vartheta^\alpha[x] = \epsilon \vartheta^\alpha[-x] = \epsilon^2 \vartheta^\alpha[x]$ showing that $\epsilon = \pm 1$.

Now put $h = 2$, $Q = \text{Id.}$, $Q' = 2 \cdot \text{Id.}$, $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ in 6.17:

$$\begin{aligned} \vartheta^{\alpha, 2}[x+y] \vartheta^{\alpha, 2}[x-y] &= c \sum_{\eta \in \frac{1}{2}T_2/T_2} U_{\tau(\eta)} \vartheta^\alpha[x] \cdot U_{\tau(\eta)} \vartheta^\alpha[y] \\ &= c \sum_{\eta \in \frac{1}{2}T_2/T_2} e(\frac{\eta}{2}, x+y) \vartheta^\alpha[-\eta+x] \vartheta^\alpha[-\eta+y] \\ &= c \sum_{\eta \in \frac{1}{2}T_2/T_2} e(\frac{\eta}{2}, x+y) e(-\eta, y) \vartheta^\alpha[-\eta+x] \vartheta^\alpha[\eta+y] \\ &= c \sum_{\eta \in \frac{1}{2}T_2/T_2} e(\frac{\eta}{2}, x-y) \vartheta^\alpha[-\eta+x] \vartheta^\alpha[\eta+y] \end{aligned}$$

(because $U_{\tau(2\eta)} \vartheta^\alpha = \vartheta^\alpha$).

Replacing y by $-y$ does not alter the L.H.S. while the R.H.S. becomes

$$\begin{aligned} &c \sum_{\eta \in \frac{1}{2}T_2/T_2} e(\frac{\eta}{2}, x+y) \vartheta^\alpha[-\eta+x] \vartheta^\alpha[\eta-y] \\ &= \epsilon c \sum_{\eta \in \frac{1}{2}T_2/T_2} e(\frac{\eta}{2}, x+y) \vartheta^\alpha[-\eta+x] \vartheta^\alpha[-\eta+y] \end{aligned}$$

which is the old R.H.S. multiplied by ϵ and this shows $\epsilon = 1$.

Take any $\xi \in \frac{1}{2}T(X)/T(X)$, with $\xi = (\xi_1, \xi_2, \dots)$. Let $\mathbf{M} = T_{\xi_1}^*(\mathbf{L})$. By 4.22D, \mathbf{M} is even or odd symmetric depending on whether $e_*(\xi)$ equals 1 or -1 . Now $2_X^* \mathbf{L} \cong 2_X^* \mathbf{M}$ so that $\widehat{\Gamma}(\mathbf{L}) = \widehat{\Gamma}(\mathbf{M})$ and $\widehat{G}(\mathbf{L}) = \widehat{G}(\mathbf{M})$ and $\sigma^{\mathbf{M}}(x) = e^{\mathbf{L}}(\xi, x) \sigma^{\mathbf{L}}(x)$. If $0 \neq s \in \Gamma(\mathbf{L})$, then $U_{\tau(\xi)} s = s' \in \Gamma(\mathbf{M})$ and therefore $\vartheta^{\alpha, \mathbf{M}}(x) = \text{const.} U_{\tau(\xi)} \vartheta^{\alpha, \mathbf{L}}(x)$.

Denoting $\vartheta^{\alpha, \mathbf{L}}$ by ϑ^α as before, we have:

$$\begin{aligned} \vartheta^{\alpha, \mathbf{M}}(x) &= e(\frac{\xi}{2}, x) \vartheta^\alpha(-\xi + x) \\ \vartheta^{\alpha, \mathbf{M}}(-x) &= e(\frac{\xi}{2}, -x) \vartheta^\alpha(-\xi - x) = e(\frac{\xi}{2}, -x) \vartheta^\alpha(\xi + x) \\ &= e(\frac{\xi}{2}, -x) e(\xi, x) e_*(\xi) \vartheta^\alpha(-2\xi + \xi + x) \\ &\quad \text{(by the quasi-periodicity of } \vartheta^\alpha) \\ &= e_*(\xi) e(\frac{\xi}{2}, x) \vartheta^\alpha(-\xi + x) \\ &= e_*(\xi) \vartheta^{\alpha, \mathbf{M}}(x). \end{aligned}$$

This finishes the proof of the Corollary because any odd symmetric $\mathbf{M} \cong T_x^* \mathbf{L}$ for some x such that $2x = 0$ and for some even symmetric \mathbf{L} from Corollary 4.24. QED

As in §5, $\vartheta^{\alpha, Q}(x)$ can be described as a matrix-coefficient:

$$\vartheta^{\alpha, Q}(x) = \ell^Q(U_{\tau(x)} s)$$

where ℓ^Q is the composite $\widehat{\Gamma}(\mathbf{L}^Q) \rightarrow \mathbf{L}^Q(0) \xrightarrow{\epsilon} k$ and $0 \neq s \in \Gamma(\mathbf{L}^Q)$, the subspace of $\widehat{\Gamma}(\mathbf{L}^Q)$ fixed by the subgroup $\sigma(TX^h) \subset \widehat{G}(\mathbf{L}^{(h)})$. Thus $\ell^Q \in \widehat{\Gamma}(\ell^Q)^*$ determines the function $\vartheta^{\alpha, Q}$. Let $\mathcal{S}(V_1^h)$ denote the space of locally constant functions on V_1^h that have compact support. Choosing an isomorphism $f : \mathcal{S}(V_1^h) \cong \widehat{\Gamma}(\mathbf{L}^Q)$, $\mu^Q = \ell^Q \circ f$ now appears as a finitely additive k -valued measure on the Boolean algebra of compact open subsets of V_1^h . Note that μ^Q is determined only up to a scalar because f and ℓ^Q are only uniquely determined up to scalars.

We want to compare

$$(A) \quad \mu^{Q_1}, \mu^{Q_2} \text{ and } \mu^Q \text{ where } Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix},$$

and

$$(B) \quad \mu^{Q'} \text{ and } \mu^Q \text{ where } Q' = {}^t A Q A.$$

(A) Let Q_i be $(h_i \times h_i)$ -matrices for $i = 1, 2$ and let $h = h_1 + h_2$. Choose isomorphisms $f_i : \mathcal{S}(V_1^{h_i}) \cong \widehat{\Gamma}(\mathbf{L}^{Q_i})$. We then have the following chain of isomorphisms of $\widehat{G}(\mathbf{L}^{(h)})$ -modules:

$$\mathcal{S}(V_1^h) \cong \mathcal{S}(V_1^{h_1}) \otimes \mathcal{S}(V_2^{h_2}) \xrightarrow{f_1 \otimes f_2} \widehat{\Gamma}(\mathbf{L}^{Q_1}) \otimes \widehat{\Gamma}(\mathbf{L}^{Q_2}) \cong \widehat{\Gamma}(\mathbf{L}^Q).$$

The last isomorphism comes from the fact that $X^Q = X^{Q_1} \times X^{Q_2}$ and $\mathbf{L}^Q = p_1^* \mathbf{L}^{Q_1} \otimes p_2^* \mathbf{L}^{Q_2}$ (which also shows that up to a non-zero constant multiple $\ell^Q = \ell^{Q_1} \otimes \ell^{Q_2}$). It follows that $\text{const. } \mu^Q = \mu^{Q_1} \times \mu^{Q_2}$ where $\mu^{Q_1} \times \mu^{Q_2}$ is the unique k -valued measure on V_1^h such that $(\mu^{Q_1} \times \mu^{Q_2})(F_1 \times F_2) = \mu^{Q_1}(F_1) \cdot \mu^{Q_2}(F_2)$ where F_1 and F_2 are compact open subsets of $V_1^{h_1}$ and $V_1^{h_2}$ respectively.

(B) Let $r = \psi_A^* : \widehat{\Gamma}(\mathbf{L}^{Q'}) \rightarrow \widehat{\Gamma}(\mathbf{L}^Q)$. Then $\ell^Q \circ r = \text{const. } \ell^{Q'}$. Choose $f : \mathcal{S}(V_1^h) \cong \widehat{\Gamma}(\mathbf{L}^Q)$ and $f' : \mathcal{S}(V_1^h) \cong \widehat{\Gamma}(\mathbf{L}^{Q'})$ to be $\widehat{G}(\mathbf{L}^{(h)})$ -module isomorphisms.

Then

$$\begin{aligned} \mu^Q &= \ell^Q \circ f \\ \text{and } \mu^{Q'} &= \ell^{Q'} \circ f' \\ &= \text{const. } \ell^Q \circ r \circ f' \\ &= \text{const. } \mu^Q \circ f^{-1} \circ r \circ f'. \end{aligned}$$

From Lemma 6.19A, $r \circ U_{h, A, g}^{Q'} = U_g^Q \circ r$ for all $g \in \widehat{G}(\mathbf{L}^{(h)})$, so if $F = f^{-1} \circ r \circ f'$, $F \circ U_{h, A, g} = U_g \circ F$ for all $g \in \widehat{G}(\mathbf{L}^{(h)})$.

But such an F is unique up to scalars because $\mathcal{S}(V_1^h)$ is irreducible. Now $f(x) \mapsto f(A^{-1}x)$ is a candidate for this isomorphism so that $Ff(x) = \text{const. } f(A^{-1}x)$. If χ_U is the characteristic function of a compact open subset $U \subset V_1^h$,

$$\begin{aligned} \mu^{Q'}(U) &\stackrel{\text{def}}{=} \mu^{Q'}(\chi_U) \\ &= \text{const. } \mu^Q(F\chi_U) \\ &= \text{const. } \mu^Q(\chi_U \circ A^{-1}) \\ &= \text{const. } \mu^Q(\chi_{AU}) \\ &= \text{const. } \mu^Q(AU). \end{aligned}$$

We summarize:

PROPOSITION 6.22. *Given a Göpel structure (V_1, V_2) on (X, \mathbf{L}) , every rational symmetric positive definite matrix Q determines, up to scalars, a finitely additive measure μ^Q on the Boolean algebra of compact open subsets of V_1^h such that*

- A. *If $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, then $\mu^Q = \text{const. } \mu^{Q_1} \times \mu^{Q_2}$, and*
 B. *If $Q' = {}^t A Q A$ with $A \in GL_h(\mathbb{Q})$, then*

$$\mu^{Q'}(U) = \text{const. } \mu^Q(AU).$$

7. Riemann's theta relation

Although the results of the previous section give us a huge class of identities satisfied by the theta functions, Riemann's theta relation is special in having some elegant reformulations. The purpose of this section is to explain these interpretations: in particular we give its formulations in terms of

(i) the Heisenberg action on the tower of spaces

$$\Gamma(L), \Gamma(L^2), \Gamma(L^3), \dots,$$

(ii) explicit formulas, and

(iii) the measures on A_f^g that define ϑ^α (see §5).

THEOREM 7.1. *Let X be an abelian variety over an algebraically closed field and L a symmetric line bundle of degree one on X . Take $n \geq 1$ and char $k \nmid n$; then trivializing L on $V(X)$ as in §6, we can consider $\widehat{\Gamma}(L^n)$ as a subspace of the functions on $V(X)$. Then*

(a) $\widehat{\Gamma}(L^n)$ is the space of function generated by polynomials of degree n in $\widehat{\Gamma}(L)$.

(b) $\widehat{\Gamma}(L^n)$ is mapped into itself by $\widehat{\mathcal{G}}(L)$ acting by:

$$\text{for } \lambda \in k, x \in V(X), (U_{\lambda\tau(x)}^{(n)} f)(y) = \lambda^n e(x/2, y)^n f(y - x)$$

(c) $\widehat{\Gamma}(L^n)$ is irreducible under this action.

PROOF: Note that (c) implies (a) because the group action on $\widehat{\Gamma}(L^n)$ is just the n^{th} symmetric product of the action on $\widehat{\Gamma}(L)$.

Choose a splitting $V(X) = V_1 \oplus V_2$ into maximal isotropic subspaces for the skew pairing induced by L . Define

$$\widehat{\mathcal{G}}(L^n) = \{(\phi_m, x_m) \mid \phi_m : m_x^* L^n \cong T_{x_m}^* m_X^* L^n\}.$$

Then $\widehat{\mathcal{G}}(L^n)$ acts on $\widehat{\Gamma}(L^n)$ and there is a map

$$\begin{array}{ccc} \widehat{\mathcal{G}}(L) & \longrightarrow & \widehat{\mathcal{G}}(L^n) \\ (\phi_m, x_m) & \longrightarrow & (\phi_m^{\otimes n}, x_m) \end{array}$$

The kernel of this map is μ_n ; hence this map is an isomorphism $\widehat{\mathcal{G}}(L^n) \cong \widehat{\mathcal{G}}(L)/\mu_n$. The commutator of $\widehat{\mathcal{G}}(L^n)$ is $e^L(x, y)^n$, so $\widehat{\mathcal{G}}(L^n)$ is still an Heisenberg group. Let $W_i = T(X) \cap V_i$; then $H_n = 1/nW_1 \oplus W_2$ is maximal isotropic for $e^L(x, y)^n$. Lift H_n to $\mathcal{H}_n = \tau(1/nW_1)\tau(W_2)$ and take the quotient of L^n by \mathcal{H}_n to get

$$\begin{array}{ccc} L^n & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & X/H_n \end{array}$$

Then $K(E) = 0$ since $1/nW_1 \oplus W_2$ is maximal isotropic; therefore $\deg E = 1$, and $\widehat{\Gamma}(L^n)^{\mathcal{H}_n}$ is one-dimensional. This implies $\widehat{\Gamma}(L^n)$ is acted on irreducibly by $\widehat{\mathcal{G}}(L^n)$. QED

LEMMA 7.2. *Let $\vartheta : A_f^{2g} \rightarrow k$ be a locally constant function such that*

$$(a) \vartheta(x+k) = e_*(k/2)e(k/2, x)\vartheta(x), \quad \forall k \in \widehat{\mathcal{Z}}^g, \text{ and}$$

$$(b) \vartheta(-x) = \vartheta(x).$$

Then there exists a finitely additive measure μ on A_f^g such that for all $x = (x_1, x_2) \in A_f^{2g}$

$$i) \vartheta(x) = \int_{x_1 + \widehat{\mathcal{Z}}^g} e(\frac{x_1 \cdot x_2}{2}) e(-x_2 \cdot u) d\mu(u), \text{ and}$$

ii) For $b \in \mathcal{Z}$,

$$\mu(a_1 + b\widehat{\mathcal{Z}}^g) = b^{-g} \sum_{a_2 \in (b^{-1}\mathcal{Z}^g/\mathcal{Z}^g)} e(\frac{a_1 \cdot a_2}{2}) \vartheta(a_1, a_2).$$

PROOF: This can be verified by a straightforward calculation. We can also proceed as follows: By Proposition 5.10 we know that the space of functions S generated by the action of the Heisenberg group on ϑ is irreducible (the action is $(U_{\lambda\tau(k)} f)(x) = \lambda e(k/2, x) f(x-k)$. Under this action ϑ is fixed by a maximal isotropic subgroup; hence there is a unique (up to a multiplicative constant) map

$$\psi : C_k^g(A_f^g) \rightarrow S$$

sending δ , the characteristic function of $\widehat{\mathcal{Z}}^g$ to ϑ . The action on $C_k^g(A_f^g)$ is $U_{\tau(y_1, y_2)} f(x) = e(\frac{y_1 \cdot y_2}{2}) e(y_2, x) f(x + y_1)$. Define a linear functional ℓ on S

by $\ell : f \rightarrow f(0)$. Then ℓ corresponds, via ψ , to a finitely additive measure on \mathbf{A}_f^g . Since ψ is equivariant

$$\begin{aligned}\vartheta(x) &= \ell(U_{\tau(-x)}\vartheta) = \int \psi^{-1}(U_{\tau(-x)}\vartheta) d\mu \\ &= \int U_{\tau(-x)}\delta d\mu = \int_{x_1+\widehat{\mathbf{Z}}^g} e(-{}^t x_2 \cdot u) e(\frac{{}^t x_1 \cdot x_2}{2}) d\mu(u).\end{aligned}$$

QED

Let ϑ satisfy properties (a) and (b) of the lemma above.

LEMMA 7.3. If there exists a function $\psi : \mathbf{A}_f^{2g} \rightarrow k$ such that

$$(B1) \quad \vartheta(x_1, x_2)\vartheta(y_1, y_2) = \sum_{\eta \in (\frac{1}{2}\widehat{\mathbf{Z}}^g/\widehat{\mathbf{Z}}^g)} e(-{}^t \eta \cdot x_2) \psi(\frac{x_1+y_1}{2} + \eta, x_2 + y_2) \psi(\frac{x_1-y_1}{2} + \eta, x_2 - y_2)$$

then

$$(B2) \quad \psi(x_1, x_2)\psi(y_1, y_2) = 2^{-g} \sum_{\zeta \in (\frac{1}{2}\widehat{\mathbf{Z}}^g/\widehat{\mathbf{Z}}^g)} e({}^t \zeta \cdot x_1) \vartheta(x_1 + y_1, \frac{x_2+y_2}{2} + \zeta) \vartheta(x_1 - y_1, \frac{x_2-y_2}{2} + \zeta)$$

and vice versa: if there exists ψ so that (B2) holds, then (B1) does also.

PROOF: We often use the quasi-periodicity of ϑ in the form

$$\vartheta(x_1 + 2\eta, x_2) = e({}^t \eta \cdot x_2) \vartheta(x_1, x_2)$$

for $\eta \in \frac{1}{2}\widehat{\mathbf{Z}}^g$. We only show that (B2) implies (B1), since the proof that (B1) implies (B2) is similar. We start with the R.H.S. of (B1) and show after a short calculation that it is, indeed, $\vartheta(x_1, x_2)\vartheta(y_1, y_2)$. First express $\psi(\frac{x_1+y_1}{2} + \eta, x_2 + y_2) \psi(\frac{x_1-y_1}{2} + \eta, x_2 - y_2)$ in terms of ϑ 's by using (B2) after changing variables:

$$\left\{ \begin{array}{ll} x_1 \rightarrow \frac{x_1+y_1}{2} + \eta & y_1 \rightarrow \frac{x_1-y_1}{2} + \eta \\ x_2 \rightarrow x_2 + y_2 & y_2 \rightarrow x_2 - y_2 \end{array} \right\}$$

One obtains for the right side of B1

$$\begin{aligned}& \sum_{\eta, \zeta} (2^{-g} e(-{}^t \eta \cdot x_2) e({}^t \zeta \cdot (\frac{x_1+y_1}{2} + \eta)) \vartheta(x_1 + 2\eta, x_2 + \zeta) \vartheta(y_1, y_2 + \zeta) \\ & \quad \text{(by quasi-periodicity)} \\ &= 2^{-g} \sum_{\eta, \zeta} e(-{}^t \eta \cdot x_2) e({}^t \zeta \cdot (\frac{x_1+y_1}{2} + \eta)) e({}^t \eta \cdot (x_2 + \zeta)) \cdot \\ & \quad \cdot \vartheta(x_1, x_2 + \zeta) \vartheta(y_1, y_2 + \zeta) \\ &= 2^{-g} \sum_{\eta, \zeta} e({}^t \zeta \cdot \frac{x_1+y_1}{2}) e(2{}^t \zeta \cdot \eta) \vartheta(x_1, x_2 + \zeta) \vartheta(y_1, y_2 + \zeta).\end{aligned}$$

Since $\zeta \in \frac{1}{2}\widehat{\mathbf{Z}}^g/\widehat{\mathbf{Z}}^g$, the sum over η is non-zero if and only if $\zeta \in 2\widehat{\mathbf{Z}}^g$. Hence the sum is, indeed, $\vartheta(x_1, x_2)\vartheta(y_1, y_2)$. QED

Choose a decomposition of $\mathbf{A}_f^{2g} = W_1 \oplus W_2$ so that e_* vanishes on W_1 and on W_2 , and $e(x, y) = e({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1)$. The main result of this section is:

THEOREM 7.4. Let $\vartheta : \mathbf{A}_f^{2g} \rightarrow k$ be a non-zero function satisfying

- (i) $\vartheta(x + k) = e_*(k/2) e(k/2, x) \vartheta(x), \quad \forall k \in T = \widehat{\mathbf{Z}}^{2g}.$
- (ii) $\vartheta(-x) = \vartheta(x).$

Let μ be the measure on \mathbf{A}_f^g associated to ϑ as in Lemma 7.2. Then the following are equivalent:

(A) Riemann's theta relation holds:

$$\begin{aligned}& \prod_{i=1}^4 \vartheta(x_i) = \\ & 2^{-g} \sum_{\eta \in (\frac{1}{2}\widehat{\mathbf{Z}}^{2g}/\widehat{\mathbf{Z}}^{2g})} e_*(\eta) e(-\eta, x_1) \vartheta(\ell_1(x) + \eta) \vartheta(\ell_2(x) + \eta) \vartheta(\ell_3(x) + \eta) \vartheta(\ell_4(x) + \eta).\end{aligned}$$

Here

$$\begin{aligned}\ell_1(x) &= (x_1 + x_2 + x_3 + x_4)/2 \\ \ell_2(x) &= (x_1 + x_2 - x_3 - x_4)/2 \\ \ell_3(x) &= (x_1 - x_2 + x_3 - x_4)/2 \\ \ell_4(x) &= (x_1 - x_2 - x_3 + x_4)/2\end{aligned}$$

and $x_i \in \mathbf{A}_f^{2g}$.

(B) There exists $\psi : \mathbb{A}_f^{2g} \rightarrow k$ so that (B1) and (B2) hold as in Lemma 7.2.

(C) Under the action of the Heisenberg group $\text{Heis}(2g, \mathbb{A}_f)$ given by

$$(U_{(\lambda, y)} f)(x) = \lambda e(y, x/2) f(x - y)$$

the span of the set of functions

$$\{U_{\lambda, y} \vartheta \cdot U_{\lambda', y'} \vartheta \mid (\lambda, y), (\lambda', y') \in \text{Heis}(2g, \mathbb{A}_f)\}$$

is an irreducible $\text{Heis}(2g, \mathbb{A}_f)$ -module.

(C') For all $n = 2^k$, $\text{Symm}^n V$, the span of

$$\left\{ \prod_{i=1}^n U_{\lambda_i, y_i} \vartheta \mid (\lambda_i, y_i) \in \text{Heis}(2g, \mathbb{A}_f) \right\}$$

is an irreducible $\text{Heis}(2g, \mathbb{A}_f)$ -module. Moreover

$$\text{Symm}^{2^k} V = 2 * (\text{Symm}^{2^{k-2}} V).$$

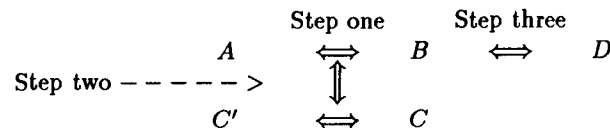
Here $2*$ is the map induced on functions by multiplication by 2 on \mathbb{A}_f^{2g} .

(D) There exists a measure ν on \mathbb{A}_f^g such that

$$A^*(\mu \times \mu) = \nu \times \nu$$

where A is the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

PROOF: We give all the details; thus the proof is long, but easy to understand. We follow the diagram:



STEP ONE: ($A \iff B$). Write $\Sigma(x, y)$ for the function on the R.H.S. of (B2) so (B2) says

$$\Sigma(x, y) = \psi(x)\psi(y).$$

This implies

$$(*) \quad \Sigma(x, y)\Sigma(u, v) = \Sigma(x, u)\Sigma(y, v).$$

On the other hand (*) implies $\Sigma(x, y)$ is the product of two functions

$$\Sigma(x, y) = \left(\frac{\Sigma(x, u)}{\Sigma(u, v)} \right) \cdot \Sigma(y, v).$$

Since ϑ can be expressed in terms of ψ , ψ can't vanish identically, so this makes sense. Now (*) implies that if $\Sigma = \psi_1 \cdot \psi_2$, then $\psi_1 = \psi_2$ up to a constant. Hence we've shown $(B) \iff (*)$. The rest of step one is given to showing that (*) is equivalent to the Riemann theta identity.

We write out (*). Set

$$\begin{aligned} z_1 &= (x_1 + y_1, (x_2 + y_2)/2) & z_3 &= (u_1 + v_1, (u_2 + v_2)/2) \\ z_2 &= (x_1 - y_1, (x_2 - y_2)/2) & z_4 &= (u_1 - v_1, (u_2 - v_2)/2). \end{aligned}$$

We get

$$\begin{aligned} \ell_1(z) &= (x_1 + u_1, (x_2 + u_2)/2) & \ell_3(z) &= (y_1 + v_1, (y_2 + v_2)/2) \\ \ell_2(z) &= (x_1 - u_1, (x_2 - u_2)/2) & \ell_4(z) &= (y_1 - v_1, (y_2 - v_2)/2). \end{aligned}$$

(*) becomes

$$\begin{aligned} & \left[\sum_{\eta} e^{(t(\eta \cdot x_1) \vartheta(x_1 + y_1, (x_2 + y_2)/2 + \eta) \vartheta(x_1 - y_1, (x_2 - y_2)/2 + \eta))} \right. \\ & \cdot \left[\sum_{\zeta} e^{(t(\zeta \cdot u_1) \vartheta(u_1 + v_1, (u_2 + v_2)/2 + \zeta) \vartheta(u_1 - v_1, (u_2 - v_2)/2 + \zeta))} \right] = \\ & \left[\sum_{\eta} e^{(t(\eta \cdot x_1) \vartheta(x_1 + u_1, (x_2 + u_2)/2 + \eta) \vartheta(x_1 - u_1, (x_2 - u_2)/2 + \eta))} \right. \\ & \cdot \left[\sum_{\zeta} e^{(t(\zeta \cdot y_1) \vartheta(y_1 + v_1, (y_2 + v_2)/2 + \zeta) \vartheta(y_1 - v_1, (y_2 - v_2)/2 + \zeta))} \right]. \end{aligned}$$

Here both η, ζ range over $\frac{1}{2}\widehat{\mathbb{Z}}^g/\widehat{\mathbb{Z}}^g$. Now substitute z 's into this, replace ζ by $(0, \zeta)$, η by $(0, \eta)$ (so we are summing over $\frac{1}{2}(W_2 \cap T)/(W_2 \cap T)$):

$$\sum_{\eta, \zeta} \prod_{i=1}^4 \vartheta(z_i + \eta) e(-\eta, (z_1 + z_2)/2) e(-\zeta, (z_3 + z_4)/2) =$$

$$\sum_{\eta, \zeta} e(-\eta, \frac{(z_1 + z_2)}{2}) e(-\zeta, \frac{(z_3 + z_4)}{2}) \prod_{i=1}^4 \vartheta(\ell_i(z) + \zeta).$$

In this formula substitute $z_1 + \kappa$ for z_1 , $z_3 + \lambda$ for z_3 , multiply by $e(-\kappa/2, z_1)$ and $e(-\lambda/2, z_3)$; sum over $\lambda, \kappa \in (W_1 \cap T)/2(W_1 \cap T)$. The L.H.S. becomes, using quasi-periodicity,

$$\sum_{\eta, \zeta, \kappa, \lambda} e(-\eta, \frac{z_1 + z_2 + \kappa}{2}) e(-\zeta, \frac{z_3 + z_4 + \lambda}{2}) e(-\kappa/2, z_1) e(-\lambda/2, z_3) \cdot$$

$$e(\kappa/2, z_1 + \eta) e(\lambda/2, z_3 + \zeta) \vartheta(z_1 + \eta) \vartheta(z_2 + \eta) \vartheta(z_3 + \zeta) \vartheta(z_4 + \zeta).$$

Simplifying, we get

$$\sum_{\eta, \zeta, \kappa, \lambda} e(\kappa, \eta) e(\lambda, \zeta) e(-\eta, (z_1 + z_2)/2) e(-\zeta, (z_3 + z_4)/2) \cdot$$

$$\vartheta(z_1 + \eta) \vartheta(z_2 + \eta) \vartheta(z_3 + \zeta) \vartheta(z_4 + \zeta).$$

If we sum over κ, λ , keeping η, ζ fixed, we see the sum is zero unless both η, ζ equal zero. Thus the L.H.S. is

$$2^{2g} \prod_{i=1}^4 \vartheta(z_i).$$

We now evaluate the R.H.S. After summing over κ, λ , we have

$$\sum_{\eta, \zeta, \kappa, \lambda} e(-\eta, (z_1 + z_2 + \kappa)/2) e(-\zeta, (z_1 - z_2 + \kappa)/2) e(-\kappa/2, z_1) e(-\lambda/2, z_3) \cdot$$

$$\vartheta(\ell_1(z) + \frac{\kappa + \lambda}{2} + \eta) \vartheta(\ell_2(z) + \frac{\kappa - \lambda}{2} + \eta) \cdot$$

$$\vartheta(\ell_3(z) + \frac{\kappa + \lambda}{2} + \zeta) \vartheta(\ell_4(z) + \frac{\kappa - \lambda}{2} + \zeta).$$

Change $-\lambda/2$ to $\lambda/2$ using quasi-periodicity. This introduces the factors

$$e(-\lambda/2, (z_1 - z_2 - z_3 + z_4 + \kappa - \lambda)/2 + \zeta) e(-\lambda/2, (z_1 + z_2 - z_3 - z_4 + \kappa - \lambda)/2 + \eta).$$

Simplifying, the R.H.S. becomes

$$\sum_{\eta, \zeta, \kappa, \lambda} e(-\lambda/2, z_1 - z_3) e(-\kappa/2, z_1) e(-\lambda/2, z_3) e(-\eta, \frac{z_1 + z_2}{2}) \cdot$$

$$e(-\zeta, \frac{z_1 - z_2}{2}) e(\frac{\kappa - \lambda}{2}, \eta) e(\frac{\kappa - \lambda}{2}, \zeta) \vartheta(\ell_1(z) + \frac{\kappa + \lambda}{2} + \eta) \cdot$$

$$\vartheta(\ell_2(z) + \frac{\kappa + \lambda}{2} + \eta) \vartheta(\ell_3(z) + \frac{\kappa + \lambda}{2} + \eta) \vartheta(\ell_4(z) + \frac{\kappa + \lambda}{2} + \eta).$$

The term $e(-\lambda/2, \kappa) = 0$ since λ, κ are both in W_1 . The first three factors give $e(-(\kappa + \lambda)/2, z_1)$; the last two e -factors give

$$\alpha := e(\kappa - \lambda)/2, \eta + \zeta).$$

Fix $\kappa + \lambda = \nu$; sum over μ with $\kappa' = \kappa + \mu$, $\lambda' = \lambda - \mu$. Note that the arguments in the ϑ -terms stay fixed. Since $\frac{\kappa - \lambda}{2} = \frac{\nu}{2} - \lambda$ we have

$$\alpha = e(\frac{\nu}{2}, \eta + \zeta) e(-\lambda, \eta + \zeta).$$

This does not stay fixed; we see that $\eta + \zeta$ must be in $W_2 \cap T$, i.e., $\eta = \zeta$, to get non-zero summands. So, summing over μ gives for the R.H.S.

$$2^g \sum_{\substack{\nu \in (W_1 \cap T)/2(W_1 \cap T) \\ \zeta \in \frac{1}{2}(W_2 \cap T)/W_2 \cap T}} [e(\nu, \zeta) e(-\nu/2, z_1) e(-\zeta, z_1) \prod_{i=1}^4 \vartheta(\ell_i(z) + \nu/2 + \zeta)].$$

Notice that the sum can be viewed as running over all $\eta \in \frac{1}{2}T/T$, so $e(\nu, \zeta) = e_*(\eta)$ and $e(-\nu/2, z_1) e(-\zeta, z_1) = e(-\eta, z_1)$. This gives the Riemann theta identity and completes the proof (A) if and only if (B).

STEP TWO: We begin by showing that (B) implies (C).

Claim. Equation (B1) implies that the span of the functions of the form $U_{\lambda, y} \vartheta \cdot U_{\lambda', y'} \vartheta$ is the same as the span of the functions of the form $U_{\lambda, y}^{(2)} \tilde{\psi}$ where $\tilde{\psi}(x_1, x_2) = \psi(x_1, 2x_2)$.

PROOF OF CLAIM: We calculate

$$(**)$$

$$(U_{(1, (y+y')/2)}^{(2)} \tilde{\psi})(x) = e((y + y')/2, x/2)^2 \tilde{\psi}(x - (y + y')/2)$$

$$= e(y + y', x/2) \tilde{\psi}((2x_1 - y_1 - y'_1)/2, (2x_2 - y_2 - y'_2)/2)$$

$$= e(y + y', x/2) \psi((2x_1 - y_1 - y'_1)/2, 2x_2 - y_2 - y'_2).$$

On the other hand, using formula B1:

$$\begin{aligned}
(U_y \vartheta \cdot U_{y'} \vartheta)(x) &= e(y, \frac{x}{2}) \vartheta(x-y) e(y', \frac{x}{2}) \vartheta(x-y') \\
&= e(y+y', \frac{x}{2}) \sum_{\eta \in 1/2\widehat{\mathbb{Z}}^g/\widehat{\mathbb{Z}}^g} e(-{}^t\eta \cdot (x_2 - y_2)) \cdot \\
&\quad \cdot \psi((2x_1 - y_1 - y'_1)/2 + \eta, 2x_2 - y_2 - y'_2) \cdot \psi((-y_1 + y'_1)/2 + \eta, -y_2 + y'_2).
\end{aligned}$$

We can ignore the last factor since the variable x does not appear there.

Since

$$U_{(-\eta, 0)}^{(2)} \tilde{\psi}(x) = e(-{}^t\eta \cdot x_2) \psi(x_1 + \eta_1, 2x_2),$$

looking at (**) we see that

$$e(-{}^t\eta \cdot x_2) \psi((2x_1 - y_1 - y'_1)/2 + \eta, 2x_2 - y_2 - y'_2)$$

is in the span of $U_y^{(2)} \tilde{\psi}$. Hence the span of the $U^{(2)} \tilde{\psi}$'s contains the span of the $U_y \vartheta U_{y'} \vartheta$'s.

From (B2) we have

$$\begin{aligned}
\psi(x_1, 2x_2) &= \\
\psi(y_1, 2y_2)^{-1} 2^{-g} \sum_{\zeta} e({}^t\zeta \cdot x_1) \vartheta(x_1 + y_1, x_2 + y_2 + \zeta) \vartheta(x_1 - y_1, x_2 - y_2 + \zeta).
\end{aligned}$$

Now

$$\begin{aligned}
(U_{(1, -y_1, -y_2 - \zeta)} \vartheta)(U_{(1, y_1, y_2 - \zeta)} \vartheta)(x) &= \\
e({}^t\zeta \cdot x_1) \vartheta(x_1 + y_1, x_2 + y_2 + \zeta) \vartheta(x_1 - y_1, x_2 - y_2 + \zeta),
\end{aligned}$$

and this gives the non-trivial part of the formula for $\tilde{\psi}$. Hence the span of the ϑ 's contains the span of the $U^{(2)} \tilde{\psi}$'s. QED for Claim

The claim implies

$$\left\{ \text{span of } \prod_{i=1}^4 U_{\lambda, y_i} \vartheta \right\} = \left\{ \text{span of } U_{\lambda, y}^{(2)} \tilde{\psi} U_{\lambda', y'}^{(2)} \tilde{\psi} \right\}.$$

This last space, by the same argument as above with ϑ and ψ interchanged, is the same as the span of $U_{\lambda, y}^{(4)} \tilde{\vartheta}$ where $\tilde{\vartheta}(x) = \vartheta(2x)$, but the span of $U_{\lambda, y}^{(4)} \tilde{\vartheta}$ is just $2 * \{ \text{span of } U_{\lambda, y} \vartheta \}$. This last space is an irreducible $\text{Heis}(2g, \mathbb{A}_f^g)$ -module. Therefore all these spaces are irreducible.

We now show that (C) implies (B). Since $\tau(T_1)\tau(T_2/2)$ is maximal isotropic for the action of $U^{(2)}$, there is a unique (up to a non-zero constant)

invariant function under $\tau(T_1)\tau(T_2/2)$. Call this function ψ . Let M be the subgroup fixing $U_{\lambda, y} \vartheta U_{\lambda', y'} \vartheta$ inside $\tau(T_1)\tau(T_2/2)$. Then

$$\sum_{\eta \in (\tau(T_1)\tau(T_2/2)/M)} U_{\eta}^{(2)}(U_{\lambda, y} \vartheta)(U_{\lambda', y'} \vartheta)(x) = c(\lambda, \lambda', y, y') \psi(x),$$

where $c(\lambda, \lambda', y, y')$ is a constant depending on the indicated variables. Set

$$\phi(x) = (U_{1, y} \vartheta)(U_{1, y'} \vartheta)(x) = e(y + y', x/2) \vartheta(x - y) \vartheta(x - y').$$

The function $\phi(x)$ is an eigenfunction for $\tau(T_1)\tau(T_2)$.

PROOF: Let $z \in T_1, T_2$; then

$$\begin{aligned}
U_{\tau(z)}^{(2)} \phi &= e(z, x/2)^2 e(y + y', (x - z)/2) \vartheta(x - y - z) \vartheta(x - y' - z) \\
&= e(z, x/2)^2 e(y + y', (x - z)/2) \vartheta(x - y) \vartheta(x - y') \\
&\quad \cdot e_*(\frac{z}{2})^2 e(-z/2, x - y) e(-z/2, x - y') \\
&= e(z, y + y') \phi(x).
\end{aligned}$$

Thus the projection of ϕ to the invariant subspace of $\tau(T_1)\tau(T_2/2)$ is zero unless $y' = -y$. Setting $y' = -y$, we sum over $\frac{1}{2}T_2/T_2$ to calculate this projection:

$$\begin{aligned}
&\sum U_{\tau(\eta)}^{(2)} (U_{\tau, y} \vartheta U_{\tau, -y} \vartheta)(x) = \\
&\sum e(\eta, x/2)^2 \vartheta(x - y - \eta) \vartheta(x + y - \eta) = \\
&\sum e(\eta, x) \vartheta(x + y + \eta) \vartheta(x - y + \eta) e(-\eta, 2x) = \\
&\sum e(-\eta, x) \vartheta(x + y + \eta) \vartheta(x - y + \eta) = \sigma(y) \psi(x),
\end{aligned}$$

where the sum is over $\eta \in \frac{1}{2}T_2/T_2$, and $\sigma(y) = c(1, 1, y, -y)$. Notice that we have the R.H.S. of (B2). We now show that this is $\psi(x)\psi(y)$. Indeed we assert that $\sigma(y)\psi(x)$ is symmetric in x and y :

$$\begin{aligned}
\sigma(y)\psi(x) &= \sum_{\eta \in (\frac{1}{2}T_2/T_2)} e(-\eta, x) \vartheta(x + y + \eta) \vartheta(y - x - \eta) \\
(\text{quasi-periodicity}) &= \sum e(-\eta, x) \vartheta(x + y + \eta) \vartheta(y - x + \eta) e(-\eta, y - x) \\
&= \sum e(-\eta, y) \vartheta(x + y + \eta) \vartheta(y - x + \eta) \\
&= \sigma(x)\psi(y).
\end{aligned}$$

This gives (B2).

STEP THREE: We show that $(D) \implies (B)$. Let $h = 2$ and write

$$\mathbf{A}_f^{2g} = \mathbf{A}_f^g \oplus \mathbf{A}_f^g = V_1 \oplus V_2 = V$$

Set $x = (x_1, x_2) \in V_1^2, y = (y_1, y_2) \in V_2^2$. Let

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Q' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

so $Q' = {}^t A Q A$. We start with a non-zero quasi-periodic function ϑ and we define

$$\vartheta^Q \begin{bmatrix} x \\ y \end{bmatrix} = \vartheta(x_1, y_1) \vartheta(x_2, y_2)$$

for $\begin{bmatrix} x \\ y \end{bmatrix} \in V$. We deduce (B2) in two steps.

ONE: We show ϑ^Q satisfies a relation of the form 6.17 for some function $\vartheta^{Q'}$.

TWO: We show there exists a function ψ on \mathbf{A}_f^{2g} so that

$$\vartheta^{Q'} \begin{bmatrix} x \\ y \end{bmatrix} = \psi(x_1, y_1) \psi(x_2, y_2)$$

STEP ONE: Define

$$\begin{aligned} \phi : \text{Heis}(2gh, \mathbf{A}_f) &\longrightarrow \text{Heis}(2gh, \mathbf{A}_f) \\ (\lambda, \begin{bmatrix} x \\ y \end{bmatrix}) &\longmapsto (\lambda, \begin{bmatrix} x {}^t A^{-1} \\ y A \end{bmatrix}). \end{aligned}$$

One checks that this is an isomorphism of groups. Let $C_k(V)$ be the space of k -valued locally constant functions on V with $\text{Heis}(2gh, \mathbf{A}_f)$ acting by

$$(U_{\lambda, z} f)(x) = \lambda e(\frac{z}{2}, x) f(x - z).$$

Define a map G_0 from $C_k(V)$ to itself by

$$G_0(f) \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} x {}^t A^{-1} \\ y A \end{bmatrix}.$$

One verifies that G_0 is a ϕ - $\text{Heis}(2gh, \mathbf{A}_f)$ -map, so $U_g(G_0 f) = G_0(U_{\phi(g)} f)$. Since ϑ is quasi-periodic, the $\text{Heis}(2gh, \mathbf{A}_f)$ -submodule Γ generated by ϑ

is irreducible by Proposition 5.10. Let $\Gamma' = G_0^{-1}(\Gamma)$ and let G_0 , when restricted to Γ' , be denoted by G . Let $\vartheta^{Q'}$ be a non-zero element in Γ' fixed by $\sigma(\widehat{\mathbf{Z}})^{2gh}$. Thus we have the diagram

$$\begin{array}{ccc} \text{Heis}(2gh, \mathbf{A}_f) & \xleftarrow{\phi} & \text{Heis}(2gh, \mathbf{A}_f) \\ \downarrow \text{ } & & \downarrow \text{ } \\ C_k(V) & \xrightarrow{G_0} & C_k(V) \\ \downarrow \text{ } & & \downarrow \text{ } \\ \Gamma' & \xrightarrow{G} & \Gamma \\ \downarrow \vartheta^{Q'} & & \downarrow \vartheta^Q \end{array}$$

Lemma 6.20 now implies there is a relationship of the form 6.17 between $G(\vartheta^{Q'})$ and ϑ^Q since ϑ^Q is fixed by $\sigma(\widehat{\mathbf{Z}})^{2gh}$:

$$(***) \quad \vartheta^{\alpha, Q'} \begin{bmatrix} x {}^t A^{-1} \\ y A \end{bmatrix} = \sum_{R, S} U_{\tau(R)} U_{\tau(S)} \vartheta^{\alpha, Q} \begin{bmatrix} x \\ y \end{bmatrix}.$$

STEP TWO: As in Lemma 7.2 we introduce measures. Let α be the unique $\text{Heis}(2gh, \mathbf{A}_f)$ -map from Γ to $S(V_1^h)$, the space of locally constant, compact support functions on V_1^h ; similarly let $\alpha' : \Gamma \mapsto S(V_1^h)'$ be an $\text{Heis}(2gh, \mathbf{A}_f)$ -isomorphism where $S(V_1^h)'$ denotes a second copy of $S(V_1^h)$. Define linear functionals on $S(V_1^h)$ and $S(V_1^h)'$ by

$$\ell(f) = (\alpha^{-1} f)(0)$$

$$\ell'(f) = [G \circ (\alpha')^{-1}(f)](0).$$

Then ℓ and ℓ' are induced by measures on V_1^h ; in particular, the measure μ from Lemma 7.2 satisfies

$$\ell(f) = \int f d\mu \times d\mu \quad f \in S(V_1^h).$$

Define the measure $\bar{\nu}$ by

$$\ell'(f) = \int f d\bar{\nu} \quad f \in S(V_1^h)'.$$

Define F by requiring that the diagram below commute

$$\begin{array}{ccc} \Gamma' & \xrightarrow{G} & \Gamma \\ \downarrow \alpha' & & \downarrow \alpha \\ S(V_1^h)' & \xrightarrow{F} & S(V_1^h). \end{array}$$

Since everything in sight is an irreducible $\text{Heis}(2gh, \mathbf{A}_f)$ -module, F is characterized, up to a constant, by being a map of $\text{Heis}(2gh, \mathbf{A}_f)$ -modules. One checks that the map from $S(V_1^h)'$ to $S(V_1^h)$ defined by

$$f \mapsto f^*(x) = f(x^t A^{-1})$$

is a map of $\text{Heis}(2gh, \mathbf{A}_f)$ -modules, so we can conclude that this is, indeed, the map F .

CLAIM: There exists a measure ν on V_1 so that $\bar{\nu} = \nu \times \nu$.

PROOF:

$$\begin{aligned} \int f(u) d\bar{\nu}(u) &= \ell \circ G \circ (\alpha')^{-1}(f) \\ &= \ell \circ \alpha^{-1}(Ff) \\ &= \int (Ff) d\mu \times d\mu \\ &= \int f A^*(d\mu \times d\mu) \\ &= \int f(d\nu \times d\nu). \end{aligned}$$

This last equality is hypothesis (D). Since $\vartheta^{Q'}$ is defined by $\bar{\nu}$ we have shown there exists ψ so that

$$\vartheta^{Q'} \begin{bmatrix} x \\ y \end{bmatrix} = \psi(x_1, y_1) \psi(x_2, y_2).$$

We show the converse. Given $Q, Q', A, \vartheta^Q, \vartheta^{Q'}$ and the function ϑ corresponding to the measure μ and we get (**). Replace x, y by $x^t A$ and $y A^{-1}$; the R.H.S. is (B2). Use (B2) to conclude there exists ψ so that $\vartheta^{\alpha, Q'} \begin{bmatrix} x \\ y \end{bmatrix} = \psi(x) \psi(y)$. Let ν be the measure corresponding to ψ . By Proposition 7.20, $\vartheta^{\alpha, Q'}$ corresponds to $A^*(\mu \times \mu)$, but it also corresponds to $\nu \times \nu$. So (D) holds. QED for Theorem 7.4

Does a function ϑ satisfying the equivalent conditions of Theorem 7.4 come from an abelian variety? This is almost true, but not precisely so. What can be proven is:

ASSERTION 7.5. Let ϑ satisfy the conditions i), ii) and A) of Theorem 7.4 and also the non-degeneracy condition:

iii) For all $x \in \mathbf{A}_f^{2g}$, there exists $\eta \in \frac{1}{2}\widehat{\mathbf{Z}}^{2g}$ such that

$$\vartheta(x + \eta) \neq 0.$$

Then there is an abelian variety X defined over k , an even symmetric degree one ample line bundle \mathbf{L} on X and an isomorphism

$$V(X) \cong \mathbf{A}_f^{2g}$$

carrying $T(X)$ to $\widehat{\mathbf{Z}}^{2g}$, $e^{\mathbf{L}}$ to e and $e_*^{\mathbf{L}}$ to e_* such that ϑ^α for (X, \mathbf{L}) is equal to ϑ .

This assertion is in unpublished notes of the senior author, and is a straightforward generalization of the results of §10 in *Equations defining abelian varieties* III, Inv. Math., vol. 3, 1967. The results of that paper deal with ϑ 's on \mathbf{Q}_2^{2g} satisfying the conditions of Theorem 7.4 and in this case, one can extend Assertion 7.5 to degenerate ϑ 's which don't satisfy iii). The result in §11 is:

THEOREM 7.6. Let $\vartheta : \mathbf{Q}_2^{2g} \rightarrow k$ satisfy the conditions i), ii) and A) of Theorem 7.4. Then there exists:

a) a subspace $W \subset \mathbf{Q}_2^{2g}$ such that $W^\perp \subset W$, where

$$W^\perp = \{y | e(x, y) = 1, \text{ all } x \in W\}$$

Let $\dim W/W^\perp = 2h$.

b) an element $\eta_0 \in \frac{1}{2}\mathbf{Z}_2^{2g}$ such that $e_*(\eta_0) = 1$ and $e_*(x) = e(x, 2\eta_0)$, all $x \in W^\perp \cap \frac{1}{2}\mathbf{Z}_2^{2g}$,

- c) an abelian variety X of dimension h and an even symmetric degree one ample line bundle L on X ,
- d) an isomorphism $V_2(X) \xrightarrow{\sim} W/W^\perp$ carrying $T_2(X)$ to $W \cap Z_2^{2g}$, e^\perp to $e|_{W \times W}$ and e_*^\perp to the form

$$e_*(x)e(x, 2\eta_0)|_{\frac{1}{2}Z_2^{2g} \cap W},$$

such that

- e) $\vartheta \equiv 0$ on $\mathbf{Q}_2^{2g} - (\eta_0 + W + Z_2^{2g})$,
- f) The function $x \mapsto e(x, \frac{\eta_0}{2})\vartheta(\eta_0 + x)$ on W is invariant modulo translation by W^\perp and induces on W/W^\perp the algebraic theta function $\vartheta_{X,L}^\alpha$ defined by X and L .

It would be very nice if Theorem 7.6 were also true for the full adelic case of Assertion 7.5, but unfortunately, this isn't true: if ϑ is a non-degenerate theta function as in 7.5, then let $U = \mathbf{Q}_2^{2g} + \widehat{Z}^{2g}$ and let χ_U be the characteristic function of U . Then $\chi_U \vartheta$ also satisfies the conditions of 7.4 but certainly isn't ϑ^α for any abelian variety. What we need are some further conditions on ϑ that imply:

- (A*) For every rational orthogonal $h \times h$ matrix A , the following identity holds:

$$\prod_{i=1}^h \vartheta(x_i) = c \cdot \sum_{\eta, \zeta} \prod_{i=1}^h e({}^t \eta \cdot \zeta) \cdot e(-\frac{(\eta_i, \zeta_i)}{2}, (Ax)_i) \vartheta((Ax)_i + \eta_i)$$

for an appropriate constant c and where η and ζ range over

$$\mathbf{Z}(g, h) \cdot {}^t A / \mathbf{Z}(g, h) \cdot {}^t A \cap \mathbf{Z}(g, h) \text{ and}$$

$$\mathbf{Z}(g, h) \cdot {}^t A^{-1} / \mathbf{Z}(g, h) \cdot {}^t A^{-1} \cap \mathbf{Z}(g, h) \quad \text{resp.}$$

and:

- (C*) For all $n \geq 1$, $\text{Symm}^n V$ is irreducible as $\text{Heis}(2g, A_f)$ -module.

This is an interesting topic for investigation.

8. The metaplectic group and the full functional equation of ϑ

In Chapter I, §7 we derived the functional equation in τ for $\vartheta(z, \tau)$ by direct methods and the same procedure can be generalized to get the functional equation for the many variable function $\vartheta(z, T)$. We shall adopt another method however. In §3, $\vartheta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (T)$ was characterized as a matrix-coefficient of the representation \mathcal{H} of $\text{Heis}(2g, \mathbf{R})$:

$$\vartheta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (T) = \langle U_{(1, x_1, x_2)} f_T, e_x \rangle,$$

and this combined with the fact that e_x is fixed by $\sigma(Z^{2g})$ allows us to deduce immediately the functional equation of ϑ in z :

$$\vartheta(z, T) = \exp(\pi i {}^t m T m + 2\pi i {}^t m z) \cdot \vartheta(z + T m + n, T).$$

In this section we shall construct a two-sheeted covering of $Sp(2g, \mathbf{R})$ which is called the metaplectic group $Mp(2g, \mathbf{R})$ and show that there is a combined action of $Mp(2g, \mathbf{R})$ and $\text{Heis}(2g, \mathbf{R})$ on \mathcal{H} . We then see that $\vartheta(z, T)$, multiplied by an exponential factor, is a matrix coefficient of this representation, enabling us to write down the full functional equation of ϑ in *both* x and T .

Roughly speaking we obtain an action of $Mp(2g, \mathbf{R})$ on \mathcal{H} , the irreducible representation of $\text{Heis}(2g, \mathbf{R})$, as follows: Let $\gamma \in Sp(2g, \mathbf{R})$, so γ defines an automorphism of $\text{Heis}(2g, \mathbf{R})$ by $(\lambda, x) \rightarrow (\lambda, \gamma(x))$. Define a new action of $\text{Heis}(2g, \mathbf{R})$ on \mathcal{H} by

$$(U'_{\lambda, x} f) = (U_{\lambda, \gamma(x)} f).$$

By the Stone-Von Neumann-Mackey theorem there must be a unitary map $A_\gamma : \mathcal{H} \rightarrow \mathcal{H}$ intertwining these two representations. The map A_γ is determined only up to a constant; indeed, there is no well-defined map $\gamma \rightarrow A_\gamma$ giving a group action of $Sp(2g, \mathbf{R})$ on \mathcal{H} ; however there is an action of a two sheeted covering of $Sp(2g, \mathbf{R})$. This is the group $Mp(2g, \mathbf{R})$. We know that

$$\vartheta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (T) = \langle U_{\tau(-x)} f_T, e_x \rangle.$$

Let $\gamma \in Sp(2g, \mathbf{R})$ act on \mathcal{H} via A_γ , then

$$\begin{aligned} \vartheta^\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (T) &= \langle A_\gamma U_{\tau(-x)} f_T, A_\gamma e_{\mathbf{Z}} \rangle \\ &= \langle U_{\tau(\gamma(-x))} A_\gamma f_T, A_\gamma e_{\mathbf{Z}} \rangle. \end{aligned}$$

The functional equation falls out of this equation once we compute $A_\gamma f_T$ and $A_\gamma e_{\mathbf{Z}}$. Unfortunately, there are many details involved in carrying out this program, although the essential idea is so simple.

In this section \mathcal{H} will always be the unique irreducible representation of $Heis(2g, \mathbf{R})$, \mathcal{H}_∞ and $\mathcal{H}_{-\infty}$ as in §2, and $f_T(y) = \exp(\pi i^t y T y) \in L^2(\mathbf{R}^g)$. Because we are working with a fixed g we shall abbreviate $Heis(2g, \mathbf{R})$, $Sp(2g, \mathbf{R})$, $\widetilde{Mp}(2g, \mathbf{R})$, and $Mp(2g, \mathbf{R})$ to $Heis$, Sp , \widetilde{Mp} , and Mp respectively.

PROPOSITION 8.1. *Let $U(\mathcal{H})$ be the group of unitary isomorphisms of \mathcal{H} and $\widetilde{Mp}(2g, \mathbf{R}) = \{A \in U(\mathcal{H}) \mid AU_{(\lambda, v)}A^{-1} = U_{(\lambda, \gamma v)} \forall (\lambda, v) \in Heis, \text{ for some } \gamma \in Sp\}$. Given $A \in \widetilde{Mp}$, the corresponding $\gamma \in Sp$ is unique; denote this γ by $\rho(A)$. Then there is an exact sequence of groups:*

$$1 \longrightarrow \mathbf{C}_1^* \longrightarrow \widetilde{Mp} \xrightarrow{\rho} Sp \longrightarrow 1.$$

PROOF: Let $A \in \widetilde{Mp}$, and let $\gamma_1, \gamma_2 \in Sp$ be such that:

$$U_{(\lambda, \gamma_i v)} = AU_{(\lambda, v)}A^{-1} \forall (\lambda, v) \in Heis.$$

Then $U_{(\lambda, \gamma_1 v)} \cdot U_{(\lambda, \gamma_2 v)}^{-1} = \text{Id.}$ and therefore it commutes with $U_h, \forall h \in Heis$, showing that $e(\gamma_1 v - \gamma_2 v, w) = 1 \forall w \in \mathbf{R}^{2g}$, and therefore, that $\gamma_1 v = \gamma_2 v, \forall v \in \mathbf{R}^{2g}$. This shows that $\rho: \widetilde{Mp} \rightarrow Sp$ is well-defined; we omit the trivial checking that \widetilde{Mp} is a group and ρ is a homomorphism.

The kernel of $\rho = \{A \in U(\mathcal{H}) \mid AU_h A^{-1} = U_h, \forall h \in Heis\} = \mathbf{C}_1^*$ by the irreducibility of \mathcal{H} .

It only remains to show that ρ is surjective. This follows from our remarks above explaining how $Sp(2g, \mathbf{R})$ can almost act on \mathcal{H} . QED

Next we shall write down explicitly members of \widetilde{Mp} sitting above a generating set of Sp for the model $\mathcal{H} = L^2(\mathbf{R}^g)$ where $U_{(1, x, 0)}$ is translation by x and $U_{(1, 0, x)}$ is multiplication by the character $\exp(2\pi i^t x y)$.

LEMMA 8.2. I. *For all $A \in GL_g(\mathbf{R})$, let $\gamma = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}$; for this γ we define*

$$Pf(y) = f(A^{-1}y) |\det A|^{-1/2}.$$

II. *For all symmetric $C \in GL_g(\mathbf{R})$ let $\gamma = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$; for this γ we define*

$$Pf(y) = f(y) \exp(-\pi i^t y C y).$$

III. *If $\gamma = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, then set*

$$Pf(y) = \widehat{f}(y) = \int f(x) \exp(-2\pi i^t x y) dx.$$

In all these cases $P \in \widetilde{Mp}$, $\gamma \in Sp$ and $\rho(P) = \gamma$.

PROOF: It suffices to check that $PU_{(1, v)}P^{-1} = U_{(1, \gamma v)}$ for $v \in \mathbf{R}^g \times 0$ and $v \in 0 \times \mathbf{R}^g$ in all three cases.

$$\begin{aligned} \text{I. } (PU_{(1, x, 0)}P^{-1}f)(y) &= (U_{(1, x, 0)}P^{-1}f)(A^{-1}y) |\det A|^{-1/2} \\ &= (P^{-1}f)(A^{-1}y + x) |\det A|^{-1/2} \\ &= f(A(A^{-1}y + x)) |\det A|^{1/2} |\det A|^{-1/2} \\ &= f(y + Ax) \\ &= (U_{(1, Ax, 0)}f)(y). \\ \text{II. } (PU_{(1, 0, x)}P^{-1}f)(y) &= (U_{(1, 0, x)}P^{-1}f)(A^{-1}y) |\det A|^{-1/2} \\ &= (P^{-1}f)(A^{-1}y) \exp(2\pi i^t x A^{-1}y) \cdot |\det A|^{-1/2} \\ &= f(AA^{-1}y) \exp(2\pi i^t ({}^t A^{-1}x)y) \\ &= (U_{(1, 0, {}^t A^{-1}x)}f)(y). \end{aligned}$$

$$\begin{aligned}
\text{II. } (PU_{(1,x,0)}P^{-1}f)(y) &= (U_{(1,x,0)}P^{-1}f)(y) \cdot \exp(-\pi i^t y C y) \\
&= (P^{-1}f)(x+y) \cdot \exp(-\pi i^t y C y) \\
&= f(x+y) \exp[\pi i^t (x+y) C (x+y)] \cdot \exp(-\pi i^t y C y) \\
&= f(x+y) \exp(2\pi i^t y C x + \pi i^t x C x) \\
&= (U_{(1,x,Cx)}f)(y).
\end{aligned}$$

Also $(1, 0, x) = (1, \gamma(0, x))$ and clearly $U_{(1,0,x)}$ commutes with P so that $PU_{(1,0,x)}P^{-1} = U_{(1,0,x)} = U_{(1,\gamma(0,x))}$.

III. is just the usual properties of the Fourier transform. QED

Because the matrix representation of $Sp(2g, \mathbf{R})$ comes from its action on the Heisenberg group, we must let it act on \mathfrak{H}_g by

$$\gamma(T) = (DT - C)(-BT + A)^{-1},$$

which is the usual action after conjugation of γ by $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. We will see below that this is necessary for everything to be compatible. In any case, $Sp(2g, \mathbf{R})$ acts transitively on \mathfrak{H}_g and the stabilizer of iI is:

$$U(g) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mid {}^tAA + {}^tBB = I, {}^tAB = {}^tBA \right\}.$$

Recall that the Lie algebra of $Heis(2g, \mathbf{R})$ acts on $\mathcal{H}_\infty \subset \mathcal{H}$; in particular, δU_{A_i} and δU_{B_i} denote the operators corresponding to the elements $(1, e_i, 0)$ and $(1, 0, e_i)$ if $\{e_1, \dots, e_g\}$ is the standard basis of \mathbf{R}^g . W_T denotes the span of the operators $\delta U_{A_i} - \Sigma T_{ij} \delta U_{B_j}$ for $i = 1, \dots, g$. Under the action of $\gamma \in Sp$ on \mathfrak{H}_g given above, W_T gets transformed into $W_{\gamma(T)}$.

Because $\mathbf{C} \cdot f_T$ is the subspace of \mathcal{H}_∞ annihilated by W_T (Theorem 3.2), if $P \in \widetilde{M}p$ and $\rho(P) = \gamma$, then Pf_T is annihilated by $W_{\gamma(T)}$ and therefore:

$$Pf_T = (\text{const. depending on } P \text{ and } T) \cdot f_{\gamma(T)}.$$

We shall compute these constants explicitly:

THEOREM 8.3. Let $P \in \widetilde{M}p$, $\rho(P) = \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Recall that in $L^2(\mathbf{R}^g)$, $f_T = e^{\pi i^t x T x}$. Then $\forall T \in \mathfrak{H}_g$,

$$Pf_T = C(P; T) f_{\gamma(T)}$$

where $C(P; T)$ is, up to a scalar of absolute value one, a branch of the holomorphic function $[\det(-BT + A)]^{-1/2}$ on $T \in \mathfrak{H}_g$.

PROOF: Let $G_1 = \{P \in \widetilde{M}p \mid Pf_T = \text{const.} \cdot f_{\rho(P)T}, \forall T \in \mathfrak{H}_g\}$.

For $P \in G_1$, let $Pf_T = C(P; T) f_{\rho(P)T}$, $\forall T \in \mathfrak{H}_g$.

Let

$$G_2 = \left\{ P \in G \mid \begin{array}{l} C(P; T) \text{ is continuous in } T \in \mathfrak{H}_g \text{ and} \\ C(P; T)^2 |\det(-BT + A)| = F(P; T) \\ \text{is independent of } T \text{ with values in } \mathbf{C}_1^* \end{array} \right\}.$$

The Theorem is equivalent to the statement: $G_1 = G_2 = \widetilde{M}p$ (that $G_1 = \widetilde{M}p$ follows from the above remarks, but we wish to deduce it from Lemma 8.2).

Clearly G_1 is a subgroup of $\widetilde{M}p$. To show that G_2 is a subgroup of G_1 it suffices to show that $C(P; T)$ and $\det(-BT + A)$ are 1-cocycles:

$$(i) \quad C(P; T)C(Q; \rho(P)T) = C(QP; T), \forall P, Q \in G_1,$$

and

$$(ii) \quad \det(-BT + A) \det(-B'\gamma(T) + A') = \det(-B''T + A'')$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix}$;

$$\text{here } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \text{ belong to } Sp.$$

But (i) follows from the definition of $C(P; T)$ and (ii) is straightforward.

Now $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$ for $A \in GL_g(\mathbf{R})$, $\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$ for symmetric $(g \times g)$ -matrices C , and $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ generate Sp and consequently the $P \in \widetilde{M}p$ given in Lemma 8.2 I, II, III and $\lambda \in \mathbf{C}_1^* \subset \widetilde{M}p$ generate $\widetilde{M}p$ and we shall show that all these elements belong to G_2 . This will conclude the proof of the theorem.

For $P = \lambda \in \mathbf{C}_1^*$, $\lambda f_T = \lambda \cdot f_{\rho(P)T}$ and therefore we get $C(P; T) = \lambda$, $F(P; T) = \lambda^2$.

For P as in I,

$$\begin{aligned} (Pf_T)(y) &= f_T(A^{-1}y) |\det A|^{-1/2} \\ &= \exp(\pi i^t y^t A^{-1} T A^{-1} y) |\det A|^{-1/2} \\ &= f_{i A^{-1} T A^{-1}}(y) |\det A|^{-1/2} \\ &= f_{\gamma(T)}(y) |\det A|^{-1/2}. \end{aligned}$$

For P as in II,

$$Pf_T(y) = \exp(\pi i^t y T y) \cdot \exp(-\pi i^t y C y) = f_{T-C}(y).$$

Here $C(P; T) = F(P; T) = 1$.

For P as in III,

$$Pf_T = \hat{f}_T = \left(\int f_T \right) f_{-T^{-1}} = \left[\det \left(\frac{T}{i} \right) \right]^{-1/2} f_{\gamma(T)}$$

so $C(P; T) = [\det(\frac{T}{i})]^{-1/2}$ with $C(P; iI) = 1$ and $F(P; T) = i^g$. QED

DEFINITION 8.4. Let $\chi(P) = F(P; T) = \det(-BT + A)C(P; T)^2$. The above shows that $\chi : \widetilde{Mp} \rightarrow \mathbf{C}_1^*$ is a character and

$$\chi(\lambda) = \lambda^2, \forall \lambda \in \mathbf{C}_1^* \subset \widetilde{Mp}.$$

The metaplectic group $Mp(2g, \mathbf{R})$ or Mp is defined as $\ker \chi$.

Note that we now have an expression of $\vartheta^\alpha[x](T)$ as a matrix coefficient of the combined representation of $Mp(2g, \mathbf{R}) \cdot Heis(2g, \mathbf{R})$ in \mathcal{H} :

$$\vartheta^\alpha[x](\gamma(iI_g)) = C(P; iI_g)^{-1} \langle U_{\tau(-x)} \cdot P \cdot f_{iI_g}, e_{\mathbf{z}} \rangle$$

where $P \in Mp(2g, \mathbf{R})$ and $\gamma = \rho(P)$.

We shall topologize \widetilde{Mp} and Mp in the following way:

STEP I: Let $C_T : \widetilde{Mp} \rightarrow \mathbf{C}^*$ be defined by $C_T(P) = C(P; T)$. Fix $T_0 \in \mathfrak{H}_g$ and give \widetilde{Mp} the weakest topology so that $\rho : \widetilde{Mp} \rightarrow Sp$ and

$C_{T_0} : \widetilde{Mp} \rightarrow \mathbf{C}^*$ are continuous. With this topology it will be shown that \widetilde{Mp} is a Hausdorff manifold.

Define $g : \widetilde{Mp} \rightarrow \mathbf{C}^* \times Sp$ by $g(P) = (C_{T_0}(P), \rho(P))$. Clearly the topology on \widetilde{Mp} is the weakest so that g is continuous. Also g is one-to-one: if $g(P) = g(Q)$, then $\rho(P) = \rho(Q)$ and thus $Q = \lambda P$ for some $\lambda \in \mathbf{C}_1^*$, but $C_T(Q) = \lambda C_T(P) = C_T(P)$ implies now that $\lambda = 1$ and $P = Q$. Let $f : \mathbf{C}^* \times Sp \rightarrow \mathbf{C}^*$ be defined by $f(\lambda, \gamma) = \lambda^2 \det(-BT_0 + A)$ where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. By Theorem 8.3, $g(\widetilde{Mp}) = f^{-1}(\mathbf{C}_1^*)$. Now $\frac{\partial f}{\partial \lambda} \neq 0$ and \mathbf{C}_1^* is a submanifold of \mathbf{C}^* implying that $g(\widetilde{Mp})$ is a submanifold of $\mathbf{C}^* \times Sp$ (and is thus automatically Hausdorff because both \mathbf{C}^* and Sp are Hausdorff).

STEP II: We show that $C(P; T)$ considered as a function from $\widetilde{Mp} \times \mathfrak{H}_g \rightarrow \mathbf{C}^*$ is continuous. Thus the topology on \widetilde{Mp} is the weakest one so that ρ and C_T are continuous, $\forall T \in \mathfrak{H}_g$.

$C(P; T)^2 \det(-BT + A) = \chi(P)$ where $\rho(P) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Putting $T = T_0$ we see that χ is continuous and therefore

$$C(P; T)^2 = \chi(P) \det(-BT + A)^{-1}$$

is continuous. To see that $C(P; T)^2$ has a continuous square-root $\mathcal{H}(P; T)$ on $\widetilde{Mp} \times \mathfrak{H}_g$, it suffices (from the lifting theorem and the fact that \mathfrak{H}_g is simply connected) to show that $C(P; T)^2$ when restricted to $\widetilde{Mp} \times \{T_0\}$ has a continuous square-root, but by definition $C_{T_0}(P)^2 = C(P; T_0)^2$ and C_{T_0} is continuous. Thus we may choose a function \mathcal{H} on $\widetilde{Mp} \times \mathfrak{H}_g$ uniquely by demanding that $\mathcal{H}(P; T_0) = C(P; T_0)$ and $\mathcal{H}(P; T)^2 = C(P; T)$. Now fixing $P \in \widetilde{Mp}$, the functions $\mathcal{H}(P; T)$ and $C(P; T)$ on \mathfrak{H}_g are both continuous, $\mathcal{H}(P; T)^2 = C(P; T)^2$ and $\mathcal{H}(P; T_0) = C(P; T_0)$; from which $\mathcal{H}(P; T) = C(P; T)$ for all $T \in \mathfrak{H}_g$. This holds for all $P \in \widetilde{Mp}$ and thus $\mathcal{H} = C$ and therefore C is continuous.

STEP III: The multiplication $m : \widetilde{Mp} \times \widetilde{Mp} \rightarrow \widetilde{Mp}$ is continuous. We have to check that

(a) $\rho \circ m$ is continuous, and

(b) $C_{T_0} \circ m$ is continuous.

But (a) follows from the fact that Sp is a topological group, and $C_{T_0} \circ m(P, Q) = C_{T_0}(PQ) = C(Q; T_0)C(P; \rho(Q)T_0)$ is continuous because $C : \widetilde{Mp} \times \mathfrak{H}_g \rightarrow \mathbf{C}^*$, $\rho : \widetilde{Mp} \rightarrow Sp$ and the map $Sp \times \mathfrak{H}_g \rightarrow \mathfrak{H}_g$ given by $(\gamma, T) \mapsto \gamma(T)$ are all continuous.

STEP IV: We now show that $h : \widetilde{Mp} \rightarrow \mathbf{C}_1^* \times Sp$ given by $h(P) = (\chi(P), \rho(P))$ defines a connected two-sheeted covering of the Lie group $\mathbf{C}_1^* \times Sp$ and thus gives \widetilde{Mp} the structure of Lie group.

$h = (\chi, \rho)$ is a continuous homomorphism and $\ker h = \ker \chi \cap \ker \rho = \ker \chi \cap \mathbf{C}_1^* = \ker(\chi|_{\mathbf{C}_1^*}) = \{\pm 1\}$. Now $h = h' \circ g$ where $h' : \mathbf{C}^* \times Sp \rightarrow \mathbf{C}^* \times Sp$ is defined by $h'(\lambda, \gamma) = (f(\lambda, \gamma), \gamma)$ and f and g are as in Step I. Clearly h' is a two-sheeted covering projection. Now $(h')^{-1}(\mathbf{C}_1^* \times Sp) = g(\widetilde{Mp})$ so h' restricted to $g(\widetilde{Mp})$ is a two sheeted covering: $g(\widetilde{Mp}) \rightarrow \mathbf{C}_1^* \times Sp$.

It only remains to prove that \widetilde{Mp} is connected; but this follows easily from the connectedness of Sp and \mathbf{C}_1^* . The following is a more subtle fact:

PROPOSITION 8.5. Mp is a closed connected subgroup of \widetilde{Mp} and $\rho|_{Mp} : Mp \rightarrow Sp$ is a covering projection with kernel $= \{\pm 1\}$.

PROOF: Except for the connectedness of Mp , everything else follows from the preceding remarks. Put $\rho|_{Mp} = q$. Recall that $U(g)$ is the stabilizer in Sp of $iI \in \mathfrak{H}_g$. The coset space of Mp with respect to $q^{-1}(U(g))$ is \mathfrak{H}_g by $g \rightarrow q(g)(iI)$. Since \mathfrak{H}_g is connected it suffices to show that $q^{-1}(U(g))$ is connected. Here $U(g)$ sits inside $Sp(2g, \mathbf{R})$ as

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid {}^tAA + {}^tBB = I, {}^tAB = {}^tBA \right\}.$$

It follows that C_T for $T = iI$, i.e. $C_{iI} : q^{-1}(U(g)) \rightarrow \mathbf{C}_1^*$, is a continuous character and

$$\left(C_{iI} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right)^2 = \det(-B(iI) + A)^{-1} = \det(A - iB)^{-1}.$$

We shall define $\det^* : U(g) \rightarrow \mathbf{C}_1^*$ by $\det^* \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det(A - iB)^{-1}$. Thus, $q^{-1}(U(g))$ along with its topology is the fibre-product of \det^* and $Sq : \mathbf{C}_1^* \rightarrow \mathbf{C}_1^*$ where $Sq(\lambda) = \lambda^2$:

$$\begin{array}{ccc} q^{-1}(U(g)) & \xrightarrow{C_{iI}} & \mathbf{C}_1^* \\ \downarrow & & \downarrow Sq \\ U(g) & \xrightarrow{\det^*} & \mathbf{C}_1^* \end{array}$$

and is therefore connected.

QED

In fact, $\pi_1(Sp) \cong \mathbf{Z}$ because $Sp \cong U(g) \times \mathfrak{H}_g$ as topological spaces and \mathfrak{H}_g is a Euclidean space and $\det^* : U(g) \rightarrow \mathbf{C}_1^*$ gives an isomorphism of fundamental group. Thus for each $n \geq 1$ there is a unique connected n -sheeted covering of Sp , and Mp is the unique connected two-sheeted covering of Sp .

COROLLARY 8.6. The exact sequence $1 \rightarrow \{\pm 1\} \rightarrow Mp \xrightarrow{q} Sp \rightarrow 1$ is non-split and $Mp = [Mp, Mp]$.

PROOF: Embed $U(1)$ in $U(g)$ by $z \mapsto \begin{pmatrix} z & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix}$, and embed $U(g)$ in $Sp(2g, \mathbf{R})$ as above. Then the sequence

$$1 \rightarrow \{\pm 1\} \rightarrow q^{-1}(U(1)) \xrightarrow{q} U(1) \rightarrow 1$$

can be identified to

$$1 \rightarrow \{\pm 1\} \rightarrow \mathbf{C}_1^* \xrightarrow{Sq} \mathbf{C}_1^* \rightarrow 1.$$

To see this use the diagram in the proof of Proposition 8.5 and note that C_{iI} restricted to $q^{-1}(U(1))$ must be one-to-one (the kernel of q is $\{\pm 1\}$ which is killed by Sq). This sequence is non-split (e.g., restricting to the torsion subgroups one gets

$$0 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{Q}/\mathbf{Z} \xrightarrow{2} \mathbf{Q}/\mathbf{Z} \rightarrow 0)$$

and thus the sequence

$$(*) \quad 1 \longrightarrow \{\pm 1\} \longrightarrow Mp \longrightarrow Sp \longrightarrow 1$$

is non-split.

Because $[Sp, Sp] = Sp$, $[Mp, Mp]$ sits in the exact sequence

$$1 \longrightarrow \{\pm 1\} \cap [Mp, Mp] \longrightarrow [Mp, Mp] \longrightarrow Sp \longrightarrow 1.$$

If $\{\pm 1\} \cap [Mp, Mp]$ is trivial, then $Sp \cong [Mp, Mp] \subset Mp$ and the sequence

(*) splits. Thus $\{\pm 1\} \cap [Mp, Mp] = \{\pm 1\}$ and $[Mp, Mp] = Mp$. QED

COROLLARY 8.7. For $T \in \mathfrak{H}_g$, let $U(T) = \{\gamma \in Sp \mid \gamma(T) = T\}$. Then

$$C_T(P)^2 = \det^* q(P), \quad \forall P \in q^{-1}(U(T)).$$

PROOF: This is easy; the case $T = \pm iI$ has already been checked while proving Proposition 8.5. QED

From now on, we shall denote elements of Mp by $\tilde{\gamma}, \tilde{\delta}, \dots$, assuming implicitly that they sit over corresponding elements γ, δ, \dots of Sp .

We shall now write down the functional equation for $\vartheta^\alpha[x](T)$ in T where $x \in \mathbf{R}^{2g}$, $T \in \mathfrak{H}_g$. Denote by $\sigma(\mathbf{Z}^{2g})$ the subgroup $(1 \times \mathbf{Z}^g \times 0) \cdot (1 \times 0 \times \mathbf{Z}^g)$ of $Heis(2g, \mathbf{R})$, and by $T_\gamma : Heis(2g, \mathbf{R}) \longrightarrow Heis(2g, \mathbf{R})$ the automorphism $(\lambda, v) \longmapsto (\lambda, \gamma v)$ where $\gamma \in Sp(2g, \mathbf{R})$. Then

$$\Gamma_{1,2} = \{\gamma \in Sp(2g, \mathbf{R}) \mid T_\gamma(\sigma(\mathbf{Z}^{2g})) = \sigma(\mathbf{Z}^{2g})\}$$

is easily checked to be

$$\left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbf{Z}) \mid {}^tCA \text{ and } {}^tBD \text{ both have even diagonal entries} \right\}.$$

Recall that $Ce_{\mathbf{Z}}$ is the subspace of $\mathcal{H}_{-\infty}$ annihilated by $U_x - 1$, $\forall x \in \sigma(\mathbf{Z}^{2g}) \subset Heis(2g, \mathbf{R})$. Thus, for $\tilde{\gamma} \in Mp$, with $\gamma \in \Gamma_{1,2}$, $\tilde{\gamma}e_{\mathbf{Z}}$ is annihilated by $U_{T_\gamma(x)} - 1 \forall x \in \sigma(\mathbf{Z}^{2g})$ and thus $\tilde{\gamma}e_{\mathbf{Z}} = \eta(\tilde{\gamma})e_{\mathbf{Z}}$ where $\eta(\tilde{\gamma}) \in \mathbf{C}^*$. It follows that $\eta : q^{-1}(\Gamma_{1,2}) \rightarrow \mathbf{C}^*$ is a character. We can now write down the functional equation of $\vartheta^\alpha[x](T)$ in terms of η :

PROPOSITION 8.8. For $x \in \mathbf{R}^{2g}$, $T \in \mathfrak{H}_g$, $\tilde{\gamma} \in q^{-1}(\Gamma_{1,2})$, and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have:

$$\vartheta^\alpha[x](T) = \overline{\eta(\tilde{\gamma})} \det(-BT + A)^{-1/2} \vartheta^\alpha[\gamma x]((DT - C)(-BT + A)^{-1}),$$

where the square root $\det(-BT + A)^{-1/2}$ is determined to be $C(\tilde{\gamma}; T)$.

PROOF:

$$\begin{aligned} \vartheta^\alpha[x](T) &= \langle U_{(1,x)} f_T, e_{\mathbf{Z}} \rangle \\ &= \langle \tilde{\gamma} U_{(1,x)} f_T, \tilde{\gamma} e_{\mathbf{Z}} \rangle \\ &= \overline{\eta(\tilde{\gamma})} \langle \tilde{\gamma} U_{(1,x)} \tilde{\gamma}^{-1} \tilde{\gamma} f_T, e_{\mathbf{Z}} \rangle \\ &= \overline{\eta(\tilde{\gamma})} \langle U_{(1,\gamma x)} c(\tilde{\gamma}; T) f_{\gamma(T)}, e_{\mathbf{Z}} \rangle \\ &= \overline{\eta(\tilde{\gamma})} c(\tilde{\gamma}; T) \langle U_{(1,\gamma x)} f_{\gamma(T)}, e_{\mathbf{Z}} \rangle \\ &= \overline{\eta(\tilde{\gamma})} c(\tilde{\gamma}; T) \vartheta^\alpha[\gamma x](\gamma T). \end{aligned}$$

QED

We immediately deduce the functional equation of $\vartheta(z, T)$ in T :

COROLLARY 8.9. For all $z \in \mathbf{C}^g$, $T \in \mathfrak{H}_g$, $\tilde{\gamma} \in q^{-1}(\Gamma_{1,2})$, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have:

$$\vartheta(z, T) = \overline{\eta(\tilde{\gamma})} \cdot \det(-BT + A)^{-1/2} \cdot \exp(\pi i {}^t z (-BT + A)^{-1} B z) \cdot \vartheta({}^t(-BT + A)^{-1} z, \gamma(T)).$$

PROOF: Starting from the functional equation for ϑ^α , and using $\vartheta^\alpha = \exp(\pi i {}^t x_1 \cdot \underline{x}) \vartheta(\underline{x}, T)$ we see that we only need to check

(A) if $z = Tx_1 + x_2$, then

$$(DT - C)(-BT + A)^{-1}(Ax_1 + Bx_2) + (Cx_1 + Dx_1) = {}^t(-BT + A)^{-1} z$$

and

$$(B) {}^t(Ax_1 + Bx_2)({}^t(-BT + A)^{-1} z - {}^t x_1 z) = {}^t z (-BT + A)^{-1} B z.$$

PROOF OF (A): $(DT - C)(-BT + A)^{-1} \in \mathfrak{H}_g$ and is therefore symmetric.

Thus

$$\begin{aligned}
& (DT - C)(-BT + A)^{-1}(Ax_1 + Bx_2) + (Cx_1 + Dx_2) \\
&= ({}^t[(DT - C)(-BT + A)^{-1}], 1) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= (-T^t B + {}^t A)^{-1}(T^t D - {}^t C, -T^t B + {}^t A) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= (-T^t B + {}^t A)^{-1}(T, 1) \begin{pmatrix} {}^t D & -{}^t B \\ -{}^t C & {}^t A \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= (-T^t B + {}^t A)^{-1}(T, 1)\gamma^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \gamma \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= {}^t(-BT + A)^{-1}(T, 1)\gamma^{-1} \cdot \gamma \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= {}^t(-BT + A)^{-1}(Tx_1 + x_2).
\end{aligned}$$

PROOF OF (B): Let $\alpha = (-BT + A)^{-1}$. Then $\alpha(-BT + A) = I$ which implies $\alpha A = \alpha BT + I$.

$$\begin{aligned}
({}^t(Ax_1 + Bx_2))\alpha z - {}^t x_1 z &= {}^t(\alpha(Ax_1 + Bx_2))z - {}^t x_1 z \\
&= {}^t((\alpha A - I)x_1 + \alpha Bx_2)z \\
&= {}^t(\alpha B(Tx_1 + x_2))z \\
&= {}^t z \cdot (-BT + A)^{-1} \cdot Bz.
\end{aligned}$$

QED

It is clear that 8.8 stands incomplete without a complete description of η . However we discuss only η^2 ; this has the advantage of being a character on $\Gamma_{1,2}$ and not just on $q^{-1}(\Gamma_{1,2})$.

PROPOSITION 8.10.

A. η surjects onto the eighth roots of unity.

B. If $\ker(\eta)^2 = \Delta$, then Δ contains $\Gamma_4 = \{\gamma \in Sp(2g, \mathbb{Z}) \mid \gamma \equiv I \pmod{4}\}$; in fact

$$\Delta \cap \Gamma_2 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 \mid \text{Tr}(D - I_g) \equiv 0 \pmod{4} \right\}.$$

C. The image of Δ in the composite $\Delta \hookrightarrow \Gamma_{1,2} \rightarrow \Gamma_{1,2}/\Gamma_2 = O(2g, \mathbb{Z}/2)$ is precisely $SO(2g, \mathbb{Z}/2)$.

We only sketch the proof and leave the details to the reader. Let ρ be the image of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times I \times \dots \times I$ under the action of the natural homomorphism from $SL_2(\mathbb{Z}) \times \dots \times SL_2(\mathbb{Z}) \rightarrow Sp(2g, \mathbb{Z})$. It is easy to see that $\eta^2(\rho) = \eta^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \eta^2(I) \cdot \eta^2(I) \dots \eta^2(I) = \eta^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The fact that the Fourier transform preserves $e_{\mathbb{Z}}$ quickly implies that $\eta^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i^{-1}$. Therefore $\eta^2(\rho) = i^{-1}$ and the image of η contains the eighth roots of unity. The proof of A will be complete if we show that $[\Gamma_{1,2} : \Delta] \leq 4$.

For $\begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}$ with $A \in SL_g(\mathbb{Z})$ the operator P from 8.2 is $Pf(y) = f(A^{-1}y) |\det A|^{-\frac{1}{2}}$; since $e_{\mathbb{Z}} = \sum \delta_n$ in $L^2(\mathbb{R}^g)$, we have $Pe_{\mathbb{Z}} = e_{\mathbb{Z}}$. The operator associated to $\begin{pmatrix} I & 0 \\ N & I \end{pmatrix}$ with N symmetric integral and even diagonal is multiplication by $\exp(\pi i {}^t y N y)$. The conditions on N insure that $Pe_{\mathbb{Z}} = e_{\mathbb{Z}}$ and hence both of these terms belong to Δ . Consequently matrices of the form $\begin{pmatrix} I & N \\ 0 & I \end{pmatrix}$ with the same restrictions on N belong to Δ (being conjugates of $\begin{pmatrix} I & 0 \\ N & I \end{pmatrix}$). Let Δ' be the subgroup of $\Gamma_{1,2}$ generated

by the above three subgroups. Writing ρ as $\begin{bmatrix} I_{g-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & I_{g-1} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and doing systematic column reduction, one sees that $\Gamma_{1,2} = \Delta' \cup \Delta' \rho \cup \Delta' \rho^2 \cup \Delta' \rho^3$. This finishes the proof of A.

B. By column reduction again one checks that $\begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}$ with $A \in SL_g(\mathbb{Z})$, $A \equiv I \pmod{2}$ and $\begin{pmatrix} I & 2N \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ 2N & I \end{pmatrix}$ with N integral generate a subgroup Δ'' of Γ_2 such that $\Delta'' \cup \Delta'' \rho^2 = \Gamma_2$. We have by our calculations in A that $\Delta'' \subseteq \Delta$; thus we have

$$\Delta'' \subseteq (\Delta \cap \Gamma_2) \subseteq \Gamma_2.$$

But $\Delta'' \subset \Delta''' = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 \mid \text{Tr}(D - I_g) \equiv 0 \pmod{4} \right\}$ which is a subgroup of index 2 in Γ_2 , and therefore $\Delta'' = \Delta'''$. Also $\eta^2(\rho^2) = -1$, which implies that $\Delta \cap \Gamma_2 = \Delta'''$.

C. It follows from A and B that the image of Δ in $O(2g, \mathbb{Z}/2)$ is a subgroup of index 2, but $SO(2g, \mathbb{Z}/2) = [O(2g, \mathbb{Z}/2), O(2g, \mathbb{Z}/2)]$ always, and this proves the result.

9. Theta Functions in Spherical Harmonics

There is still another extremely natural and important generalization of the theta function! In §6, we introduced theta functions with quadratic forms Q , which arise inevitably either by considering the algebraic identities on ϑ or by seeking more general modular forms represented by theta series. In this section, we introduce theta functions defined using *both* a quadratic form Q and a spherical harmonic polynomial P . These can be motivated and then analyzed either by considering the derivatives of ϑ or by carrying further the representation-theoretic definition of ϑ , yielding still more modular forms given by theta series. We shall split our discussion in three; first defining these new functions by differentiating analytic theta functions, then by matrix coefficients, and finally by differentiating sections of line bundles on abelian varieties, leading to an algebraic theory.

VIEWPOINT I: Differentiating analytic thetas

As usual, $T \in \mathfrak{H}_g$, Q is a positive definite symmetric rational $(h \times h)$ -matrix, and Z is a complex $(g \times h)$ matrix. As in §6, let

$$\vartheta^Q \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (Z; T) = \sum_{N \in \mathbf{Q}(g, h)} \chi \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (N) \exp \pi i \operatorname{Tr} ({}^t N T N Q + 2 {}^t N Z)$$

where

$$\begin{aligned} \chi \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (N) &= 0 \text{ if } N - A \notin \mathbf{Z}(g, h) \\ &= \exp 2\pi i \operatorname{Tr} {}^t N B \text{ otherwise.} \end{aligned}$$

Let $R = \mathbb{C}[z_{ij}; 1 \leq i \leq g, 1 \leq j \leq h]$, the ring of polynomial functions on the Z -space of complex $(g \times h)$ -matrices. For any homogeneous polynomial $P \in R$, let $P(\partial) = P\left(\frac{\partial}{\partial z_{ij}}\right)$. Then differentiating the $\vartheta^Q \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right]$ termwise, we get

$$P(\partial) \vartheta^Q \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (Z; T) = \sum_{N \in \mathbf{Q}(g, h)} \chi \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (N) P(2\pi i N) \exp \pi i \operatorname{Tr} ({}^t N T N Q + 2 {}^t N Z).$$

Omitting the $2\pi i$ factors, we give these functions names:

DEFINITION 9.1. For T, Q, Z as above and for $P \in R$ homogeneous and $A, B \in \mathcal{Q}(g, h)$,

- (a) $\vartheta^{P, Q} \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (Z; T) = \sum_{N \in \mathcal{Q}(g, h)} \chi \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (N) P(N) \exp \pi i \text{Tr} ({}^t N T N Q + 2 {}^t N Z)$.
- (b) $\vartheta^{P, Q} (Z; T) = \vartheta^{P, Q} \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] (Z; T)$.
- (c) $\vartheta^{P, Q} \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (T) = \vartheta^{P, Q} \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (0; T)$.

The above shows that when P is homogeneous of degree k , then

$$\vartheta^{P, Q} \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (Z; T) = (2\pi i)^{-k} P(\partial) \vartheta^Q \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (Z; T).$$

We want to find out for which polynomials $P \in R$, the $\vartheta^{P, Q} \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \right] (T)$ are modular forms (in a generalized sense which we shall explain later) for all $A, B \in \mathcal{Q}(g, h)$. The answer is: if and only if P is pluri-harmonic in the following sense:

DEFINITION 9.2. Let $S = (s_{pq})$ be the inverse of Q . Then $P \in R$ is pluri-harmonic with respect to Q if and only if

$$\sum_{p, q} \frac{\partial^2 P}{\partial z_{ip} \partial z_{jq}} \cdot s_{pq} = 0 \quad \text{for all } (i, j).$$

Let's work out some examples of what pluri-harmonic means:

A. When $g = 1$, put $z_{1p} = z_p$. Then

$$P \text{ is pluri-harmonic} \iff \left(\sum_{p, q} s_{pq} \frac{\partial^2}{\partial z_p \partial z_q} \right) P = 0.$$

But $\Delta = \sum_{p, q} s_{pq} \frac{\partial^2}{\partial z_p \partial z_q}$ is just the Laplacian operator for Q (and hence invariant under the orthogonal group of the quadratic form Q). In this case P is pluri-harmonic iff P is harmonic in the usual sense.

B. When $h = 1$, put $z_{i1} = z_i$. Then P is pluri-harmonic iff $\frac{\partial^2 P}{\partial z_i \partial z_j} = 0$ for all i, j ; i.e., if P has the form: $P = c + \sum_{i=1}^g a_i z_i$ for some $c, a_1, a_2, \dots, a_g \in \mathbb{C}$.

C. When $h = 2$ and $Q = I_2$, put $z_{i1} = x_i$ and $z_{i2} = y_i$, and $z_i = x_i + \sqrt{-1} \cdot y_i$, $\bar{z}_i = x_i - \sqrt{-1} \cdot y_i$. Then $R = \mathbb{C}[z_{ij}] = \mathbb{C}[x_1, \dots, x_g, y_1, \dots, y_g] = \mathbb{C}[z_1, \dots, z_g, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_g]$. $P \in R$ is pluri-harmonic iff $\frac{\partial^2 P}{\partial z_i \partial \bar{z}_j} + \frac{\partial^2 P}{\partial \bar{z}_j \partial z_i} = 0$ for all i and j . It is not hard to see that this is so if and only if

$$P = h_1(z_1, z_2, \dots, z_g) + h_2(\bar{z}_1, \bar{z}_2, \dots, \bar{z}_g) + \sum_{i < j} c_{ij} (z_i \bar{z}_j - z_j \bar{z}_i),$$

where h_1 and h_2 are polynomials and the $c_{ij} \in \mathbb{C}$.

Next we put $g = h = 1$ and $Q = 1$, and use the results of Chapter I to get some idea of when $\vartheta^{P, Q}$ is a modular form. The functional equation of Chapter I, §7 says:

$$\vartheta \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] (z, \tau) = (c\tau + d)^{-1/2} \cdot \exp \pi i \left(-\frac{cz^2}{c\tau + d} \right) \cdot \vartheta \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in some congruence subgroup of $SL_2(\mathbb{Z})$ with the sign of $(c\tau + d)^{-1/2}$ carefully chosen. Differentiating with respect to z :

$$\begin{aligned} \frac{\partial}{\partial z} \vartheta \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] (z, \tau) = \\ (c\tau + d)^{-1/2} \cdot \left(\frac{-2\pi icz}{c\tau + d} \right) \cdot \exp \frac{-\pi icz^2}{c\tau + d} \cdot \vartheta \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \\ + (c\tau + d)^{-3/2} \exp \left(\frac{-\pi icz^2}{c\tau + d} \right) \vartheta' \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \vartheta \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] (z, \tau) = \\ (c\tau + d)^{-1/2} \left(\frac{-2\pi ic}{c\tau + d} \right) \exp \frac{-\pi icz^2}{c\tau + d} \cdot \vartheta \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \\ + (c\tau + d)^{-1/2} \left(\frac{2\pi icz}{c\tau + d} \right)^2 \exp \frac{-\pi icz^2}{c\tau + d} \cdot \vartheta \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \\ + 2(c\tau + d)^{-3/2} \left(\frac{-2\pi icz}{c\tau + d} \right) \exp \frac{-\pi icz^2}{c\tau + d} \cdot \vartheta' \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \\ + (c\tau + d)^{-5/2} \exp \left(\frac{-\pi icz^2}{c\tau + d} \right) \cdot \vartheta'' \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right). \end{aligned}$$

Putting $z = 0$, we get:

$$\vartheta' \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] (0, \tau) = (c\tau + d)^{-3/2} \vartheta' \left[\begin{smallmatrix} p \\ q \end{smallmatrix} \right] \left(0, \frac{a\tau + b}{c\tau + d} \right),$$

$$\vartheta'' \begin{bmatrix} p \\ q \end{bmatrix} (0, \tau) = (c\tau + d)^{-1/2} \cdot \left(\frac{-2\pi ic}{c\tau + d} \right) \vartheta \begin{bmatrix} p \\ q \end{bmatrix} \left(0, \frac{a\tau + b}{c\tau + d} \right) \\ + (c\tau + d)^{-5/2} \cdot \vartheta'' \begin{bmatrix} p \\ q \end{bmatrix} \left(0, \frac{a\tau + b}{c\tau + d} \right)$$

which shows that $\vartheta^{z, I} \begin{bmatrix} p \\ q \end{bmatrix} (\tau)$ is a modular form of weight 3/2, whereas $\vartheta^{z^2, I} \begin{bmatrix} p \\ q \end{bmatrix} (\tau)$ is not a modular form. This is in agreement with the fact that $P(z) = z$ is pluri-harmonic and $P(z) = z^2$ is not. In explicit series:

$$\sum \chi(n) \cdot n \cdot e^{\pi i n^2 \tau}$$

is a modular form, but

$$\sum \chi(n) \cdot n^2 \cdot e^{\pi i n^2 \tau}$$

is not.

Note that $\vartheta^{z, I} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau) = \sum_{n \in \mathbb{Z}} n \exp \pi i n^2 \tau = 0$ for all τ so that the modular forms obtained this way are not all non-zero. However, for any P, Q and T , $\vartheta^{P, Q}(Z; T)$ is not identically zero as a function of Z . And since $A + B \cdot T$ are dense in $\mathbb{C}(g, h)$, there are A and B such that $\vartheta^{P, Q} \begin{bmatrix} A \\ B \end{bmatrix} (T) \neq 0$, i.e., for suitable A and B , these series are non-zero.

We shall now show that the functional equation of the $\vartheta^{P, Q} \begin{bmatrix} R \\ S \end{bmatrix} (T)$ can be deduced exactly as above: by differentiating the functional equation of $\vartheta^Q \begin{bmatrix} R \\ S \end{bmatrix} (Z, T)$. Having done this, we shall proceed to interpret the $\vartheta^{P, Q} \begin{bmatrix} R \\ S \end{bmatrix} (T)$ as modular forms in a suitable generalized sense. The two functional equations are:

THEOREM 9.4. For rational $(g \times h)$ -matrices R and S ,

$$\vartheta^Q \begin{bmatrix} R \\ S \end{bmatrix} (Z, T) = \det(CT + D)^{-h/2} \exp(-\pi i \text{Tr}(^t Z(CT + D)^{-1} CZQ^{-1})) \\ \cdot \vartheta^Q \begin{bmatrix} R \\ S \end{bmatrix} (^t(CT + D)^{-1} Z, (AT + B)(CT + D)^{-1})$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in a suitable subgroup of $Sp(2g, \mathbb{Z})$.

THEOREM 9.5. Let P be a pluri-harmonic polynomial with respect to the quadratic form Q . Then, for rational $(g \times h)$ -matrices R and S ,

$$\vartheta^{P, Q} \begin{bmatrix} R \\ S \end{bmatrix} (T) = \det(CT + D)^{-h/2} \vartheta^{P', Q} \begin{bmatrix} R \\ S \end{bmatrix} ((AT + B)(CT + D)^{-1})$$

with $P'(Z) = P((CT + D)^{-1} Z)$, for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in a suitable congruence subgroup of $Sp(2g, \mathbb{Z})$.

Theorem 9.4 will be proved in Viewpoint II.

First we need some generalities on pluri-harmonic polynomials.

LEMMA 9.6. For $A \in \mathbb{C}[X_1, X_2, \dots, X_n]$, we let $A(\partial)$ denote the operator $A\left(\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right)$. For $A, B \in \mathbb{C}[X_1, X_2, \dots, X_n]$, $\langle A, B \rangle = (A(\partial)B)(0)$ is a symmetric non-degenerate bilinear form on $\mathbb{C}[X_1, X_2, \dots, X_n]$ which satisfies: $\langle A, BC \rangle = \langle B(\partial)A, C \rangle$ for all $A, B, C \in \mathbb{C}[X_1, X_2, \dots, X_n]$.

PROOF: $\langle X_1^{h_1} X_1^{h_2} \dots X_n^{h_n}, X_1^{g_1} X_2^{g_2} \dots X_n^{g_n} \rangle = 0$ if the exponents satisfy $(h_1, h_2, \dots, h_n) \neq (g_1, g_2, \dots, g_n)$ and equals $h_1! h_2! \dots h_n!$ otherwise. This proves that $\langle A, B \rangle$ is symmetric bilinear non-degenerate (in fact, positive definite when restricted to $\mathbb{R}[X_1, X_2, \dots, X_n]$), and similarly $\langle A, BC \rangle = \langle B(\partial)A, C \rangle$ is checked for monomials A, B and C , from which the lemma follows. QED

LEMMA 9.7. Let $\mathbf{H} \subset R = [z_{ij}; 1 \leq i \leq g, 1 \leq j \leq h]$ be the space of all pluri-harmonic polynomials with respect to Q and $I \subset R$ be the ideal generated by the $h_{ij} = \sum_{p, q} s_{pq} z_{ip} z_{jq}$ for all i and j ($S = (s_{pq}) = Q^{-1}$ as before). Then $\mathbf{H} = I^\perp$ with respect to the pairing \langle, \rangle introduced in the previous lemma, and $R = \mathbf{H} \oplus I$.

PROOF: $\langle fh_{ij}, P \rangle = (f(\partial)h_{ij}(\partial)P)(0) = 0$ for all $f \in R$, for all i and j , if and only if the $h_{ij}(\partial)P$ vanishes along with all its repeated partial derivatives at 0, i.e., if and only if $h_{ij}(\partial)P \equiv 0$ for all i, j , i.e., if and only

if P is pluri-harmonic. Thus $\mathbf{H} = I^\perp$. The same argument shows that $\mathbf{H}_\mathbf{R} = I_\mathbf{R}^\perp$ where $\mathbf{H}_\mathbf{R} = \mathbf{R}[z_{ij}] \cap \mathbf{H}$ and $I_\mathbf{R} = I \cap \mathbf{R}[z_{ij}]$. But \langle, \rangle is positive definite on $\mathbf{R}[z_{ij}]$ and therefore $I_\mathbf{R} \oplus \mathbf{H}_\mathbf{R} = \mathbf{R}[z_{ij}]$; thus $\mathbf{R} = \mathbf{H} \oplus I$. QED

LEMMA 9.8. If P is pluri-harmonic, then $P(\partial)[g(Z)\exp \text{Tr}({}^t Z C Z Q^{-1})]$ and $P(\partial)g(Z)$ coincide at $Z = 0$, where C is any complex $(g \times g)$ -matrix and g is any function analytic in a neighborhood of $Z = 0$.

PROOF: Let $h(Z) = \text{Tr}({}^t Z C Z Q^{-1})$ and $S = (s_{pq}) = Q^{-1}$. Clearly it suffices to prove this for polynomials $g(Z)$. Expanding, we see that:

$$P(\partial)g(Z)\exp h(Z) = \sum_{n=0}^{\infty} \frac{1}{n!} P(\partial)g(Z)h(Z)^n$$

when $Z = 0$ since only finitely many terms of the summation are non-zero.

Therefore $P(\partial)g(Z)\exp h(Z)$ at $Z = 0$ equals

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle P, g \cdot h^n \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} (h(\partial)^n P, g) = \langle P, g \rangle = P(\partial)g$$

at $Z = 0$, provided we show $h(\partial)P = 0$.

Now

$$\begin{aligned} h(Z) &= \sum_{i,j,k,\ell} z_{ij} c_{ik} z_{k\ell} s_{\ell j} \\ &= \sum_{i,k} c_{ik} \left(\sum_{j,\ell} z_{ij} z_{k\ell} s_{\ell j} \right) \\ &= \sum_{i,k} c_{ik} h_{ik}, \end{aligned}$$

and therefore since P is pluri-harmonic $h(\partial)P = \sum_{i,k} c_{ik} (h_{ik}(\partial)P) = 0$,

which finishes the proof. QED

The following will be used in Viewpoint II.

COROLLARY 9.9. If P is pluri-harmonic,

$$P(\partial)\exp \text{Tr}({}^t Z C Z Q^{-1}) = P(2CZQ^{-1})\exp \text{Tr}({}^t Z C Z Q^{-1}).$$

PROOF: Put $h(Z) = \exp \text{Tr}({}^t Z C Z Q^{-1})$. For any $A \in \mathbf{C}(g, h)$, let $f(Z) = h(Z + A) = h(Z)h(A)g(Z)$ where $g(Z) = \exp \text{Tr}(2{}^t Z C A Q^{-1})$. Then

$P(\partial)h(Z)$ evaluated at $Z = A$

$= P(\partial)f(Z)$ evaluated at $Z = 0$

$= h(A) \cdot P(\partial)g(Z)$ evaluated at $Z = 0$, by the above lemma.

But

$$\frac{\partial}{\partial z_{ij}} g(Z) = \frac{\partial}{\partial z_{ij}} \exp 2 \left(\sum_{p,q} z_{pq} (C A Q^{-1})_{pq} \right) = g(Z) (2 C A Q^{-1})_{ij}.$$

A repeated application of this shows that $P(\partial)g(Z) = P(2C A Q^{-1})g(Z)$, and when $Z = 0$ this equals $P(2C A Q^{-1})$. QED

LEMMA 9.10. Let $f'(Z) = f(AZB)$ and $P'(Z) = P({}^t A Z {}^t B)$, where A and B are complex $(g \times g)$ and $(h \times h)$ -matrices respectively. Then $P(\partial)f'(Z) = (P'(\partial)f)(AZB)$. In particular, $\langle P, f' \rangle = \langle P', f \rangle$.

We omit the proof.

COROLLARY 9.11. For the action of $GL_g(\mathbf{C}) \times O(Q)$ on R given by $(A, B)P(Z) = P(A^{-1}ZB)$ for all $A \in GL_g(\mathbf{C})$, $B \in O(Q)$, $P \in R$, the space of pluri-harmonic polynomials, \mathbf{H} , is an invariant subspace.

PROOF: By 9.7 the orthogonal complement of \mathbf{H} is the ideal I generated by $h_{ij} = \sum s_{pq} z_{ip} z_{jq}$. So by 9.10 it suffices to check that $h_{ij}(AZ)$ and $h_{ij}(ZB)$ belong to I for all $A \in GL_g(\mathbf{C})$, $B \in O(Q)$, where $h_{ij}(Z) = \sum s_{pq} z_{ip} z_{jq}$.

$$\begin{aligned} h_{ij}(AZ) &= \sum_{\ell, m, p, q} a_{i\ell} z_{\ell p} a_{jm} z_{mq} s_{pq} \\ &= \sum_{\ell, m} a_{i\ell} a_{jm} \left(\sum_{p, q} z_{\ell p} z_{mq} s_{pq} \right) \\ &= \sum_{\ell, m} a_{i\ell} a_{jm} h_{\ell m}(Z) \in I. \end{aligned}$$

$$\begin{aligned} h_{ij}(ZB) &= \sum_{\ell, m, p, q} z_{i\ell} b_{\ell p} z_{jm} b_{mq} s_{pq} \\ &= \sum_{\ell, m} z_{i\ell} z_{jm} (B Q^{-1} \cdot {}^t B)_{\ell m}. \end{aligned}$$

But ${}^tBQB = Q$; therefore $B^{-1}Q^{-1}{}^tB^{-1} = Q^{-1}$; thus $Q^{-1} = BQ^{-1}{}^tB$. Therefore $h_{ij}(ZB) = h_{ij}(Z)$. QED

We shall now prove Theorem 9.5. Let P be a homogeneous pluri-harmonic polynomial of degree k . Apply the operator $(2\pi i)^{-k}P(\partial)$ to Theorem 9.4 and put $Z = 0$. By Lemma 9.8, we get:

$$\vartheta^{P,Q} \begin{bmatrix} R \\ S \end{bmatrix} (T) = (2\pi i)^{-k} \det(CT + D)^{-h/2} \cdot P(\partial) \vartheta^Q \begin{bmatrix} R \\ S \end{bmatrix} ({}^t(CT + D)^{-1}Z, (AT + B)(CT + D)^{-1})$$

evaluated at $Z = 0$. By Lemma 9.10 this simplifies to:

$$\det(CT + D)^{-h/2} \vartheta^{P',Q} \begin{bmatrix} R \\ S \end{bmatrix} ((AT + B)(CT + D)^{-1})$$

where $P'(Z) = P((CT + D)^{-1}Z)$. QED

In the functional equation for $\vartheta^{P,Q}$, $\vartheta^{P,Q}$ is related to $\vartheta^{P',Q}$ for a P' obtained from P by the action of $GL_g(\mathbf{C})$ on the space of pluri-harmonic polynomials. A clearer way to express what's happening is to group the $\vartheta^{P,Q}$'s together into vector-valued functions which transform into themselves if we view these vectors as points in a homogeneous vector bundle over \mathfrak{H}_g .

Recall that a homogeneous vector bundle \mathbf{E} on \mathfrak{H}_g is a holomorphic vector bundle over \mathfrak{H}_g together with a lifting of the action of $Sp(2g, \mathbf{R})$ from \mathfrak{H}_g to \mathbf{E} . Actually, we want to allow for the usual ambiguity of sign in the functional equation, so we generalize this slightly and ask for an action of the double cover $Mp(2g, \mathbf{R})$ of $Sp(2g, \mathbf{R})$ on \mathbf{E} which makes the projection $\mathbf{E} \rightarrow \mathfrak{H}_g$ equivariant. As usual, these bundles are obtained from finite dimensional representations of the subgroup of $Sp(2g, \mathbf{R})$ (or $Mp(2g, \mathbf{R})$) which fixes a base point, e.g., iI_g , in \mathfrak{H}_g . This subgroup is the unitary group $U(g)$ and its finite dimensional representations extend to holomorphic representations of $GL_g(\mathbf{C})$. Then homogeneous vector bundles for $Sp(2g, \mathbf{R})$ are given by:

$$\mathbf{E}^{(\rho)} = E \times \mathfrak{H}_g$$

where $\rho : GL_g(\mathbf{C}) \rightarrow \text{Aut}(E)$ is a representation of $GL_g(\mathbf{C})$ and $Sp(2g, \mathbf{R})$ acts by

$$\gamma(X, T) = (\rho(CT + D)X, (AT + B)(CT + D)^{-1})$$

for all $X \in E, T \in \mathfrak{H}_g, \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbf{R})$. As in §8, to be consistent we are sometimes forced to let $Sp(2g, \mathbf{R})$ act after conjugating by $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, in which case it come out as:

$$\gamma(X, T) = (\rho(-BT + A) \cdot X, (DT + C) \cdot (-BT + A)^{-1}).$$

To include metaplectic actions, we need to define a double cover $GL_g^\sim(\mathbf{C})$ of $GL_g(\mathbf{C})$ by:

$$GL_g^\sim(\mathbf{C}) = \{(\alpha, A) \mid \alpha^2 = \det A, \alpha \in \mathbf{C}^*, A \in GL_g(\mathbf{C})\}.$$

In what follows, α will simply be denoted as $\sqrt{\det A}$. Then if

$$\rho : GL_g^\sim(\mathbf{C}) \rightarrow \text{Aut}(E)$$

is a representation, $\mathbf{E}^{(\rho)} = E \times \mathfrak{H}_g$ as before and $Mp(2g, \mathbf{R})$ acts by:

$$\tilde{\gamma}(X, T) = (\rho(C(\tilde{\gamma}; T)^{-1}, (-BT + A)) \cdot X, (DT + C) \cdot (-BT + A)^{-1})$$

where $\tilde{\gamma}$ maps to $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbf{R})$ and $C(\tilde{\gamma}; T)$ was defined in (8.3). The simplest example is given by the one-dimensional representation in which $\rho(\alpha, A) = \alpha = \sqrt{\det A}$. This defines a homogeneous line bundle on \mathfrak{H}_g which we write as $L^{1/2}$, i.e.,

$$\begin{aligned} L^{1/2} &= \mathbf{C} \times \mathfrak{H}_g \quad \text{with action :} \\ \tilde{\gamma}(x, T) &= (C(\tilde{\gamma}, T)^{-1} \cdot x, \gamma(T)) \\ &= (\det(-BT + A)^{1/2} \cdot x, (DT - C)(-BT + A)^{-1}) \end{aligned}$$

Its square L is the homogeneous line bundle for $Sp(2g, \mathbf{R})$ associated to the determinant.

As usual, a vector-valued modular form on \mathfrak{H}_g is a holomorphic section of such a bundle $\mathbf{E}^{(\rho)}$, which is invariant under some $\tilde{\Gamma} \subset Mp(2g, \mathbf{R})$

whose image Γ in $Sp(2g, \mathbf{R})$ is a congruence subgroup. Such a form is a holomorphic map $f : \mathfrak{H}_g \rightarrow E$ such that

$$f(T) = \rho(C(\tilde{\gamma}; T), (-BT + A)^{-1}) \cdot f((DT - C)(-BT + A)^{-1})$$

for all $\tilde{\gamma} \in \tilde{\Gamma}$.

Now let W be some $GL_g(\mathbf{C})$ -stable subspace of \mathbf{H} , the space of pluri-harmonic polynomials. Define $\vartheta_W \begin{bmatrix} R \\ S \end{bmatrix} : \mathfrak{H}_g \rightarrow W^*$ as follows:

$$\left(\vartheta_W \begin{bmatrix} R \\ S \end{bmatrix} (T) \right) (P) = \vartheta^{P, Q} \begin{bmatrix} R \\ S \end{bmatrix} (T)$$

for all $P \in W \subset \mathbf{H}$.

Theorem 9.5 now reads as:

$$\vartheta_W \begin{bmatrix} R \\ S \end{bmatrix} (T)(P) = \det(CT + D)^{-h/2} \cdot \vartheta_W \begin{bmatrix} R \\ S \end{bmatrix} ((AT + B)(CT + D)^{-1})(P')$$

where $P'(Z) = P((CT + D)^{-1}Z)$. Denoting by ρ the action of $GL_g(\mathbf{C})$ on $W \subset \mathbf{H}$ given by $(\rho(A)P)(z) = P(A^{-1}z)$, P' is simply $\rho(CT + D)P$. The action ρ^* of $GL_g(\mathbf{C})$ on W^* is given, as usual, by: $(\rho^*(A)\ell)(P) = \ell(\rho(A)^{-1}P)$ for all $A \in GL_g(\mathbf{C})$, $\ell \in W^*$, $P \in W$. The above formula now reads:

$$\vartheta_W \begin{bmatrix} R \\ S \end{bmatrix} (T) = \det(CT + D)^{-h/2} \rho^*(CT + D)^{-1} \vartheta_W \begin{bmatrix} R \\ S \end{bmatrix} ((AT + B)(CT + D)^{-1}).$$

Combining this with the remarks at the beginning of the section, we have:

THEOREM 9.12. *The $\vartheta_W \begin{bmatrix} R \\ S \end{bmatrix} (T)$ defined above is a modular form with values in $W^* \otimes L^{h/2}$. For any W and T , it is non-zero for suitable R and S .*

This is the “modular form interpretation” of Theorem 9.5.

We want to give an interesting application that shows the usefulness of these series. The following theorem is a version of results of Freitag and Stillman:

THEOREM 9.13. *For all $g \geq 2$ and $1 \leq r \leq g - 1$, there are congruence subgroups $\Gamma \subset Sp(2g, \mathbf{Z})$ and Γ -invariant non-zero holomorphic k -forms on \mathfrak{H}_g where $k = \frac{g(g+1)}{2} - \frac{r(r+1)}{2}$.*

This is a basic insight into the complex geometry of the Siegel modular varieties \mathfrak{H}_g/Γ : for $r = g - 1, g - 2, \dots$, this means that we get g -forms, $(2g - 1)$ -forms, $(3g - 3)$ -forms, $(4g - 6)$ -forms, \dots on these varieties.

The first step in the proof is to identify holomorphic forms as sections of a homogeneous vector bundle:

LEMMA 9.14. *If V denotes the g -dimensional identity representation of $GL_g(\mathbf{C})$, then there is an $Sp(2g, \mathbf{R})$ -invariant isomorphism:*

$$\text{Symm}^2(\mathbf{V}) \cong \Omega_{\mathfrak{H}_g}^1$$

where Ω^1 is the cotangent bundle or bundle of 1 forms.

PROOF: Use the elementary identity:

$$\begin{aligned} (A(T + \delta T) + B)(C \cdot (T + \delta T) + D)^{-1} = \\ (AT + B)(CT + D)^{-1} + {}^t(C(T + \delta T) + D)^{-1} \cdot \delta T \cdot (CT + D)^{-1}. \end{aligned}$$

This implies that a *tangent* vector to \mathfrak{H}_g , given by a symmetric $g \times g$ matrix X , transforms by the rule:

$$\gamma(X, T) = ({}^t(CT + D)^{-1} \cdot X \cdot (CT + D)^{-1}, (AT + B)(CT + D)^{-1}).$$

Therefore the dual action on cotangent vectors is given by:

$$\gamma(\omega, T) = ((CT + D) \cdot \omega \cdot {}^t(CT + D), (AT + B)(CT + D)^{-1})$$

where ω is also given by a symmetric $g \times g$ matrix, (paired with a vector X via $\text{Tr}(\omega \cdot X)$.) But \mathbf{V} is defined by the action

$$\gamma(x, T) = ((CT + D) \cdot x, (AT + B)(CT + D)^{-1})$$

where x is a column vector. So if we write the elements of $\text{Symm}^2(\mathbf{V})$ as $\sum x_i \otimes {}^t x_i$, then we get the same rule:

$$\gamma(\sum x_i \otimes {}^t x_i, T) = ((CT + D)(\sum x_i \otimes {}^t x_i) \cdot {}^t(CT + D), (AT + B)(CT + D)^{-1}).$$

QED

COROLLARY 9.15. *There is an isomorphism of homogeneous bundles:*

$$\Lambda^k(\text{Symm}^2(V)) \cong \Omega_{\mathfrak{h}_g}^k.$$

Now recall the classification theorem for irreducible representations ρ of $GL_g(\mathbb{C})$: for every sequence $d_1 \geq d_2 \geq \dots \geq d_g$, there is a unique irreducible representation E of $GL_g(\mathbb{C})$ which contains a vector x such that

$$(*) \quad \rho \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1g} \\ & a_{22} & \dots & a_{2g} \\ & & \dots & \\ O & & & a_{gg} \end{pmatrix} x = (a_{11}^{d_1} a_{22}^{d_2} \dots a_{gg}^{d_g}) x$$

and every irreducible representations arises in this way. (d_1, \dots, d_g) is called the highest weight vector of E .

The representation $\Lambda^k(\text{Symm}^2(V))$ of $GL_g(\mathbb{C})$ has a rather complicated decomposition into irreducibles. However, if

$$k = g(g+1)/2 - r(r+1)/2,$$

there is one quite simple piece: namely, if e_1, \dots, e_n are the unit vectors in V , consider

$$x = \bigwedge_{\substack{1 \leq i \leq j \leq g \\ i \leq g-r}} (e_i \circ e_j).$$

Here $e_i \circ e_j \in \text{Symm}^2 V$ and exactly k terms are wedged. It's easy to see that x satisfies $(*)$ with $d_1 = \dots = d_{g-r} = g+1$ and $d_{g-r+1} = \dots = d_g = g-r$. This proves:

LEMMA 9.16. *If $E^{(m,n)}$ is the irreducible representation of $GL_g(\mathbb{C})$ with highest weight $d_1 = \dots = d_m = n$, $d_{m+1} = \dots = d_g = 0$, then there is an embedding of homogeneous bundles:*

$$\mathbf{E}^{(g-r, r+1)} \otimes \mathbf{L}^{g-r} \subset \Omega_{\mathfrak{h}_g}^{\frac{g(g+1)}{2} - \frac{r(r+1)}{2}}.$$

Finally, we need some pluri-harmonic polynomials in order to get global sections of this bundle via theta series. Take Q to be the identity matrix of size $2(g-r) \times 2(g-r)$, so our harmonic polynomials P are functions of a $g \times 2(g-r)$ -matrix X_{ij} . By the examples given in the beginning of the section, all polynomials:

$$P(\dots, X_{i, 2j-1} + \sqrt{-1} X_{i, 2j}, \dots)$$

are pluri-harmonic. In particular, look at

$$P(X) = \left[\det_{\substack{1 \leq j \leq g-r \\ r+1 \leq i \leq g}} (X_{i, 2j-1} + \sqrt{-1} \cdot X_{i, 2j}) \right]^{r+1}.$$

Under the action of $GL_g(\mathbb{C})$, it is easy to see that P satisfies the highest weight condition $(*)$ with $d_1 = \dots = d_r = 0$, $d_{r+1} = \dots = d_g = -(r+1)$. Therefore \mathbf{H} contains the corresponding irreducible representation W of $GL_g(\mathbb{C})$. But W^* has highest weight $d_1 = \dots = d_{g-r} = (r+1)$, $d_{g-r+1} = \dots = d_g = 0$, i.e., $W^* \cong E^{(g-r, r+1)}$. Proposition 9.12 plus Lemma 9.16 therefore imply Theorem 9.13.

VIEWPOINT II: Representation theory

Let V be a real vector space with a non-degenerate alternating form A and let W be a real vector space with a positive definite quadratic form B . Then $V \otimes W$ has a natural non-degenerate alternating form $A \otimes B$: $(A \otimes B)(v \otimes w, v' \otimes w') = A(v, v') \cdot B(w, w')$. If $\varphi \in Sp(V, A)$ and $\sigma \in O(W, B)$, then $\varphi \otimes \sigma \in Sp(V \otimes W, A \otimes B)$. This gives a homomorphism

$$Sp(V, A) \times O(W, B) \longrightarrow Sp(V \otimes W, A \otimes B).$$

Let \mathcal{H}_V and $\mathcal{H}_{V \otimes W}$ be the irreducible unitary representations respectively of $Heis(V)$ and $Heis(V \otimes W)$ such that $U_\lambda = \lambda \cdot Id$, $\forall \lambda \in \mathbb{C}_1^*$. The action of $Mp(V \otimes W, A \otimes B)$ on $\mathcal{H}_{V \otimes W}$ restricts to an action of a double-covering of $Sp(V, A) \times O(W, B)$ on $\mathcal{H}_{V \otimes W}$. In fact, when $h = \dim W$ is

odd, the double covering is almost $Mp(V, A) \times O(W, B)$ but not quite, and it is $\{\pm 1\} \times Sp(V, A) \times O(W, B)$ when h is even. To see this, note:

- (a) Let $V = V_1 \oplus V_2$, with V_1 and V_2 isotropic spaces, then $V \otimes W = (V_1 \otimes W) \oplus (V_2 \otimes W)$ and the $V_i \otimes W$ are clearly isotropic spaces. In the model $\mathcal{H}_{V \otimes W} \cong L^2(V_1 \otimes W)$, for $\sigma \in O(W, B)$, $f \in L^2(V_1 \otimes W)$, $\rho(\sigma)f = f \circ (1_{V_1} \otimes \sigma^{-1})$ defines an action of $O(W, B)$ on $L^2(V_1 \otimes W)$. From Lemma 8.2I, it follows that $\rho(\sigma) \in Mp(V \otimes W, A \otimes B)$ if $\sigma \in SO(W, B)$ or if $\sigma \in O(W, B)$ and $\dim V_1$ is even.

- (b) Now put W in standard form: $W = \mathbf{R}^h$ with $B(x, y) = {}^t x y$. Then

$$\begin{aligned} L^2(V_1 \otimes W) &\cong L^2(V_1 \oplus V_1 \dots \oplus V_1) \\ &\cong L^2(V_1) \hat{\otimes} L^2(V_1) \dots \hat{\otimes} L^2(V_1). \end{aligned}$$

For $\tilde{\gamma} \in Mp(V_1, A)$, define $\rho(\tilde{\gamma}) = \tilde{\gamma} \otimes \tilde{\gamma} \otimes \dots \otimes \tilde{\gamma}$. This defines $\rho : Mp(V, A) \longrightarrow Mp(V \otimes W, A \otimes B)$.

In any case, (a) and (b) combine to give

$$\rho : Mp(V, A) \times O(W, B) \longrightarrow \widetilde{Mp}(V \otimes W, A \otimes B) \hookrightarrow \text{Aut}(\mathcal{H}_{V \otimes W}).$$

The complete decomposition of $\mathcal{H}_{V \otimes W}$ under the action of the group $Mp(V, A) \times O(W, B)$ is studied in Kashiwara and Vergne, *On the Segal-Shale-Weil representation and harmonic polynomials*, Inv. Math., Vol. 44, 1978.

Let $2g = \dim V$ and $h = \dim W$. Their main result is that we have two decompositions. Firstly, the space of pluri-harmonic polynomials decomposes:

$$(9.17) \quad \mathbf{H} = \bigoplus_{\alpha} (E_{\alpha} \otimes F_{\alpha})$$

where E_{α} are distinct irreducible $GL_g(\mathbf{C})$ -modules and F_{α} are distinct irreducible $O(W, B)$ -modules, while

$$(9.18) \quad \mathcal{H}_{V \otimes W} = \widehat{\bigoplus_{\alpha} [\Gamma(\mathfrak{H}_g, \mathbf{E}^{(\alpha)} \otimes \mathbf{L}^{h/2}) \otimes F_{\alpha}]}$$

(here Γ refers to the space of L^2 -sections of the homogeneous bundle as defined in Viewpoint I, considered as a representation of $Mp(V, A)$.) They also give a detailed algorithm for describing which representations of GL_g and of $O(W, B)$ are paired in this sum.

We do not discuss all this but go just far enough to write down the functional equations for $\vartheta^{P, Q} \begin{bmatrix} R \\ S \end{bmatrix}$. This will involve the transformation laws for $P(Y) \exp \pi i {}^t Y T Y Q$ and the invariance of $e \begin{bmatrix} R \\ S \end{bmatrix} \in \mathcal{H}_{V \otimes W}^{-\infty}$.

As we have seen in §8, the functional equation for ϑ comes from

- (a) defining ϑ as a matrix coefficient of $f_T = e^{\pi i {}^t y {}^t T y}$ with respect to $e \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and
- (b) studying the action of Sp (or rather Mp) on f_T and $e \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We proceed in exactly that fashion here. Of course we do not want to determine how all of $Mp(V \otimes W)$ acts, rather just how $Mp(V)$ acts on $P(Y) f_T(Y)$, where T^* corresponds to $T \otimes Q$ as in §6 (note that there is an embedding $i : \mathfrak{H}_V \longrightarrow \mathfrak{H}_{V \otimes W}$ sending T to T^*). We shall now proceed to work everything out in detail in matrix notation.

I. Let $V = \mathbf{R}^g \oplus \mathbf{R}^g$ and $W = \mathbf{R}^h$, the members of V and W being thought of as columns and rows respectively. Then $V \otimes W$ is naturally identified with $\mathbf{R}(g, h) \oplus \mathbf{R}(g, h)$ where $\mathbf{R}(g, h)$ is the space of all $(g \times h)$ -matrices with real entries. Let $A(x, y) = {}^t x_1 y_2 - {}^t x_2 y_1$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and $x_1, x_2, y_1, y_2 \in \mathbf{R}^g$. Let $B(x, y) = x \cdot Q^{-1} \cdot {}^t y$ where Q is a positive definite rational symmetric matrix. Then $(A \otimes B)(X', Y') = \text{Tr}({}^t X'_1 Y'_2 - {}^t X'_2 Y'_1) \cdot Q^{-1}$, where $X' = (X'_1, X'_2)$ and $Y' = (Y'_1, Y'_2)$ and $X'_1, Y'_1, X'_2, Y'_2 \in \mathbf{R}(g, h)$. However we want $A \otimes B$ in standard form, so it is prudent to put $(X'_1, X'_2) = (X_1 Q, X_2)$ so that the pairing is

$$(A \otimes B)(X', Y') = \tilde{A}(X, Y) = \text{Tr}({}^t X_1 Y_2 - {}^t X_2 Y_1).$$

II. Let $Sp(2(g, h); \mathbf{R})$ be the group of all linear transformations from the space $\mathbf{R}(g, h) \oplus \mathbf{R}(g, h)$ to itself that preserve the pairing $\tilde{A}(X, Y) =$

$Tr({}^tX_1Y_2 - {}^tX_2Y_1)$. We write down $j : Sp(2g, \mathbf{R}) \longrightarrow Sp(2(g, h); \mathbf{R})$. If $\psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbf{R})$, then $j(\psi)(X'_1, X'_2) = (AX'_1 + BX'_2, CX'_1 + DX'_2)$. Changing coordinates by $(X'_1, X'_2) = (X_1Q, X_2)$, we get:

$$j(\psi)(X_1, X_2) = (AX_1 + BX_2Q^{-1}, CX_1Q + DX_2).$$

III. We have

$$\begin{array}{ccc} Mp(2g, \mathbf{R}) & \xrightarrow{\rho} & Mp(2(g, h); \mathbf{R}) \\ \downarrow & & \downarrow \\ Sp(2g, \mathbf{R}) & \xrightarrow{j} & Sp(2(g, h); \mathbf{R}). \end{array}$$

We compute ρ explicitly on a set of generators for the model of the representation $L^2(\mathbf{R}(g, h)) \cong \mathcal{H}_{V \otimes W}$. For $\gamma \in Sp(2g, \mathbf{R})$ and $\tilde{\gamma} \in Mp(2g, \mathbf{R})$ over it:

(a) If $\gamma = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$,

$$(\rho(\tilde{\gamma})f)(Y) = f(A^{-1}Y)(\det A)^{-h/2}, \text{ where } C(\tilde{\gamma}; T) = (\det A)^{-1/2}.$$

(b) If $\gamma = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$ with C symmetric and $C(\tilde{\gamma}; T) = 1$, then

$$\rho(\tilde{\gamma})f(Y) = f(Y) \exp \pi i Tr({}^tYCYQ).$$

(c) $\gamma = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, then $j(\gamma)(X_1, X_2) = (-X_2Q^{-1}, X_1Q)$, hence

$$\rho(\tilde{\gamma})f(Y) = \hat{f}(YQ) \cdot i^{-h/2} \text{ where } i^{-1/2} = C(\tilde{\gamma}; iI).$$

(d) If $\tilde{\gamma} = -1, \gamma = Id. \in Sp(2g, \mathbf{R})$, then

$$\rho(\tilde{\gamma}) = (-1)^h.$$

These follow immediately from Lemma 8.2 and the formula in II above.

IV. We discuss $i : \mathfrak{H}_g \longrightarrow \mathfrak{H}_{(g, h)}$. Let $T \in \mathfrak{H}_g$. The complex structure on $\mathbf{R}^g \oplus \mathbf{R}^g$ is given by $h : \mathbf{R}^g \oplus \mathbf{R}^g \longrightarrow \mathbf{C}^g$, $h(x_1, x_2) = Tx_1 + x_2$. It follows that the complex structure on $\mathbf{R}(g, h) \oplus \mathbf{R}(g, h)$ is given by $(X'_1, X'_2) \longmapsto TX'_1 + X'_2$. Putting $(X'_1, X'_2) = (X_1Q, X_2)$, the isomorphism $\mathbf{R}(g, h) \oplus$

$\mathbf{R}(g, h) \longrightarrow \mathbf{C}(g, h)$ given by $(X_1, X_2) \longmapsto TX_1Q + X_2$ gives the complex structure. This corresponds to $T \longmapsto T^*$ of §6. Thus

$$f_{i(T)}(Y) = \exp \pi i Tr({}^tYTYQ) \in L^2(\mathbf{R}(g, h)).$$

V. We now come to the main calculation: the transformation laws for $f_{i(T)} \in L^2(\mathbf{R}(g, h))$, and more generally, of $P(Y)f_{i(T)}(Y) \in L^2(\mathbf{R}(g, h))$ where P is pluri-harmonic with respect to Q , under the action of the metaplectic group $Mp(2g, \mathbf{R})$. Put $f(Y) = P(Y)f_{i(T)}(Y)$.

(a) If $\gamma = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$, then apply III(a) from above:

$$\begin{aligned} \rho(\tilde{\gamma})f(Y) &= (\det A)^{-h/2} f(A^{-1}Y) \\ &= (\det A)^{-h/2} P(A^{-1}Y) \exp \pi i Tr({}^tY{}^tA^{-1}TA^{-1}YQ) \\ &= (\det A)^{-h/2} P(A^{-1}Y) f_{i_{\gamma(T)}}(Y). \end{aligned}$$

(b) If $\gamma = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$ and $C(\tilde{\gamma}; T) = 1$, then

$$\begin{aligned} \rho(\tilde{\gamma})f(Y) &= P(Y)f_{i(T)}(Y) \cdot \exp(-\pi i Tr({}^tYCYQ)) \\ &= P(Y)f_{i(T-C)}(Y) = P(Y)f_{i_{\gamma(T)}}(Y). \end{aligned}$$

(c) If $\gamma = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, then we have defined $\rho(\tilde{\gamma})f(Y) = i^{-h/2} \hat{f}(YQ)$.

We compute this. For all $T \in \mathfrak{H}_g$, $\hat{f}_T(y) = \det(\frac{T}{i})^{-1/2} \cdot f_{-T^{-1}}(y)$. This holds for any g , and in particular,

$$\begin{aligned} \hat{f}_{i(T)}(Y) &= \det(\frac{T \otimes Q}{i})^{-1/2} \exp \pi i Tr({}^tY(-T^{-1})YQ^{-1}) \\ &= \det(\frac{T}{i})^{-h/2} \cdot \det(Q)^{-g/2} \cdot \exp \pi i Tr({}^tY(-T^{-1})YQ^{-1}). \end{aligned}$$

Assume now that P is a homogeneous pluri-harmonic polynomial of degree k . Since $f(Y) = P(Y)f_{i(T)}(Y) = (2\pi i)^{-k} P(2\pi i Y)f_{i(T)}(Y)$, and since the Fourier transform interchanges differentiation and multiplication by the variable, it follows that

$$\hat{f}(Y) = (2\pi i)^{-k} P(\partial) \hat{f}_{i(T)}(Y)$$

$$\begin{aligned}
&= (2\pi i)^{-k} P(\partial) \exp \pi i \operatorname{Tr}({}^t Y (-T^{-1}) Y Q^{-1}) \det\left(\frac{T}{i}\right)^{-h/2} \cdot \det(Q)^{-g/2} \\
&= P(-T^{-1} Y Q^{-1}) \exp \pi i \operatorname{Tr}({}^t Y (-T^{-1}) Y Q^{-1}) \det\left(\frac{T}{i}\right)^{-h/2} \cdot \det(Q)^{-g/2}
\end{aligned}$$

from Corollary 9.9. Therefore

$$\begin{aligned}
\rho(P)f(Y) &= i^{-h/2} \hat{f}(YQ) \\
&= i^{-h/2} P(-T^{-1} Y) \exp(\pi i \operatorname{Tr}({}^t Q^t Y (-T^{-1}) Y) \det\left(\frac{T}{i}\right)^{-h/2} \cdot \det(Q)^{-g/2} \\
&= f_{i\gamma(T)} P(-T^{-1} Y) \det(-T)^{-\frac{h}{2}}.
\end{aligned}$$

Finally,

$$(d) \text{ If } \tilde{\gamma} = -1, \gamma = Id. \in Sp(2g, \mathbf{R}), \rho(\tilde{\gamma})f_{i(T)} = (-1)^h f_{i(T)}.$$

Using exactly the same method as in Theorem 8.3, we deduce:

PROPOSITION (9.19). For all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbf{R}), T \in \mathfrak{H}_g$ and P pluri-harmonic,

$$\rho(\tilde{\gamma})f_{i(T)}(Y)P(Y) = f_{i\gamma(T)}(Y)P((-BT + A)^{-1}Y) \det(-BT + A)^{-h/2}.$$

VI. We now investigate how Mp acts on $e \begin{bmatrix} R \\ S \end{bmatrix} \in \mathcal{H}_{V \otimes W}^{-\infty}$ where

$R, S \in \mathbf{Q}(g, h)$. Now $e \begin{bmatrix} R \\ S \end{bmatrix}$ is defined as a distribution by:

$$\langle f, e \begin{bmatrix} R \\ S \end{bmatrix} \rangle = \sum_{N \in \mathbf{Q}(g, h)} \chi \begin{bmatrix} R \\ S \end{bmatrix} (N) f(N)$$

where

$$\begin{aligned}
\chi \begin{bmatrix} R \\ S \end{bmatrix} (N) &= 0 \quad \text{if } N - R \notin \mathbf{Z}(g, h) \\
&= \exp 2\pi i \operatorname{Tr}^t NS \quad \text{otherwise.}
\end{aligned}$$

We have:

$$\begin{aligned}
\langle f, e \begin{bmatrix} R \\ S \end{bmatrix} \rangle &= \sum_{N \in \mathbf{Z}(g, h)} \exp 2\pi i \operatorname{Tr}^t (R + N) S \cdot f(R + N) \\
&= \sum_{N \in \mathbf{Z}(g, h)} (U_{1,0,S} f)(R + N) \\
&= \sum_{N \in \mathbf{Z}(g, h)} (U_{1,R,0} U_{1,0,S} f)(N) \\
&= \langle U_{1,R,0} U_{1,0,S} f, e \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rangle \\
&= \langle f, U_{1,0,-S} U_{1,-R,0} e \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rangle.
\end{aligned}$$

As usual, $e \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ will be denoted by $e_{\mathbf{Z}}$.

We shall show that the $\hat{\gamma} \in Mp(2(g, h); \mathbf{R})$ that satisfy $\hat{\gamma} e \begin{bmatrix} R \\ S \end{bmatrix} = e \begin{bmatrix} R \\ S \end{bmatrix}$ form a group G so that $q(G) \subset Sp(2(g, h); \mathbf{R})$ contains a congruence subgroup. Let

$$\Delta = \{ \hat{\gamma} \in Mp(2(g, h); \mathbf{R}) \mid \hat{\gamma} e_{\mathbf{Z}} = e_{\mathbf{Z}} \}.$$

By 8.10, $q(\Delta) \supseteq$ the principal congruence subgroup of level 4. Let

$$\Delta' = \{ \hat{\gamma} \in \Delta \mid q(\hat{\gamma}) \equiv 1 \pmod{2k^2} \}$$

where k is chosen so that kR and kS are integral. Then, for

$\tilde{\gamma} \in \Delta', q(\tilde{\gamma}) = \gamma \in Sp(2(g, h); \mathbf{Z})$,

$$\begin{aligned}
\tilde{\gamma} e \begin{bmatrix} R \\ S \end{bmatrix} &= \tilde{\gamma} U_{(1,0,-S)} U_{(1,-R,0)} e_{\mathbf{Z}} \\
&= U_{(1,\gamma(0,-S))} U_{(1,\gamma(-R,0))} \tilde{\gamma} e_{\mathbf{Z}} \\
&= U_{(1,2kZ,2kW-S)} U_{(1,2kZ'-R,2kW')} e_{\mathbf{Z}} \\
&= U_{(1,0,-S)} U_{(1,-R,0)} e_{\mathbf{Z}} \\
&= e \begin{bmatrix} R \\ S \end{bmatrix}.
\end{aligned}$$

From the explicit formula for $Sp(2g, \mathbf{R}) \rightarrow Sp(2(g, h); \mathbf{R})$, we see finally that

$$q \left\{ \tilde{\gamma} \in Mp(\mathbf{R}) \mid \rho(\tilde{\gamma}) e \begin{bmatrix} R \\ S \end{bmatrix} = e \begin{bmatrix} R \\ S \end{bmatrix} \right\}$$

contains $\{ \gamma \in Sp(2g, \mathbf{Z}) \mid \gamma \equiv 1 \pmod{2k^2 n} \}$ where n is chosen so that nQ and nQ^{-1} are integral.

VII. The functional equations now follow easily: the relation

$$\vartheta^{P,Q} \begin{bmatrix} R \\ S \end{bmatrix} (T) = \langle P(Y) f_{i(T)}(Y), e \begin{bmatrix} R \\ S \end{bmatrix} \rangle$$

combined with V and VI above allows us immediately to deduce Theorem 9.5.

Since

$$\langle U_{1,X_1,X_2} f_{i(T)}(Y), e \begin{bmatrix} R \\ S \end{bmatrix} \rangle = \exp \pi i T r^t X_1 Z \cdot \vartheta^Q \begin{bmatrix} R \\ S \end{bmatrix} (Z, T)$$

where $Z = T X_1 Q + X_2$, using the results of V and VI we can deduce Theorem 9.4 exactly as in 8.8.

The idea of treating theta functions as matrix coefficients is due to Weil in his classic paper, *Sur certains groupes d'opérateurs unitaires*, Acta Math., vol. 111, 1964. From this point of view, it is natural to view the whole construction of theta functions as embodied in a basic map, which is often called the *Weil map*. Abstractly, the situation is as follows: suppose

$$G = \text{any algebraic group defined over } \mathbb{Q},$$

and suppose that one comes up somehow with a representation

$$r : G_{\mathbb{R}} \longrightarrow \text{Aut}(\mathcal{H})$$

and a vector

$$e \in \mathcal{H}_{-\infty} \text{ such that}$$

$$\gamma(e) = e, \quad \text{all } \gamma \in \text{some arithmetic subgroup } \Gamma \subset G_{\mathbb{Q}}.$$

Then define the Weil map:

$$w : \mathcal{H}_{\infty} \longrightarrow \mathbb{C}(\Gamma \backslash G_{\mathbb{R}})$$

by

$$w(f)(\gamma) = \langle r(\gamma)f, e \rangle,$$

In other words, one gets automorphic functions of some kind from any such representation r and vector e . Moreover, if

$$\mathcal{H}_f = \{e' | r(\gamma)e' = e' \text{ for all } \gamma \text{ in some congruence subgroup}\},$$

then we can extend w to:

$$w : \mathcal{H}_{\infty} \otimes \mathcal{H}_f \longrightarrow \bigcup_{\substack{\text{congruence subgroups} \\ \Gamma' \subset \Gamma}} \mathbb{C}(\Gamma' \backslash G_{\mathbb{R}})$$

$$w(f \otimes e')(\gamma) = (r(\gamma)f, e').$$

This is almost exactly what theta functions do for us: let $G = Sp(2g)$, $\mathcal{H} = L^2(\mathbb{R}^{(g,h)})$ and let ρ be the 2-valued representation:

$$\rho : Sp(2g, \mathbb{R}) \longrightarrow \text{Aut}(\mathcal{H})$$

(single-valued if h is even). Let $e_{\mathbf{z}} = e \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then for some congruence subgroup $\Gamma \subset Sp(2g, \mathbb{R})$, lifted to Mp , we get

$$w : S(\mathbb{R}^{(g,h)}) \longrightarrow \mathbb{C}(\Gamma \backslash Mp(2g, \mathbb{R}))$$

$$w(f)(\gamma) = \langle \rho(\gamma)f, e_{\mathbf{z}} \rangle$$

and using the $e \begin{bmatrix} R \\ S \end{bmatrix}$'s, we can extend this map to:

$$(9.19) \quad w : S(\mathbb{R}^{(g,h)}) \otimes S(A_f^{(g,h)}) \longrightarrow \bigcup_{\Gamma} \mathbb{C}(\Gamma \backslash Mp(2g, \mathbb{R})).$$

This can be reformulated adelically if one wants. In order to get classical modular forms from this map, we need only to specialize to particular elements $f \in S(\mathbb{R}^{(g,h)})$ and then combine the resulting functions of γ with elementary factors (to make them right invariant by the maximal compact subgroup $U(g)$ instead of left-invariant by Γ): start with

$$F_{P_{\alpha}, Q}(Z, T) = P_{\alpha}(Z) \cdot e^{\pi i T r({}^t Z \cdot T Z Q)}$$

where the P_{α} are pluriharmonic and transform under GL_g by the representation τ , i.e., $P_{\alpha}(AZ) = \Sigma \tau_{\alpha\beta}(A) \cdot P_{\beta}(Z)$. Let $f_{P_{\alpha}, Q}(Z) = F_{P_{\alpha}, Q}(Z, iI_g)$.

Then

$$\begin{aligned} w(f_{P_{\alpha}, Q} \otimes e \begin{bmatrix} R \\ S \end{bmatrix})(\tilde{\gamma}) &= \langle \rho(\tilde{\gamma})f_{P_{\alpha}, Q}, e \begin{bmatrix} R \\ S \end{bmatrix} \rangle \\ &= \det(A - iB)^{-h/2} \sum_{\beta} \tau_{\alpha\beta}((A - iB)^{-1})(F_{P_{\beta}, Q}(\cdot, \gamma(iI_g)), e \begin{bmatrix} R \\ S \end{bmatrix}), \end{aligned}$$

which we may invert to show:

$$\begin{aligned}
& \det(A - iB)^{h/2} \sum_{\beta} \tau_{\alpha\beta} (A - iB) w(f_{P_{\beta}, Q} \otimes e \begin{bmatrix} R \\ S \end{bmatrix}) (\tilde{\gamma}) \\
&= \langle F_{P_{\alpha}, Q}(\cdot, \gamma(iI_g)), e \begin{bmatrix} R \\ S \end{bmatrix} \rangle \\
&= \vartheta^{P_{\alpha}, Q} \begin{bmatrix} R \\ S \end{bmatrix} (\gamma(iI_g)), \quad \text{a modular form on } \mathfrak{H}_g/K.
\end{aligned}$$

I don't want to pursue the applications of these ideas any further, but I would like to make a few conjectures which highlight exactly how little beyond the definitions we know. There is a vast unknown area concerning the image and kernel of the Weil map. Let's specialize to the very simplest case $g = 1$, $Q = I_h$, $P = 1$, so $f = e^{-\pi(z_1^2 + \dots + z_h^2)}$. Then

$$(a - ib)^{h/2} \cdot w(f \otimes e \begin{bmatrix} r \\ s \end{bmatrix}) \tilde{\gamma} = \vartheta^{1, I_h} \begin{bmatrix} r \\ s \end{bmatrix} (\gamma(i)).$$

Fixing f , but letting r and s vary over h -tuples of rationals, this extends to a map

$$(a - ib)^{h/2} \cdot w : \mathcal{S}(A_f^h) \longrightarrow \bigcup_k \left\{ \begin{array}{l} \text{modular forms on } \mathfrak{H} \text{ of wt.} \\ h/2, \text{ w.r.t. congruence} \\ \text{subgroup of level } k \text{ of} \\ SL(2, \mathbb{Z}) \end{array} \right\}.$$

Explicitly, this is the map

$$\chi \longmapsto \sum_{n_1, \dots, n_h \in \mathbb{Q}} \chi(\vec{n}) e^{\pi i (\Sigma n_i^2) \tau}$$

where χ is supported on $\frac{1}{m} \mathbb{Z}^h$ and is constant modulo $m \mathbb{Z}^h$, for some integer m . Here are two questions:

QUESTION 1: Does the range of this map include all cuspidal modular forms if h even, $h > 4$?

QUESTION 2: Is the kernel of this map spanned by the obvious relations $\chi \circ A - \chi$, $A \in O(h, \mathbb{Q})$?

To go a little further, the question arises whether pluriharmonic P 's actually add new scalar modular forms, or just vector-valued ones. To get scalar modular forms, we need pluriharmonic P 's such that:

$$P(AZ) = \det(A)^t \cdot P(Z)$$

for some Z and all $A \in GL(g)$. The first case is $h = g$, $P(Z) = \det Z$. Then we get the family of power series (for $\chi_1 \in \mathcal{S}(A_f^h)$):

$$f_1(T) = \sum_{N \in \mathbb{Q}^{(g, g)}} \chi_1(N) \cdot \det(N) \cdot e^{\pi i \text{Tr}(^t N \cdot T \cdot N)}$$

which are modular forms for the homogeneous line bundle defined by the representation $(\det)^{1+(g/2)}$, i.e., $L^{(g+2)/2}$. On the other hand, multiplying $g + 2$ ordinary theta series, we get the modular forms:

$$f_2(T) = \sum_{N \in \mathbb{Q}^{(g, g+2)}} \chi_2(N) \cdot e^{\pi i \text{Tr}(^t N \cdot T \cdot N)}$$

which are also sections of $L^{(g+2)/2}$.

QUESTION 3: For which χ_1 and χ_2 are f_1 and f_2 equal?

This has a long history: for $g = 1$, a particular case is the famous identity of Jacobi

$$\begin{aligned}
& \sum (-1)^n (2n+1) e^{\pi i (n+1/2)^2 \tau} \\
&= \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot e^{\pi i n^2 \tau} \cdot \sum_{n \in \mathbb{Z}} e^{\pi i (n+1/2)^2 \tau}
\end{aligned}$$

proven in Ch. 1. For $g = 2, 3$ and 4, Riemann and Frobenius found generalizations of this identity (some proofs were discovered in his unpublished papers by Edwards!). Fay (*On the Riemann-Jacobi Formula*, Nachr. Akad. Wiss. Göttingen, 1979) found a generalization to $g = 5$ analyzed the situation for higher g and found that not all forms of type f_1 were equal to some f_2 .

VIEWPOINT III: The algebraic version

A purely algebraic method of defining the analytic modular forms $\vartheta^{P,Q} \begin{bmatrix} A \\ B \end{bmatrix} (T)$, for pluri-harmonic P and positive definite rational Q can be found by following the beautiful ideas of I. Barsotti, contained in his paper, *Considerazioni sulle funzioni theta*, Symp. Math. **3** (1970), p. 247. In characteristic p , the story is more complex and has its own twists – as does any construction involving differentiating – so here we develop the method only for characteristic 0. Barsotti's theory has been developed in many interesting ways by Cristante, to whose papers we refer the reader.

First, recall the basic result:

THEOREM 9.20. *Let X be an abelian variety and let $p_{i_1 \dots i_k} : X^n \rightarrow X$ denote the map $(x_1, \dots, x_n) \rightarrow x_{i_1} + \dots + x_{i_k}$. Let \mathbf{L} be any line bundle on X . Then there is a canonical isomorphism ψ between the line bundle on $X \times X \times X$:*

$$\tilde{\mathbf{L}} = p_{123}^* \mathbf{L} \otimes p_{12}^* \mathbf{L}^{-1} \otimes p_{13}^* \mathbf{L}^{-1} \otimes p_{23}^* \mathbf{L}^{-1} \otimes p_1^* \mathbf{L} \otimes p_2^* \mathbf{L} \otimes p_3^* \mathbf{L} \otimes \mathbf{L}(0)^{-1}$$

and the trivial line bundle.

This is an immediate consequence of the theorem of the cube: if we restrict $\tilde{\mathbf{L}}$ to $X \times X \times (0)$, we get

$$p_{12}^* \mathbf{L} \otimes p_{12}^* \mathbf{L}^{-1} \otimes p_1^* \mathbf{L}^{-1} \otimes p_2^* \mathbf{L}^{-1} \otimes p_1^* \mathbf{L} \otimes p_2^* \mathbf{L} \otimes \mathbf{L}(0) \otimes \mathbf{L}(0)^{-1}$$

which is trivial. The same holds on $X \times (0) \times X$ and $(0) \times X \times X$, so $\tilde{\mathbf{L}}$ is trivial. Moreover, at $(0,0,0)$, $\tilde{\mathbf{L}}$ is defined as the tensor product $\mathbf{L}(0)^4 \otimes \mathbf{L}(0)^{-4}$, i.e., it is canonically trivial, so the “trivialization” of $\tilde{\mathbf{L}}$ is canonical too.

Barsotti's result concerns trivializing an arbitrary line bundle \mathbf{L} over an abelian variety X in a neighborhood of $O \in X$ in the sense of *formal power series*. Over an arbitrary field k , since X is smooth at O , we can start with arbitrary functions x_1, \dots, x_g in the local ring $\mathcal{O}_{O,X}$ of rational

functions regular at O . Then if $x_i(O) = 0$ and the differentials dx_1, \dots, dx_g are independent at O (algebraically, this means $x_i \in \mathcal{M}_{O,X}$ and $\bar{x}_i \in \mathcal{M}_{O,X}/\mathcal{M}_{O,X}^2$ are independent over k), an arbitrary function $f \in \mathcal{O}_{O,X}$ can be expanded as a formal power series in x_1, \dots, x_g . The algebraic way of saying this is that the completion $\hat{\mathcal{O}}_{O,X}$ of $\mathcal{O}_{O,X}$ is isomorphic to the ring $k[[x_1, \dots, x_g]]$ of formal power series in x_1, \dots, x_g .

When X is abelian and $\text{char } k = 0$, there are power series coordinates $t_1, \dots, t_g \in \hat{\mathcal{O}}_{O,X}$ in terms of which the group law on X is just addition, i.e.,

$$\hat{\mathcal{O}}_{O,X} \cong k[[t_1, \dots, t_g]]$$

and

$$t_i(P+Q) = t_i(P) + t_i(Q)$$

(for any R -valued points P, Q of X factoring through $\text{Spec}(\hat{\mathcal{O}}_{O,X})$). In other words, these t_i play the role algebraically of the *linear* coordinates t_1, \dots, t_g on the universal cover \mathbf{C}^g of X , when $k = \mathbf{C}$. We won't develop this theory here at any length, except to show how the t_i are constructed: starting with any local coordinates x_1, \dots, x_g in $\mathcal{O}_{O,X}$, the dx_i span the cotangent space to X at O , and their duals $\partial/\partial x_i$ span the tangent space to X at O . Translating by the group law, $\partial/\partial x_i$ extends to an invariant vector field D_i on X . Algebraically, D_i is a derivation from \mathcal{O}_X to \mathcal{O}_X . We form the expression

$$\psi = \sum_{k=0}^{\infty} (t_1 D_1 + \dots + t_g D_g)^k / k!$$

which defines a ring homomorphism from $\hat{\mathcal{O}}_{O,X}$ to $\hat{\mathcal{O}}_{O,X}[[t_1, \dots, t_g]]$, hence to $k[[t_1, \dots, t_g]]$ (by evaluating functions on X at O). This is an isomorphism of $\hat{\mathcal{O}}_{O,X}$ with $k[[t_1, \dots, t_g]]$ and the inverse images of the t_i are the desired additive coordinates. To see what we're doing, recall the dictionary:

vector field X	\longleftrightarrow	derivation D of a ring
1-parameter group of automorphisms generated by X	\longleftrightarrow	automorphisms e^{tD} of a ring.

Thus $\Sigma t_i D_i$ are the invariant vector fields on X and ψ is dual to the map

$$\mathbb{C}^g \times X \longrightarrow X$$

obtained by integrating simultaneously the commuting vector fields D_i .

Thus

$$\mathbb{C}^g \times (0) \hookrightarrow \mathbb{C}^g \times X \longrightarrow X$$

gives additive coordinates, and this is dual to (eval. at O) $\circ \psi$.

Now fix such additive coordinates $t_1, \dots, t_g \in \widehat{\mathcal{O}}_{O,X}$.

Barsotti's basic result is the following:

THEOREM 9.21. *Suppose X is an abelian variety in char. 0 and \mathbb{L} is any line bundle on it. Let $\widehat{\mathcal{O}}_{O,X}$ be the completion of the local ring of X at O . Then there is an isomorphism*

$$\phi : \mathbb{L} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{O,X} \xrightarrow{\sim} \widehat{\mathcal{O}}_{O,X}$$

such that the induced isomorphism

$$\tilde{\phi} : \tilde{\mathbb{L}} \otimes \widehat{\mathcal{O}}_{O,X \times X \times X} \xrightarrow{\sim} \widehat{\mathcal{O}}_{O,X \times X \times X}$$

is the completion of the canonical trivialization ψ of Theorem 9.20. Moreover, any two such ϕ 's differ by:

$$\phi_1 = c \cdot e^{Q(t)} \cdot \phi_2$$

where Q is a homogeneous quadratic polynomial in additive coordinates t_1, \dots, t_g at $O \in X$.

We can re-phrase Barsotti's theorem in terms of formal power series as follows. Let

$$\phi : \mathbb{L} \otimes \widehat{\mathcal{O}}_{O,X} \longrightarrow \widehat{\mathcal{O}}_{O,X}$$

be any formal trivialization of \mathbb{L} at O . Then using ϕ we obtain trivializations of all bundles $p_{i_1, \dots, i_k}^* \mathbb{L}$ at $(0, 0, 0)$, and hence a trivialization

$$\tilde{\phi} : \tilde{\mathbb{L}} \otimes \widehat{\mathcal{O}}_{O, X \times X \times X} \xrightarrow{\sim} \widehat{\mathcal{O}}_{O, X \times X \times X}.$$

Our aim is to choose ϕ so that $\tilde{\phi}$ equals the canonical trivialization ψ . Denote the power series $\psi \circ \tilde{\phi}^{-1}$ by

$$f(x, y, z) \in \widehat{\mathcal{O}}_{O, X \times X \times X}.$$

What we must show is that if ϕ is modified by multiplication by some $g(x) \in \widehat{\mathcal{O}}_{O,X}$, $g(0) \neq 0$, then f becomes 1. But changing ϕ by $g(x)$ changes $p_{123}^* \phi$ by $g(x+y+z)$, $p_{12}^* \phi$ by $g(x+y)$, etc., so it changes the trivialization of $\tilde{\mathbb{L}}$ by

$$\frac{g(x+y+z)g(x)g(y)g(z)}{g(x+y)g(x+z)g(y+z)g(0)}.$$

We need to show that for suitable g :

$$f(x, y, z) = \frac{g(x+y+z)g(x)g(y)g(z)}{g(x+y)g(x+z)g(y+z)g(0)}.$$

Obviously we need some identities on f to do so. What can we say about f ? We claim:

$$(a) \quad f(x, y, 0) = 1$$

$$(b) \quad f(x, y, z) \text{ is symmetric in } x, y, z$$

$$(c) \quad f(x, y, u+v) \cdot f(x, u, v) = f(x, y+u, v) \cdot f(x, y, u)$$

(a) follows because the factors in $\tilde{\mathbb{L}}|_{X \times X \times (0)}$ all cancel out, so both the restriction of ψ and trivialization induced by ϕ are canonical, hence equal.

(b) follows similarly because both the canonical trivialization ψ and that induced by ϕ are invariant under permuting the factors of $X \times X \times X$.

To prove (c), we consider $p_{1234}^* \mathbb{L}$ on $X \times X \times X \times X$. Now p_{i_1, \dots, i_k} denotes the appropriate projection from $X \times X \times X \times X$. Let's abbreviate $p_{i_1, \dots, i_k}^* \mathbb{L}$ to $\mathbb{L}_{i_1, \dots, i_k}$. Then we will construct a diagram of bundles on $X \times X \times X \times X$:

$$\begin{array}{ccc}
L_{1234} \otimes L_{234}^{-1} \otimes L_{12}^{-1} \otimes L_{13}^{-1} \otimes L_{14}^{-1} \otimes L_1^2 \otimes L_2 \otimes L_3 \otimes L_4 \otimes L(0)^{-1} & & \\
a \swarrow & & \searrow b \\
L_{134} \otimes L_{34}^{-1} \otimes L_{13}^{-1} \otimes L_{14}^{-1} \otimes L_1 \otimes L_3 \otimes L_4 & L_{123} \otimes L_{23}^{-1} \otimes L_{12}^{-1} \otimes L_{13}^{-1} \otimes L_1 \otimes L_2 \otimes L_3 & \\
c \searrow & & \swarrow d \\
\mathcal{O}_{X \times X \times X \times X} \otimes L(0) & &
\end{array}$$

Here d is just the trivialization ψ of 9.20, and c is the analogous trivialization on the 1st, 3rd and 4th factors, i.e.,

$$\text{map } c = (p_1, p_3, p_4)^* \psi$$

$$\text{map } d = (p_1, p_2, p_3)^* \psi.$$

But consider the morphism

$$(p_1, p_2, p_{34}) : X \times X \times X \times X \longrightarrow X \times X \times X$$

(i.e., $(x, y, u, v) \longrightarrow (x, y, u + v)$). Then $(p_1, p_2, p_{34})^* \psi$ is a trivialization of the bundle:

$$L_{1234} \otimes L_{12}^{-1} \otimes L_{134}^{-1} \otimes L_{234}^{-1} \otimes L_1 \otimes L_2 \otimes L_{34} \otimes L(0)^{-1}.$$

Thus, checking that the factors cancel appropriately, we see that

$$\text{map } a = (p_1, p_2, p_{34})^* \psi.$$

Likewise:

$$\text{map } b = (p_1, p_{23}, p_4)^* \psi.$$

Now the diagram must commute, because it certainly commutes up to an automorphism of $\mathcal{O}_{X \times X \times X \times X}$, i.e., up to a scalar, and, at 0, all 4 bundles have fibre $L(0)$ and the maps are the identity. Finally, the trivialization ϕ of $L \otimes \hat{\mathcal{O}}_{\mathcal{O}, X}$ also gives us a commutative diagram, and comparing the two diagrams, we get the identity

$$f \circ (p_1, p_2, p_{34}) \cdot f \circ (p_1, p_3, p_4) = f \circ (p_1, p_{23}, p_4) \cdot f \circ (p_1, p_2, p_3)$$

which is (c).

This is the first step of the proof. The second is the lemma:

LEMMA 9.22. Let x, y, z stand for g -tuples of variables (x_1, \dots, x_g) , etc. Then for all power series $f(x, y)$ over a field k of char. 0, if f satisfies:

$$a) f(0, 0) = 0$$

$$b) f(x, y + z) + f(y, z) \equiv f(x + y, z) + f(x, y),$$

then f has the form:

$${}^t x \cdot A \cdot y + g(x + y) - g(x) - g(y)$$

for some skew-symmetric $g \times g$ matrix A and some power series $g(x)$ without constant or linear terms.

PROOF OF LEMMA: Note that because of (b), f can have no linear terms. f can have an arbitrary bilinear term however, which we can express by ${}^t x A y$, A skew and by $g(x + y) - g(x) - g(y)$, g quadratic. So let's assume these parts are dealt with. We now assume f has no linear or bilinear terms.

Take identity (b), differentiate with respect to z_i and set $z = 0$. If $f_{2,i}$ denotes the partial of f with respect to the i^{th} component of its 2nd argument, we get:

$$f_{2,i}(x, y) + f_{2,i}(y, 0) = f_{2,i}(x + y, 0).$$

Write $h_i(x) = f_{2,i}(x, 0)$, so that

$$(*) \quad f_{2,i}(x, y) = h_i(x + y) - h_i(y).$$

Now if $h_{i,j}$ denotes the partial of h_i with respect to its j^{th} component

$$\begin{aligned}
\frac{\partial^2}{\partial y_i \partial y_j} (f(x, y)) &= h_{i,j}(x + y) - h_{i,j}(y) \\
&\text{and} \quad = h_{j,i}(x + y) - h_{j,i}(y).
\end{aligned}$$

Therefore

$$h_{i,j}(x) = h_{j,i}(x) + h_{i,j}(0) - h_{j,i}(0).$$

But by the absence of bilinear terms in f , differentiating (*) with respect to x_j , and evaluating at 0, we get:

$$0 = \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial y_i} (f(x, y)) \right) (0, 0) = \frac{\partial}{\partial x_j} f_{2,i}(x, y)(0, 0) = h_{i,j}(0).$$

Thus $h_{i,j} = h_{j,i}$. Therefore there exists a power series $g(x)$ such that

$$h_i(x) = \frac{\partial}{\partial x_i} g(x), \quad g(0) = 0.$$

But then (*) says

$$\frac{\partial}{\partial y_i} f(x, y) = \frac{\partial}{\partial y_i} g(x + y) - \frac{\partial}{\partial y_i} g(y), \quad \text{all } i,$$

$$\text{or} \quad f(x, y) = g(x + y) - g(y) - h(x),$$

for some $h(x)$ with $h(0) = 0$. Putting this back into (b) and setting $y = 0$, we get immediately $h(x) = g(x)$. QED

Finally we put these together. Let $f(x, y, z)$ be the power series satisfying (a), (b), (c) in the beginning of the proof of 9.21. Let $f^*(x, y, z) = \log(f(x, y, z))$ to make everything additive. Considering x as constant, f^* satisfies the conditions of the lemma. As the lemma is just a formal manipulation of power series, it tells us that

$$(**) \quad f^*(x, y, z) = {}^t y \cdot A(x) \cdot z + g(x, y + z) - g(x, y) - g(x, z)$$

for some skew matrix A with power series entries and some power series $g(x, y)$ without linear terms in y . Since f^* is symmetric in y and z , $A(x) = 0$. Now let $f_{3,i}^*$ be the partial of f^* with respect to the i^{th} component of its 3rd argument, $g_{2,i}$ similarly. Differentiating (**) with respect to z_i and setting $z = 0$, we get:

$$f_{3,i}^*(x, y, 0) = g_{2,i}(x, y) - g_{2,i}(x, 0) = g_{2,i}(x, y).$$

But by the symmetry of f^* , $f_{3,i}^*(x, y, 0) = f_{3,i}^*(y, x, 0)$. Likewise, f^* satisfies the cocycle condition in its 1st two variables, so $f_{3,i}^*$ does too:

$$f_{3,i}^*(x + y, z, 0) + f_{3,i}^*(x, y, 0) = f_{3,i}^*(x, y + z, 0) + f_{3,i}^*(y, z, 0).$$

Therefore $g_{2,i}$ is symmetric and satisfies the cocycle condition so by the lemma again:

$$(***) \quad g_{2,i}(x, y) = h_i(x + y) - h_i(x) - h_i(y)$$

for some h_i without constant or linear terms. But

$$\frac{\partial^2}{\partial y_i \partial y_j} (g(x, y)) = \frac{\partial}{\partial y_j} (g_{2,i}(x, y)) = h_{i,j}(x + y) - h_{i,j}(y)$$

$$\text{and} \quad = \frac{\partial}{\partial y_i} (g_{2,j}(x, y)) = h_{j,i}(x + y) - h_{j,i}(y).$$

Therefore setting $y = 0$, we get

$$h_{i,j}(x) = h_{j,i}(x) + h_{i,j}(0) - h_{j,i}(0) = h_{j,i}(x).$$

Therefore there is a power series $h(x)$ such that

$$h_i(x) = \frac{\partial}{\partial x_i} h(x), \quad h(0) = 0.$$

Therefore by (***)

$$\frac{\partial}{\partial y_i} (g(x, y) - h(x + y) + h(y)) = -h_i(x),$$

or

$$g(x, y) = h(x + y) - h(y) - \sum h_i(x) \cdot y_i - k(x).$$

Putting this into (**) and recalling that $g(x, y)$ is a power series without linear terms in y , we see that

$$\begin{aligned} f^*(x, y, z) &= [h(x + y + z) - h(y + z) - k(x)] \\ &\quad - [h(x + y) - h(y) - k(x)] \\ &\quad - [h(x + z) - h(z) - k(x)] \\ &= h(x + y + z) - h(x + y) - h(x + z) - h(y + z) \\ &\quad + k(x) + h(y) + h(z). \end{aligned}$$

Setting $y = z = 0$, we see that $h(x) = k(x)$ and that if $g^*(x) = e^{h(x)}$, then

$$f(x, y, z) = \frac{g^*(x + y + z) \cdot g^*(x) \cdot g^*(y) \cdot g^*(z)}{g^*(x + y) \cdot g^*(x + z) \cdot g^*(y + z)}$$

as required.

QED for Th. 9.21

Barsotti's basic result leads directly to the algebraic construction of the theta functions with pluri-harmonic polynomials. Now analytically these functions are vector-valued modular forms in T , when $z = 0$, so what do we expect to be able to define algebraically? Firstly, we need an analog of homogeneous vector bundles \mathbf{E} on \mathfrak{H}_g . These are functorial ways of assigning a k -vector space:

$$\mathbf{E}(X, \mathbf{L})$$

to a pair:

$$X = \text{abelian variety over } k$$

$$\mathbf{L} = \text{ample, degree 1, symmetric line bundle on } X,$$

which, moreover, "glue" together to vector bundles:

$$\mathbf{E}(\mathcal{X}, \mathcal{L}) \text{ over } S$$

whenever

$$\mathcal{X} = \text{abelian scheme over } S$$

$$\mathcal{L} = \text{relatively ample, degree 1, symmetric line bundle on } \mathcal{X}.$$

Then an \mathbf{E} -valued algebraic modular form ϕ is a functorial rule for defining elements

$$\phi(X, \mathbf{L}) \in \mathbf{E}(X, \mathbf{L})$$

which glue together to sections:

$$\phi(\mathcal{X}, \mathcal{L}) \in \Gamma(\mathbf{E}(\mathcal{X}, \mathcal{L})).$$

We don't want to develop the abstract theory of these at all, but only use these definitions as a setting in which to draw out the consequences of Barsotti's theorem.

Let's first of all consider the case $g = 1$. Start with an elliptic curve X/k . Then $\Gamma(X, \mathbf{L})$ is one-dimensional and let $\vartheta \in \Gamma(X, \mathbf{L})$ be a non-zero

section. Let $t \in \hat{\mathcal{O}}_{O, X}$ be an additive coordinate. Then the theorem tells us that there is a canonical trivialization

$$\phi : \mathbf{L} \otimes \hat{\mathcal{O}}_{O, X} \xrightarrow{\sim} \hat{\mathcal{O}}_{O, X}$$

unique up to

$$\phi^* = c \cdot e^{at^2} \cdot \phi$$

for $a, c \in k, c \neq 0$. Therefore $\hat{\vartheta}(t) = \phi(\vartheta)$ is a power series in t , unique up to transformations.

$$\hat{\vartheta} \longmapsto c \cdot e^{at^2} \cdot \hat{\vartheta}(t),$$

i.e., modulo representations of the multiplicative group \mathbf{G}_m :

$$\hat{\vartheta} \longmapsto c \cdot \hat{\vartheta}, \quad c \in \mathbf{G}_m$$

and the additive group \mathbf{G}_a :

$$\hat{\vartheta} \longmapsto e^{at^2} \cdot \hat{\vartheta}, \quad a \in \mathbf{G}_a.$$

Expand

$$\hat{\vartheta}(t) = \vartheta_0 + \vartheta_1 \cdot t + \vartheta_2 \cdot \frac{t^2}{2} + \vartheta_3 \cdot \frac{t^3}{6} + \vartheta_4 \frac{t^4}{24} + \cdots$$

Let $R(\vartheta_0, \dots, \vartheta_e)$ be any polynomial which is homogeneous of degree h .

Then polarizing R , write

$$R(\vartheta_0, \dots, \vartheta_e) = \tilde{R}(\vartheta_0^{(1)}, \dots, \vartheta_e^{(1)}; \dots; \vartheta_0^{(h)}, \dots, \vartheta_e^{(h)})$$

where \tilde{R} is linear in each set of variables $\vartheta_0^{(i)}, \dots, \vartheta_e^{(i)}$. Then

$$\begin{aligned} (\dagger) \quad R(\vartheta_0, \dots, \vartheta_e) &= \tilde{R}\left(\hat{\vartheta}(z_1), \dots, \frac{\partial^e \hat{\vartheta}}{\partial z_1^e}; \dots; \hat{\vartheta}(z_h), \dots, \frac{\partial^e \hat{\vartheta}}{\partial z_h^e}\right) \Big|_{z_1=\dots=z_h=0} \\ &= \tilde{R}\left(1, \frac{\partial}{\partial z_1}, \dots, \frac{\partial^e}{\partial z_1^e}; \dots; 1, \frac{\partial}{\partial z_h}, \dots, \frac{\partial^e}{\partial z_h^e}\right) (\hat{\vartheta}(z_1) \cdots \hat{\vartheta}(z_h)) \Big|_{z_1=\dots=z_h=0} \\ &= P\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_h}\right) (\hat{\vartheta}(z_1) \cdots \hat{\vartheta}(z_h)) \Big|_{z_1=\dots=z_h=0} \end{aligned}$$

where $P(x_1, \dots, x_h) = \tilde{R}(1, x_1, \dots, x_1^e; \dots; 1, x_h, \dots, x_h^e)$. Now we can also expand

$$\begin{aligned} (e^{at^2} \cdot \hat{v}(t)) &= v_0 + v_1 t + (v_2 + 2av_0) \frac{t^2}{2} + (v_3 + 6av_1) \frac{t^3}{6} \\ &\quad + (v_4 + 12av_2 + 12a^2v_0) \frac{t^4}{24} + \dots \\ &= \sum v_k^* \cdot \frac{t^k}{k!} \end{aligned}$$

giving a representation of the additive group in a on the vector space of sequences (v_0, v_1, \dots) . Then

$$\begin{aligned} R(v_0^*, \dots, v_e^*) &= \tilde{R}(e^{az_1^2} \hat{v}(z_1), \dots, \frac{\partial^e}{\partial z_1^e} (e^{az_1^2} \hat{v}(z_1)); \dots) \Big|_{z_1=\dots=z_h=0} \\ &= P\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_h}\right) (e^{az_1^2} \hat{v}(z_1) \dots e^{az_h^2} \hat{v}(z_h)) \Big|_{z_1=\dots=z_h=0}. \end{aligned}$$

Therefore $R(v_0, \dots, v_e) = R(v_0^*, \dots, v_e^*)$ for all a if and only if

$$P\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_h}\right) (e^{a(z_1^2 + \dots + z_h^2)} \hat{v}(z_1) \dots \hat{v}(z_h)) \Big|_{z_1=\dots=z_h=0}$$

is independent of a . By Lemma 9.8, this means that P is a harmonic polynomial. Taking low degree harmonic polynomials P we get the following R 's:

P	R
1	v_0
z_1	v_1
$z_1^3 - 3z_1z_2^2$	$v_3v_0 - 3v_1v_2$
$z_1^4 - 6z_1^2z_2^2 + z_2^4$	$2(v_4v_0 - 3v_2^2)$
$z_1^5 - 10z_1^3z_2^2 + 5z_1z_2^4$	$v_5v_0 - 10v_3v_2 + 5v_1v_4$
$z_1^5z_2 - \frac{10}{3}z_1^3z_2^3 + z_1z_2^5$	$2(v_5v_1 - \frac{5}{3}v_3^2)$

Thus any harmonic polynomial $P(z_1, \dots, z_h)$ in h variables defines by (†) an algebraic modular form. How does this work: first of all v_0 is just the image of $v \in \Gamma(X, L)$ by evaluation at 0:

$$\Gamma(X, L) \longrightarrow L(0).$$

To put this in terms of modular forms, the 1-dimensional vector spaces

$$\text{Hom}(\Gamma(X, L), L(0))$$

glue together into the basic "theta" functorial line bundle on \mathfrak{H}_1 , which we will write $\mathcal{L}_\theta(X, L)$. Then v_0 is a section of this line bundle \mathcal{L}_θ :

$$v_0 \in \Gamma(\mathcal{L}_\theta).$$

How about v_1 ? v_1 is the differential of v at 0, i.e., the image of $v \in \Gamma(X, L)$ by:

$$\Gamma(X, L) \longrightarrow L(0) \otimes \Omega_X^1(0).$$

Thus if ω is the functorial line bundle defined by

$$\omega(X, L) = \Omega_X^1(0)$$

$$\text{or} \quad \omega(\mathcal{X}, \mathcal{L}) = \Omega_{\mathcal{X}/S}^1(0)$$

then

$$v_1 \in \Gamma(\mathcal{L}_\theta \otimes \omega).$$

Similarly, v_k looks like a section of $\mathcal{L}_\theta \otimes \omega^k$ — except that it depends on the trivialization ϕ . To eliminate this dependence, start with a harmonic polynomial $P(z_1, \dots, z_h)$. Then as above, P defines a polynomial R in the coefficients v_i invariant under replacing ϕ by $e^{at^2} \cdot \phi$. If P is homogeneous of degree e , then one checks that:

$$P\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_h}\right) (\tilde{v}(z_1) \dots \tilde{v}(z_h)) \Big|_{z_1=\dots=z_h=0} \in \Gamma(\mathcal{L}_\theta^h \otimes \omega^e).$$

This still looks a bit more complicated than the analytic theory. The final step is to construct a canonical isomorphism of the line bundles

$$\mathcal{L}_\theta^2 \cong \omega$$

over some finite covering of the moduli space \mathfrak{H}_1 , i.e., for $(\mathcal{X}, \mathcal{L})$'s with some finite extra structure. In terms of this fundamental isomorphism, a section of $\mathcal{L}_\theta^h \otimes \omega^e$ becomes a section of $\mathcal{L}_\theta^{h+2e}$, which we call a modular form of degree $h/2 + e$. This is an algebraic construction of the modular forms $v^{P, Q}$ for $g = 1$, $Q = \text{identity } h \times h \text{ matrix}$.

For $g > 1$, the same construction works. Barsotti's theory gives us a nearly canonical trivialization:

$$\phi : \mathbb{L} \otimes \widehat{\mathcal{O}}_{0,X} \xrightarrow{\sim} \widehat{\mathcal{O}}_{0,X}$$

hence a power series $\widehat{\vartheta} = \phi(\vartheta)$ unique up to multiplication by a quadratic exponential. As above, for any pluriharmonic polynomial $P(z_{ij})$ in a $g \times h$ -matrix Z

$$P\left(\frac{\partial}{\partial z_{ij}}\right)(\vartheta(z_{11}, \dots, z_{g1}) \cdots \vartheta(z_{1h}, \dots, z_{gh})) \Big|_{z_{ij}=0}$$

is a polynomial in the coefficients of $\widehat{\vartheta}$ invariant under such transformations. Thus we can construct algebraic modular forms for $g > 1$. Where do they lie? As in the case $g = 1$,

$$\mathcal{L}_\theta(X, \mathbb{L}) = \text{Hom}(\Gamma(X, \mathbb{L}), \mathbb{L}(0))$$

defines a functorial line bundle for g -dimensional abelian schemes and

$$\omega(X, \mathbb{L}) = \Omega_X^1(0)$$

defines a functorial vector bundle of rank g . They are connected by a canonical isomorphism:

$$\mathcal{L}_\theta^2 \cong \Lambda^g \omega$$

defined for (X, \mathcal{L}) 's with some finite extra structure (cf. Morel-Baily, *Pinceaux de Variétés Abeliennes*, Astérisque 129, 1985). If P is homogeneous of degree e , then it is easy to see that

$$P\left(\frac{\partial}{\partial z_{ij}}\right)(\vartheta(z_{i1}, \dots, z_{ih})) \Big|_{z_{ij}=0} \in \Gamma(\mathcal{L}_\theta^h \otimes \text{Sym}^e(\omega)).$$

These sections give an algebraic construction for the modular forms $\vartheta^{P,Q}$'s with $Q = h \times h$ -identity matrix (which can readily be generalized to arbitrary Q).

10. The homogeneous coordinate ring of an abelian variety

One of the main applications of theta functions is to provide *explicit bases* for linear systems $\Gamma(X, \mathcal{L})$ on abelian varieties X . The goal of this section is to study the consequences of having such explicit bases. Now one of the main elements of structure of these vector spaces is the set of multiplication maps:

$$\Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{M}) \longrightarrow \Gamma(X, \mathcal{L} \otimes \mathcal{M})$$

and, in particular, the *ring* structure on

$$R(X, \mathcal{L}) = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{L}^n).$$

The theta identities described in §6 and §7 allow us in many cases to express these multiplications in terms of the theta bases with coefficients given by the values at 0 of other theta functions. Moreover, we can hope that the polynomial identities defining the ring, i.e., the kernel I_n of:

$$\Phi : S^n(\Gamma(X, \mathcal{L})) \longrightarrow \Gamma(X, \mathcal{L}^n)$$

will be spanned by linear combinations of suitable theta identities as found in §6 and §7. In this case, the whole algebra of the equations defining abelian varieties can be determined from the theory of theta functions.

Let us now be more precise. The basic case is where \mathcal{L} is an ample symmetric degree 1 line bundle on the complex abelian variety X_T . Then a basis of $\Gamma(X, \mathcal{L}^n)$ is given by:

$$\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (nz, nT), \quad a \in \frac{1}{n} \mathbb{Z}^g / \mathbb{Z}^g.$$

An analog of Cor. 6.10 will give the multiplication map explicitly in terms of this basis: to determine

$$(*) \quad \Gamma(\mathcal{L}^n) \otimes \Gamma(\mathcal{L}^m) \longrightarrow \Gamma(\mathcal{L}^{n+m})$$

we use the equivalence of the quadratic forms $Q' = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$ and $Q = (n+m) \begin{pmatrix} 1 & 0 \\ 0 & nm \end{pmatrix}$ given by:

$$\begin{pmatrix} 1 & -m \\ 1 & n \end{pmatrix}^t \cdot \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix} \cdot \begin{pmatrix} 1 & -m \\ 1 & n \end{pmatrix} = \begin{pmatrix} n+m & 0 \\ 0 & nm(n+m) \end{pmatrix}$$

and deduce formulae of the type:

$$\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (nz, nT) \cdot \vartheta \begin{bmatrix} b \\ 0 \end{bmatrix} (mz, mT) =$$

$$\sum_{\eta} \vartheta \left[\frac{na+mb}{n+m} + n\eta \right] ((n+m)z, (n+m)T) \cdot \vartheta \left[\frac{a-b}{n+m} + \eta \right] (0, (n+m)T),$$

where the sum is over all $\eta \in \frac{1}{n+m} \mathbb{Z}^g / \mathbb{Z}^g$. (This is Remark 3 preceding Cor. 6.6. Set $A = \begin{pmatrix} 1 & -m \\ 1 & n \end{pmatrix}^{-1}$, $N = 0$, $Z = (nz, mz)$, $Q = \begin{pmatrix} n+m & 0 \\ 0 & nm(n+m) \end{pmatrix}$ and $Q' = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$.) Note, however, that different theta functions are used for each n making the story a bit complex. We do know that the two theta functions

$$\vartheta(nz, nT) \quad \text{and} \quad \theta(n\ell^2 z, n\ell^2 T)$$

are essentially the same (because the two quadratic forms nX^2 and $n\ell^2 X^2$ are equivalent over \mathbb{Q} and we use the second fundamental identity). Therefore, the full description is in terms of the entire family of theta functions $\vartheta(nz, nT)$, for all square-free n .

One way to get a handle on this algebra is to stick to the linear systems

$$\Gamma(X, \mathcal{L}^{2^n})$$

and the products:

$$(**) \quad \Gamma(X, \mathcal{L}^{2^n}) \otimes \Gamma(X, \mathcal{L}^{2^n}) \longrightarrow \Gamma(X, \mathcal{L}^{2^{n+1}})$$

In this case, all bases are written using one of two theta functions $\vartheta(z, T)$ or $\vartheta(2z, 2T)$, depending on whether n is even or odd and the multiplication comes from use of the simplest identity:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^t \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

applied in Cor. 6.8 and §7. In the paper *Equations Defining Abelian Varieties*, this was the approach taken. In particular, this approach allows one to prove that (**) is surjective and to give its kernel explicitly if $n \geq 2$.

Subsequently, the questions of when the more general map (*) is surjective and the description of its kernel have been investigated at length by Koizumi, Sekiguchi and Kempf. Two methods have emerged: the explicit description of the map via theta functions and the use of more abstract cohomological arguments together with the use of the finite Heisenberg group. The latter gives more powerful results and we want to show here how these methods work. We follow Kempf, *Linear Systems on Abelian Varieties*, Am. J. Math. (1989), closely.

We will first prove:

THEOREM 10.1. *If \mathcal{L} is an ample line bundle on an abelian variety over any field k , then*

$$\Gamma(\mathcal{L}^n) \otimes \Gamma(\mathcal{L}^m) \longrightarrow \Gamma(\mathcal{L}^{n+m})$$

is surjective if $n \geq 2, m \geq 3$.

The main new tool we need is the concept of the dual abelian variety \hat{X} . A general reference is D. Mumford, *Abelian Varieties*, Oxford Univ. Press. \hat{X} classifies the line bundles on X which are deformations of the trivial bundle. In fact, there is a "universal" bundle \mathcal{P} over $X \times \hat{X}$ called the Poincaré bundle such that the restrictions \mathcal{P}_a of \mathcal{P} to $X \cong X \times \{a\}$, for various $a \in \hat{X}$, run through all such line bundles exactly once: i.e.

$$i) \quad \mathcal{P}_a \cong \mathcal{P}_b \quad \text{iff } a = b$$

$$ii) \quad \text{all deformations of the trivial bundle occur.}$$

If \mathcal{L} is any line bundle on X , then $\{T_x^*(\mathcal{L})\}_{x \in X}$ gives us a family of deformations of \mathcal{L} itself, parametrized by X . Thus $\{T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}\}_{x \in X}$ is a family of deformations of the trivial bundle and we can define a map:

$$\phi_{\mathcal{L}} : X \longrightarrow \hat{X}$$

by

$$\phi_{\mathcal{L}}(x) = [\text{point } a \text{ of } \hat{X} \text{ such that } T_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{P}_a]$$

This map $\phi_{\mathcal{L}}$ is a morphism of varieties, and it can be used to construct \hat{X} .

In fact, if \mathcal{L} is ample

$$\begin{aligned} \ker(\phi_{\mathcal{L}}) &= \{x | T_x^* \mathcal{L} \cong \mathcal{L}\} \\ &= K(\mathcal{L}) \end{aligned}$$

so we may define \hat{X} as the quotient $X/K(\mathcal{L})$. (In char. p , we must use the full group scheme $\underline{K}(\mathcal{L})$ defined at the end of §3). Finally, the addition on \hat{X} is a result of the tensor product operation on line bundles, i.e.,

$$\mathcal{P}_{a+b} \cong \mathcal{P}_a \otimes \mathcal{P}_b.$$

In the above construction, the "theorem of the square":

$$T_{x+y}^* \mathcal{L} \cong T_x^* \mathcal{L} \otimes T_y^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

implies then that $\phi_{\mathcal{L}}$ is a homomorphism:

$$\begin{aligned} \mathcal{P}_{\phi_{\mathcal{L}}(x) + \phi_{\mathcal{L}}(y)} &\cong \mathcal{P}_{\phi_{\mathcal{L}}(x)} \otimes \mathcal{P}_{\phi_{\mathcal{L}}(y)} \\ &\cong (T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}) \otimes (T_y^* \mathcal{L} \otimes \mathcal{L}^{-1}) \\ &\cong T_{x+y}^* \mathcal{L} \otimes \mathcal{L}^{-1} \quad (\text{by the theorem of the square}) \\ &\cong \mathcal{P}_{\phi_{\mathcal{L}}(x+y)} \end{aligned}$$

hence $\phi_{\mathcal{L}}(x) + \phi_{\mathcal{L}}(y) = \phi_{\mathcal{L}}(x+y)$.

Our first step in proving 10.1 is the

LEMMA 10.2. Let R, S be invertible sheaves on X such that $\Gamma(R) \neq 0$ and $\Gamma(S) \neq 0$, and $R \otimes S$ is ample. Then

$$\sum_{a \in \hat{X}} \Gamma(R \otimes \mathcal{P}_a) \otimes \Gamma(S \otimes \mathcal{P}_{-a}) \longrightarrow \Gamma(R \otimes S)$$

is surjective. Indeed, there exists an open dense subset U so that if you sum over $a \in U$ the assertion remains true.

This result is shown in D. Mumford's lectures in the C.I.M.E. Summer School, 1969, *Questions Concerning Algebraic Varieties* (these notes, unfortunately, are almost unobtainable).

PROOF: Let W denote the image; we show that W is invariant under the action of the Heisenberg group $\mathcal{G}(R \otimes S)$. This suffices: Since $\Gamma(R \otimes S)$ is $\mathcal{G}(R \otimes S)$ -irreducible either $W = (0)$ or $W = \Gamma(R \otimes S)$. The first possibility does not occur since $\Gamma(R)$ and $\Gamma(S)$ have non-zero elements.

Let $r \in \Gamma(R \otimes \mathcal{P}_a)$, $s \in \Gamma(S \otimes \mathcal{P}_{-a})$ with divisors D, E . Thus $r \otimes s$ has divisor $D + E$. If $\phi \in \mathcal{G}(R \otimes S)$ gives an isomorphism $R \otimes S \cong T_x^*(R \otimes S)$, then ϕ acts on $r \otimes s$ and gives a section of $R \otimes S$ with divisor $T_x^*(D + E)$. We need to find elements $r', s' \in \Gamma(R \otimes \mathcal{P}_\beta)$, $\Gamma(S \otimes \mathcal{P}_\beta)$ so that $r' \otimes s'$ has divisor $T_{-x}^*(D + E)$.

Since $x \in \ker(\phi_{R \otimes S})$, $\phi_R(x) + \phi_S(x) = 0$. If $\beta = a + \phi_R(x) = a - \phi_S(x)$, then

$$T_x^*(R \otimes \mathcal{P}_a) = R \otimes (T_x^* R \otimes R^{-1}) \otimes \mathcal{P}_a = R \otimes \mathcal{P}_{\phi_R(x)} \otimes \mathcal{P}_a = R \otimes \mathcal{P}_\beta.$$

Similarly, $T_x^*(S \otimes \mathcal{P}_{-a}) = S \otimes \mathcal{P}_{-\beta}$. Thus $T_x^* r = r' \in \Gamma(R \otimes \mathcal{P}_\beta)$,

$T_x^* s = s' \in \Gamma(S \otimes \mathcal{P}_{-\beta})$ and $r' \cdot s'$ has divisor $T_{-x}^* D + T_{-x}^* E$. QED

Using this, we follow Kempf and prove:

THEOREM 10.3. Fix $y \in X$. For x in any dense open set of X , the multiplication

$$m(x) : \Gamma(\mathcal{L}^2 \otimes \mathcal{P}_{-x}) \otimes \Gamma(\mathcal{L}^2 \otimes \mathcal{P}_{x+y}) \longrightarrow \Gamma(\mathcal{L}^4 \otimes \mathcal{P}_y)$$

is surjective. If $\dim \Gamma(\mathcal{L}) = 1$, then $m(x)$ is an isomorphism.

PROOF: If $\dim \Gamma(\mathcal{L}) = 1$, the source and target of $m(x)$ have the same dimension; hence surjective implies isomorphism.

The proof goes as follows. First we find a group \mathcal{K} acting on the domain and range of $m(x)$ in such a way that $m(x)$ is equivariant. The key point here is to show that the target has a finite number of maximal \mathcal{K} -invariant subspaces, so as x varies the images of $m(x)$ cannot span the target unless almost all $m(x)$ are themselves surjective.

Let $\mathcal{G} = \mathcal{G}(\mathcal{L}^2)$. Then \mathcal{G} acts on $\mathcal{L}^2 \otimes \mathcal{P}_{-x}$: note that $\mathcal{L}^2 \otimes \mathcal{P}_{-x} \cong T_x^*(\mathcal{L}^2)$ for some $z \in X$ since $\phi_{\mathcal{L}^2}$ is surjective, hence if $\varphi \in \mathcal{G}$, then φ acts on

$\mathcal{L}^2 \otimes \mathcal{P}_{-x}$ by $T_z^*(\varphi)$. Similarly \mathcal{G} acts on $\mathcal{L}^2 \otimes \mathcal{P}_{x+y}$. Tensoring these, we get an action of \mathcal{G} on $\mathcal{L}^4 \otimes \mathcal{P}_y$ so that $m(x)$ is equivariant.

We restrict the action of \mathcal{G} to an abelian subgroup. Let K_0 be a maximal isotropic subgroup of $K(\mathcal{L})$ and set

$$\mathcal{K} = \{x \in X \mid 2x \in K_0\}.$$

CLAIM: $e^{\mathcal{L}^2}(\mathcal{K}, \mathcal{K}) \subset \mu_2$.

PROOF OF CLAIM: $e^{\mathcal{L}^2}(\mathcal{K}, \mathcal{K})^2 = e^{\mathcal{L}}(2\mathcal{K}, 2\mathcal{K}) = e^{\mathcal{L}}(K_0, K_0) = 1$.

This has two consequences. First it implies that \mathcal{K} is maximal isotropic in $K(\mathcal{L}^4)$ since $e^{\mathcal{L}^4}(\mathcal{K}, \mathcal{K}) = e^{\mathcal{L}^2}(\mathcal{K}, \mathcal{K})^2 = 1$ and the rank of \mathcal{K} , as a finite group scheme, is correct. Second, note that the action of $\mathcal{G}(\mathcal{L}^2)$ on the space $\Gamma(\mathcal{L}^2 \otimes \mathcal{P}_{-x}) \otimes \Gamma(\mathcal{L}^2 \otimes \mathcal{P}_{y-x})$ and on $\Gamma(\mathcal{L}^4 \otimes \mathcal{P}_y)$ factors through $\mathcal{G}(\mathcal{L}^2)/\mu_2 = \mathcal{G}'$. The claim implies that we can split \mathcal{G}' over \mathcal{K} :

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{K}(\mathcal{L}^{\otimes 2}) \longrightarrow 0 \\ & & & & \nwarrow & \nearrow & \\ & & & & & \mathcal{K} & \end{array}$$

Thus \mathcal{K} acts on the domain and range of $m(x)$. We now study $m(x)$ as a map of \mathcal{K} -modules.

LEMMA 10.4. If $\mathcal{K} \subset K(\mathcal{L})$ is a maximal isotropic subgroup then there are only a finite number of maximal \mathcal{K} -invariant subspaces W in $\Gamma(X, \mathcal{L})$.

PROOF: The irreducible Heisenberg representation of $\mathcal{G}(\mathcal{L})$ can be constructed as the space of functions on $K(\mathcal{L})/\mathcal{K}$.¹ By the Heisenberg property,

¹If $K(\mathcal{L})$ is an ordinary finite group, this is clear. In the case where $K(\mathcal{L})$ is a group scheme, we prove this by considering a splitting σ of $\mathcal{G}(\mathcal{L})$ over \mathcal{K} and showing that

$$V_\sigma = \left\{ \text{functions } f \text{ on } \mathcal{G}(\mathcal{L}) \mid \begin{array}{l} f(\lambda x) = \lambda f(x), \lambda \in \mathbf{G}_m, \\ f(x \cdot \sigma(k)) = f(x), \text{ all } k \in \mathcal{K} \end{array} \right\}$$

is the Heisenberg representation of $\mathcal{G}(\mathcal{L})$, cf. Moret-Bailly, *Pinceaux de Variétés Abéliennes*, Astérisque 129, 1985.

this is the dual abelian group $\hat{\mathcal{K}}$ to \mathcal{K} and \mathcal{K} operates here by multiplication by characters. Therefore $\Gamma(X, \mathcal{L})$ is isomorphic to the affine ring of $\hat{\mathcal{K}}$ and we need only show that this ring contains only a finite number of maximal subspaces invariant under multiplication by all characters. But a subspace is invariant like this if and only if it is an ideal in the affine ring of $\hat{\mathcal{K}}$ and any finite dimensional commutative ring contains only a finite number of maximal ideals.

Note that this proof is valid in char. p where $\mathcal{G}(\mathcal{L})$ and $K(\mathcal{L})$ may be group schemes. One should notice that even if, e.g., \mathcal{K} consisted of only one point, so $\hat{\mathcal{K}}$ had only the trivial character, for $W \subset \Gamma(\hat{\mathcal{K}})$ to be invariant under the character action of the scheme \mathcal{K} still implies that W is an ideal. This is because the ring multiplication in $\Gamma(\hat{\mathcal{K}})$:

$$(*) \quad \Gamma(\hat{\mathcal{K}}) \otimes \Gamma(\hat{\mathcal{K}}) \longrightarrow \Gamma(\hat{\mathcal{K}})$$

is dual to the character action of the group scheme \mathcal{K} on $\Gamma(\hat{\mathcal{K}})$:

$$\Gamma(\hat{\mathcal{K}}) \longrightarrow \Gamma(\mathcal{K}) \otimes \Gamma(\hat{\mathcal{K}})$$

via the duality of $\Gamma(\mathcal{K})$ and $\Gamma(\hat{\mathcal{K}})$. More explicitly, let g be an R -valued point of \mathcal{K} for some k -algebra R . Then g is given by a k -homomorphism

$$\Gamma(\mathcal{K}) \longrightarrow R,$$

which may be viewed as an element of $\text{Hom}(\Gamma(\mathcal{K}), k) \otimes_k R$ which is $\Gamma(\hat{\mathcal{K}}) \otimes_k R$. In this way, the set $\mathcal{K}(R)$ of R -valued points is a subset of $\Gamma(\hat{\mathcal{K}}) \otimes_k R$, and using the ring multiplication $(*)$, we get an action:

$$\mathcal{K}(R) \times (\Gamma(\hat{\mathcal{K}}) \otimes_k R) \longrightarrow (\Gamma(\hat{\mathcal{K}}) \otimes_k R).$$

Since, as R varies, the points $\mathcal{K}(R)$ span $\Gamma(\hat{\mathcal{K}})$, we see that a \mathcal{K} -invariant subspace of $\Gamma(\hat{\mathcal{K}})$ is indeed an ideal. QED

Apply this to our \mathcal{K} in $K(\mathcal{L}^4)$. It follows that for each x the image $W(x)$ of $m(x)$ is either all of $\Gamma(\mathcal{L}^4 \otimes \mathcal{P}_y)$ or is contained in one of a finite set U_1, \dots, U_N of subspaces of $\Gamma(\mathcal{L}^4 \otimes \mathcal{P}_y)$. Now as x varies, the vector spaces

$$\Gamma(\mathcal{L}^2 \otimes \mathcal{P}_{-x}), \quad (\text{resp. } \Gamma(\mathcal{L}^2 \otimes \mathcal{P}_{x+y}))$$

fit together into vector bundles, and the maps $m(x)$ vary continuously. Therefore, for all x in a dense open set, $m(x)$ will have a fixed maximal rank. Therefore, either $W(x) = \Gamma(\mathcal{L}^4 \otimes \mathcal{P}_y)$ for all x in a dense open set, or $W(x) \subseteq$ some fixed U_i , all x . But Lemma 10.2 says that

$$\sum_x W(x) = \Gamma(\mathcal{L}^4 \otimes \mathcal{P}_y).$$

Therefore $W(x)$ must equal $\Gamma(\mathcal{L}^4 \otimes \mathcal{P}_y)$ for almost all x .

QED for Theorem 10.3.

To prove Theorem 10.1, note that by Theorem 10.3,

$$\Gamma(\mathcal{L}^2) \otimes \Gamma(\mathcal{L}^2 \otimes \mathcal{P}_x) \longrightarrow \Gamma(\mathcal{L}^4 \otimes \mathcal{P}_x)$$

is surjective for all x in a dense open set U of \widehat{X} . (Take $y = 0$ in 10.3 and note that the maps

$$\Gamma(\mathcal{L}^2 \otimes \mathcal{P}_{-x}) \otimes \Gamma(\mathcal{L}^2 \otimes \mathcal{P}_x) \longrightarrow \Gamma(\mathcal{L}^4)$$

$$\Gamma(\mathcal{L}^2) \otimes \Gamma(\mathcal{L}^2 \otimes \mathcal{P}_{2x}) \longrightarrow \Gamma(\mathcal{L}^4 \otimes \mathcal{P}_{2x})$$

are just translates of each other, so have the same rank.)

By Lemma 10.2,

$$\sum_{x \in U} \Gamma(\mathcal{L}^4 \otimes \mathcal{P}_x) \otimes \Gamma(\mathcal{L} \otimes \mathcal{P}_x) \longrightarrow \Gamma(\mathcal{L}^5)$$

is surjective. But it factors through $\Gamma(\mathcal{L}^2) \otimes \Gamma(\mathcal{L}^3)$! This proves 10.1 for $n = 2, n = 3$. The higher cases follow similarly. QED for Theorem 10.1.

In particular, 10.1 implies:

COROLLARY 10.5. If $\mathcal{M} = \mathcal{L}^n$ and

$$R = \bigoplus_{k=0}^{\infty} \Gamma(X, \mathcal{M}^k),$$

then R is generated by $\Gamma(X, \mathcal{M})$ if $n \geq 3$, hence \mathcal{M} is very ample.

We now turn to the relations in the homogenous coordinate ring of an abelian variety. The main result is this:

THEOREM 10.6. If \mathcal{L} is an ample line bundle on an abelian variety X over any field k and $\mathcal{M} = \mathcal{L}^n$, let I be the kernel of:

$$\bigoplus_{k=0}^{\infty} S^k[\Gamma(X, \mathcal{M})] \longrightarrow \bigoplus_{k=0}^{\infty} \Gamma(X, \mathcal{M}^k).$$

Then the ideal I is generated by its quadratic and cubic polynomials I_2, I_3 if $m \geq 3$ and by its quadratic polynomials alone if $n \geq 4$.

In fact, if $n \geq 4$ and n is even, then we can give a basis of I_2 by theta relations resulting from the Riemann theta relation. The simplest case is the pair of quadratic equations defining elliptic curves embedded in \mathbf{P}^3 by $\Gamma(X, \mathcal{L}^4)$ with $\deg \mathcal{L} = 1$, given in Chapter I. Let's describe these quadratic relations for general g when $X = X_T$ is a principally polarized complex abelian variety and \mathcal{L} is the basic degree 1 line bundle. Then as above

$$\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (nz, nT), \quad a \in \frac{1}{n} \mathbf{Z}^g$$

is a basis of $\Gamma(X, \mathcal{M})$, and instead of writing multiplication by

$$\begin{aligned} & \vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (nz, nT) \cdot \vartheta \begin{bmatrix} b \\ 0 \end{bmatrix} (nz, nT) \\ &= \sum_{\eta \in \frac{1}{2} \mathbf{Z}^g / \mathbf{Z}^g} \vartheta \begin{bmatrix} \frac{a+b}{2} + \eta \\ 0 \end{bmatrix} (0, 2nT) \cdot \vartheta \begin{bmatrix} \frac{a+b}{2} + \eta \\ 0 \end{bmatrix} (2nz, 2nT) \end{aligned}$$

as above, we replace a and b by $a + \eta_1, b + \eta_1$ and multiply by a character

of η_1 . This gives us (compare Ch. II, (6.5), (6.6) and (6.7)):

$$\begin{aligned}
 & \sum_{\eta \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} e^{4\pi i' c(a+\eta)} \vartheta \begin{bmatrix} a+\eta \\ 0 \end{bmatrix} (nz, nT) \cdot \vartheta \begin{bmatrix} b+\eta \\ 0 \end{bmatrix} (nz, nT) \\
 &= e^{4\pi i' c \cdot a} \cdot \sum_{\eta \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} e^{4\pi i' c \cdot \eta} \cdot \vartheta \begin{bmatrix} \frac{a-b}{2} + \eta \\ 0 \end{bmatrix} (0, 2nT) \\
 (*) \quad & \cdot \sum_{\eta \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} e^{4\pi i' c \cdot \eta} \cdot \vartheta \begin{bmatrix} \frac{a+b}{2} + \eta \\ 0 \end{bmatrix} (2nz, 2nT) \\
 &= \vartheta \begin{bmatrix} a-b \\ c \end{bmatrix} (0, \frac{n}{2}T) \cdot \vartheta \begin{bmatrix} a+b \\ c \end{bmatrix} (nz, \frac{n}{2}T)
 \end{aligned}$$

for any $c \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$. The last equality is verified by using the standard Fourier expansion for ϑ . Here we use the alternate basis:

$$\vartheta \begin{bmatrix} a \\ c \end{bmatrix} (nz, \frac{n}{2}T), \quad a \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g, \quad c \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$$

of $\Gamma(X, \mathcal{M}^2)$. Since $\Gamma(X, \mathcal{M}) \otimes \Gamma(X, \mathcal{M})$ maps onto $\Gamma(X, \mathcal{M}^2)$, the RHS's of (*) must span $\Gamma(X, \mathcal{M}^2)$, so taking $d = a + b$, a Corollary is the non-vanishing result:

(**)

$$\text{For all } d \in \frac{1}{n}\mathbb{Z}^g, c \in \frac{1}{2}\mathbb{Z}^g, \vartheta \begin{bmatrix} d-2b \\ c \end{bmatrix} (0, \frac{n}{2}T) \neq 0 \quad \text{for some } b \in \frac{1}{n}\mathbb{Z}^g.$$

Now since the equations (*) are a basis for the full multiplication table $\Gamma(X, \mathcal{M}) \otimes \Gamma(X, \mathcal{M}) \rightarrow \Gamma(X, \mathcal{M}^2)$, they tell us that a basis of the space I_2 of quadratic relations is given by

$$\begin{aligned}
 & \vartheta \begin{bmatrix} a'-b' \\ c \end{bmatrix} (0, \frac{n}{2}T) \sum_{\eta} e^{\dots} \cdot \vartheta \begin{bmatrix} a+\eta \\ 0 \end{bmatrix} (nz, nT) \vartheta \begin{bmatrix} b+\eta \\ 0 \end{bmatrix} (nz, nT) \\
 &= \vartheta \begin{bmatrix} a-b \\ c \end{bmatrix} (0, \frac{n}{2}T) \cdot \sum_{\eta} e^{\dots} \cdot \vartheta \begin{bmatrix} a'+\eta \\ 0 \end{bmatrix} (nz, nT) \vartheta \begin{bmatrix} b'+\eta \\ 0 \end{bmatrix} (nz, nT),
 \end{aligned}$$

if $a+b \equiv a'+b' \pmod{\mathbb{Z}^g}$. We multiply this by

$$\vartheta \begin{bmatrix} a+b \\ c \end{bmatrix} (nz_0, \frac{n}{2}T) = e^{\dots} \vartheta \begin{bmatrix} a+b+2d \\ c \end{bmatrix} (0, \frac{n}{2}T),$$

where $z_0 = T \cdot d$, $d \in \frac{1}{n}\mathbb{Z}^g$; this factor can be made non-zero by suitable choice of d . Then using (*) and putting nz_0 into the characteristic, the identity becomes:

$$\begin{aligned}
 & \left(\sum_{\eta} s_{\eta} \vartheta \begin{bmatrix} a'+d+\eta \\ 0 \end{bmatrix} (0, nT) \cdot \vartheta \begin{bmatrix} b'+d+\eta \\ 0 \end{bmatrix} (0, nT) \right) \\
 & \cdot \left(\sum_{\eta} s_{\eta} \vartheta \begin{bmatrix} a+\eta \\ 0 \end{bmatrix} (nz, nT) \cdot \vartheta \begin{bmatrix} b+\eta \\ 0 \end{bmatrix} (nz, nT) \right) \\
 (10.7) \quad &= \left(\sum_{\eta} s_{\eta} \vartheta \begin{bmatrix} a+d+\eta \\ 0 \end{bmatrix} (0, nT) \cdot \vartheta \begin{bmatrix} b+d+\eta \\ 0 \end{bmatrix} (0, nT) \right) \\
 & \cdot \left(\sum_{\eta} s_{\eta} \vartheta \begin{bmatrix} a'+\eta \\ 0 \end{bmatrix} (nz, nT) \cdot \vartheta \begin{bmatrix} b'+\eta \\ 0 \end{bmatrix} (nz, nT) \right)
 \end{aligned}$$

where $s_{\eta} = (-1)^{(2c) \cdot (2\eta)}$. In this form, we have a basis of I_2 whose coefficients are the value at 0 of the basis of $\Gamma(X, \mathcal{M})$. Using algebraic theta functions, this result extends to any field k if $\text{char}(k) \nmid n$. Also note that if we set $z = 0$ in (10.7), we get quartic identities on the theta-nulls $\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (0, nT)$: these are just another form of Riemann's quartic theta relation.

Our next goal is to describe Kempf's proof of Theorem 10.6 (implying that in the case n even, $n \geq 4$ the relations above generate the whole of I). It is based on ideas similar to the proof of Theorem 10.1, except that Heisenberg groups are not used, but instead we use more cohomology. The point is to look at *families* of maps where the line bundles involved are tensored with \mathcal{P}_a 's, and put these together into maps of bundles over the space \hat{X} of all a 's. At a key place, we will make use of a basic calculation of the higher cohomology, groups of \mathcal{P} itself, which can be found in Mumford, *Abelian Varieties*.

First, some notation: If V is a vector space, \hat{V} is its dual and if \mathcal{S} is an \mathcal{O}_Y -sheaf for a scheme Y , then $\mathcal{S}^{\wedge} = \text{Hom}(\mathcal{S}, \mathcal{O}_Y)$. Let π and $\hat{\pi}$ denote the projections of $X \times \hat{X}$ to X and \hat{X} respectively. As we will be dealing

with many powers \mathcal{L}^{ℓ_i} of the basic ample \mathcal{L} (with $\ell_i > 0$), we abbreviate this to \mathcal{L}_i . Finally, the family of vector spaces

$$\{\Gamma(X, \mathcal{L}_i \otimes \mathcal{P}_{\pm a})\}_{a \in \hat{X}}$$

fits together into a vector bundle $\mathcal{W}^{\pm}(\mathcal{L}_i)$ on \hat{X} . Formally:

$$\mathcal{W}^{\pm}(\mathcal{L}_i) = \hat{\pi}_*(\pi^* \mathcal{L}_i \otimes \mathcal{P}^{\pm 1})$$

Since tensor product maps $\mathcal{L}_1 \otimes \mathcal{P}_a$ times $\mathcal{L}_2 \otimes \mathcal{P}_{-a}$ to $\mathcal{L}_1 \otimes \mathcal{L}_2$, we get a map of bundles:

$$\widetilde{M} : \mathcal{W}^+(\mathcal{L}_1) \otimes \mathcal{W}^-(\mathcal{L}_2) \longrightarrow \Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \mathcal{O}_{\hat{X}}.$$

BIG LEMMA 10.8. The map M induced by \widetilde{M} :

$$M : \Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2)^{\wedge} \longrightarrow \Gamma(\hat{X}, (\mathcal{W}^+(\mathcal{L}_1) \otimes \mathcal{W}^-(\mathcal{L}_2))^{\wedge})$$

is an isomorphism.

PROOF: We introduce the sheaf \mathcal{F} on $X \times X \times \hat{X}$ defined by:

$$\mathcal{F} = \pi_1^* \mathcal{L}_1^{-1} \otimes \pi_{13}^* \mathcal{P}^{-1} \otimes \pi_2^* \mathcal{L}_2^{-1} \otimes \pi_{23}^* \mathcal{P}$$

where π_{ij} (resp. π_i) is the projection onto the $(i, j)^{\text{th}}$ factors (resp. i^{th} factor). The proof goes by calculating the cohomology of \mathcal{F} in two different ways: via the restriction of \mathcal{F} to the fibres of π_3 and via the restriction of \mathcal{F} to the fibres of π_{12} .

CLAIM I:

$$R^i \pi_{3,*}(\mathcal{F}) = \begin{cases} 0 & \text{if } i \neq 2g \\ (\mathcal{W}^+(\mathcal{L}_1) \otimes \mathcal{W}^-(\mathcal{L}_2))^{\wedge} & \text{if } i = 2g \end{cases}$$

PROOF: The restriction of \mathcal{F} to $X \times X \times \{a\}$ is

$$\pi_1^* (\mathcal{L}_1^{-1} \otimes \mathcal{P}_{-a}) \otimes \pi_2^* (\mathcal{L}_2^{-1} \otimes \mathcal{P}_a),$$

so its k^{th} cohomology is the tensor product of terms:

$$\bigoplus_{i+j=k} [H^i(X, \mathcal{L}_1^{-1} \otimes \mathcal{P}_{-a}) \otimes H^j(X, \mathcal{L}_2^{-1} \otimes \mathcal{P}_a)].$$

But the \mathcal{L}_i are ample, hence $H^i(X, \mathcal{L}_i \otimes \mathcal{P}_b) = (0)$ if $i > 0$, hence by Serre duality $H^i(X, \mathcal{L}_i^{-1} \otimes \mathcal{P}_b) = (0)$ if $i < g$. Thus the only non-zero group here is

$$\begin{aligned} H^{2g}(X \times X \times [a], \mathcal{F}|_{X \times X \times \{a\}}) &= H^g(\mathcal{L}_1^{-1} \otimes \mathcal{P}_{-a}) \otimes H^g(\mathcal{L}_2^{-1} \otimes \mathcal{P}_a) \\ &\cong [H^0(\mathcal{L}_1 \otimes \mathcal{P}_a) \otimes H^0(\mathcal{L}_2 \otimes \mathcal{P}_{-a})]^{\wedge} \end{aligned}$$

by Serre duality. Therefore all $R^i \pi_{3,*}$ are zero except for the $2g^{\text{th}}$ one, which is the dual of the bundle

$$\bigcup_a [H^0(\mathcal{L}_1 \otimes \mathcal{P}_a) \otimes H^0(\mathcal{L}_2 \otimes \mathcal{P}_{-a})],$$

i.e., the dual of $\mathcal{W}^+(\mathcal{L}_1) \otimes \mathcal{W}^-(\mathcal{L}_2)$.

CLAIM II:

$$R^i \pi_{12,*}(\mathcal{F}) = \begin{cases} 0 & \text{if } i \neq g \\ (\mathcal{L}_1^{-1} \otimes_{\mathcal{O}_{X \times X}} \mathcal{L}_2^{-1}) \otimes \mathcal{O}_{\Delta} & \text{if } i = g \end{cases}$$

where \mathcal{O}_{Δ} is the structure sheaf of the diagonal in $X \times X$.

PROOF: Here's where we need the result

$$R^i \pi_* \mathcal{P} = \begin{cases} (0), & i \neq g \\ \mathcal{O}_e, & i = g \end{cases}$$

for \mathcal{P} on $X \times \hat{X}$ and $\pi : X \times \hat{X} \rightarrow X$. (See *Abelian Varieties*, §13). To reduce the claim to this, note that

$$\pi_{13}^* \mathcal{P}^{-1} \otimes \pi_{23}^* \mathcal{P} \cong (\pi_2 - \pi_1, \pi_3)^* \mathcal{P}$$

(here $(\pi_2 - \pi_1, \pi_3)$ is the map $(x, y, z) \mapsto (y - x, z)$ from $X \times X \times \hat{X}$ to $X \times \hat{X}$). This follows from the theorem of the cube (*Abelian Varieties*,

§10) since the bundles are isomorphic on $\{0\} \times X \times \widehat{X}$, $X \times \{0\} \times \widehat{X}$ and $X \times X \times \{0\}$. Therefore substituting the expression for \mathcal{F} , we get:

$$\begin{aligned} R^i \pi_{12,*}(\mathcal{F}) &\cong (\mathcal{L}_1^{-1} \otimes_{(\mathcal{O}_{X \times X})} \mathcal{L}_2^{-1}) \otimes R^i \pi_{12,*}((\pi_2 - \pi_1, \pi_3)^* \mathcal{P}) \\ &\cong (\mathcal{L}_1^{-1} \otimes_{(\mathcal{O}_{X \times X})} \mathcal{L}_2^{-1}) \otimes (\pi_2 - \pi_1)^*(R^i \pi_* \mathcal{P}) \end{aligned}$$

(the second is flat base change for the diagram:

$$\begin{array}{ccc} X \times X \times \widehat{X} & \xrightarrow{(\pi_2 - \pi_1, \pi_3)} & X \times \widehat{X} \\ \downarrow \pi_{12} & & \downarrow \pi \\ X \times X & \xrightarrow{\pi_2 - \pi_1} & X \end{array} .)$$

This gives Claim II immediately.

We now finish the proof of the big lemma. Apply the Leray spectral sequence for π_{12} :

$$H^p(X \times X, R^q \pi_{12,*}(\mathcal{F})) \implies H^{p+q}(X \times X \times \widehat{X}, \mathcal{F}).$$

By Claim II, this sequence degenerates to:

$$\begin{aligned} H^k(X \times X \times \widehat{X}, \mathcal{F}) &\cong H^{k-g}(X \times X, R^g \pi_{12,*}(\mathcal{F})) \\ &\cong H^{k-g}(\Delta, (\mathcal{L}_1 \otimes \mathcal{L}_2)^{-1}) \\ &\cong H^{2g-k}(X, \mathcal{L}_1 \otimes \mathcal{L}_2)^\wedge \quad \text{by Serre duality.} \end{aligned}$$

Thus

$$(*) \quad H^k(X \times X \times \widehat{X}, \mathcal{F}) = \begin{cases} (0) & \text{if } k \neq 2g \\ \Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2)^\wedge & \text{if } k = 2g. \end{cases}$$

On the other hand, by Claim I the Leray Spectral sequence for π_3 degenerates to:

$$(**) \quad H^k(X \times X \times \widehat{X}, \mathcal{F}) \cong H^{k-2g}(\widehat{X}, (\mathcal{W}^+(\mathcal{L}_1) \otimes \mathcal{W}^-(\mathcal{L}_2))^\wedge).$$

Comparing (*) and (**), we get the lemma.

To prove the theorem, it is helpful to have a notation for "relations" in more general contexts: for all sheaves \mathcal{F}, \mathcal{G} on X , we denote the kernel of multiplication

$$\Gamma(\mathcal{F}) \otimes \Gamma(\mathcal{G}) \longrightarrow \Gamma(\mathcal{F} \otimes \mathcal{G})$$

by $R(\mathcal{F}, \mathcal{G})$.

LEMMA 10.9. If $\ell_1, \ell_2 \geq 2$ and $\ell_1 + \ell_2 \geq 5$, then

$$\sum_{\alpha \in \widehat{X}} R(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathcal{P}_\alpha) \cdot \Gamma(X, \mathcal{L}_3 \otimes \mathcal{P}_{-\alpha}) = R(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathcal{L}_3).$$

PROOF: Let R be the left hand side of this equality. By definition, we have an exact sequence

$$0 \longleftarrow R(\mathcal{L}_1, \mathcal{L}_2, \otimes \mathcal{L}_3)^\wedge \longleftarrow (\Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2 \otimes \mathcal{L}_3))^\wedge \longleftarrow \Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3)^\wedge$$

so to prove the lemma, we just have to show that if λ is a linear functional on $\Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2 \otimes \mathcal{L}_3)$ that vanishes on R , then λ is induced by a linear functional on $\Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3)$.

Since

$$\sum_{\alpha \in \widehat{X}} \Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2 \otimes \mathcal{P}_\alpha) \otimes \Gamma(\mathcal{L}_3 \otimes \mathcal{P}_{-\alpha}) \longrightarrow \Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2 \otimes \mathcal{L}_3)$$

is surjective by Lemma 10.2, λ is determined by its restrictions λ_α to $\Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2 \otimes \mathcal{P}_\alpha) \otimes \Gamma(\mathcal{L}_3 \otimes \mathcal{P}_{-\alpha})$. We have the diagram:

$$\begin{array}{ccccccc} 0 \rightarrow & R(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathcal{P}_\alpha) & \otimes & \Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2 \otimes \mathcal{P}_\alpha) & \otimes & \Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{P}_\alpha) & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & \Gamma(\mathcal{L}_3 \otimes \mathcal{P}_{-\alpha}) & \rightarrow & \Gamma(\mathcal{L}_3 \otimes \mathcal{P}_{-\alpha}) & \rightarrow & \Gamma(\mathcal{L}_3 \otimes \mathcal{P}_{-\alpha}) & \rightarrow 0 \\ & & & & & \downarrow \lambda_\alpha & \\ & & & & & k & \end{array}$$

$\swarrow \mu_\alpha$

Surjectivity in the top line follows from Theorem 10.1. Since λ_α vanishes on $R(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathcal{P}_\alpha) \cdot \Gamma(\mathcal{L}_3 \otimes \mathcal{P}_{-\alpha})$, λ_α induces μ_α as in the diagram. Now let's put all these maps together into maps of bundles over \widehat{X} . Let \mathcal{S} be the kernel of the map

$$\Gamma(\mathcal{L}_1) \otimes \widehat{\pi}_*(\mathcal{L}_2 \otimes \mathcal{P}) \longrightarrow \widehat{\pi}_*(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{P}).$$

This map of bundles is surjective, so \mathcal{S} is a bundle too.

Consider the exact sequence of sheaves on \hat{X} :

$$\begin{array}{ccccccc}
 0 \rightarrow \mathcal{S} \otimes \hat{\pi}_*(\mathcal{L}_3 \otimes \mathcal{P}^{-1}) & \rightarrow & \begin{array}{c} \Gamma(\mathcal{L}_1) \otimes \hat{\pi}_*(\mathcal{L}_2 \otimes \mathcal{P}) \\ \otimes \\ \hat{\pi}_*(\mathcal{L}_3 \otimes \mathcal{P}^{-1}) \end{array} & \rightarrow & \begin{array}{c} \hat{\pi}_*(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{P}) \\ \otimes \\ \hat{\pi}_*(\mathcal{L}_3 \otimes \mathcal{P}^{-1}) \end{array} & \rightarrow & 0 \\
 & & \downarrow \tilde{\lambda} & \swarrow \tilde{\mu} & & & \\
 & & \hat{\mathcal{O}}_X & & & &
 \end{array}$$

Here $\tilde{\lambda}$ is just the globalization of the λ_α 's, i.e., it arises from

$$\Gamma(\mathcal{L}_1) \otimes \hat{\pi}_*(\mathcal{L}_2 \otimes \mathcal{P}) \otimes \hat{\pi}_*(\mathcal{L}_3 \otimes \mathcal{P}^{-1}) \rightarrow \Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2 \otimes \mathcal{L}_3) \otimes \mathcal{O}_{\hat{X}} \xrightarrow{\lambda \otimes 1} \mathcal{O}_{\hat{X}}.$$

$\hat{\lambda}$ restricts to zero on each fibre of the bundle $\mathcal{S} \otimes \hat{\pi}_*(\mathcal{L}_3 \otimes \mathcal{P}^{-1})$, hence it is zero on the whole bundle. Consequently $\tilde{\mu}$ exists.

We apply the Big Lemma: Since the multiplication

$$\Gamma(\hat{X}, (\hat{\pi}_*(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{P}) \otimes \hat{\pi}_*(\mathcal{L}_3 \otimes \mathcal{P}^{-1}))^\wedge) \rightarrow \Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3)^\wedge$$

is an isomorphism, $\tilde{\mu}$ corresponds to an element of $\Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3)^\wedge$. QED

THEOREM 10.10. If $\ell_1 \geq 3, \ell \geq 4$ (or $\ell_1 \geq 2, \ell_2 \geq 5$), and $\ell_3 \geq 2$, then

$$R(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathcal{L}_3) = R(\mathcal{L}_1, \mathcal{L}_2) \Gamma(\mathcal{L}_3).$$

PROOF: Write $\mathcal{L}_2 = \mathcal{L}_4 \otimes \mathcal{L}_5$ with $\ell_5 = 2$, so $\ell_4 \geq 2$. By the above lemma

$$R(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathcal{L}_3) = \sum_{\alpha} R(\mathcal{L}_1, \mathcal{L}_4 \otimes \mathcal{P}_{\alpha}) \Gamma(\mathcal{L}_5 \otimes \mathcal{L}_3 \otimes \mathcal{P}_{-\alpha}).$$

By our surjectivity results

$$\Gamma(\mathcal{L}_5 \otimes \mathcal{P}_{-\alpha}) \Gamma(\mathcal{L}_3) = \Gamma(\mathcal{L}_5 \otimes \mathcal{L}_3 \otimes \mathcal{P}_{-\alpha});$$

therefore

$$R(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathcal{L}_3) = \sum_{\alpha} \underbrace{R(\mathcal{L}_1, \mathcal{L}_4 \otimes \mathcal{P}_{\alpha}) \Gamma(\mathcal{L}_5 \otimes \mathcal{P}_{-\alpha})}_{\substack{| \\ \cap \\ R(\mathcal{L}_1, \mathcal{L}_2)}} \Gamma(\mathcal{L}_3)$$

This gives

$$R(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathcal{L}_3) = R(\mathcal{L}_1, \mathcal{L}_2) \Gamma(\mathcal{L}_3). \quad QED$$

In particular, if $\ell_1 \geq 4$,

$$R(\mathcal{L}_1, \mathcal{L}_1^{n+1}) = R(\mathcal{L}_1, \mathcal{L}_1) \Gamma(\mathcal{L}_1^n) \quad \text{if } n \geq 0$$

and if $\ell_1 \geq 3$

$$R(\mathcal{L}_1, \mathcal{L}_1^{\otimes(n+2)}) = R(\mathcal{L}_1, \mathcal{L}_1^{\otimes 2}) \Gamma(\mathcal{L}_1^{\otimes n}).$$

This proves Theorem 10.6.

Let us recapitulate the results so far: let $\mathcal{M} = \mathcal{L}^n$, $n \geq 4$, \mathcal{L} ample of degree one. Let H_n denote $\frac{1}{n} \mathbb{Z}^g / \mathbb{Z}^g$. Then the basis $\{\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (nz, nT)\}_{a \in H_n}$ of $\Gamma(X, \mathcal{M})$ defines an embedding

$$i_{\theta} : X \hookrightarrow \mathbb{P}^{H_n}$$

into the fixed projective space whose coordinates are indexed by H_n . Here we have taken the complex case for simplicity, but the same construction can be made using algebraic theta functions so long as $\text{char}(k) \nmid n$. We have shown

- a) the ideal of $i_{\theta}(X)$ is generated by quadrics,
- b) if n is even, the quadrics can be given explicitly by (10.7) in terms of the coordinates of $i_{\theta}(0)$.

We can go further and give a geometric construction of the quadrics containing $i_{\theta}(X)$ using only the point $i_{\theta}(0)$ which is valid for *any* $n \geq 4$, even or odd. To do this, let $K_n = \frac{1}{n} \mathbb{Z}^{2g} / \mathbb{Z}^{2g}$ and let K_n act on \mathbb{P}^{H_n} by the maps:

$$X'_a = e^{2\pi i n^t a \cdot b_2} \cdot X_{a+b_1}, \quad a \in H_n$$

for all $(b_1, b_2) \in K_n$. This is obviously the projective version of the irreducible action of the finite Heisenberg group $\text{Heis}(2g, (\mathbb{Z}/n\mathbb{Z}))$ and the usual formulae for θ -functions show that:

PROPOSITION 10.11. K_n maps $i_\theta(X)$ to itself and restricts on $i_\theta(X)$ to translating by the n -torsion subgroup X_n .

In particular, $i_\theta(X_n) = K_n(i_\theta(0))$ and we claim:

PROPOSITION 10.12. If $n \geq 3$, a quadric $Q \subset \mathbf{P}^{H_n}$ contains $i_\theta(X)$ if and only if it contains $i_\theta(X_n)$.

PROOF: Note that $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (nz, T)$ is a basis of $\Gamma(\mathcal{L}^{n^2})$. Choose some $y \in X$ where none of these functions are zero. Assume Q is a quadric which contains $i_\theta(X_n)$. Then Q defines a section $s \in \Gamma(\mathcal{L}^{2n})$ which is zero on X_n . Then $T_{-y}^*(s) \in \Gamma(T_{-y}^* \mathcal{L}^{2n})$ is zero on $y + X_n$. Since $n \geq 3$, $\mathcal{L}^{n^2} \otimes T_{-y}^* \mathcal{L}^{-2n}$ is ample so it has a non-zero section t . Then $T_y^*(s) \otimes t$ is a section of \mathcal{L}^{n^2} zero on $y + X_n$. Now X_n is an isotropic subgroup of $K(\mathcal{L}^{n^2}) = X_{n^2}$ so translation by X_n lifts to an action of X_n on \mathcal{L}^{n^2} and on $\Gamma(X, \mathcal{L}^{n^2})$. Therefore, write

$$T_y^*(s) \otimes t = \sum_{\lambda \in \widehat{X}_n} c_\lambda \cdot s_\lambda$$

where s_λ are eigenfunctions for X_n . There s_λ are just $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (nz)$, as is immediate by calculating $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (n(z + \text{pt of order } n))$. Since each $c_\lambda s_\lambda$ is a linear combination of translates of $T_y^*(s) \otimes t$ by X_n , $c_\lambda s_\lambda$ is zero at y . But all $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}$'s are non-zero at y , so $c_\lambda = 0$ all λ . Therefore $T_{-y}^* s \otimes t = 0$, hence $s = 0$, hence Q contains $i_\theta(X)$ QED

COROLLARY 10.13. For all $n \geq 4$,

$$i_\theta(X) = \{\cap Q \mid \text{all quadrics } Q \text{ such that } Q \supset K_n(i_\theta(0))\}.$$

To end this section, we want to rephrase the fact that the coordinates of $i_\theta(0)$ determine X and \mathcal{M} by saying instead that $\{\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (0, nT)\}_{a \in H_n}$ are homogeneous coordinates on a suitable *moduli space* of abelian varieties. The precise statement is a little technical because, before the algebraic analogs of $\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (0, nT)$ can be defined on an arbitrary abelian variety X ,

some labelling of its points of finite order must be done. We will sketch the result. Whenever $\text{char}(k) \nmid n$, the moduli space \mathcal{A}_n is defined to be the variety which classifies up to isomorphism triples (X, \mathcal{L}, ϕ_n) , where X is an abelian variety, \mathcal{L} is a degree one, ample line bundle on X and ϕ_n is an isomorphism of X_n with $(\mathbf{Z}/n\mathbf{Z})^{2g}$ carrying the skew-symmetric form e_n^L on X_n to the standard form on $(\mathbf{Z}/n\mathbf{Z})^{2g}$ (a primitive n^{th} root of 1 in k must be fixed to define this). ϕ_n is called a "level n structure". When $n|m$, there is a map $\mathcal{A}_m \rightarrow \mathcal{A}_n$ because a level m structure determines a level n structure, i.e., we have the usual tower $\{\mathcal{A}_n\}$ of moduli spaces. To work out the meaning of Cor. 10.13, we need a moduli space intermediate between \mathcal{A}_n and \mathcal{A}_{2n} , which Igusa named $\mathcal{A}_{n,2n}$. We assume n is even. $\mathcal{A}_{n,2n}$ is the variety which classifies up to isomorphism triples (X, \mathcal{M}, α) , where X is an abelian variety, \mathcal{M} is an ample symmetric line bundle on X with $e_*^{\mathcal{M}} \equiv 1$ and α is a symmetric² isomorphism

$$\mathcal{G}(\mathcal{M}) \xrightarrow{\sim} \text{Heis}(2g, \mathbf{Z}/n\mathbf{Z}).$$

Then α induces an isomorphism (unique up to scalars) between the Heisenberg representation $\Gamma(X, \mathcal{M})$ of $\mathcal{G}(\mathcal{M})$ and any of the standard realizations of this representation for $\text{Heis}(2g, \mathbf{Z}/n\mathbf{Z})$. This gives us algebraically a basis s_a , $a \in (\mathbf{Z}/n\mathbf{Z})^g$, of $\Gamma(X, \mathcal{M})$ generalizing the basis $\vartheta \begin{bmatrix} a \\ 0 \end{bmatrix}$ in the complex case, hence an embedding $i_\theta : X \hookrightarrow \mathbf{P}^{H_n}$. The map

$$(X, \mathcal{M}, \alpha) \mapsto i_\theta(0) \in \mathbf{P}^{H_n}$$

gives us a map

$$\Theta_n : \mathcal{A}_{n,2n} \rightarrow \mathbf{P}^{H_n}$$

which is itself one-to-one because $i_\theta(0)$ allows us to reconstruct X, \mathcal{M} and α . The final result is:

²i.e., the symmetry $i^{\mathcal{M}}$ of $\mathcal{G}(\mathcal{M})$ corresponds to the involution $(\lambda, x, y) \mapsto (\lambda, -x, -y)$.

THEOREM 10.14. a) For all $n \geq 4$, Θ_n is an immersion of the scheme $\mathcal{A}_{n,2n}$ in \mathbf{P}^{H_n} .

b) If n is even and $n \geq 6$, then $\text{Im}(\Theta_n)$ is a Zariski-open subset of the closed subscheme in \mathbf{P}^{H_n} defined by the quartic polynomials given by the equations (10.7) with $z = 0$:

$$\left(\sum_{\eta} s_{\eta} X_{a'+d+\eta} X_{b'+d+\eta} \right) \cdot \left(\sum_{\eta} s_{\eta} X_{a+\eta} X_{b+\eta} \right) = \left(\sum_{\eta} s_{\eta} X_{a+d+\eta} X_{b+d+\eta} \right) \cdot \left(\sum_{\eta} s_{\eta} X_{a'+\eta} X_{b'+\eta} \right)$$

where $s_{\eta} = (-1)^{2c \cdot 2\eta}$, $a, b, a', b', d \in (\frac{1}{n}\mathbf{Z}^g / \mathbf{Z}^g)$ satisfying $a + b = a' + b'$ and the sum being over $\eta \in \frac{1}{2}\mathbf{Z}^g / \mathbf{Z}^g$.

PROOF: See Mumford, *Equations Defining Abelian Varieties II*, §6 for the case $8|n$ and Kempf, *Linear systems on abelian varieties*, Am. J. Math., (1989), for the general case. It would be really nice if this were also true for $n = 4$, but this is open.

Progress in Mathematics

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