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A RANK 2 VECTOR BUNDLE ON $\mathbb{P}^4$ WITH 15,000 SYMMETRIES

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THE MOTIVATION for this paper was to look for rank 2 vector bundes $\mathscr{F}$ on $\mathbb{P}^n$ for $n \geqq 4$ which are not direct sums of lines bundles. Schwarzenberg [14], found many such bundles on $\mathbb{P}^2$ and one of us [4] found quite a few on $\mathbb{P}^3$ although already they seem to be "rarer". In this paper, we construct *one* on $\mathbb{P}^4$. It seems quite plausible that there are none on $\mathbb{P}^n$ if n is large enough. The question is closely related to the existence of non-singular subvarieties $X^{n-2} \subset \mathbb{P}^n$ of dimension $n-2$ which are not complete intersections:

$$X = H_1.H_2.$$

If $\mathscr{F}$ is an indecomposable rank 2 vector bundle and $n \geq 3$ then for $k \gg 0$, a general section $s \in \Gamma(\mathscr{F}(k))$ will vanish on a non-singular X^{n-2} which is not a complete intersection; conversely, if $X^{n-2} \subset \mathbb{P}^n$ is non-singular and $n \geq 6$, a recent result of Barth and Larsen ([1] and [9]) shows that the line bundle Ω_X^{n-2} is isomorphic to $\mathcal{O}_X(k)$ for some k, from which it follows readily that X is the zero-set of a section of a rank 2 bundle $\mathscr{F}$. And if X is not a complete intersection, then $\mathscr{F}$ is indecomposable. Now interestingly enough, it seems as far as we know that classical procedures and classical examples yield non-singular X^{n-2}'s in $\mathbb{P}^n$, which are not complete intersections, *only if* $n \leq 5$.

The vector bundle constructed here has a 4-dimensional space of sections almost all of which vanish on a non-singular $X_s \subset \mathbb{P}^4$ which is an abelian surface. We first found the bundle by establishing that such X_s's had to exist and then constructing $\mathscr{F}$ from X_s as an extension. However by then applying the general "Postnikov" construction of [3], we found a much more direct description of $\mathscr{F}$. The theory of the bundle $\mathscr{F}$ and of the surfaces X_s is united by the fact that both are acted on by the Heisenberg group H (an irreducible 2-step nilpotent subgroup of $SL_5(\mathbb{C})$ of order 125: cf. §1) which is well known from the theory of theta functions; $\mathscr{F}$ is acted on also by the normalizer N of H, of order 15,000. We have developed all our results by keeping track of the action of N at every stage and using the character table of N where necessary. This is a quick efficient method although unfortunately not very illuminating. Our main results are as follows: we construct the bundle $\mathscr{F}$ in §2 and note immediately by examining its Chern classes that it is indecomposable; in §4 we find the cohomology of $\mathscr{F}(n)$ for every n; in §5, we prove that the zero-sets X_s of its general sections are abelian and that conversely all abelian surfaces in $\mathbb{P}^4$ arise in this way; in §6, we show that as a corollary we get an explicit birational map between a certain moduli space of abelian surfaces and $\mathbb{P}^3$. We have put the character table of $SL_2(\mathbb{Z}_5)$ in an appendix for easy reference.

§1. THE HEISENBERG GROUP IN DIMENSION FIVE

The purpose of this section is to review in a special case a configuration of groups studied recently by Weil [16] (cf. also Igusa [7, Chap. 1]; Mumford [12, §1]) and closely related to the theory of theta functions and abelian varieties. Weil's construction starts from an arbitrary locally compact abelian group A, but we take $A = \mathbb{Z}_5$, the cyclic group of order 5, and proceed as follows:

Let

$$V = \mathrm{Map}(\mathbb{Z}_5, \mathbb{C}),$$

be the vector-space of complex-valued functions on $\mathbb{Z}_5$. Note that V has a natural $\mathbb{Q}$-rational structure given by the $\mathbb{Q}$-subspace $\mathrm{Map}(\mathbb{Z}_5, \mathbb{Q})$. Let $\varepsilon = e^{2\pi i/5} \in \boldsymbol{\mu}_5$, the group of 5th roots of 1. The *Heisenberg group*

$$H \subset SL_5(\mathbb{C})$$

is the subgroup generated by σ and τ, given by

$$\sigma x(i) = x(i+1)$$
$$\tau x(i) = \varepsilon^i x(i)$$

for all $x \in V$. Explicitly, H is the set of matrices

$$A_{ij} = (\varepsilon^{ai+b} \cdot \delta_{i,j+c})$$

and has order 125. An an algebraic group, H is defined over $\mathbb{Q}$, but it only splits over $\mathbb{Q}(\varepsilon)$. The Galois group Θ of $\mathbb{Q}(\varepsilon)$ over $\mathbb{Q}$ acts on H. Let $\theta \in \Theta$ be the generator given by $\theta(\varepsilon) = \varepsilon^2$ (so that $\theta^2 =$ complex conjugation). We shall sometimes use the notation $'$ to indicate the action of θ. The group H has center C equal to $\boldsymbol{\mu}_5 \cdot I_V$ and is a central extension:

$$1 \to \boldsymbol{\mu}_5 \to H \to \mathbb{Z}_5 \times \mathbb{Z}_5 \to 1, \tag{1.1}$$

where σ, τ in H are mapped to (1, 0), (0, 1). The action of Θ preserves this sequence and θ acts on $\mathbb{Z}_5 \times \mathbb{Z}_5$ by $(n, m) \to (n, 2m)$.

V is clearly an irreducible H-module, and it gives rise to three more by the action of Θ: let V_i be the representation obtained from V by composing $H \to \mathrm{Aut}\, V$ with θ^i. The trace $\langle h, V_i \rangle$ of an element $h \in H$ on V_i is given by:

$$\langle \varepsilon^r I_V, V_i \rangle = 5 \cdot \varepsilon^{2^i r}, \qquad \langle h, V_i \rangle = 0 \qquad (h \in H - C). \tag{1.2}$$

It follows that the four representations V_i are inequivalent. These plus the 25 characters of $\mathbb{Z}_5 \times \mathbb{Z}_5$ exhaust the irreducible representations of H since the sum of squares of their degrees is 125, the order of H.

Let ϕ: $\mathbb{Z}_5 \times \mathbb{Z}_5 \to H$ be the section of (1.1) given by:

$$\phi(m, n) = \varepsilon^{2mn} \sigma^m \tau^n,$$

and define ω: $\boldsymbol{\mu}_5 \times (\mathbb{Z}_5 \times \mathbb{Z}_5) \to H$ by:

$$\omega(\alpha, z) = \alpha \cdot \phi(z).$$

Then ω is bijective and the group law on H goes over to the law of composition:

$$(a, z) \cdot (a', z') = (aa'B(z, z'), z + z')$$

where

$$B(m, n; m', n') = \varepsilon^{3(mn' - m'n)}$$

is a μ_5-valued skew-symmetric form on $\mathbb{Z}_5 \times \mathbb{Z}_5$. Note that all automorphisms of H preserve the sequence (1.1) and since $B(z, z')^2 \cdot I_V$ is the commutator of (a, z) and (a', z'), they preserve the form B.

Let N be the normalizer of H in $SL_5(\mathbb{C})$. Each element of N induces by conjugation an automorphism of H, hence an automorphism of $\mathbb{Z}_5 \times \mathbb{Z}_5$ preserving B. But the group of such automorphisms is isomorphic to $SL_2(\mathbb{Z}_5)$, hence we get a homomorphism:

$$\alpha: N \to SL_2(\mathbb{Z}_5).$$

The kernel of α is just H itself because (a) any automorphism of H which is the identity on C and on H/C is in fact inner, and (b) since the representation V is irreducible, C is the centralizer of H in $SL_2(\mathbb{Z}_5)$. Moreover α is surjective. If $x \in SL_2(\mathbb{Z}_5)$, define $\gamma_x: H \to H$ by

$$\gamma_x \omega(a, z) = \omega(a, x(z)).$$

Since x preserves B, the mapping γ_x is an automorphism. The new representation of H on V obtained by composing with γ_x is equivalent to V since γ_x is the identity on C and so leaves the character fixed. So x is induced by an element of N, in fact an element of $N \cap SL_5(\mathbb{Q}(\varepsilon))$. Thus α is surjective and $N \subset SL_5(\mathbb{Q}(\varepsilon))$, hence Θ acts on N and the action of N on V induces actions on each V_i.

Since $\gamma_x \cdot \gamma_y = \gamma_{xy}$ and γ_x is induced by a member of N determined up to multiplication by elements of C, it follows that N/C is a semi-direct product $(H/C) \cdot SL_2(\mathbb{Z}_5)$. Let X be the inverse image in N of the factor $SL_2(\mathbb{Z}_5)$. Then X is a central extension of $SL_2(\mathbb{Z}_5)$ by C. But the group of Schur multipliers $H^2(SL_2(\mathbb{Z}_5), \mathbb{C}^*)$ is zero [5, p. 645], hence X is a product $C \cdot SL_2(\mathbb{Z}_5)$ and the full group N is a semi-direct product $H \cdot SL_2(\mathbb{Z}_5)$.

Next, look at the dual representation V_i^* of V_i: since N acts on each V_i by unitary representations, V_i^* is isomorphic as N-module to the complex conjugate $\overline{V}_i$, i.e. to V_{i+2}.

Finally, look at the representation of N in $V_i \otimes V_i^*$. C acts trivially here so we have a representation of N/C. For all $x \in V_i$, $l \in V_i^*$, put

$$F_{x \otimes l}(h) = l(hx).$$

This gives a map

$$F: V_i \otimes V_i^* \to \mathrm{Map}(H, \mathbb{C})$$

which is easily seen to be injective, with image the space W_i of functions f on H such that

$$f(\alpha h) = \alpha^{2i} \cdot f(h), \qquad \alpha \in \mu_5 I_V.$$

Moreover, for every $n \in N$, let n induce by conjugation the automorphism n^* of H. Then

$$\begin{aligned} F_{n(x \otimes l)}(h) &= F_{nx \otimes l \cdot n^{-1}}(h) \\ &= l \cdot n^{-1}(hnx) \\ &= l(n^*(h)x) \\ &= F_{x \otimes l}(n^*h) \end{aligned}$$

so F transforms the cation of n on $V_i \otimes V_i^*$ to the cation $f \mapsto f \cdot n^*$ on $\mathbb{C}$-valued functions on H. Now it is easy to check that if

$$\tilde{f}(\omega(\alpha, z)) = \alpha^{2^i} \cdot f(\omega(\alpha, 2z)),$$

then $f \mapsto \tilde{f}$ is an isomorphism of W_i and W_{i+1} commuting with the action of N: therefore the four representations $V_i \otimes V_i^*$ are all equivalent, so we may as well work with $V \otimes V^*$. This space has a decomposition $\mathbb{C} \oplus Z$ where Z is the subspace of trace zero. One sees immediately that as an H/C-module Z is the sum of the 24 non-trivial (linear) characters of H/C. Since N acts transitively on these, Z is irreducible as N/C-module. Its character ζ has values in $\mathbb{Q}$ since Z is equivalent to all its conjugates.

To summarize our conclusions, we have found groups:

$$\begin{array}{ccccccc} \text{order } 5 & & \text{order } 125 & & \text{order } 15{,}000 & & \\ C & \subset & H & \subset & N & \subset & SL_5(\mathbb{Q}(\varepsilon)). \\ \wr\Vert & & H/C & & \wr\Vert & & \\ \boldsymbol{\mu}_5 & & \wr\Vert & & H \cdot SL_2(\mathbb{Z}_5) & & \\ & & \mathbb{Z}_5 \times \mathbb{Z}_5 & & & & \end{array} \qquad (1.3.)$$

It is not hard to work out the explicit matrices representing elements of N. They turn out to be of two types:

$$A_{ij} = \pm \frac{\varepsilon^{ai^2 + bij + cj^2 + di + ej + f}}{\sqrt{5}} \qquad (a, \ldots, f \in \mathbb{Z}_5, b \neq 0)$$

and

$$A_{ij} = \pm \varepsilon^{ai^2 + bi + c} \delta_{i, dj+e} \qquad (a, \ldots, e \in \mathbb{Z}_5, d \neq 0)$$

(the sign being adjusted to make the determinant $+1$). It will be necessary to identify some special elements of N of the second type for the purpose of computation. Look at the elements $\iota, \mu, \nu \in SL_5(\mathbb{Q}(\varepsilon))$ given by

$$\iota x(i) = x(-i)$$
$$\mu x(i) = -x(2i)$$
$$\nu x(i) = \varepsilon^{i^2} \cdot x(i)$$

$$\begin{aligned} &\langle \iota, V_i \rangle = 1 \\ &\langle \mu, V_i \rangle = -1, \langle \mu^2, V_i \rangle = 1 \\ &\langle \nu, V_i \rangle = \theta^i(\eta - \eta'), \langle \nu^2, V_i \rangle = \theta^i(\eta' - \eta) \end{aligned} \qquad (1.4)$$

where $\eta = \varepsilon + \varepsilon^4$, $\eta' = \varepsilon^2 + \varepsilon^3$. Conjugating σ and τ, we find

$$\begin{aligned} \iota^{-1}\sigma\iota &= \sigma^{-1}, & \iota^{-1}\tau\iota &= \tau^{-1} \\ \mu^{-1}\sigma\mu &= \sigma^2, & \mu^{-1}\tau\mu &= \tau^3 \\ \nu^{-1}\sigma\nu &= \sigma\tau^2 \bmod C, & \nu^{-1}\tau\nu &= \tau. \end{aligned}$$

Thus ι, μ, $\nu \in N$ and their images $\bar{\iota}$, $\bar{\mu}$, $\bar{\nu}$ in $\mathrm{SL}_2(\mathbb{Z}_5)$ are:

$$\bar{\iota} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\bar{\mu} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \tag{1.5}$$
$$\bar{\nu} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

§2. THE BUNDLE $\mathscr{F}$

Let $\mathbb{P}$ be the projective space representing the one-dimensional subspaces of V^i. Since V is given an underlying rational vector space, $\mathbb{P}$ is to be regarded as the complexification of a scheme over $\mathbb{Q}$. In particular it is meaningful to speak of coherent sheaves and their homomorphisms as being defined over specified subfields of $\mathbb{C}$.

Write $\mathcal{O}$ for the sheaf of local rings of $\mathbb{P}$ and $\mathcal{O}(1)$ for the canonical positive invertible sheaf on $\mathbb{P}$. The general linear group acts on $\mathcal{O}(1)$ and the space of sections $\Gamma(\mathcal{O}(1))$ is canonically isomorphic to V^* the dual of V. Regard V as a sheaf over Spec $\mathbb{C}$. The external tensor product $\mathcal{O}(1) \otimes_{\mathbb{C}} V$ is a sheaf on $\mathbb{P}$ and $\Gamma(\mathcal{O}(1) \otimes_{\mathbb{C}} V)$ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(V, V)$. Let ∂ in $\Gamma(\mathcal{O}(1) \otimes_{\mathbb{C}} V)$ correspond to I_V. The Koszul complex $\mathscr{K}$ is the exterior algebra $\Lambda^*(\mathcal{O}(1) \otimes_{\mathbb{C}} V)$ with multiplication by ∂ as differential:

$$0 \to \mathcal{O} \to \mathcal{O}(1) \otimes V \to \mathcal{O}(2) \otimes \Lambda^2 V \to \mathcal{O}(3) \otimes \Lambda^3 V \to \mathcal{O}(4) \otimes \Lambda^4 V \to \mathcal{O}(5) \otimes \Lambda^5 V \to 0.$$

The quotient $\mathcal{O}(1) \otimes V/\mathcal{O}$ is isomorphic to the tangent sheaf $\mathscr{T}$ to $\mathbb{P}$ and the sheaf of cycles $\mathrm{Im}(\mathcal{O}(i) \otimes \Lambda^i V) \subset \mathcal{O}(i+1) \otimes \Lambda^{i+1} V$ is isomorphic to the ith exterior power $\Lambda^i \mathscr{T}$ of $\mathscr{T}$. For the construction of the bundle $\mathscr{F}$ the relevant part of $\mathscr{K}$ is:

$$\overbrace{\mathcal{O}(2) \otimes_{\mathbb{C}} \Lambda^2 V}^{rk\ 10} \xrightarrow{p_0} \overbrace{\Lambda^2 \mathscr{T}}^{rk\ 6} \xrightarrow{q_0} \overbrace{\mathcal{O}(3) \otimes_{\mathbb{C}} \Lambda^3 V}^{rk\ 10}. \tag{2.1}$$

Note also that $\mathscr{K}$ has a symmetric pairing

$$\mathscr{K}^i \otimes \mathscr{K}^{5-i} \to \mathcal{O}(5) \otimes_{\mathbb{C}} \Lambda^5 V \cong \mathcal{O}(5),$$

given by $x \otimes y \to (x \wedge y)_5$, and that this induces the natural pairing $\Lambda^i \mathscr{T} \otimes \Lambda^{4-i} \mathscr{T} \to \mathcal{O}(5)$ and is compatible with the action of $SL_5(\mathbb{C})$. Note that with respect to these pairings $q_0 = p_0{}^*(5)$.

The H-modules $\Lambda^2 V$ and $2V_1$ are isomorphic since $\langle \varepsilon I_V, \Lambda^2 V \rangle = 10\varepsilon^2$ and by (1.2) $2V_1$ is the only representation of degree 10 for which this is possible. In identifying $\Lambda^2 V$ and other such spaces as N-modules, the reader should use the general observation:

(2.2) *Let Y, Z be representation spaces for a group G and let K be a normal subgroup of G. Suppose that Y is irreducible as a K-module and that $Z \cong nY$ as K-modules, then* $\mathrm{Hom}_K(Y, Z)$ *is a G/K-module, the evaluation mapping $Y \otimes \mathrm{Hom}_K(Y, Z) \to Z$ is an isomorphism of G-modules, and Z is irreducible as a G-module if and only if* $\mathrm{Hom}_K(Y, Z)$ *is irreducible as a G/K-module.*

In the present case, put $W = \mathrm{Hom}_H(V_1, \Lambda^2 V)$. It is a representation of N/H of degree 2, and the trace of $\bar{v}$ (the image of v in N/H) is

$$\langle \bar{v}, W \rangle = (3 + \varepsilon + \varepsilon^4)/(1 + 2\varepsilon^2 + 2\varepsilon^3) = \eta'.$$

It follows that W has character χ_2 (cf. character table in Appendix).

Since N/H is perfect, W is unimodular. So W has an invariant skew symmetric pairing defined over $\mathbb{Q}$ and this form is unique up to a scale factor. Let

$$f\colon V_1 \to \Lambda^2 V \otimes W$$

be the N-homomorphism determined by this form, and let

$$g\colon \Lambda^3 V \otimes W \to V_3 (= V_1^*)$$

be the dual of f composed with the canonical mapping $\Lambda^3 V \otimes W \cong \Lambda^3 V \otimes W^*$. Combining these with (2.1) gives the sequence of sheaf homomorphisms

$$\begin{aligned} \mathcal{O}(2) \otimes V_1 \xrightarrow{1_{\mathcal{O}} \otimes f} \mathcal{O}(2) \otimes \Lambda^2 V \otimes W \xrightarrow{p_o \otimes 1_W} \Lambda^2 \mathcal{T} \otimes W \xrightarrow{q_o + 1_W} \\ \mathcal{O}(3) \otimes \Lambda^3 V \otimes W \xrightarrow{1_{\mathcal{O}} \otimes g} \mathcal{O}(3) \otimes V_3. \end{aligned} \tag{2.3}$$

This sequence is defined over $\mathbb{Q}$.

Let

$$p\colon \overbrace{\mathcal{O}(2) \otimes V_1}^{rk\ 5} \to \overbrace{\Lambda^2 \mathcal{T} \otimes W}^{rk\ 12}, \qquad q\colon \overbrace{\Lambda^2 \mathcal{T} \otimes W}^{rk\ 12} \to \overbrace{\mathcal{O}(3) \otimes V_3}^{rk\ 5}$$

be the composites of the first two and last two morphisms in 2.3. Note that $q \cong p^*(5)$. We shall prove that $qp = 0$ and p, q are locally split. From this it follows that $\mathcal{F} = \mathrm{Ker}\, q/\mathrm{Im}\, p$ is locally free of rank 2 and defined over $\mathbb{Q}$. The bundle $\mathcal{F}$ is our goal.

To prove that $qp = 0$ it is sufficient, since $\mathcal{O} \otimes V_1$ is generated by its sections, to show that $\Gamma(qp(-2)) = 0$ and to prove that p, q are locally split it is sufficient that p is – for then q is locally split by duality. The first assertion follows immediately from

LEMMA 2.4. *Let U be the symmetric square representation $S^2 W$ of degree* 3, *and let W' be the representation obtained by acting on W with the Galois automorphism θ.*

(i) $\Gamma(\Lambda^2 \mathcal{T}(-2)) \cong \Lambda^2 V \cong V_1 \otimes W$, $\Gamma(\Lambda^2 \mathcal{T}(-2) \otimes W) \cong V_1 \oplus (V_1 \otimes U)$, *and* $\Gamma(p(-2))$ *is equivalent to the inclusion* $V_1 \to V_1 \oplus (V_1 \otimes U)$.

(ii) $\Gamma(\mathcal{O}(1) \otimes V_3) \cong (V_1 \otimes U) \oplus (V_1 \otimes W')$ *and* $\Gamma(q(-2))$ *is equivalent to the homomorphism* $V_1 \oplus (V_1 \otimes U) \to (V_1 \otimes U) \oplus (V_1 \otimes W')$ *induced by the identity on* $V_1 \otimes U$.

Proof. (i) The first isomorphism follows from (2.1), and the second is just the evaluation mapping. The third isomorphism now follows from the decomposition $W \otimes W = \mathbb{C} \oplus U$. Finally, since $\Gamma(\Lambda^2 \mathcal{T}(-2)) \cong \Lambda^2 V$, the mapping $\Gamma(p(-2))$ is equivalent to f which is just the mapping induced by $\mathbb{C} \to \mathbb{C} \oplus U$.

(ii) First note that $\Gamma(\mathcal{O}(1) \otimes V_3) \cong V^* \otimes V_3$. The character of $V^* \cong V_3$ as an H-module is given by

$$\langle h, V^* \otimes V_3 \rangle = 0 (h \in H - C), \quad \langle \varepsilon^i 1, V^* \otimes V_3 \rangle = 25\varepsilon^{2i}.$$

So $V^* \otimes V_3 \cong 5V_1$ as an H-module. Put $X = \mathrm{Hom}_H(V_1, V^* \otimes V_3)$. The formulas (1.4) show that

$$\langle \bar{\mu}, X \rangle = -1, \langle \bar{\mu}^2, X \rangle = 1, \langle \bar{v}, X \rangle = \eta - \eta'.$$

The first of these shows that X must have an irreducible component X^1 of degree 3, the second that the remaining component X^2 is irreducible of degree 2, and the third that X^1, X^2 have characters χ_3, χ_2'. Since $\chi_2{}^2 = \chi_3 + \chi_1$ and $\chi_2' = \theta\chi_2$ it follows that $X^1 \cong U$ and $X^2 \cong W'$.

To prove the statement about $\Gamma q(-2)$ it is sufficient to show that $\Gamma q(-2) \neq 0$, for V_1, $V_1 \otimes U$, $V_1 \otimes W$ are irreducible. But $q \cong p^*(5)$ and $p \neq 0$ by (i). So $q \neq 0$, and $\Gamma q(-2) \neq 0$ since $\Gamma(\mathcal{O}(1) \otimes V_3)$ generates $\mathcal{O}(1) \otimes V_3$. *Q.E.D.*

It remains to be proven that $p(-2)$ and hence p splits locally. Let v be a non-zero element of V and write $\hat{v}$ for the corresponding point of $\mathbb{P}$. We must show that the induced map on the vector bundle fibres:

$$p(-2)_{\hat{v}}: V_1 \to \Lambda^2 T_{\hat{v}} \otimes \mathcal{O}(-2)_{\hat{v}} \otimes W$$

is injective ($T_{\hat{v}}$ = tangent space to $\mathbb{P}$ at $\hat{v}$). But via p_o

$$\Lambda^2 T_{\hat{v}} \otimes \mathcal{O}(-2)_{\hat{v}} \cong \Lambda^{2+}/v \wedge V,$$

so in view of (2.3), the injectivity of $p(-2)_{\hat{v}}$ is equivalent to:

LEMMA 2.5. *For all nonzero $v \in V$ and $t \in V_1$, the element $f(t) \notin (v \wedge V) \otimes W$.*

Proof. Let v_i be the element of V defined by $v_i(j) = \delta_{ij}$ and put

$$z_0{}^+ = v_2 \wedge v_3, z_1{}^+ = v_3 \wedge v_4, z_2{}^+ = v_4 \wedge v_0, z_3{}^+ = v_0 \wedge v_1, z_4{}^+ = v_1 \wedge v_2$$
$$z_0{}^- = v_1 \wedge v_4, z_1{}^- = v_2 \wedge v_0, z_2{}^- = v_3 \wedge v_1, z_3{}^- = v_4 \wedge v_2, z_4{}^- = v_0 \wedge v_3.$$

The linear mappings $w^+: V_1 \to \Lambda^2 V$, $w^-: V_1 \to \Lambda^2 V$ defined by $w^+(v_i) = z_i{}^+$, $w^-(v_i) = z_i{}^-$ are H-homomorphisms, and form a base for W. We wish to show that for non-zero $v \in V$, $t \in V_1$, the equations

$$w^+(t) = v \wedge y^+, w^-(t) = v \wedge y^-$$

are contradictory. But these imply that

$$w^+(t) \wedge w^-(t) = 0,$$

and if $t = \sum a_i v_i$, then one computes that

$$w^+(t) \wedge w^-(t) = \sum_{i=0}^{4} (-1)^i a_i{}^2 v_0 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_4$$

hence $t = 0$. *Q.E.D.*

This completes the proof that $\mathscr{F}$ is a locally free sheaf of rank 2. To show that $\mathscr{F}$ is indecomposable it is sufficient to verify that its total Chern class $c(\mathscr{F})$ is irreducible. By definition

$$c(\mathscr{F}) = c(\Lambda^2 \mathscr{T})^2 \cdot c(\mathcal{O}(2))^{-5} \cdot c(\mathcal{O}(3))^{-5}.$$

Let h be the positive generator for the Chow ring of $\mathbb{P}$. It follows that

$$c(\mathscr{F}) = ((1+2h)^{10}(1+h)^{-5})^2(1+2h)^{-5}(1+3h)^{-5}$$

(where the first factor comes from the resolution of $\Lambda^2\mathscr{T}$ given by the Koszul complex). Hence

$$c(\mathscr{F}) = 1 + 5h + 10h^2.$$

Applying the Riemann–Roch theorem or directly from the definition of $\mathscr{F}$, one also computes the Hilbert polynomial:

$$\chi(\mathscr{F}(n-5)) = \tfrac{1}{12}(n^2-1)(n^2-24).$$

§3. THE INVARIANT QUINTICS

This section is preliminary to the computation of $\Gamma(\mathscr{F})$ and the proof of the non-singularity of the zero set of a general section. The main results are the determination of the N/H-module $\Gamma_H(\mathcal{O}(5))$ of H-invariants of $\Gamma(\mathcal{O}(5))$ and the sheaf of ideals $\mathscr{L}$ in $\mathcal{O}$ generated by the subspace $\Gamma_H(\mathcal{O}(5))$. In the next section we show that this subspace is isomorphic to the second exterior power of $\Gamma(\mathscr{F})$, however the present section does not depend on this fact.

Write a_i for the character of $\Lambda^i V$ as an N-module, h_i for the character of S^iV, and a_i^*, h_i^* for the characters of the duals. Then

$$\begin{aligned} a_i^* &= \bar{a}_i = \theta^2 a_i \\ h_i^* &= \bar{h}_i = \theta^2 h_i \end{aligned} \tag{3.1}$$

since the representations are unitary. Also

$$a_i^* = a_{5-i} \tag{3.2}$$

since the representation is unimodular. As in §1, decompose $V \otimes V^*$ into $\mathbb{C} \oplus Z$ and let ζ be the character of Z. It follows that $h_1 \cdot \theta^2 h_1 = \theta h_1 \cdot \theta^3 h_1 = 1 + \zeta$.

LEMMA 3.3. (i) $a_1 = h_1$

(ii) $a_2 = \chi_2 \cdot \theta h_1$

(iii) $a_3 = \chi_2 \cdot \theta^3 h_1$

(iv) $a_4 = \theta^2 h_1$

(v) $h_2 = \chi_3' \cdot \theta h_1$

(vi) $h_3 = (\chi_5 + \chi_2') \cdot \theta^3 h_1$

(vii) $h_4 = (\chi_4 + \chi_4^{\#} + \chi_3 + \chi_3') \cdot \theta^2 h_1$

(viii) $h_5 = (\chi_3 + \chi_3') + \zeta \cdot (\chi_3 + \chi_3' - 1)$

(ix) $h_1 \theta h_1 = (\chi_3 + \chi_2') \cdot \theta^3 h_1$.

Proof. (ii) follows from Lemma (2.4) and then (i), (iii) and (iv) follows from (3.1) and (3.2). To prove (*v*), note that (2.2) implies $h_2 = \chi \cdot \theta h_1$ for some character χ of N/H. A simple computation shows that $\chi(\iota) = 3$ and $\chi(\nu) = -\eta$. So since χ nas degree 3, $\chi = \chi_3'$. Now use the well-known formula:

$$h_i = a_1 h_{i-1} - a_2 h_{i-2} + \cdots - (-1)^i a_i h_0,$$

plus the identity (ix) proven in (2.4) and (vi), (vii) and (viii) follow by computing characters via the character table in the Appendix. *Q.E.D.*

The first of the main results of this section follows at once from part (viii) of this lemma:

THEOREM 3.5. *The character of* $\Gamma_H(\mathcal{O}(5))$ *is* $\chi_3 + \chi_3'$ *and its dimension is* 6.

Let y_i be the ith coordinate function on V, $(y_i(x) = x(i))$. The monomials

$$y_0^5,\ y_0^3 y_1 y_4,\ y_0^3 y_2 y_3,\ y_0^2 y_2^2 y_1,\ y_0^2 y_1^2 y_3,\ \prod_{i=0}^{4} y_i,$$

are invariants of τ, and the six forms

$$S = \sum y_i^5,\ Q,\ Q',\ R,\ R',\ Y = 5\prod y_i$$

obtained by summing these monomials over the powers of σ are invariants of H. Since they are linearly independent they form a base for $\Gamma_H(\mathcal{O}(5))$.

Another natural basis of $\Gamma_H(\mathcal{O}(5))$ is obtained as follows: the group H/C has six proper subgroups and these subgroups are permuted triply transitively by N. The fixed point set in $\mathbb{P}$ of the subgroup $\{C, \tau C, \tau^2 C, \tau^3 C, \tau^4 C\}$ is just the simplex of reference. The six simplexes determined in this way by the six subgroups we call the *fundamental simplexes*. Each of them determines, up to a scalar multiple, the quintic whose zero set consists of five three-dimensional faces of the simplex. The subspace of $\Gamma_H(\mathcal{O}(5))$ that these quintics span is invariant under both N and Θ. So Theorem 3.5 shows that these six quintics also form a base for $\Gamma_H(\mathcal{O}(5))$.

Now let L be the set of common zeros of the polynomials in $\Gamma_H(\mathcal{O}(5))$, and let L_i be its intersection with $y_i = 0$. But

$$\begin{aligned}
S(0, y_1, \ldots, y_4) &= y_1^5 + y_2^5 + y_3^5 + y_4^5 \\
Q(0, y_1, \ldots, y_4) &= y_2 y_3 (y_2^2 y_1 + y_3^2 y_4) \\
Q'(0, y_1, \ldots, y_4) &= y_1 y_4 (y_1^2 y_3 + y_4^2 y_2) \\
R(0, y_1, \ldots, y_4) &= y_2 y_3 (y_1^2 y_3 + y_4^2 y_2) \\
R'(0, y_1, \ldots, y_4) &= y_1 y_4 (y_2^2 y_1 + y_3^2 y_4).
\end{aligned}$$

These equations define the set L_0 and it is straightforward to check that it consists of precisely the five lines

$$y_2 + \varepsilon^r y_3 = \varepsilon^{2r} y_1 + y_4 = 0 \qquad (*)$$

plus the 20 points

$$\begin{aligned}
y_1 &= y_2 = 0,\ y_3 = \varepsilon^r y_4 \\
y_1 &= y_3 = 0,\ y_2 = \varepsilon^r y_4 \\
y_4 &= y_2 = 0,\ y_3 = \varepsilon^r y_1 \\
y_4 &= y_3 = 0,\ y_2 = \varepsilon^r y_1.
\end{aligned}$$

The set of points of L_i is just $\sigma^i L_0$, the same set with $y_i = 0$ after a cyclic permutation of the coordinates. Taking the union over all i, it follows that the set L consists of the 25 skew lines:

$$y_i = y_{i+2} = \varepsilon^r y_{i+3} = \varepsilon^{2r} y_{i+1} + y_{i+4} = 0, \qquad (0 \leqq i, r \leqq 4). \tag{3.6}$$

We claim that the *scheme* $\mathcal{O}/\mathcal{L}$ is this set of 25 skew lines with reduced structure sheaf, hence is a regular scheme. If $x \in L$ lies on only one face of each of the fundamental simplices, then the ideal $\mathcal{L}_x$ is defined by six linear forms in the y_i's, so $\sqrt{\mathcal{L}_x} = \mathcal{L}_x$. On the other hand, say $x \in L$ lies on a 2-dimensional face of at least one fundamental simplex. Then x is necessarily on a 1-dimensional face or edge of this simplex (as you see by intersecting the line (3.6) with a typical 2-dimensional face $y_0 = y_1 = 0$ on the simplex of reference). But the edges of two distinct fundamental simplices do not intersect: in fact N permutes the fundamental simplices triply transitively and one may readily calculate that the edges of $\prod_i y_i = 0$ and $\prod_i (y_0 + \varepsilon^i y_1 + \varepsilon^{2i} y_2 + \varepsilon^{3i} y_3 + \varepsilon^{4i} y_4) = 0$ do not intersect. Therefore x is singular on at most one fundamental simplex. This shows that $\mathcal{L}_x$ is generated by five linear forms and one of higher degree. If these five linear forms met in a plane, L would contain more than one line in this plane, i.e. L would have two components that met. Since this is false, $\mathcal{L}_x$ is generated by the five linear forms and $\sqrt{\mathcal{L}_x} = \mathcal{L}_x$ again.

§4. THE SPACES $H^i(\mathcal{F}(n))$

Let $\psi^i(n)$ be the character of $H^i(\mathcal{F}(n))$ as a representation of N. To determine these characters we write $\mathcal{G}$ for the cokernel of p (cf. §2) and consider the exact sequences

$$\begin{aligned} &0 \to \mathcal{O}(2) \otimes V_1 \to \Lambda^2 \mathcal{T} \otimes W \to \mathcal{G} \to 0 \\ &0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{O}(3) \otimes V_3 \to 0. \end{aligned} \tag{4.1}$$

Using the well-known values for the cohomology of $\mathcal{O}(n)$ and $\Lambda^2(\mathcal{T}(n))$ (which is just $\Omega^2(5+n)$, Ω^i being the shear of i-forms on $\mathbb{P}$) we get

$$\begin{aligned} &0 \to \Gamma(\mathcal{O}(n+2)) \otimes V_1 \to \Gamma(\Lambda^2 \mathcal{T}(n)) \otimes W^n \to \Gamma(\mathcal{G}(n)) \to 0; \\ &H^1(\mathcal{G}(n)) = (0); \\ &H^2(\mathcal{G}(n)) = (0) \text{ if } n \neq -5,\ H^2(\mathcal{G}(-5)) \cong W; \\ &0 \to \Gamma(\mathcal{F}(n)) \to \Gamma(\mathcal{G}(n)) \to \Gamma(\mathcal{O}(n+3)) \otimes V_3 \to H^1(\mathcal{F}(n)) \to 0; \\ &\psi^2(n) = 0 \text{ if } n \neq -5,\ \psi^2(-5) = \chi_2. \end{aligned} \tag{4.2}$$

Now since $\Lambda^2 \mathcal{F} \cong \mathcal{O}(5) \cong \Omega^4(10)$, Serre duality asserts that $H^i(\mathcal{F}(n))$ and $H^{4-i}(\mathcal{F}(-10-n))$ are dual. We deduce using (4.2)

$$\begin{aligned} \psi^1(n) &= 0 \quad \text{if} \quad n \leqq -4, \\ \psi^3(n) &= 0 \quad \text{if} \quad n \geqq -6. \end{aligned} \tag{4.3}$$

Since $\Gamma(\Lambda^2 \mathcal{T}(-3)) = (0)$ we further deduce from (4.2)

$$\psi^1(-3) = \theta^3 h_1,\ \psi^3(-7) = \theta h_1. \tag{4.4}$$

From Lemma 2.4 (ii) the image of $\Gamma(\mathcal{G}(-2))$ in $\Gamma(\mathcal{O}(1)) \otimes V_3$ has dimension 15. But the first exact sequence of (4.2) shows that the dimension of $\Gamma\mathcal{G}(-2)$ is 15. So from the second exact sequence and Serre duality we deduce that

$$\begin{aligned} \psi^0(n) &= 0 \quad \text{if} \quad n \leqq -2 \\ \psi^4(n) &= 0 \quad \text{if} \quad n \geqq -8. \end{aligned} \tag{4.5}$$

We can now calculate $\psi^1(-2)$ from the exact sequences of (4.2) together with (4.5) and we find

$$\psi^1(-2) = \chi_2' \cdot \theta h_1, \psi^3(-8) = \chi_2' \cdot \theta^3 h_1. \tag{4.6}$$

Consider now the characters $\psi^i(j)$ for $i = 0$, 1 and $j = 0, -1$. The exact sequences of (4.2) together with Lemma 3.3 give

$$\begin{aligned} \psi^0(-1) - \psi^1(-1) &= -\chi_2' \cdot h_1, \\ \psi^0(0) - \psi^1(0) &= \chi_4 - \chi_2', \end{aligned} \tag{4.7}$$

and in particular it follows that

$$\psi^0(0) \geqq \chi_4, \psi^1(0) \geqq \chi_2', \tag{4.8}$$

where the inequality means that the difference between the two sides is the character of a representation. Also, since $\Gamma(\mathscr{F}(n))$ is a subspace of $\Gamma(\mathscr{G}(n))$, the exact sequences show that

$$\begin{aligned} \psi^0(-1) &\leqq (\chi_4^{\#} + \chi_4 + \chi_5)h_1, \\ \psi^0(0) &\leqq \chi_4 + \chi_5 + (\chi_4 + \chi_5 - \chi_2)\zeta. \end{aligned} \tag{4.9}$$

Since $\Lambda^2 \mathscr{F} \cong \mathcal{O}(5)$ there are homomorphisms

$$\lambda(-1)\colon \Lambda^2\Gamma(\mathscr{F}(-1)) \to \Gamma(\mathcal{O}(3)), \lambda\colon \Lambda^2\Gamma(\mathscr{F}) \to \Gamma\mathcal{O}(5).$$

LEMMA 4.10 *Let $\mathscr{S}$ be a locally free sheaf of rank 2 on $\mathbb{P}$ such that $\Gamma(\mathscr{S}(-1)) = 0$, let $\gamma\colon \Lambda^2\Gamma(\mathscr{S}) \to \Gamma\Lambda^2(\mathscr{S})$ be the canonical homomorphism, and let A be any subspace of $\Gamma(\mathscr{S})$. Then*

$$\dim \gamma(\Lambda^2 A) \geqq 2 \dim A - 3.$$

Proof. Since the Grassman cone in $\Lambda^2 A$ has dimension $2 \dim A - 3$ it is sufficient to show that the only element of this cone in Ker γ is the zero.

Suppose that $\gamma(s \wedge t) = 0$ for some s, t in A. Assume s, t are not both zero, then they generate a subsheaf $\mathscr{L}$ of $\mathscr{S}$ with rank 1. Since $\mathscr{L}$ is torsion-free its bidual is an invertible sheaf. So s, t are contained in a subsheaf isomorphic to $\mathcal{O}(r)$ for some r. As $\Gamma(\mathscr{S}(-1)) = 0$, it follows that $r \leqq 0$. Hence s, t are proportional and $s \wedge t = 0$. *Q.E.D*

First consider $\lambda(-1)$. Suppose that $\psi^0(-1) \neq 0$. Since the terms of the first inequality of (4.9) are irreducible characters with degrees at least 20, it follows that $\dim \Gamma(\mathscr{F}(-1)) \geqq 20$ and so by the lemma $\dim \Gamma(\mathcal{O}(3)) \geqq 37$. Since $\dim \Gamma\mathcal{O}(3) = 35$ it follows that

$$\psi^0(-1) = 0, \psi^1(-1) = \chi_2' \cdot h_1. \tag{4.11}$$

Now consider λ. Take A to be the subspace $\Gamma_H(\mathscr{F})$ of H-invariant sections. From Theorem 3.5, $\dim \Gamma_H(\mathcal{O}(5)) = 6$. So the lemma shows that $\dim A < 5$. Together with (4.8) and (4.9) this shows that the character of $\Gamma_H(\mathscr{F})$ is χ_4. Applying the lemma again shows that $\dim \lambda(A) \geqq 5$. So since $\Gamma_H(\mathcal{O}(5))$ has character $\chi_3 + \chi_3'$ it follows that λ is an isomorphism from $\Lambda^2\Gamma_H(\mathscr{F})$ to $\Gamma_H(\mathcal{O}(5))$.

We claim that in fact $\Gamma_H(\mathscr{F}) = \Gamma(\mathscr{F})$. If this is not true, then as a representation of

H, the space $\Gamma(\mathscr{F})$ must contain all the non-trivial characters of H/C at least once (by (4.9)), hence $\dim \Gamma(\mathscr{F}) \geqq 28$. Let $A, B \subseteq \mathbb{P}^4$ be two hyperplanes with homogeneous equations a, b and consider the exact sequence

$$0 \to \mathscr{F}(-2) \xrightarrow{(a,b)} \mathscr{F}(-1) \oplus \mathscr{F}(-1) \xrightarrow{(b,-a)} \mathscr{F} \to \mathscr{F}_{A\cdot B} \to 0.$$

We find, since $\Gamma(\mathscr{F}(-1)) = 0$,

$$\dim \Gamma(\mathscr{F}_{A\cdot B}) \geqq \dim \Gamma(\mathscr{F}) - \dim \operatorname{Ker}[(a, b) \text{ on } H^1(\mathscr{F}(-2))].$$

But from (4.2), $H^1(\mathscr{F}(-1))$ is generated by $H^0(\mathcal{O}(1)) \otimes H^1(\mathscr{F}(-2))$. Note that $h^1(\mathscr{F}(-2)) = 10$, $h^1(\mathscr{F}(-1)) = 10$ (by (4.6), (4.11)). So if a, b are sufficiently generic the image $a \cdot H^1(\mathscr{F}(-2)) + b \cdot H^1(\mathscr{F}(-2))$ in $H^1(\mathscr{F}(-1))$ has dimension at least 4. Therefore $\dim \Gamma \mathscr{F}_{A\cdot B} \geqq 28 - 10 + 4 = 22$. But let s be a non-zero section of $\mathscr{F}$. Its zero set X_s is a surface (since $\Gamma\mathscr{F}(-1) = 0$) and non-empty (since $c_2(\mathscr{F}) \neq 0$), so we can choose A, B so that $A \cdot B \cdot X_s$ is a non-empty finite set of points. Then $\mathscr{F}_{A\cdot B}/s\mathcal{O}_{A\cdot B}$ is a torsion-free rank 1 sheaf on $A \cdot B$, and hence isomorphic to $\mathscr{J} \cdot \mathcal{O}_{A\cdot B}(n)$ for some sheaf of ideals $\mathscr{J}$ defining $A \cdot B \cdot X_s$. Computing Chern classes we find $n = 5$. So, since $A \cdot B \cdot X_s$ is non-empty, $\dim \Gamma(\mathscr{F}_{A\cdot B}/s\mathcal{O}_{A\cdot B}) < 21$, and finally

$$21 \leqq \dim \Gamma(\mathscr{F}_{A\cdot B}) - 1 \leqq \dim \Gamma(\mathscr{F}_{A\cdot B}/s\mathcal{O}_{A\cdot B}) < 21,$$

which is a contradiction. So, taking account of (4.7), we have

$$\psi^0(0) = \chi_4, \; \psi^1(0) = \chi_2'. \tag{4.12}$$

Finally we claim $\psi^1(n) = 0$ if $n \geqq 1$. By Castelnuovo's lemma [11], it suffices to prove that $\psi^1(1) = 0$. By (4.2) the cup product $\alpha: \Gamma(\mathcal{O}(1)) \otimes H^1(\mathscr{F}) \to H^1(\mathscr{F}(1))$ is surjective. On the other hand N acts irreducibly on $\Gamma(\mathcal{O}(1)) \otimes H^1(\mathscr{F})$ by (2.2). Therefore either $\psi^1(1) = 0$ or α is an isomorphism. But $\Gamma(\mathcal{O}(1)) \otimes H^1(\mathscr{F}(-1)) \to H^1(\mathscr{F})$ is also surjective so for some $a \in \Gamma(\mathcal{O}(1))$, $\sigma \in H^1(\mathscr{F}(-1))$, it follows that $a \cup \sigma \neq 0$. Since $\dim \Gamma(\mathcal{O}(1)) > \dim H^1(\mathscr{F})$, it follows that $b \cup \sigma = 0$ for some other non-zero b. Therefore $\alpha(b, a \cup \sigma) = 0$ and α is not injective.

We summarize our calculations as follows:

TABLE OF $\dim H^i(\mathscr{F}(n-5))$

n	H^0	H^1	H^2	H^3	H^4
$n \geqq 6$	$\frac{(n^2-1)(n^4-24)}{12}$	0	0	0	0
5	4	2	0	0	0
4	0	10	0	0	0
3	0	10	0	0	0
2	0	5	0	0	0
1	0	0	0	0	0
0	0	0	2	0	0
−1	0	0	0	0	0
−2	0	0	0	5	0
−3	0	0	0	10	0
−4	0	0	0	10	0
−5	0	0	0	2	4
$n \leqq -6$	0	0	0	0	$\frac{(n^2-1)(n^2-24)}{12}$

§5. THE ZERO SETS X_s, $s \in \Gamma(\mathscr{F})$

THEOREM 5.1. *For almost all $s \in \Gamma(\mathscr{F})$, the zero set of s is a non-singular surface $X_s \subset \mathbb{P}$ of degree 10; when X_s is nonsingular, it is an abelian surface.*

Proof. Let Q be the projective space associated to $\Gamma(\mathscr{F})$, and let Z be the subvariety of $Q \times \mathbb{P}$ represented by pairs (s, x) $(s \in \Gamma(\mathscr{F}),\ x \in \mathbb{P})$ such that $s(x) = 0$. Since $\Lambda^2\Gamma(\mathscr{F}) \cong \Gamma_H(\mathcal{O}(5))$ the sheaf $\mathscr{F}$ is generated by $\Gamma(\mathscr{F})$ except at points of the set of 25 skew lines L whose ideal is generated by $\Gamma_H(\mathcal{O}(5))$ (see §3). It follows that Z is a fibre-bundle over $\mathbb{P} - L$, in particular it is non-singular over $\mathbb{P} - L$. Applying Sard's theorem [15] to the projection $Z \to Q$ shows that the zero variety $X_s = \{x \mid s(x) = 0\}$ of a general section s is a surface that is non-singular except possibly at points of L.

Let $x \in L$ and let e_1, e_2 be a basis of the free rank two $\mathcal{O}_x$-module $\mathscr{F}_x$. Each $s \in \psi(\mathscr{F})$ can be written

$$s = s_1 e_1 + s_2 e_2, \qquad s_i \in \mathcal{O}_x,$$

so that if s, $t \in \Gamma(\mathscr{F})$,

$$s \wedge t = (s_1 t_2 - s_2 t_1) e_1 \wedge e_2 .$$

If for every $s \in \Gamma(\mathscr{F})$, $s_1(x) = s_2(x) = 0$, then for every s and t, $s \wedge t$ would vanish to second order at x. Using again the fact that $\Lambda^2\Gamma(\mathscr{F}) \cong \Gamma_H(\mathcal{O}(5))$ and that $\Gamma_H(\mathcal{O}(5))$ generates the ideal of L, this is impossible. We may therefore choose e_1, e_2 so that e_1 is an element of $\Gamma(\mathscr{F})$. Write out a basis of $\Gamma(\mathscr{F})$ locally:

$$\begin{aligned} s &= e_1 \\ t &= f e_1 + u e_2 \\ t' &= f' e_1 + u' e_2 \qquad \text{where} \quad f, f', f'', u, u', u'' \in \mathcal{O}_x . \\ t'' &= f'' e_1 + u'' e_2 \end{aligned}$$

Then

$$\begin{aligned} s \wedge t &= u \cdot e_1 \wedge e_2 \\ s \wedge t' &= u' \cdot e_1 \wedge e_2 \\ s \wedge t'' &= u'' \cdot e_1 \wedge e_2 \end{aligned}$$

and $t^{(i)} \wedge t^{(j)}$ vanishes at x to 2nd order.

Therefore u, u', u'' must generate the ideal of L at x, i.e. their differentials are independent at x. But if $\lambda s + \mu t + \mu' t' + \mu'' t''$ is a general section in $\Gamma(\mathscr{F})$, so that $(\lambda, \mu, \mu', \mu'')$ are homogeneous coordinates in Q, then Z is described above points near x by the equations:

$$\begin{aligned} \lambda &= -(\mu f + \mu' f' + \mu'' f'') \\ 0 &= \mu u + \mu' u' + \mu'' u'' \end{aligned}$$

which are easily seen to define a non-singular subvariety of $Q \times \mathbb{P}$. Thus Z is everywhere non-singular, hence by Sard's theorem so is the set X_s of zeros of a generic section s of $\mathscr{F}$.

To prove that $X(= X_s)$ is abelian of degree 10 note that its normal bundle N in $\mathbb{P}$ is isomorphic to $\mathscr{F} \otimes \mathcal{O}_X$. So the Chern class $c(N)$ of N (in the Chow ring of X) is just the restriction to X of $1 + 5h + 10h^2$. Since $\mathbb{P}$ has Chern class $1 + 5h + 10h^2 + \cdots$, the Chern class of X is 1. So the canonical class K_X is zero and the Euler characteristic $c_2(X)$ is zero. This characterizes abelian surfaces [8, §6]. Since $c_2(N)$ is just the self intersection of X, the degree of X is 10. *Q.E.D*

THEOREM 5.2. *Every abelian surface $Z \subset \mathbb{P}$ is projectively equivalent to the zero set of some section s of $\mathscr{F}$.*

Proof. Let $\mathscr{D} = \mathcal{O}(1) \otimes \mathcal{O}_Z$. Since the Chern class of Z is 1 that of its normal bundle is the restriction of $1 + 5h + 10h^2$. As above it follows that Z has degree 10. Choose an origin on Z and let $H(\mathscr{D})$ be the subgroup

$$\{z \in Z \mid T_z^* \mathscr{D} \cong \mathscr{D}\}$$

where T_z is just the translation by z (cf. [13, §13]). Since $H(\mathscr{D})$ has order $(\deg Z/2)^2$ and carries a non-degenerate alternating form,

$$H(\mathscr{D}) \cong \mathbb{Z}_5 \times \mathbb{Z}_5 .$$

Further the Riemann–Roch theorem for abelian varieties (*ibid.*) shows that dim $\Gamma(\mathscr{D}) = 5$, and Lefschetz's theorem implies that Z cannot lie in a subspace of $\mathbb{P}$ (otherwise Z would be simply-connected). So the mapping

$$\phi : \Gamma\mathcal{O}(1) \to \Gamma(\mathscr{D})$$

is necessarily an isomorphism. Applying the results of [12, §1] it follows that when Z is embedded in $\mathbb{P}^4$ by the complete linear system $\Gamma(\mathscr{L})$ and a suitable isomorphism is chosen between $\mathbb{P}^4$ of §2 then Z is invariant under the action of the Heisenberg group introduced in §1. But since ϕ is an isomorphism this is just the composition of our given embedding and a projective transformation, i.e. after a projective transformation we may assume that Z is invariant under H. Actually we can go a bit further: if we choose an origin O in Z with respect to which $\mathscr{L}$ is symmetric then the map $x \mapsto -x$ for this origin extends to a projective transformation $\imath_0$ of $\mathbb{P}$, leaving Z fixed, normalizing the action of H/C and so that $\imath_0 \cdot \eta \cdot \imath_0^{-1} = \eta^{-1}$ for $\eta \in H/C$. Therefore $\imath_0$ must be induced by the element $\imath$ of N introduced in §1, and Z is invariant under H and $\imath$.

Next, look at the natural map:

$$\psi : \Gamma_H(\mathbb{P}, \mathcal{O}(5)) \to \Gamma_H(Z, \mathscr{D}^5).$$

The group $H(\mathscr{D})$ acts on the line bundle $\mathscr{D}^5$, hence there is a line bundle $\mathscr{M}$ on $Y = Z/H(\mathscr{D})$ such that $\pi^*\mathscr{M} \cong \mathscr{D}^5$ ($\pi : Z \to Y$ the natural homomorphism). Then $\Gamma(Y, \mathscr{M}) \cong \Gamma_H(Z, \mathscr{D}^5)$ and $\deg \mathscr{M} = \deg \mathscr{D}^5/\deg \pi = 5$, so dim $\Gamma(Y, \mathscr{M}) = 5$. In fact, under the symmetry $x \mapsto -x$, the space $\Gamma(Y, \mathscr{M})$ breaks up into the sum of an eigenspace of dimension 3 and one of dimension 2 (cf. [12, §2]; note that the action of $x \mapsto -x$ on $\Gamma(Y, \mathscr{M})$ is only well determined up to sign, so we have no obvious way of labelling one eigenspace "even" and the other "odd"). Since $\imath$ is the identity on $\Gamma_H(\mathbb{P}, \mathcal{O}(5))$, the image of ψ is contained in one of these eigenspaces. Therefore dim $\ker(\psi) \geqq 3$, *i.e.* at least three independent quintics of §3 contain Z.

Consider the map

$$\Lambda^2\Gamma(\mathscr{F}) \xrightarrow{\approx} \Gamma_H(\mathcal{O}(5)).$$

We have proven that there is a subspace $K \subset \Lambda^2\Gamma(\mathscr{F})$ of dim $\geqq 3$ consisting of elements that are mapped to zero in $\Gamma(\Lambda^2(\mathscr{F} \otimes \mathcal{O}_Z))$. It follows with a little linear algebra that there are two possibilities:

for some basis s_1, s_2, s_3, s_4 of $\Gamma(\mathscr{F})$, either

(a) $s_1 \wedge s_2, s_3 \wedge s_4, s_1 \wedge s_3 - s_2 \wedge s_4 \in K$, or

(b) $s_1 \wedge s_2, s_1 \wedge s_3$ (and a 3rd independent elt.) $\in K$.

Now if $s \wedge t \in K$, and $\bar{s}$, $\bar{t}$ are the restrictions of s, t to Z, then $\bar{s} = f \cdot \bar{t}$ for some $f \in \mathbb{C}(Z)$. Therefore in case (a),

$$\bar{s}_1 = f \cdot \bar{s}_2 \qquad \bar{s}_4 = f \cdot \bar{s}_3.$$

Let D be the divisors of poles of f and let $\mathscr{M} = \mathcal{O}_Z(D)$. Define

$$\alpha\colon \mathscr{M} + \mathscr{M} \to \mathscr{F} \otimes \mathcal{O}_Z \quad \text{by} \quad (g_1, g_2) \mapsto g_1\bar{s}_2 + g_2\bar{s}_3.$$

Then all four sections $\bar{s}_i$ of $\mathscr{F} \otimes \mathcal{O}_Z$ are images by α of sections of $\mathscr{M} + \mathscr{M}$ (i.e. $\alpha(1, 0)$, $\alpha(0, 1)$, $\alpha(f, 0)$ and $\alpha(0, f)$). But the s_i generate $\mathscr{F}$ everywhere except at the 25 lines L. Since Z is abelian, none of these lines is contained in Z, hence the $\bar{s}_i$ generate $\mathscr{F} \otimes \mathcal{O}_Z$ at all but a finite set of points. But α is a homomorphism of rank 2 bundles, so the support of its cokernel is defined by the principal ideal (det α), and has codimension 1. Therefore α must be an isomorphism. But then $\mathscr{M}^2 \cong \Lambda^2\mathscr{F} \otimes \mathcal{O}_Z \cong \mathscr{D}^5$, hence

$$4(D^2) = 25 \cdot c_1(\mathscr{D})^2 = 25 \cdot \deg Z = 250,$$

contradiction.

In case (b), either $\bar{s}_2 = f \cdot \bar{s}_1, \bar{s}_3 = g \cdot \bar{s}_1, f, g \in \mathbb{C}(Z)$ or $\bar{s}_1 = 0$. In the 1st case, as above, we get a homomorphism:

$$\alpha\colon \mathscr{M} \to \mathscr{F} \otimes \mathcal{O}_Z,$$

with three out of the four $\bar{s}_i$'s in the image $\alpha\Gamma(\mathscr{M})$. Then $\bar{s}_4$ generates the cokernel except at a finite set of points:

$$0 \to \mathcal{O}_Z \xrightarrow{\bar{s}_4} \mathscr{F} \otimes \mathcal{O}_Z/\alpha\mathscr{M} \to \underset{\text{finite support}}{\mathscr{G}} \to 0.$$

By elementary homological algebra, extensions of $\mathscr{G}$ by a line bundle split. Using this twice we find $\mathscr{G}$ must be zero, and we have

$$0 \to \mathscr{M} \xrightarrow{\alpha} \mathscr{F} \otimes \mathcal{O}_Z \to \mathcal{O}_Z \to 0,$$

hence $c_2(\mathscr{F} \otimes \mathcal{O}_Z) = Z \cdot c_2(\mathscr{F}) = 0$, which is absurd. Thus $\bar{s}_1 = 0$, i.e. $Z \subset$ zeroes of s_1. Since $\deg Z = 10 = \deg X_{s_1}$, it follows that $Z = X_{s_1}$. *Q.E.D.*

§6. CONNECTIONS WITH MODULI

The bundle $\mathscr{F}$ can be used to give an explicit representation of a certain moduli space for 2-dimensional abelian varieties. We first recall some standard results in the theory of moduli of abelian varieties:

(a) Let $g \geq 1$ be the dimension;

(b) Let $\mathfrak{H}_g$ = Siegel upper $\frac{1}{2}$-space of $g \times g$ symmetric matrices Ω, $\operatorname{Im} \Omega > 0$,

$$\cong \mathrm{Sp}(2g, \mathbb{R})/\text{maximum compact } K;$$

(c) Fix a sequence δ of positive integers $\delta_1, \ldots, \delta_g$ such that δ_i divides δ_{i+1};

(d) Sp $(2g, \mathbb{Q})$ acts on $\mathbb{Q}^{2g}$, fixing the form

$$A(e_i, e_{g+j}) = \delta_{ij}, \ A(e_i, e_j) = A(e_{g+i}, e_{g+j}) = 0, \qquad 1 \leq i, j \leq g;$$

(e) Let L_δ = the sublattice $\mathbb{Z}^g \times \prod_{i=1}^g \delta_i \mathbb{Z}$ of $\mathbb{Z}^{2g}$,

$L_\delta{}^\perp$ = the lattice $\prod_{i=1}^g (1/\delta_i)\mathbb{Z} \times \mathbb{Z}^g$, characterized as the set of $x \in \mathbb{Q}^{2g}$ such that $A(x, y) \in \mathbb{Z}$, all $y \in L_\delta$;

(f) On $L_\delta{}^\perp/L_\delta$, put the multiplicative symplectic form

$$e_\delta(x, y) = e^{2\pi i A(x, y)}.$$

(g) Let $\Gamma(\delta_1, \ldots, \delta_g)_0 = \{X \in \mathrm{Sp}(2g, \mathbb{Q}) \mid X(L_\delta) = L_\delta\}$;

$\Gamma(\delta_1, \ldots, \delta_g) = \{X \in \mathrm{Sp}(2g, \mathbb{Q}) \mid X(L_\delta) = L_\delta \text{ and } X = \text{id. on } L_\delta{}^\perp/L_\delta\}$;

(h) Then the analytic quotient spaces have the significance

$$\mathfrak{U}_\delta{}^{(0)} \underset{\text{def}}{=} \mathfrak{H}_g/\Gamma(\delta_1, \ldots, \delta_g)_0 = \left\{\begin{array}{c}\text{moduli space of pairs } (X, \lambda),\ X \text{ a} \\ g\text{-dimensional abelian variety, } \lambda\text{:} \\ X \to \hat{X} \text{ a polarization such that} \\ \ker(\lambda) \cong \prod_{i=1}^g (\mathbb{Z}/\delta_i\mathbb{Z})^2.\end{array}\right\}$$

$$\mathfrak{U}_\delta \underset{\text{def}}{=} \mathfrak{H}_g/\Gamma(\delta_1, \ldots, \delta_g) = \left\{\begin{array}{c}\text{moduli space of triples } (X, \lambda, \alpha), \\ (X, \lambda) \text{ as above, and} \\ \alpha\colon \ker(\lambda) \xrightarrow{\cong} L_\delta{}^\perp/L_\delta \\ \text{a symplectic isomorphism with} \\ \text{respect to } e_\lambda \text{ and } e_\delta.\end{array}\right\}$$

(i) $\mathfrak{U}_\delta$ and $\mathfrak{U}_\delta{}^{(0)}$ have natural structures of quasi-projective varieties;

(j) Note that the finite "symplectic" group $\Gamma(\delta)_0/\Gamma(\delta)$ acts on $\mathfrak{U}_\delta$ and $\mathfrak{U}_\delta{}^{(0)}$ is the quotient $\mathfrak{U}_\delta/[\Gamma(\delta)_0/\Gamma(\delta)]$.

Now if $\lambda : X \to \hat{X}$ is a polarization, let L_λ denote one of the corresponding invertible sheaves—all such are isomorphic after a translation. The result can now be stated:

THEOREM 6.1. *Let*

$$\mathfrak{U}^*_{(5,1)} = \left\{\begin{array}{l}\textit{the Zariski-open set of points of } \mathfrak{U}_{(5,1)} \\ \textit{corresponding to triples } (X, \lambda, \alpha) \textit{ such} \\ \textit{that } L_\lambda \textit{ is very ample}\end{array}\right\}.$$

Let

$$\mathbb{P}(\Gamma(\mathscr{F}))^* = \left\{\begin{array}{l}\textit{the Zariski-open subset of } \mathbb{P}(\Gamma(\mathscr{F})) \textit{ of spaces} \\ \textit{of sections } \mathbb{C}\cdot s, \textit{ whose zero sets } X_s \textit{ are non-} \\ \textit{singular}\end{array}\right\}.$$

Then $\mathfrak{U}^*_{(5,1)} \cong \mathbb{P}(\Gamma(\mathscr{F}))^*$, *the action of* $\Gamma(5,1)_0/\Gamma(5,1) \cong SL_2(\mathbb{Z}_5)$ *on* $\mathfrak{U}_{(5,1)}$ *corresponding to the action of* $N/H \cong SL_2(\mathbb{Z}_5)$ *on* $\mathbb{P}(\Gamma(\mathscr{F}))$.

Proof. The idea is to set up a set-theoretic map from $\mathbb{P}(\Gamma(\mathscr{F}))^*$ to $\mathfrak{U}^*_{(5,1)}$; verify that it is a morphism and is bijective; and apply Zariski's Main Theorem. To define the map, start with a one-dimensional subspace $\mathbb{C} \cdot s \subset \Gamma(\mathscr{F})$. This determines uniquely its zero-set X_s. This variety carries a line bundle, $\mathcal{O}_{X_s}(1)$, and is invariant under the group H/C. Strictly speaking, X_s is not yet an abelian variety, since it has no distinguished origin. We can either choose any point $x \in X_s$ as origin, or if we wish to be canonical, replace X_s by its "double dual":

$$X_s' = \mathrm{Pic}^0(\mathrm{Pic}^0 X_s),$$

(Pic^0 = connected component of Grothendieck's Picard scheme).

In this case, X_s is canonically a principal homogeneous space over X_s'. In both cases, $\mathcal{O}_{X_s}(1)$ induces a polarization λ on X_s (or X_s'). And the automorphisms induced by H/C are the translations by the points of $\ker(\lambda)$, so we get an isomorphism

$$\alpha\colon \ker(\lambda) \xrightarrow{\approx} H/C \underset{\mathrm{def}}{=} \mathbb{Z}_5 \times \mathbb{Z}_5 = L^{\perp}_{(5,1)}/L_{(5,1)}.$$

This is a point of $\mathfrak{U}^*_{(5,1)}$. The fact that this is a morphism comes from checking that the above construction can be carried out universally leading to an abelian scheme $\mathfrak{X}$ over $\mathbb{P}(\Gamma(\mathscr{F}))^*$, plus a polarization $\Lambda\colon \mathfrak{X} \to \hat{\mathfrak{X}}$ plus an isomorphism of $\ker(\Lambda)$ with the constant group scheme $\mathbb{Z}_5 \times \mathbb{Z}_5$. This induces a morphism from $\mathbb{P}(\Gamma(\mathscr{F}))^*$ to $\mathfrak{U}^*_{(5,1)}$ by the universal property characterizing coarse moduli spaces (cf. [10, p. 96]). To check that this map is injective, say $\mathbb{C} \cdot s_1$ and $\mathbb{C} \cdot s_2$ lead to isomorphic triples (X, λ, α). It follows that there is an isomorphism

$$\phi\colon X_{s_1} \to X_{s_2}$$

such that $\phi^* \mathcal{O}_{X,s_2}(1)$ is algebraically equivalent to $\mathcal{O}_{X,s_1}(1)$ and such that for all $\sigma \in H/C$, if σ induces on X_{s_i} translation by $x_i \in X_{s_i}$, then $\phi T_{x_1} = T_{x_2}\phi$. But then changing ϕ by a translation, we can assume that $\phi^*(\mathcal{O}_{X,s_2}(1)) \cong \mathcal{O}_{X,s_1}(1)$, hence ϕ is the restriction to X_{s_1} of a projective transformation τ. Moreover τ satisfies $\sigma\tau \equiv \tau\sigma$ on X_{s_1}, all $\sigma \in H/C$, hence $\tau\sigma = \sigma\tau$ in $PGL_5(\mathbb{C})$. But H/C is its own centralizer so $\tau \in H/C$. Therefore $X_{s_2} = \tau(X_{s_1}) = X_{s_1}$, hence $\mathbb{C} \cdot s_1 = \mathbb{C} \cdot s_2$. Finally surjectivity follows from (5.2). *Q.E.D.*

A natural question is to analyze how the isomorphism above goes wrong outside the open sets *. We have not worked this out completely, but we state without proof two pretty facts about this:

(a) If the abelian variety X tends to $E_1 \times E_2$, E_i an elliptic curve, so that the polarization tends to $\lambda = 5\lambda_1 + \lambda_2$ (λ_i: $E_i \xrightarrow{\approx} \hat{E}_i$ the canonical isomorphism), then λ is *not* very ample. In fact L_λ has a fixed component F and defines the morphism

$$X - F \xrightarrow{p_1} E_1 \xrightarrow{\phi_1} \mathbb{P}^4$$

where ϕ_1 is the morphism defined by L_{λ_1}. Let $C_1 = \phi_1(E_1)$, an elliptic quintic curve. Then while X approaches $E_1 \times E_2$, the corresponding section s of $\mathscr{F}$ has a well-defined limit s_0

and X_{s_0} is the singular ruled surface with C_1 as cuspidal double curve equal to the union of the tangent lines to C_1. It follows that at the points $(E_1 \times E_2, 5\lambda_1 + \lambda_2, \alpha) \in \mathfrak{U}_{(5,1)}$ the correspondence with $\mathbb{P}(\Gamma(\mathscr{F}))$ given in (6.1) is still regular, but not biregular, since the image does not depend on E_2.

(b) Suppose we compactify $\mathfrak{U}_{(5,1)}$ following Igusa [6] [i.e. take his compactification of $\mathfrak{U}_{(1,1)}$ and normalize it in $\mathbb{C}(\mathfrak{U}_{(5,1)})$ via any of the canonicalmorphisms $\mathfrak{U}_{(5,1)} \to \mathfrak{U}_{(1,1)}$]. Then at some of the points at ∞ lying even on the 0-dimensional piece of Satake's compactification, the correspondence remains biregular. The corresponding X_s's depend on one parameter $\alpha \in \mathbb{C} - (0)$ and are unions of five non-singular quadric surfaces as follows:

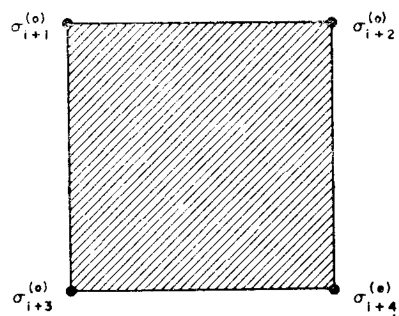
$$X_s = Q_0 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4$$
$$Q_i = (\text{locus } Y_i = \alpha Y_{2+i} Y_{3+i} + Y_{1+i} Y_{4+i} = 0).$$

The 10 lines $Y_i = Y_j = Y_k = 0$ $(0 \leqq i < j < k \leqq 4)$ are double lines on X_s, and the five points P_i given by $Y_j(P_i) = \delta_{ij}$ are 4-fold points of X_s. The whole configuration is readily visualized if you form a CW-complex Σ as follows:

(a) take a point $\sigma_i^{(0)}$ for each point P_i;

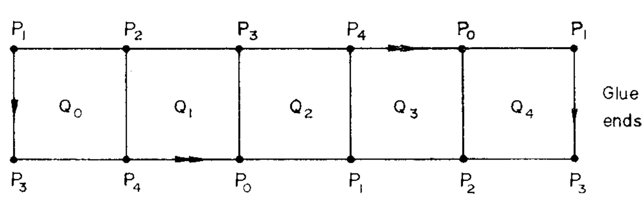
(b) joint $\sigma_i^{(0)}$ and $\sigma_j^{(0)}$ by a 1-simplex $\sigma_{ij}^{(1)}$ corresponding to the double line $\overline{P_i P_j}$, for each $i < j$;

(c) glue in a square $\sigma_i^{(2)}$ corresponding to Q_i filling in the loop



for each $0 \leqq i \leqq 4$ (read the subscripts mod 5).

Then a point or line is on a line or a quadric in $\mathbb{P}$ if and only if the corresponding 0-simplex or 1-simplex is on the corresponding 1-simplex or square in Σ. The nice thing is that the Σ you get is homeomorphic to a 2-dimensional real torus:



Glue top and bottom with horizontal shift

REFERENCES

1. W. Barth and M. E. Larsen: On the homotopy-groups of complex projective-algebraic varieties, *Math. Scand.* (to appear).
2. A. Borel, R. Carter, T. Springer *et al.*: *Seminar on Algebraic Groups and Related Finite Groups.* Springer Lecture Notes 131.
3. G. Horrocks: Vector bundles on the punctured spectrum of a local ring, *Proc. Lond. math. Soc.* **14** (1964), 689.
4. G. Horrocks: A construction for locally free sheaves, *Topology* **7** (1968), 117.
5. B. Huppert; *Endliche Gruppen I.* Springer, Berlin (1967).
6. J.-I. Igusa: A desingularization problem in the theory of Siegel modular functions, *Math. Annal.*
7. J.-I. Igusa: *Theta Functions.* Springer, Berlin (1972).
8. K. Kodaira: On the structure of compact, complex analytic surfaces I, *Am. J. Math.* **86** (1964), 751.
9. M. E. Larsen: On the vanishing of certain homotopy-groups of complex projective-algebraic manifolds. Preprint, Math. Inst. Copenhagen University (1971).
10. D. Mumford: *Geometric Invariant Theory.* Springer, Berlin (1965).
11. D. Mumford: Lectures on curves on an algebraic surface, *Ann. Math. Studs.* (1966).
12. D. Mumford: On the equations defining abelian varieties I, *Inv. Math.* **1** (1966), 287.
13. D. Mumford: *Abelian Varieties.* Oxford University Press (1971).
14. R. L. E. Schwarzenberger: Vector bundles on the projective plane, *Proc. Lond. math. Soc.* **11** (1961), 623.
15. S. Sternberg, *Lectures on Differential Geometry.* Prentice-Hall, N.J. (1962).
16. A. Weil: Sur certains groupes d'opérateurs unitaires, *Acta Math.* **111** (1964), 143.

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APPENDIX

The character table of $SL_2(\mathbb{Z}_5)$ [2, p. 160]

Put $\varepsilon = \exp 2\pi\sqrt{-1}/5$, $\eta = \varepsilon + \varepsilon^4$, $\eta' = \varepsilon^2 + \varepsilon^3$, and let ω be a primitive root of $x^6 = 1$ over $\mathbb{Z}_5$.

			$\bar{\mu}$				$\bar{\nu}$			
	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}2&0\\0&3\end{pmatrix}$	$\begin{pmatrix}\omega^2&0\\0&\omega^{-2}\end{pmatrix}$	$\begin{pmatrix}\omega&0\\0&\omega^{-1}\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&2\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&2\\0&-1\end{pmatrix}$	Symbols used in text for these representations
χ_1	1	1	1	1	1	1	1	1	1	I
χ_5	5	5	1	-1	-1	0	0	0	0	
χ_6	6	-6	0	0	0	1	1	-1	-1	
χ_4	4	4	0	1	1	-1	-1	-1	-1	
$\chi_4^{\#}$	4	-4	0	1	-1	-1	-1	1	1	
χ_3	3	3	-1	0	0	$-\eta$	$-\eta'$	$-\eta$	$-\eta'$	U
χ_3'	3	3	-1	0	0	$-\eta'$	$-\eta$	$-\eta'$	$-\eta$	U
χ_2	2	-2	0	-1	$+1$	η	η'	$-\eta$	$-\eta'$	W
χ_2'	2	-2	0	-1	$+1$	η'	η	$-\eta'$	$-\eta$	W'