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(with an appendix by George Kempf)

VARIETIES DEFINED BY QUADRATIC EQUATIONS

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Introduction

First of all, let me fix my terminology and set-up. I will always be working over an algebraically closed ground field k. We will be concerned almost entirely with projective varieties over k (although many of our results generalize immediately to arbitrary projective schemes). By a projective variety, I will understand a topological space X all of whose points are closed, plus a sheaf $\mathcal{O}_X$ of k-valued functions on X isomorphic to some subvariety of $\mathbb{P}^n$ for some n. By a subvariety of $\mathbb{P}^n$, I will mean the subset $X \subset \mathbb{P}^n(k)$ defined by some homogeneous prime ideal $\wp \subset k[X_0,\cdots,X_n]$, with its Zariski-topology and with the sheaf $\mathcal{O}_X$ of functions from X to k induced locally by polynomials in the affine coordinates. Note that our varieties have only k-rational points — no generic points. In this, we depart slightly from the language of schemes. Note too that a projective variety can be isomorphic to many different subvarieties of $\mathbb{P}^n$. An isomorphism of X with a subvariety of $\mathbb{P}^n$ will be called an immersion of X in $\mathbb{P}^n$.

Let me begin with an elementary but somewhat startling result:

Definition: For all d, the d-ple immersion of $\mathbb{P}^n$ is the morphism:

$$s_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N, \qquad N = \binom{n+d}{d} - 1$$

given by:

$$s_d(a_0,\cdots,a_n) = (a^{\alpha^{(0)}},\cdots,a^{\alpha^{(N)}})$$

where $\alpha^{(0)},\cdots,\alpha^{(N)}$ runs through the (n+1)-tuples $\alpha = (\alpha_0,\cdots,\alpha_n)$, such that $\alpha_i \geq 0$, $\Sigma\alpha_i = d$, and

$$a^{\alpha} = \prod_{i=0}^{n} a_i^{\alpha_i} .$$

Theorem 1: Let $X \subset \mathbb{P}^n$ be a subvariety, and let d_0 be the degree of X. For all $d \geq d_0$, consider the new projective embedding:

$$X \subset \mathbb{P}^n \xrightarrow{s_d} \mathbb{P}^N .$$

Then the subvariety of $\mathbb{P}^N$ so obtained is an intersection of quadrics.*

Proof: Let r = dim X. For all linear spaces L of dimension n-r-2, disjoint from X, let H_L be the join of X and L, i.e., the locus of lines joining X and L. H_L is a hypersurface of degree $\leq d_0$. Then it is easy to see that

$$X = \bigcap_{L\cap X=\emptyset} H_L .$$

In fact, if $x \in \mathbb{P}^n - X$, let

$$\pi: \mathbb{P}^n - \{x\} \longrightarrow \mathbb{P}^{n-1}$$

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*When we talk about an r-dimensional subvariety X of $\mathbb{P}^n$ being an intersection of quadrics, we never mean an intersection of only n-r quadrics (called usually a "complete intersection"). We just mean that there is a large set of quadrics Q_α, $\alpha \in S$, such that $X = \bigcap Q_\alpha$. Of course, S can be assumed finite.

be projection with center x. Then $\pi(X)$ is an r-dimensional subvariety of $\mathbb{P}^{n-1}$ so there exists a linear subspace $M \subset \mathbb{P}^{n-1}$ disjoint from $\pi(X)$ of dimension $(n-1)-r-1$. Choose L such that $\pi(L) = M$. Then $x \notin H_L$.

Thus X is an intersection of hypersurfaces of degree $\leq d_o$. Therefore, for all $d \geq d_o$, X is the intersection of those hypersurfaces of degree d that contain it. But by definition of s_d, if $H_1 \subset \mathbb{P}^n$ is a hypersurface of degree d, there is a hyperplane $H_2 \subset \mathbb{P}^N$ such that

$$H_1 = s_d^{-1}(H_2).$$

Therefore, there is a linear space $K \subset \mathbb{P}^N$ such that $X = s_d^{-1}(K)$, or

$$s_d(X) = K \cap s_d(\mathbb{P}^n).$$

To prove the theorem, it remains to check that $s_d(\mathbb{P}^n)$ is an intersection of quadrics. This follows from the remark:

For all $b_o, \cdots, b_N$,

$$(*) \qquad \left[\begin{array}{l}\text{There exists } a_o, \cdots, a_n \\ \text{such that } b_i = a^{\alpha(i)}\end{array}\right] \Longleftrightarrow \left[\begin{array}{l} b_i b_j = b_k b_\ell \text{ whenever} \\ \alpha(i) + \alpha(j) = \alpha(k) + \alpha(\ell)\end{array}\right]$$

We leave this to the reader.

QED

I want to make 2 remarks. Suppose by the <u>rank</u> r of quadric we mean the rank of the corresponding symmetric matrix. Then the proof of this theorem shows that X is actually an intersection of quadrics of rank ≤ 4. Suppose we make the definition:

<u>Definition</u>: A subvariety $X \subset \mathbb{P}^n$ is <u>ideal-theoretically</u> an intersection of hypersurfaces $H_1, \cdots, H_m$ if set-theoretically:

$$X = H_1 \cap \cdots \cap H_m$$

and moreover, every $x \in X$ has an affine open neighborhood $U \subset \mathbb{P}^n$ such that the ideal $I(X)$ of $X \cap U \subset U$ is generated by the affine equations $f_1, \cdots, f_n$ of $H_1, \cdots, H_n$.

<u>Lemma</u>: If X is non-singular, then X is ideal-theoretically the intersections of $H_1, \cdots, H_n$ if and only if

1) $X = \bigcap_{i=1}^{n} H_i$

2) for all $x \in X$,

$$T_{x,X} = \bigcap_{i=1}^{n} T_{x,H_i}$$

(the intersection being taken in $T_{x,\mathbb{P}^n}$; here T means Zariski tangent space).

We leave the proof to the reader. Using this, we can then prove a variant of Theorem 1 to the effect that if X is non-singular, then $s_d(X)$ is ideal-theoretically an intersection of quadrics.

§1. The cohomological method.

In setting up the concepts of linear systems and ampleness and in the construction of projective embeddings, we have to make a choice between 3 equivalent formulations — that of divisor classes, of line bundles, or of invertible sheaves. It is well known that on any variety X, the group of (Cartier) divisors mod linear equivalence, the group of line bundles and the group of invertible sheaves are all canonically isomorphic. For our purposes, it is most convenient to use the sheaves:

Definition: An invertible sheaf L on X is a sheaf of $\mathcal{O}_X$-modules, locally isomorphic to $\mathcal{O}_X$ itself.

Two such sheaves L_1, L_2 can be tensored to form a 3^{rd} $L_1 \otimes L_2$; $\mathcal{O}_X$ itself is an invertible sheaf forming a unit for this multiplication; and for any L, $L^{-1} = \mathcal{H}om(L, \mathcal{O}_X)$ is an inverse since $L \otimes L^{-1} \cong L^{-1} \otimes L \cong \mathcal{O}_X$. The set of all invertible sheaves, mod isomorphisms, thus forms an abelian group, called Pic(X).

$\Gamma(L)$ or $H^0(L)$ will be the vector space of global sections of L. If $s \in \Gamma(L)$, and $x \in X$, then via an isomorphism $L|_U \cong \mathcal{O}_X|_U$ in some neighborhood U of x, we can find a value s(x); and the conditions $s(x) = 0$ or $s(x) \neq 0$ are independent of this local isomorphism.

<u>Definition</u>: The <u>base points</u> of $\Gamma(L)$ are the points $x \in X$ such that for all $s \in \Gamma(L)$, $s(x) = 0$.

If $\Gamma(L)$ is base point free, L defines a canonical morphism into projective space. Let $\mathbb{P}(\Gamma(L))$ be the projective space of hyperplanes in $\Gamma(L)$. Then define

$$\phi_L : X \longrightarrow \mathbb{P}(\Gamma(L))$$

by $$\phi_L(x) = \{s \in \Gamma(L) \mid s(x) = 0\}.$$

This is easily checked to be a morphism. More explicitly, let $s_0, s_1, \cdots, s_n$ be a basis of $\Gamma(L)$. Define:

$$\phi_L : X \longrightarrow \mathbb{P}^n$$

by $$\phi_L(x) = \text{pt. with homog. coord. } (s_0(x), s_1(x), \cdots, s_n(x))$$

<u>Definition</u>: L is <u>very ample</u> if $\Gamma(L)$ is base point free and ϕ_L is an immersion (= an isomorphism of X with $\phi_L(X)$). L is <u>ample</u> if L^n is very ample for some $n \geq 1$.

Write $\mathbb{P}$ for $\mathbb{P}(\Gamma(L))$ and suppose L is very ample. Then the vector space $\Gamma(L)$ is <u>canonically</u> isomorphic to the space of homogeneous coordinate functions on the projective space $\mathbb{P}$, i.e,

$$\Gamma(L) \cong \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)).$$

And the k^{th} symmetric power of $\Gamma(L)$, which we write $S^k\Gamma(L)$, is canonically isomorphic to the space of homogeneous polynomials of degree k in the homogeneous coordinates on $\mathbb{P}$, i.e.,

$$S^k\Gamma(L) \cong \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)).$$

Thus the vector space of homogeneous polynomials of degree k that vanish on $\phi_L(X)$ is nothing but the kernel of the canonical map:

$$S^k\Gamma(L) \longrightarrow \Gamma(L^k).$$

A strengthening of the assertion that $\phi_L(X)$ is an intersection of quadrics is that its homogeneous ideal is generated by quadrics. This is the same as asking whether the canonical map:

$$S^{k-2}\Gamma(L) \otimes \mathrm{Ker}\left[S^2\Gamma(L) \to \Gamma(L^2)\right] \longrightarrow \mathrm{Ker}\left[S^k\Gamma(L) \longrightarrow \Gamma(L^k)\right]$$

is surjective for all $k \geq 2$.

Our basic definition is this:

Definition: Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X. Define $\mathcal{R}(\mathcal{F},\mathcal{G})$, $\mathcal{S}(\mathcal{F},\mathcal{G})$ as the kernel and cokernel of the canonical map α:

$$0 \longrightarrow \mathcal{R}(\mathcal{F},\mathcal{G}) \longrightarrow \Gamma(\mathcal{F})\otimes\Gamma(\mathcal{G}) \xrightarrow{\alpha} \Gamma(\mathcal{F}\otimes\mathcal{G}) \longrightarrow \mathcal{S}(\mathcal{F},\mathcal{G}) \longrightarrow 0.$$

Thus if L is a very ample invertible sheaf, $\mathcal{R}(L,L)$ is the space (a) of alternating elements of $\Gamma(L)\otimes\Gamma(L)$, and (b) of the quadratic relations holding on $\phi_L(X)$.

Definition: Let L be an ample sheaf on X. Then L is normally generated if

$$\Gamma(L)^{\otimes k} \longrightarrow \Gamma(L^k)$$

is surjective, all $k \geq 1$.

This is clearly equivalent to the condition $\mathcal{S}(L^i, L^j) = (0)$, $i, j \geq 1$. Note that if L is normally generated then L is necessarily very ample too! In fact, consider the 2 morphisms:

$$\begin{array}{ccc} & & \mathbb{P}(\Gamma(L^n)) \\ & \nearrow^{\phi_{L^n}} & \\ X & & \\ & \searrow_{\phi_L} & \\ & & \mathbb{P}(\Gamma(L)) \end{array}$$

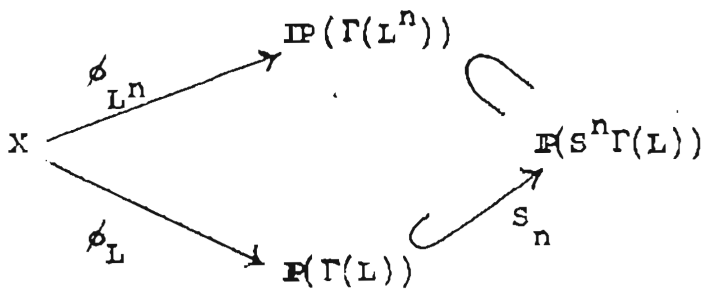
The n-ple embedding of the projective space $\mathbb{P}(V)$ of hyperplanes for any vector space V is canonically a map

$$s_n : \mathbb{P}(V) \longrightarrow \mathbb{P}(S^n V).$$

Moreover, via the surjection

$$S^n \Gamma(L) \longrightarrow \Gamma(L^n),$$

we can identify $\mathbb{P}(\Gamma(L^n))$ canonically with a linear subspace of $\mathbb{P}(S^n\Gamma(L))$. Putting this together, we get a diagram:



It is easy to check that this commutes. Now for large n, L^n is very ample, hence ϕ_{L^n} is an immersion, so it follows from the diagram that ϕ_L is an immersion too, i.e., L is very ample.

Definition: Let L be a normally generated invertible sheaf. Then L is normally presented if one of the 4 equivalent conditions holds:

(A) $\mathrm{Ker}[S^2\Gamma(L) \longrightarrow \Gamma(L^2)] \otimes \Gamma(L^{k-2}) \longrightarrow \mathrm{Ker}[S^k\Gamma(L) \longrightarrow \Gamma(L^k)]$

is surjective, all $k \geq 2$

(B) $\bigoplus_{1 \leq i < j \leq n} [\mathcal{R}(L,L) \otimes \Gamma(L)^{k-2}] \longrightarrow \mathrm{Ker}[\Gamma(L)^{\otimes k} \longrightarrow \Gamma(L^k)]$

is surjective, all $k \geq 2$.

The above homomorphism maps an element $a \otimes b$ in the $(i,j)^{th}$ factor to the element of $\Gamma(L)^{\otimes k}$ whose i^{th} and j^{th} components are determined by a, and the rest by b.

(C) $\Gamma(L^{i-1}) \otimes \mathcal{R}(L,L) \otimes \Gamma(L^{j-1}) \longrightarrow \mathcal{R}(L^i, L^j)$

is surjective, if $i,j \geq 1$.

Here, if $\Sigma a_i \otimes b_i \in \mathcal{R}(L,L) \subset \Gamma(L) \otimes \Gamma(L)$, and $c \in \Gamma(L^{i-1})$, $d \in \Gamma(L^{j-1})$, then we map $c \otimes (\Sigma a_i \otimes b_i) \otimes d$ to $\Sigma(a_i c) \otimes (b_i d) \in \Gamma(L^i) \otimes \Gamma(L^j)$.

(D) $\mathcal{R}(L^i,L^j) \otimes \Gamma(L^k) \longrightarrow \mathcal{R}(L^i,L^{j+k})$

is surjective if $i,j,k \geq 1$.

It is not so obvious that all these properties are equivalent! Thus to see (A)$\Longleftrightarrow$(B), note that $\mathcal{R}(L,L) \subset \Gamma(L) \otimes \Gamma(L)$ contains the alternating tensors, so the image of

$$\bigoplus_{1 \leq i < j \leq n} [\mathcal{R}(L,L) \otimes \Gamma(L)^{k-2}]$$

in $\Gamma(L)^k$ contains all the alternating tensors. So the image equals $\mathrm{Ker}(\Gamma(L)^k \longrightarrow \Gamma(L^k))$ if and only if its image in $S^k\Gamma(L)$ equals $\mathrm{Ker}(S^k\Gamma(L) \longrightarrow \Gamma(L^k))$. But its image in $S^k\Gamma(L)$ is the same as the image of the map in (A).

(C)$\Longrightarrow$(D) follows immediately using normal <u>generation</u> and (D)$\Longrightarrow$(C) follows by factoring the map in (C) thus:

$$\Gamma(L^{i-1}) \otimes \mathcal{R}(L,L) \otimes \Gamma(L^{j-1}) \longrightarrow \Gamma(L^{i-1}) \otimes \mathcal{R}(L,L^j) \longrightarrow \mathcal{R}(L^i,L^j).$$

Next, to prove (C)$\Longrightarrow$(B), factor $\Gamma(L)^k \longrightarrow \Gamma(L^k)$ as follows:

$$\Gamma(L) \otimes \Gamma(L)^{k-1} \xrightarrow{\text{onto}} \Gamma(L^2) \otimes \Gamma(L)^{k-2} \xrightarrow{\text{onto}} \cdots \xrightarrow{\text{onto}} \Gamma(L^k) .$$

To prove (B), it is enough to show that $\oplus[\mathcal{R}(L,L)\otimes\Gamma(L)^{k-2}]$ goes onto the kernel at each stage of this sequence. Thus it is enough if $\Gamma(L)^{i-1}\otimes\mathcal{R}(L,L)$ is mapped <u>onto</u> $\mathrm{Ker}[\Gamma(L^i)\otimes\Gamma(L)\longrightarrow\Gamma(L^{i+1})]$. This last space is $\mathcal{R}(L^i,L)$, so this ontoness is part of (C).

Finally, to prove (B) $\Longrightarrow$ (C), factor $\Gamma(L)^k \longrightarrow \Gamma(L^k)$ when $k = i+j$, as follows:

$$\Gamma(L)^{i+j} \xrightarrow[\alpha]{\text{onto}} \Gamma(L^i)\otimes\Gamma(L^j) \xrightarrow[\beta]{\text{onto}} \Gamma(L^{i+j}).$$

It follows from normal generation that we get a surjection:

$$\mathrm{Ker}(\beta\cdot\alpha) \xrightarrow{\text{onto}} \mathrm{Ker}(\beta) = \mathcal{R}(L^i,L^j).$$

But $\mathrm{Ker}(\beta\,\alpha)$ is generated by $\oplus[\mathcal{R}(L,L)\otimes\Gamma(L)^{i+j-2}]$. The image of this last space in $\Gamma(L^i)\otimes\Gamma(L^j)$ is the same as the image of $\Gamma(L^{i-1})\otimes\mathcal{R}(L,L)\otimes\Gamma(L^{j-1})$, so (C) follows.

This at least gives us a nice definition to work with! It seems easier to prove things about $\mathcal{S}$ first, and then to use these results to obtain things about $\mathcal{R}$. Our first result is:

<u>Theorem 2</u> (Generalized lemma of Castelnuovo): Suppose L is an ample invertible sheaf on a variety X such that $\Gamma(L)$ has no base points. Suppose $\mathfrak{F}$ is a coherent sheaf on X such that

$$H^i(\mathfrak{F}\otimes L^{-i}) = (0), \quad i \geq 1.$$

Then (a) $H^i(\mathfrak{F}\otimes L^j) = (0)$ if $i+j \geq 0$, $i \geq 1$

and (b) $\mathcal{S}(\mathfrak{F}\otimes L^i, L) = (0)$, $i \geq 0$.

To motivate this, look at the case of Castelnuovo's original lemma: X = non-singular curve, $\mathfrak{A}$, $\mathfrak{B}$ divisors on X, $\mathfrak{I} = \mathcal{O}_X(\mathfrak{B})$, $L = \mathcal{O}_X(\mathfrak{A})$. In classical language:

$$\left(\begin{array}{c}\Gamma(L) \text{ has no} \\ \text{base points}\end{array}\right) \underset{\text{def}}{=} \left(\begin{array}{c}|\mathfrak{A}| \text{ is base point} \\ \text{free}\end{array}\right)$$

$$\left(H^1(\mathfrak{I} \otimes L^{-1}) = (0)\right) \underset{\text{def}}{=} \left(|\mathfrak{B} - \mathfrak{A}| \text{ is non-special}\right)$$

Translating the conclusion, we find:

$$\left(\mathcal{S}(\mathfrak{I},L) = (0)\right) \underset{\text{def}}{=} \left(|\mathfrak{A} + \mathfrak{B}| = \begin{array}{c}\text{the minimal sum} \\ |\mathfrak{A}| + |\mathfrak{B}|\end{array}\right)$$

<u>Proof of Theorem 2</u>: Use induction on $\dim(\operatorname{Supp} \mathfrak{I})$. If $\dim(\operatorname{Supp} \mathfrak{I}) = 0$, then choose $s \in \Gamma(L)$ such that $s(x) \neq 0$ for all $x \in \operatorname{Supp}(\mathfrak{I})$. Then

$$\Gamma(\mathfrak{I}) \underset{k}{\otimes} (s^i \cdot k) \xrightarrow{\quad \approx \quad} \Gamma(\mathfrak{I} \otimes L^i)$$

is an isomorphism, so certainly

$$\Gamma(\mathfrak{I}) \underset{k}{\otimes} \Gamma(L^i) \longrightarrow \Gamma(\mathfrak{I} \otimes L^i)$$

is surjective. Therefore $\mathcal{S}(\mathfrak{I},L^i) = (0)$. Also, all groups $H^i(\mathfrak{I} \otimes \text{anything})$, $i \geq 1$, vanish.

Now suppose we are given an $\mathfrak{I}$, and we have proven the theorem for all $\mathfrak{I}^*$'s with $\dim(\operatorname{Supp} \mathfrak{I}^*) < \dim(\operatorname{Supp} \mathfrak{I})$. I claim that there is an element $s \in \Gamma(L)$ sufficiently "generic" so that for every $x \in X$, if we choose an isomorphism $L|_U \cong \mathcal{O}_X|_U$ near x, so that s can be considered as a function, then s is <u>not</u> a 0-divisor in the stalk $\mathfrak{F}_x$ of $\mathfrak{F}$. To see, recall that by the Noetherian decomposition theorems, for any coherent $\mathfrak{F}$, there is a finite set of irreducible subsets $Z_1, \cdots, Z_n \subset \mathrm{Supp}(\mathfrak{F})$ (including the components of $\mathrm{Supp}(\mathfrak{F})$, but possibly including some "embedded components" too) such that the support of any element

$$\alpha \in \Gamma(U, \mathfrak{F})$$

is a union of some of the sets $U \cap Z_i$. For each i, not all sections $s \in \Gamma(L)$ vanish identically on Z_i. Therefore there is an element $s \in \Gamma(L)$ not identically zero on any Z_i. If $\alpha \in \Gamma(U, \mathfrak{F})$, then s must be non-zero at at least one point x of $\mathrm{Supp}(\alpha)$, hence $\alpha \otimes s \in \Gamma(U, \mathfrak{F} \otimes L)$ is not zero near x. Thus s has the required property and the map $\mathfrak{F} \longrightarrow \mathfrak{F} \otimes L$, defined by $\alpha \longmapsto \alpha \otimes s$, is injective.

It is more convenient to use the map $\mathfrak{F} \otimes L^{-1} \longrightarrow \mathfrak{F}$, defined by $\alpha \longmapsto \alpha \otimes s$. Let $\mathfrak{F}^*$ be the cokernel. Then for all i, we have exact sequences:

$$(*)_i \qquad 0 \longrightarrow \mathfrak{F} \otimes L^{-i-1} \xrightarrow{\otimes s} \mathfrak{F} \otimes L^{-i} \longrightarrow \mathfrak{F}^* \otimes L^{-i} \longrightarrow 0.$$

Note that $\dim(\mathrm{Supp}\, \mathfrak{F}^*) < \dim(\mathrm{Supp}\, \mathfrak{F})$. In fact, for all i, $\otimes s$ is an isomorphism on almost all of Z_i, hence $Z_i \not\subset \mathrm{Supp}(\mathfrak{F}^*)$. Therefore every component of $\mathrm{Supp}(\mathfrak{F}^*)$ is a proper closed subset of some component of $\mathrm{Supp}(\mathfrak{F})$. By $(*)_i$, we get an exact sequence:

$$\begin{array}{ccccc} H^i(\mathfrak{F}\otimes L^{-i}) & \longrightarrow & H^i(\mathfrak{F}^*\otimes L^{-i}) & \longrightarrow & H^{i+1}(\mathfrak{F}\otimes L^{-i-1}), \quad i \geq 1 \\ \| & & & & \| \\ (0) & & & & (0) \end{array}$$

hence $H^i(\mathfrak{F}^*\otimes L^{-i}) = (0)$. Thus the hypothesis of the theorem is valid for $\mathfrak{F}^*$, so by our induction hypothesis, so is the conclusion. Going back from $\mathfrak{F}^*$ to $\mathfrak{F}$, use the exact sequence:

$$H^i(\mathfrak{F} \otimes L^{-i}) \longrightarrow H^i(\mathfrak{F} \otimes L^{-i+1}) \longrightarrow H^i(\mathfrak{F}^* \otimes L^{-i+1}).$$

The 1st group is (0) by the hypothesis on $\mathfrak{F}$; the 3rd group is (0) by the theorem for $\mathfrak{F}^*$: so the 2nd is (0). Replacing $\mathfrak{F}$ by $\mathfrak{F} \otimes L$, we continue in this way and prove by induction on $i+j$ that

$$H^i(\mathfrak{F} \otimes L^j) = (0), \quad i+j \geq 0, \quad i \geq 1.$$

As for the $\mathcal{S}$'s, look at the diagram of solid arrows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\mathfrak{F}\otimes L^{-1}) \otimes \Gamma(L) & \longrightarrow & \Gamma(\mathfrak{F}) \otimes \Gamma(L) & \longrightarrow & \Gamma(\mathfrak{F}^*) \otimes \Gamma(L) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(\mathfrak{F}) & \longrightarrow & \Gamma(\mathfrak{F}\otimes L) & \longrightarrow & \Gamma(\mathfrak{F}^*\otimes L) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathcal{S}(\mathfrak{F}\otimes L^{-1},L) & \xrightarrow{\alpha} & \mathcal{S}(\mathfrak{F},L) & \longrightarrow & \mathcal{S}(\mathfrak{F}^*,L) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

It has exact rows since $H^1(\mathfrak{F}\otimes L^{-1}) = H^1(\mathfrak{F}) = (0)$, and exact columns by definition. Define the dotted arrow by $\alpha \longmapsto \alpha\otimes s$. Then the shaded triangle commutes, which proves that the map α is zero! Since $\mathcal{S}(\mathfrak{F}^*,L) = (0)$, it follows that $\mathcal{S}(\mathfrak{F},L) = (0)$. As we may replace $\mathfrak{F}$ by $\mathfrak{F}\otimes L^i$, $i \geq 1$, the rest of (b) follows too.

QED

A useful remark is that a close examination of this proof shows a slightly more precise result. Namely, that if $n = \dim(\mathrm{Supp}\ \mathfrak{F})$; and if $s_o,\cdots,s_n \in \Gamma(L)$ are sufficiently "generic" elements, then in fact $\Gamma(\mathfrak{F}\otimes L)$ is spanned by the images of $\Gamma(\mathfrak{F}) \otimes_k (s_i\cdot k)$, for $0 \leq i \leq n$.

<u>Theorem 3</u>: Let L be an ample invertible sheaf on an n-dimensional variety X. Suppose $\Gamma(L)$ has no base points and

$$H^i(L^j) = (0),\quad i \geq 1,\ j \geq 1.$$

Then $\mathcal{S}(L^i,L^j) = (0)$ if $i \geq n+1$, $j \geq 1$.

In particular, if $i \geq n+1$, L^i is ample with normal generation, hence very ample.

<u>Proof</u>: Apply Theorem 2 to $\mathfrak{F} = L^{n+1}$. It follows that $\mathcal{S}(L^i,L) = (0)$, if $i \geq n+1$. Explicitly $\Gamma(L^i) \otimes \Gamma(L) \longrightarrow \Gamma(L^{i+1})$ is surjective if $i \geq n+1$. Composing these maps, $\Gamma(L^i) \otimes \Gamma(L)^j \longrightarrow \Gamma(L^{i+j})$ is surjective if $i \geq n+1$. Therefore $\Gamma(L^i) \otimes \Gamma(L^j) \longrightarrow \Gamma(L^{i+j})$ is surjective too, if $i \geq n+1$.

QED

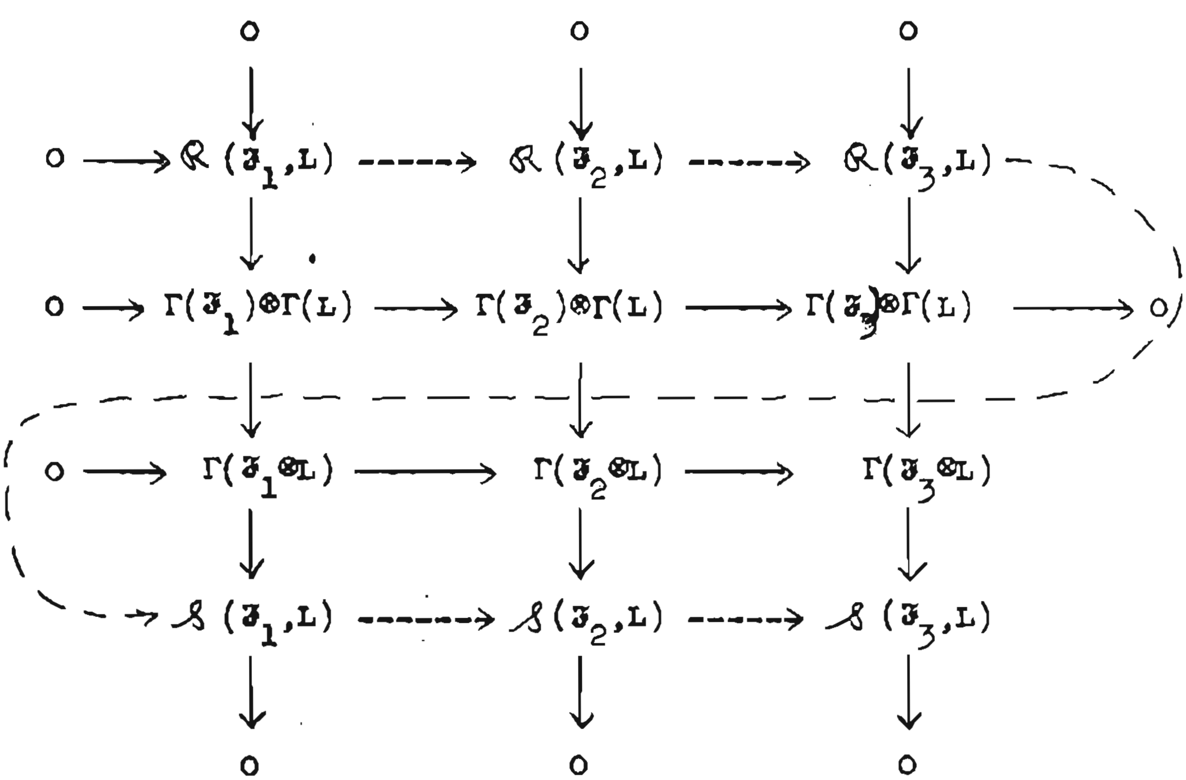
Next we want to prove similar results about $\mathcal{R}$. We need the preliminary result:

<u>6-lemma</u>: If $0 \longrightarrow \mathfrak{F}_1 \longrightarrow \mathfrak{F}_2 \longrightarrow \mathfrak{F}_3 \longrightarrow 0$ is an exact sequence of coherent sheaves, and $\Gamma(\mathfrak{F}_2) \longrightarrow \Gamma(\mathfrak{F}_3)$ is surjective — e.g., if $H^1(\mathfrak{F}_1) = (0)$ — then for all invertible sheaves L there is an exact sequence:

$$0 \longrightarrow \mathcal{R}(\mathfrak{F}_1,L) \longrightarrow \mathcal{R}(\mathfrak{F}_2,L) \longrightarrow \mathcal{R}(\mathfrak{F}_3,L) \longrightarrow \mathcal{S}(\mathfrak{F}_1,L) \longrightarrow \mathcal{S}(\mathfrak{F}_2,L) \longrightarrow \mathcal{S}(\mathfrak{F}_3,L).$$

Also, even if $\Gamma(\mathfrak{F}_2) \longrightarrow \Gamma(\mathfrak{F}_3)$ is not surjective, the 1st 3 terms form an exact sequence.

<u>Proof</u>: Look at the diagram of solid arrows:



The rows and columns are exact, by hypothesis. By the so-called "serpent" argument, you get an exact sequence indicated by the dotted arrows.

QED

We apply this to prove:

Theorem 4: Let L and M be ample invertible sheaves on a projective variety X, let $\mathfrak{F}$ be a coherent sheaf on X, and assume:

i) $\Gamma(L)$, $\Gamma(M)$ have no base points,

ii) $H^{i+j-1}(\mathfrak{F}\otimes L^{-i}\otimes M^{-j}) = (0)$ if $i,j \geq 1$.

Then the natural map:

$$\mathcal{R}(\mathfrak{F},L) \otimes \Gamma(M) \longrightarrow \mathcal{R}(\mathfrak{F} \otimes M, L)$$

is surjective.

[One can also check that the hypotheses imply that

$$H^k(\mathfrak{F}\otimes L^{-i}\otimes M^{-j}) = (0) \text{ if } k \geq 1,\ i+k \geq 0,\ j+k \geq 0,\ i+j+k \geq -1.$$

Therefore the hypotheses are stable under the substitution $\mathfrak{F} \longmapsto \mathfrak{F} \otimes L$ or $\mathfrak{F} \otimes M$. However, we may as well stick to the simplest case of the theorem.]

Proof: As in Theorem 2, we use induction on $\dim(\mathrm{Supp}\ \mathfrak{F})$. If $\dim(\mathrm{Supp}\ \mathfrak{F}) = 0$, we get the diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{R}(\mathfrak{J},L)\otimes\Gamma(M) & \longrightarrow & \mathfrak{R}(\mathfrak{J}\otimes M,\ L) & \underset{\alpha}{\cong} & \mathfrak{R}(\mathfrak{J},L) \\
\downarrow & & \downarrow & & | \\
\Gamma(\mathfrak{J})\otimes\Gamma(L)\otimes\Gamma(M) & \longrightarrow & \Gamma(\mathfrak{J}\otimes M)\otimes\Gamma(L) & \underset{\beta}{\cong} & \Gamma(\mathfrak{J})\otimes\Gamma(L) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(\mathfrak{J}\otimes L)\otimes\Gamma(M) & \longrightarrow & \Gamma(\mathfrak{J}\otimes L\otimes M) & \underset{\gamma}{\cong} & \Gamma(\mathfrak{J}\otimes L)
\end{array}$$

where the isomorphisms β and γ are obtained by choosing a section $s \in \Gamma(M)$ non-zero at all points of Supp($\mathfrak{J}$), hence an isomorphism of M and $\mathfrak{O}_X$ in a neighborhood of Supp($\mathfrak{J}$). β and γ induce an isomorphism α. But in the map on the top row, if $p \in \mathfrak{R}(\mathfrak{J},L)$, then $p\otimes s \in \mathfrak{R}(\mathfrak{J},L)\otimes\Gamma(M)$ is taken to $p \in \mathfrak{R}(\mathfrak{J},L)$, so this map is surjective. This proves the theorem when dim(Supp $\mathfrak{J}$) = 0 .

In the general case, choose a good section $s \in \Gamma(M)$ as in the proof of Theorem 2 so as to obtain an exact sequence:

$$0 \longrightarrow \mathfrak{J}\otimes M^{-1} \xrightarrow{\otimes s} \mathfrak{J} \longrightarrow \mathfrak{J}^* \longrightarrow 0$$

with dim(Supp $\mathfrak{J}^*$) < dim(Supp $\mathfrak{J}$). We obtain exact sequences:

$$H^{i+j-1}(\mathfrak{J}\otimes L^{-i}\otimes M^{-j}) \longrightarrow H^{i+j-1}(\mathfrak{J}^*\otimes L^{-i}\otimes M^{-j}) \longrightarrow H^{i+j}(\mathfrak{J}\otimes L^{-i}\otimes M^{-j-1}).$$

The 1st and 3rd groups are 0 by hypothesis, so the 2nd is also. This shows that $\mathfrak{J}^*$ satisfies the hypotheses of the theorem too. So by the induction hypothesis, $\mathfrak{R}(\mathfrak{J}^*,L)\otimes\Gamma(M) \longrightarrow \mathfrak{R}(\mathfrak{J}^*\otimes M,\ L)$ is surjective. Moreover, by Castelnuovo's lemma (Theorem 2), applied to

$\mathfrak{F} \otimes M^{-1}$ and L, $\mathcal{S}(\mathfrak{F} \otimes M^{-1}, L) = (0)$ and $H^1(\mathfrak{F} \otimes M^{-1}) = (0)$. Applying the 6-lemma, we deduce that:

$$0 \longrightarrow \mathcal{R}(\mathfrak{F} \otimes M^{-1}, L) \longrightarrow \mathcal{R}(\mathfrak{F}, L) \longrightarrow \mathcal{R}(\mathfrak{F}^*, L) \longrightarrow 0$$

is exact. Now consider the diagram of solid arrows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{R}(\mathfrak{F} \otimes M^{-1}, L) \otimes \Gamma(M) & \longrightarrow & \mathcal{R}(\mathfrak{F}, L) \otimes \Gamma(M) & \longrightarrow & \mathcal{R}(\mathfrak{F}^*, L) \otimes \Gamma(M) & \longrightarrow 0 \\
& \downarrow & & \downarrow \beta & & \gamma \downarrow & \\
0 \longrightarrow & \mathcal{R}(\mathfrak{F}, L) & \xrightarrow{\ \alpha\ } & \mathcal{R}(\mathfrak{F} \otimes M, L) & \longrightarrow & \mathcal{R}(\mathfrak{F}^* \otimes M, L) &
\end{array}$$

If you define the dotted arrow by $a \longmapsto a \otimes s$, it is clear that the shaded triangle commutes. Therefore $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$ and using the surjectivity of γ, the surjectivity of β follows.

QED

To apply this Theorem, we need another result:

Proposition: Let $\mathfrak{F}$ be a coherent sheaf, and L, M invertible sheaves on X. If

a) $\mathcal{R}(\mathfrak{F}, L) \otimes \Gamma(M) \longrightarrow \mathcal{R}(\mathfrak{F} \otimes M, L)$ is surjective

b) $\mathcal{S}(\mathfrak{F}, L) = (0)$,

then

c) $\mathcal{R}(\mathfrak{F}, M) \otimes \Gamma(L) \longrightarrow \mathcal{R}(\mathfrak{F} \otimes L, M)$ is surjective.

Proof: Use the diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \Gamma(L) \otimes \mathcal{R}(\mathfrak{J},M) & \xrightarrow{\ \gamma\ } & \mathcal{R}(L \otimes \mathfrak{J}, M) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{R}(L,\mathfrak{J})\otimes\Gamma(M) & \longrightarrow & \Gamma(L)\otimes\Gamma(\mathfrak{J})\otimes\Gamma(M) & \xrightarrow{\ \beta\ } & \Gamma(L\otimes\mathfrak{J}) \otimes \Gamma(M) \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{R}(L, \mathfrak{J} \otimes M) & \longrightarrow & \Gamma(L) \otimes \Gamma(\mathfrak{J}\otimes M) & \longrightarrow & \Gamma(L \otimes \mathfrak{J} \otimes M)
\end{array}$$

By assumption, α and β are surjective. "Chasing" the diagram, one sees quickly that γ is surjective too.

<u>QED</u>

<u>Theorem 5</u>: Let L be an ample invertible sheaf on an n-dimensional variety X. Assume:

i) $\Gamma(L)$ is base point free,

ii) $H^i(L^j) = (0)$ if $i,j \geq 1$.

Then it follows that:

$$\mathcal{R}(L^i,L^j) \otimes \Gamma(L^k) \longrightarrow \mathcal{R}(L^{i+k}, L^j)$$

is surjective, if $i \geq n+2$, $j,k \geq 1$. In particular, if $i \geq n+2$, L^i is ample with normal presentation.

<u>Proof</u>: By Theorem 4,

$$\mathcal{R}(L^i,L) \otimes \Gamma(L) \longrightarrow \mathcal{R}(L^{i+1},L)$$

is surjective, if $i \geq n+2$. Iterating, we find that:

$$\mathcal{R}\,(L^i, L) \otimes \Gamma(L^j) \longrightarrow \mathcal{R}\,(L^{i+j}, L)$$

is surjective, if $i \geq n+2$, $j \geq 1$. Since $\mathcal{S}\,(L^i, L) = (0)$, $i \geq n+2$ apply the Proposition to prove that:

$$\mathcal{R}\,(L^i, L^j) \otimes \Gamma(L) \longrightarrow \mathcal{R}\,(L^{i+1}, L^j)$$

is surjective, if $i \geq n+2$, $j \geq 1$. Iterating again, we get the required assertion.

<u>QED</u>

§2. The case of curves.

For the whole of this section, X will be assumed to be a non-singular complete curve of genus g. We want to strengthen the results of §1 in this case. We need some more concepts and definitions. A divisor $\mathfrak{A}$ is a formal linear combination $\Sigma n_i x_i$ of points of X. For all divisors $\mathfrak{A}$, $\mathcal{O}(\mathfrak{A})$ is the invertible sheaf of functions f which are regular except at the x_i's, and at x_i have at most an n_i-fold pole, if $n_i \geq 0$, or must have at least a $(-n_i)$-fold zero if $n_i \leq 0$. A fact that we need is that if an invertible sheaf L has a section s with zeroes exactly at $x_1, \cdots, x_k$ of multiplicities $n_1, \cdots, n_k$, then $L \cong \mathcal{O}(\mathfrak{A})$, with $\mathfrak{A} = \Sigma n_i x_i$. If L is an invertible sheaf, $L(\mathfrak{A})$ stands for $L \otimes \mathcal{O}(\mathfrak{A})$. Ω will be the sheaf of regular differentials on X.

Theorem 6: Let L,M be invertible sheaves on X such that $\deg L \geq 2g+1$, $\deg M \geq 2g$. Then $\mathcal{S}(L,M) = (0)$.

Proof: Let $d = \deg L$. $\mathfrak{A}$ is to be a positive divisor of degree $d-(g+1)$ which will be chosen later. Then $L(-\mathfrak{A})$ is naturally a subsheaf of L, and we get an exact sequence:

$$0 \longrightarrow L(-\mathfrak{A}) \longrightarrow L \longrightarrow L^* \longrightarrow 0$$

where $\operatorname{Supp} L^* = \operatorname{Supp} \mathfrak{A}$. The 1st requirement on $\mathfrak{A}$ is that $H^1(L(-\mathfrak{A})) = (0)$. Assuming for the moment that $\mathfrak{A}$ has this property, by the 6-lemma of §1, we get an exact sequence:

$$\mathcal{S}(L(-\mathfrak{A}),M) \longrightarrow \mathcal{S}(L,M) \longrightarrow \mathcal{S}(L^*,M)$$

But it is well known that if K is an invertible sheaf on X with deg $K \geq 2g$, $\Gamma(K)$ has no base points. In particular, $\Gamma(M)$ has no base points, and $\mathrm{Supp}(L^*)$ is 0-dimensional. So by Castelnuovo's lemma, $\mathcal{S}(L^*,M) = (0)$.

Next, apply the Riemann-Roch theorem to $L(-\mathfrak{A})$:

$$\dim H^0(L(-\mathfrak{A})) = \deg L(-\mathfrak{A}) - (g-1) + \dim H^1(L(-\mathfrak{A}))$$

$$= 2.$$

Thus $\Gamma(L(-\mathfrak{A}))$ is a "pencil" and the 2nd requirement on $\mathfrak{A}$ is that it is base point free. Finally we want to apply Castelnuovo's lemma to deduce that $\mathcal{S}(L(-\mathfrak{A}),M) = (0)$. For this we need only that

$$H^1(M \otimes L(-\mathfrak{A})^{-1}) = H^1(M \otimes L^{-1}(\mathfrak{A})) = (0).$$

This is the 3rd requirement on $\mathfrak{A}$ Putting all this together, it will follow that $\mathcal{S}(L,M) = (0)$.

Can we find an $\mathfrak{A}$ with these 3 properties? Since $\mathfrak{A}$ consists in $d-(g+1) \geq g$ points all of which can be chosen arbitrarily, it is well known that for a suitable choice of $\mathfrak{A}$, $\mathfrak{o}(\mathfrak{A})$ will be isomorphic to any invertible sheaf K of degree $d-(g+1)$. Now the set of all invertible sheaves K of degree $d-(g+1)$ forms a projective variety J, which is exactly the Jacobian of X except that J does not have any natural base point on it to serve as the origin. It suffices to find a K such that

i) $H^1(L \otimes K^{-1}) = (0)$

ii) for all $x \in X$, $\dim H^0(L \otimes K^{-1}(-x)) = 1$

iii) $H^1(M \otimes L^{-1} \otimes K) = (0)$.

Now if (i) is false, $\dim H^0(L \otimes K^{-1}) \geq 2$ by Riemann-Roch, hence (ii) will be false for <u>all x</u>! Therefore it is enough to check (ii) and (iii) for all x. But by Riemann-Roch,

$$\begin{aligned}
\dim H^0(L \otimes K^{-1}(-x)) > 1 &\Longleftrightarrow \dim H^1(L \otimes K^{-1}(-x)) > 0 \\
&\Longleftrightarrow \dim H^0(\Omega \otimes L^{-1} \otimes K(x)) > 0 \\
&\Longleftrightarrow \exists\, y_1, \ldots, y_{g-2} \text{ such that } \\
&\qquad \Omega \otimes L^{-1} \otimes K(x) \cong \mathcal{O}(\Sigma y_i) \\
&\Longleftrightarrow \exists\, y_1, \ldots, y_{g-2} \text{ such that } \\
&\qquad K \cong \Omega \otimes L^{-1}(x - \Sigma y_i).
\end{aligned}$$

We have only g-1 variable points here, so the locus of K's not satisfying (ii) has dimension at most g-1. Similarly if $\deg M = e + 2g$, we find:

$$H^1(M \otimes L^{-1} \otimes K) \neq (0) \Longleftrightarrow H^0(\Omega \otimes M^{-1} \otimes L \otimes K^{-1}) \neq (0)$$

$$\Longleftrightarrow \exists\ y_1,\ldots,y_k \text{ where}$$

$$k = \deg(\Omega \otimes M^{-1} \otimes L \otimes K^{-1}) = g-1-e$$

such that

$$\Omega \otimes M^{-1} \otimes L \otimes K^{-1} \cong \mathcal{O}_C(\Sigma y_i)$$

$$\Longleftrightarrow \exists\ y_1,\ldots,y_k \text{ such that}$$

$$K \cong \Omega \otimes M^{-1} \otimes L(-\Sigma y_i).$$

Again there are at most g-1 variable points here, so the locus of K's not satisfying (iii) has dimension at most g-1. Since dim J = g, almost all K's do satisfy (ii) and (iii). Thus an $\mathcal{U}$ with the required properties exists.

QED

Corollary: If L is an invertible sheaf of degree $\geq 2g+1$, then L is ample with normal generation.

If the argument in the above proof is traced through, it is not hard to show that it proves the following:

$$\exists\ s_1, s_2 \in \Gamma(L)$$
$$\exists\ \quad t \in \Gamma(M) \quad \text{such that}$$

$$[k s_1 \otimes_k \Gamma(M) + k.s_2 \otimes_k \Gamma(M) + \Gamma(L) \otimes_k k.t]$$

$$\longrightarrow \Gamma(L \otimes M)$$

is surjective.

Our argument is essentially the same as the classical argument used to prove that if X is not hyperelliptic, then Ω is normally generated (See Hensel-Landsberg). We can paraphrase this argument in our language as follows:

We begin as before with an exact sequence:

$$0 \longrightarrow \Omega(-\mathfrak{A}) \longrightarrow \Omega \longrightarrow \Omega^* \longrightarrow 0$$

where we now assume that $\mathfrak{A}$ is a positive cycle of degree g-2. In order to apply the 6-lemma, it is not necessary that $H^1(\Omega(-\mathfrak{A})) = (0)$. In fact, it is enough if:

i) $H^1(\Omega(-\mathfrak{A})) \longrightarrow H^1(\Omega)$ is an isomorphism.

This is the 1^{st} requirement on $\mathfrak{A}$. We then deduce as before that

$$\mathcal{S}(\Omega(-\mathfrak{A}),\Omega) \longrightarrow \mathcal{S}(\Omega,\Omega) \longrightarrow \mathcal{S}(\Omega^*,\Omega)$$

is exact. Since $\Gamma(\Omega)$ has no base points, we know that $\mathcal{S}(\Omega^*,\Omega) = (0)$ by Castelnuovo's lemma. By the Riemann-Roch theorem, it follows as before that $\Gamma(\Omega(-\mathfrak{A}))$ is a pencil and our 2^{nd} requirement is that it is base point free. Unfortunately, we cannot apply Castelnuovo's lemma to prove $\mathcal{S}(\Omega(-\mathfrak{A}),\Omega) = (0)$, since $H^1(\Omega \otimes \Omega(-\mathfrak{A})^{-1}) = H^1(\mathcal{O}(\mathfrak{A}))$ is never (0). We use instead a direct computation of dimensions to prove $\mathcal{S}(\Omega(-\mathfrak{A}),\Omega) = (0)$! Let s_1, s_2 be a basis of $\Gamma(\Omega(-\mathfrak{A}))$. Look at the map:

$$\underbrace{\Gamma(\Omega)s_1 \oplus \Gamma(\Omega)s_2}_{\dim\,=\,2g} \xrightarrow{\ \alpha\ } \underbrace{\Gamma(\Omega^2(-\mathfrak{A}))}_{\dim\,=\,2g-1}$$

(The dimension on the right is computed by the Riemann-Roch theorem.) We want α to be surjective. But the kernel will be isomorphic to the spaces of pairs $\omega_1, \omega_2 \in \Gamma(\Omega)$ such that $\omega_1 \otimes s_1 = -\omega_2 \otimes s_2$. Since s_1 and s_2 have no common zeroes, this implies that ω_1 is zero at the zeroes $\mathfrak{b}_2$ of s_2, i.e., $\omega_1 = \eta \otimes s_2$ where $\eta \in \Gamma(\Omega(-\mathfrak{b}_2))$. Then ω_2 is necessarily $-\eta \otimes s_1$, so

$$\operatorname{Ker}(\alpha) \cong \Gamma(\Omega(-\mathfrak{b}_2)).$$

Since $\mathfrak{o}(\mathfrak{b}_2) \cong \Omega(-\mathfrak{A})$, it follows that

$$\begin{aligned} \dim \operatorname{Ker}(\alpha) &= \dim \Gamma(\Omega \otimes \Omega(-\mathfrak{A})^{-1}) \\ &= \dim \Gamma(\mathfrak{o}(\mathfrak{A})) \\ &= \dim H^1(\Omega(-\mathfrak{A})) = 1. \end{aligned}$$

Therefore α is surjective, hence $\mathcal{S}(\Omega,\Omega) = (0)$.

Now let $\Omega(-\mathfrak{A}) = K$. K is a sheaf of degree g, and conversely every sheaf K of degree g such that $\dim \Gamma(K) \geq 2$ has the property $\dim \Gamma(\Omega \otimes K^{-1}) \geq 1$ by Riemann-Roch, hence $\Omega \otimes K^{-1} \cong \mathfrak{o}(\mathfrak{A})$, some $\mathfrak{A}$, hence $K \cong \Omega(-\mathfrak{A})$, some $\mathfrak{A}$. Therefore we have proven:

<u>Theorem 7</u>: If X carries an invertible sheaf K of degree g such that $\Gamma(K)$ is a base point free pencil, then $\mathcal{S}(\Omega,\Omega) = (0)$.

The existence of such a K is not hard to show whenever X is <u>not</u> hyperelliptic, but we omit this. The proof that $\mathcal{S}(\Omega,\Omega^i) = (0)$ if $i \geq 2$, is even easier.

Theorem 6 for the vanishing of $\mathcal{S}$ is definitely the best possible unless further restrictions are placed on L and M. For example, if $L = \Omega(P+Q)$, then although L is ample and $\Gamma(L)$ has no base points, $\phi_L(P) = \phi_L(Q)$, so L is not very ample. Since L^2 is very ample, there must be sections $s \in \Gamma(L^2)$ such that $s(P) = 0$, $s(Q) \neq 0$, hence $s \notin \mathrm{Im}(\Gamma(L) \otimes \Gamma(L))$. Therefore $\mathcal{S}(L,L) \neq (0)$!

We now go on to results about $\mathcal{R}$ for curves. I don't think, unfortunately, that my results here are best possible. I shall prove:

<u>Theorem 8</u>: **Let L,M,N be invertible sheaves on X such that $\deg L \geq 3g+1$, $\deg M$, $\deg N \geq 2g+2$. Then**

$$\mathcal{R}(L,M) \otimes \Gamma(N) \longrightarrow \mathcal{R}(L \otimes N, M)$$

is surjective.

From this we deduce immediately:

<u>Corollary</u>: Let L be an invertible sheaf on X such that $\deg L \geq 3g+1$. Then L is normally presented.

<u>Proof of the Theorem</u>: We shall use the following lemma:

<u>Lemma</u>: For all invertible sheaves N on X such that $\deg N \geq 2g+2$ and $\Gamma(N)$ has no base points, there is a decomposition:

$$N = N_1 \otimes \cdots \otimes N_k, \qquad k \geq 2$$

where

(1) $\deg N_i = g+1, \quad 1 \leq i \leq k-1$

$$g+1 \leq \deg N_k \leq 2g+1,$$

(2) $\Gamma(N_i)$ has no base points.

(3) If J_1 (resp. J^*) is the variety of invertible sheaves of degree = degree N_1 (resp. deg N_1 - deg N_2), then for all open sets $U_1 \subset J_1$, $U^* \subset J^*$, we may assume

$$N_1 \in U_1$$

$$N_1 \otimes N_2^{-1} \in U^* .$$

Proof: If $\deg N \leq 2g+1$, then let $k = 1$, $N_1 = N$. Now suppose $\deg N = e + (g+1)$, $g+1 \leq e \leq 2g+1$. Then $k = 2$ and we must decompose $N = N_1 \otimes N_2$, $\deg N_1 = g+1$, $\deg N_2 = e$. Let J_2 be the variety of invertible sheaves of degree = e. Let $V_i \subset J_i$ be the set of invertible sheaves K such that $H^1(K) = (0)$ and $\Gamma(K)$ has no base points. It is well known that V_i is open and non-empty. Consider the maps:

$$f: \quad J_1 \longrightarrow J_2, \quad \text{given by} \quad N_1 \longmapsto N \otimes N_1^{-1}$$

$$g: \quad J_1 \longrightarrow J^*, \quad \text{given by} \quad N_1 \longmapsto N_1^2 \otimes N^{-1} .$$

If identity points are chosen arbitrarily on J_1, J_2, J^*, then all these varieties are canonically the same, and are nothing but the jacobian of X. Then in terms of the group law on the jacobian f becomes a map of the form $x \longmapsto a-x$, and g is of the form $x \longmapsto 2x+b$. Thus both f and g are surjective. In particular, $f^{-1}(V_2)$ and $g^{-1}(U^*)$ are non-empty. Now choose $N_1 \in U_1 \cap V_1 \cap f^{-1}(V_2) \cap g^{-1}(U^*)$, and let $N_2 = N \otimes N_1^{-1}$. Then N_1 and N_2 have all the required properties.

If $k > 2$, the proof is similar, but even simpler.

QED

To prove Theorem 8, begin by decomposing the N in the Theorem by the method of the lemma. It clearly will suffice to prove:

$$\mathcal{R}(L \otimes N_1 \otimes \cdots \otimes N_i, M) \otimes \Gamma(N_{i+1}) \longrightarrow \mathcal{R}(L \otimes N_1 \otimes \cdots \otimes N_{i+1}, M)$$

is surjective, for every i with $0 \leq i \leq k-1$. Checking degrees here, we find that we have reduced the Theorem to:

(A) If $\Gamma(N)$ has no base points, $g+1 \leq \deg N \leq 2g+1$, $\deg M \geq 2g+2$, and $\deg L - \deg N \geq 2g$, then

$$\mathcal{R}(L,M) \otimes \Gamma(N) \longrightarrow \mathcal{R}(L \otimes N, M)$$

is surjective.

We now want to apply the Proposition in §1 to interchange M and N in (A). Since $H^1(L \otimes N^{-1}) = (0)$, hence $\mathcal{S}(L,N) = (0)$, (A) is implied by:

(B) If $\Gamma(N)$ has no base points, $g+1 \leq \deg N \leq 2g+1$, $\deg M \geq 2g+2$, and $\deg L - \deg N \geq 2g$, then

$$\mathcal{R}(L,N) \otimes \Gamma(M) \longrightarrow \mathcal{R}(L \otimes M, N)$$

is surjective.

Now decompose M by the method of the lemma. To prove (B) it will suffice to prove:

(i) $\mathcal{R}(L,N) \otimes \Gamma(M_1) \longrightarrow \mathcal{R}(L \otimes M_1, N)$ surjective

(ii) $\mathcal{R}(L\otimes M_1,N) \otimes \Gamma(M_2) \longrightarrow \mathcal{R}(L \otimes M_1 \otimes M_2, N)$ surjective

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(k) $\mathcal{R}(L \otimes M_1 \otimes \ldots \otimes M_{k-1}, N) \otimes \Gamma(M_k) \longrightarrow \mathcal{R}(L \otimes M_1 \otimes \cdots \otimes M_k, N)$ surjective.

We want to apply Theorem 4 to prove these facts. Since $\Gamma(N)$ and $\Gamma(M_1)$ are base point free, we need only check:

(i) $H^1(L \otimes N^{-1} \otimes M_1^{-1}) = (0)$

(ii) $H^1(L \otimes N^{-1} \otimes M_1 \otimes M_2^{-1}) = (0)$

(k) $H^1(L \otimes N^{-1} \otimes M_1 \otimes \cdots \otimes M_{k-1} \otimes M_k^{-1}) = (0)$.

Now $\deg(L\otimes N^{-1}\otimes M_1^{-1}) \geq 2g-(g+1) = g-1$, so if M_1 lies in a suitable open subset of the Jacobian, (i) will hold. Secondly, $\deg(L\otimes N^{-1}\otimes M_1\otimes M_2^{-1}) \geq 2g + (g+1)-(2g+1) = g$, so if $M_1 \otimes M_2^{-1}$ lies in a suitable open subset of the Jacobian, (ii) will hold. Since the lemma allows us to choose M_1 and $M_1\otimes M_2^{-1}$ in any open sets, (i) and (ii) can be achieved. As for the rest, if, for instance, $k \geq 3$,

$$\deg(L\otimes N^{-1}\otimes M_1\otimes M_2\otimes M_3^{-1}) \geq 2g + (g+1) + (g+1) - (2g+1) = 2g+1$$

so (iii) is automatic. The same holds for all the rest. Thus (B) is proven, hence (A), hence the Theorem.

QED

§3. Abelian varieties: the method of theta-groups.

By definition, an abelian variety is a projective variety with a structure of a group such that $(x,y) \longmapsto x+y$ and $x \longmapsto -x$ are morphisms $X \times X \longrightarrow X$ and $X \longrightarrow X$ respectively. We first recall various basic facts about invertible sheaves on such varieties.

(I.) For every X, there is a 2nd abelian variety $\hat{X}$, called its dual, and an invertible sheaf P on $X \times \hat{X}$, called the Poincaré sheaf such that $P|_{X \times \{0\}} \cong \mathcal{O}_X$, $P|_{\{0\} \times \hat{X}} \cong \mathcal{O}_{\hat{X}}$, which is characterized by the non-degeneracy properties:

(a) If $Z \subset \hat{X}$ is a subscheme such that $P|_{X \times Z} \cong \mathcal{O}_{X \times Z}$, then $Z = \{0\}$ with reduced structure,

(b) $Z \subset X$ is a subscheme such that $P|_{Z \times \hat{X}} \cong \mathcal{O}_{Z \times \hat{X}}$, then $Z = \{0\}$ with reduced structure.

(II.) If $\mathrm{Pic}(X)$ is the group of all invertible sheaves on X, there is a subgroup $\mathrm{Pic}^0(X)$ characterized by the property:

$$L \in \mathrm{Pic}^0(X) \iff T_x^* L \cong L, \quad \text{all} \quad x \in X$$

where $T_x: X \longrightarrow X$ is the map $T_x(y) = x+y$. For all $a \in \hat{X}$, let $P_a = P|_{X \times \{a\}}$, an invertible sheaf on X. Then for all $a \in \hat{X}$, $P_a \in \mathrm{Pic}^0(X)$, and $a \longmapsto P_a$ defines an isomorphism of groups:

$$\hat{X} \cong \mathrm{Pic}^0(X).$$

(III.) For all invertible sheaves L on X, and $x,y \in X$,

$$T_{x+y}^{*}L \otimes L \cong T_x^{*}L \otimes T_y^{*}L.$$

Therefore $T_x^{*}L \otimes L^{-1} \in \mathrm{Pic}^0(X)$ and there is a unique homomorphism $\phi_L : X \longrightarrow \hat{X}$ characterized by:

$$P_{\phi_L(x)} \cong T_x^{*}L \otimes L^{-1} .$$

(IV.) The Riemann-Roch theorem for abelian varieties asserts: if $L = \mathcal{O}(D)$, D a divisor on X, then

$$\chi(L) = (D^g)/g! = \pm\sqrt{\deg \phi_L}$$

If this number is not 0, L is said to be non-degenerate. Then there is exactly one i, called the index of L, for which $H^i(L) \neq (0)$. In particular, if L is ample, then

$$\chi(L) = \dim \Gamma(L) > 0,$$
$$H^i(L) = (0),\ i \geq 1.$$

These facts are all more or less well known. Detailed proofs can be found, for example, in my book "Abelian Varieties", to be published by Oxford University Press in the series "Tata Institute Studies in Mathematics." We require, in addition, another invariant of invertible sheaves, which I call its theta-group. We treat this group first set-theoretically:

Definition: $\mathcal{G}(L)$ = the set of all pairs (x,ϕ), where $x \in X$ and $\phi: L \longrightarrow T_x^*L$ is an isomorphism.

The group law is given by:

$$(x,\phi) \cdot (y,\psi) = (x+y, T_y^*\phi \cdot \psi)$$

$$L \xrightarrow{\psi} T_y^*L \xrightarrow{T_y^*\phi} T_{x+y}^*L$$

It is easy to see that if $K(L) = \ker(\phi_L)$, then this groups fits into an exact sequence:

$$1 \longrightarrow k^* \xrightarrow{i} \mathcal{G}(L) \xrightarrow{\pi} K(L) \longrightarrow 1.$$

if
$$i(\lambda) = (0, \text{mult. by } \lambda),$$
$$\pi(x,\phi) = x.$$

Moreover, $i(k^*)$ commutes with everything in $\mathcal{G}(L)$. If, instead of using invertible sheaves, we spoke of line bundles, $\mathcal{G}(L)$ would be just the group of automorphisms of L that cover translations of X. Or if we use the language of divisors and divisor classes, then:

$$\mathcal{G}(\mathcal{O}_X(D)) = \text{the set of pairs } (x,f), \quad f \in k(X), \text{ such that}$$
$$T_x^{-1}D = D + (f)$$
$$((f) = \text{divisor of poles and zeroes of } f).$$

The group law in this version is:

$$(x,f) \quad (y,g) = (x+y, T_y^*f \cdot g).$$

This group is well known in one case: if $L \in \mathrm{Pic}^o(X)$. In this case, $\phi_L \equiv 0$, so $K(L) = X$ and $\mathcal{G}(L)$ is an extension:

$$1 \longrightarrow k^* \longrightarrow \mathcal{G}(L) \longrightarrow X \longrightarrow 1.$$

Serre has studied this case, and has shown that $\mathcal{G}(L)$ is abelian, has a natural structure of algebraic group itself, and that

$$L \longmapsto \mathcal{G}(L)$$

defines an isomorphism:

$$\mathrm{Pic}^o(X) \longrightarrow \mathrm{Ext}^1(X, \mathbb{G}_m).$$

One can describe the non-commutativity of $\mathcal{G}(L)$ conveniently as follows: look at the commutators $xyx^{-1}y^{-1}$. Since $K(L)$ is abelian, $\pi(xyx^{-1}y^{-1}) = 1$, and $xyx^{-1}y^{-1} \in k^*$. Moreover, since $k^* \subset$ center $(\mathcal{G}(L))$, if we alter x or y by an element of k^*, $xyx^{-1}y^{-1}$ does not change. Therefore there is a map:

$$e_L: \ K(L) \times K(L) \longrightarrow k^*$$

$$\text{such that } \ xyx^{-1}y^{-1} = e(\pi x, \pi y), \text{ all } x,y \in \mathcal{G}(L).$$

It is easy to check that e_L is bi-multiplicative and skew-symmetric.

In treating characteristic p, we need more than a set-theoretic group $\mathcal{G}(L)$ we need a full group <u>scheme</u> $\mathcal{G}(L)$. This is defined by asking that the S-valued points of $\mathcal{G}(L)$, for every scheme S/k should be functorially isomorphic to the groups of pairs (x,ϕ), where x is an S-valued point of X, and if $T_x: X\times S \longrightarrow X\times S$ is translating by x, then

$$\phi: L \otimes \mathcal{O}_S \longrightarrow T_x^*(L \otimes \mathcal{O}_S)$$

is an isomorphism. It fits into an exact sequence of group schemes:

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\ i\ } \mathcal{G}(L) \xrightarrow{\ \pi\ } K(L) \longrightarrow 1$$

where π is smooth and surjective, and $\mathbb{G}_m$ is the kernel of ϕ_L in the category of group schemes. For details, see the last § of my book on Abelian Varieties.

The theta-group $\mathcal{G}(L)$ acts in a natural way on the cohomology groups $H^i(L)$. In fact, if $(x,\phi) \in \mathcal{G}(L)$, then define the automorphism of $H^i(L)$:

$$U_{(x,\phi)}: H^i(L) \xrightarrow[\approx]{T_x^*} H^i(T_x^*L) \xleftarrow[\approx]{H^i(\phi)} H^i(L).$$

This gives a representation of $\mathcal{G}(L)$ and it works equally well for group schemes or for ordinary groups.

I propose to divide the rest of this section in half: I shall look first in characteristic 0, where only the set-theoretic $\mathcal{G}(L)$'s are needed, and prove a theorem for these; I will then discuss the extension to characteristic p.

So let char(k) = 0 now. First we need some pure group theory: Let K be a finite abelian group, and let $\mathcal{G}$ be a central extension:

$$1 \longrightarrow k^* \longrightarrow \mathcal{G} \longrightarrow K \longrightarrow 1.$$

Call $\mathcal{G}$ <u>non-degenerate</u> if k^* is exactly the center of $\mathcal{G}$. Then if $\mathcal{G}$ is non-degenerate:

(1) explicitly, $\mathcal{G}$ has the form $\mathcal{G} \cong k^* \times A \times \hat{A}$, where A is a finite abelian group, $\hat{A} = \mathrm{Hom}(A,k^*)$, and multiplication is

$$(\lambda,x,\xi)\cdot(\mu,y,\eta) = (\lambda\mu\eta(x),\ x+y,\ \xi+\eta).$$

(2) $\mathcal{G}$ has a unique irreducible representation V in which k^* acts by its natural character. All such representations are sums of V with itself. For $k^* \times A \times \hat{A}$, this representation can be realized by:

$$V = k\text{-valued functions on } A$$

$$U_{(\lambda,x,\xi)} f\,(y) = \lambda\cdot\xi(y)\cdot f(x+y),\quad \forall\ f \in V.$$

(3) If $H \subset \mathcal{G}$ is an abelian subgroup such that $H \cap k^* = \{1\}$, then we can decompose the irreducible representation V in (2) according to the characters of H:

$$V = \bigoplus_{\lambda \in \hat{H}} V_\lambda .$$

Then each V_λ is non-empty, and if $\mathcal{G}'$ is the centralizer of H in $\mathcal{G}$, then

$$\mathcal{G}'/\{\lambda(x)^{-1}\cdot x \mid x \in H\}$$

acts on V_λ, is again a non-degenerate extension, and V_λ is its irreducible representation.

This is all elementary group theory and is easy enough to prove. (See my paper "On the equations defining abelian varieties", Inv. Math., vol. 1). The key result is:

<u>Theta-structure theorem</u>: If L is non-degenerate of index i, then $\mathcal{G}(L)$ is a non-degenerate extension and $H^i(L)$ is its unique irreducible representation, with k^* acting naturally.

We now prove in characteristic 0:

<u>Theorem 9</u>: Let L be an ample invertible sheaf on an abelian variety X. Then for all $\alpha, \beta \in \hat{X}$, all $n, m \geq 4$

$$\mathcal{S}(L^n \otimes P_\alpha, L^m \otimes P_\beta) = (0).$$

<u>Proof</u>: We require the preliminary fact:

<u>Lemma</u>: Let L and M be invertible sheaves on an abelian variety such that $\Gamma(L) \neq (0)$, $\Gamma(M) \neq (0)$, and $L \otimes M$ is ample. Then

$$\sum_{\alpha \in \hat{X}} \Gamma(L \otimes P_\alpha) \otimes \Gamma(M \otimes P_{-\alpha}) \longrightarrow \Gamma(L \otimes M)$$

is surjective.

<u>Proof of lemma</u>: If W is the image, let us show that W is invariant under the action of $\mathcal{G}(L\otimes M)$. Note that if $x \in K(L\otimes M)$, then

$$\phi_L(x) + \phi_M(x) = \phi_{L\otimes M}(x) = 0.$$

Therefore if $\beta = \alpha + \phi_L(x) = \alpha - \phi_M(x)$,

$$\begin{aligned} T_x^*(L \otimes P_\alpha) &\cong L \otimes [T_x^* L \otimes L^{-1}] \otimes P_\alpha \\ &\cong L \otimes P_{\phi_L(x)} \otimes P_\alpha \\ &\cong L \otimes P_\beta \end{aligned}$$

and

$$\begin{aligned} T_x^*(M \otimes P_{-\alpha}) &\cong M \otimes [T_x^* M \otimes M^{-1}] \otimes P_{-\alpha} \\ &\cong M \otimes P_{\phi_M(x)} \otimes P_{-\alpha} \\ &\cong M \otimes P_{-\beta} . \end{aligned}$$

Therefore we get a diagram:

$$\begin{array}{ccc} \Gamma(L\otimes P_\alpha) \otimes \Gamma(M\otimes P_{-\alpha}) & \longrightarrow & \Gamma(L\otimes M) \\ \Big\downarrow T_x^* \qquad \Big\downarrow T_x^* & & \Big| T_x^* \\ \Gamma(T_x^*(L\otimes P_\alpha))\otimes\Gamma(T_x^*(M\otimes P_{-\alpha})) & \longrightarrow & \Gamma(T_x^*(L\otimes M)) \\ \wr\| \qquad\qquad \wr\| & & \wr\| \\ \Gamma(L\otimes P_\beta) \otimes \Gamma(M\otimes P_{-\beta}) & \longrightarrow & \Gamma(L\otimes M) \end{array}$$

In other words, under the action of an element $(x,\phi) \in \mathcal{G}(L\otimes M)$ on $\Gamma(L\otimes M)$, the image of $\Gamma(L\otimes P_\alpha) \otimes \Gamma(M\otimes P_{-\alpha})$ is taken into the image of $\Gamma(L\otimes P_\beta) \otimes \Gamma(M\otimes P_{-\beta})$. Therefore W is $\mathcal{G}(L\otimes M)$-invariant.

Now since $\Gamma(L\otimes M)$ is $\mathcal{G}(L\otimes M)$-irreducible, either $W = (0)$ or $W = \Gamma(L\otimes M)$. But if $s \in \Gamma(L)$, $s \neq 0$ and $t \in \Gamma(M)$, $t \neq 0$, then $s\otimes t \in \Gamma(L\otimes M)$ is not 0: so $W \neq (0)$.

QED

Returning to the theorem, we use the lemma to reduce the proof of the theorem to the special case $n = m = 4$. In fact, consider the diagram:

$$\begin{array}{ccc}
\sum_{\gamma \in \hat{X}} \Gamma(L^n \otimes P_\alpha) \otimes \Gamma(L^{m-1} \otimes P_{\beta+\gamma}) \otimes \Gamma(L \otimes P_{-\gamma}) & \xrightarrow{a} & \sum_{\gamma \in \hat{X}} \Gamma(L^{n+m-1} \otimes P_{\alpha+\beta+\gamma}) \otimes \Gamma(L \otimes P_{-\gamma}) \\
\Big\downarrow b & & \Big\downarrow c \\
\Gamma(L^n \otimes P_\alpha) \otimes \Gamma(L^m \otimes P_\beta) & \xrightarrow[d]{} & \Gamma(L^{n+m} \otimes P_{\alpha+\beta}).
\end{array}$$

By the lemma, c is surjective. By induction on n and m, a is surjective. Therefore d is surjective.

Now assume $n = m = 4$. We must show that the map:

$$\tau: \Gamma(L^4 \otimes P_\alpha) \otimes \Gamma(L^4 \otimes P_\beta) \longrightarrow \Gamma(L^8 \otimes P_{\alpha+\beta})$$

is surjective. We need first some simple remarks. One is that if L is any non-degenerate sheaf, then there is a natural isomorphism:

$$\mathcal{G}(L \otimes P_\alpha) \cong \mathcal{G}(L), \qquad \text{all } \alpha \in \hat{X}.$$

In fact, consider the diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(P_\alpha) & \xrightarrow{\pi} & X & \longrightarrow & 1 \\
 & & & & \nwarrow & & \cup & & \\
 & & & & & \rho_\alpha & K(L) & &
\end{array}$$

Since k^* is a divisible group, and $\mathcal{G}(P_\alpha)$ is abelian, it is easy to check that there is a homomorphism ρ_α such that $\pi \cdot \rho_\alpha = id^*$. In other

*If $0 \to A \to B \to C \to 0$ is an extension of abelian groups, then it splits whenever A is divisible.

words, the extension $\mathcal{G}(P_\alpha)$ <u>splits</u> over $K(L)$. Then for all $(x,\phi) \in \mathcal{G}(L)$, where $\phi: L \longrightarrow T_x^* L$ is an isomorphism, we get an isomorphism

$$L \otimes P_\alpha \xrightarrow{\phi \otimes \rho_\alpha(x)} T_x^*(L \otimes P_\alpha),$$

hence an element $(x, \phi \otimes \rho_\alpha(x)) \in \mathcal{G}(L \otimes P_\alpha)$.

The second remark is that τ is, in a certain sense, $\mathcal{G}(L^4)$-linear. In fact, define $\delta: \mathcal{G}(L^4) \longrightarrow \mathcal{G}(L^8)$ by

$$\delta(x,\phi) = (x, \phi^{\otimes 2})$$

where $\phi^{\otimes 2}: L^8 \longrightarrow T_x^* L^8$ is just $\phi \otimes \phi$.

Note that δ fits into a diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(L^4) & \longrightarrow & K(L^4) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \lambda \mapsto \lambda^2} & & \downarrow{\scriptstyle \delta} & & \cap & & \\
1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(L^8) & \longrightarrow & K(L^8) & \longrightarrow & 1 \\
 & & & & & & \cap & & \\
 & & & & & & X & &
\end{array}$$

Now choose splittings:

$$\rho_\alpha: K(L^8) \longrightarrow \mathcal{G}(P_\alpha)$$
$$\rho_\beta: K(L^8) \longrightarrow \mathcal{G}(P_\beta)$$

and let ρ_α, ρ_β induce a 3^{rd} splitting:

$$\rho_{\alpha+\beta}: K(L^8) \longrightarrow \mathcal{G}(P_{\alpha+\beta}).$$

Use $\rho_\alpha, \rho_\beta, \rho_{\alpha+\beta}$ to define isomorphisms $\mathcal{G}(L^4) \cong \mathcal{G}(L^4 \otimes P_\alpha) \cong \mathcal{G}(L^4 \otimes P_\beta)$

and $\mathcal{G}(L^8) \cong \mathcal{G}(L^8 \otimes P_{\alpha+\beta})$. Then it is immediate that via δ, τ is $\mathcal{G}(L^4)$-linear.

The next step is to split $\mathcal{G}(L^4)$ over X_2, the group of points of X of order 2:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(L^4) & \xrightarrow{\ \pi\ } & K(L^4) & \longrightarrow & 1 \\ & & & & & \nwarrow{\scriptstyle \rho} & \cup & & \\ & & & & & & X_2 & & \end{array}$$

As in the case of $\mathcal{G}(P_\alpha)$, this is possible if we check that the subgroup $\pi^{-1}(X_2)$ is abelian. But $K(L^4) = \mathrm{Ker}(\phi_{L^4}) = \mathrm{Ker}(4\cdot\phi_L)$, so $x \in K(L^4)$ if and only if $4x \in K(L)$. In particular, $X_4 \subset K(L^4)$. Therefore, if $x_1, x_2 \in X_2$ and $x_2 = 2y_2$, $y_2 \in X_4$, and

$$\begin{aligned} e_{L^4}(x_1, x_2) &= e_{L^4}(x_1, 2y_2) \\ &= e_{L^4}(2x_1, y_2) \\ &= e_{L^4}(0, y_2) = 1. \end{aligned}$$

Thus $\pi^{-1}(X_2)$ is abelian and ρ exists. We may now decompose all 3 vector spaces under the action of the abelian group $\rho(X_2)$:

$$\Gamma(L^4 \otimes P_\alpha) = \bigoplus_{\ell \in \hat{X}_2} E_\ell$$

$$\Gamma(L^4 \otimes P_\beta) = \bigoplus_{\ell \in \hat{X}_2} F_\ell$$

$$\Gamma(L^8 \otimes P_{\alpha+\beta}) = \bigoplus_{\ell \in \hat{X}_2} G_\ell$$

Note that $\tau(E_\ell \otimes F_m) \subset G_{\ell+m}$, since τ is, in particular, X_2-linear.

Next, I claim that in $\mathcal{G}(L^8)$, $\delta(\mathcal{G}(L^4))$ is the centralizer of $\delta(\rho(X_2))$. Since $\delta(\mathcal{G}(L^4))$ is exactly the inverse image $\pi^{-1}(K(L^4))$ in $\mathcal{G}(L^8)$, and since e_{L^8} computes the commutators in $\mathcal{G}(L^8)$, this is equivalent to saying: $\forall\, x \in K(L^8)$

$$(*) \qquad x \in K(L^4) \iff e_{L^8}(x,y) = 1 \qquad \forall y \in X_2.$$

But if $y \in X$, then $y \in K(L^8) \iff 2y \in K(L^4)$. Since X is divisible, $K(L^4) = 2\cdot K(L^8)$. Therefore, if we abbreviate $K(L^8) = K$, (*) comes down to the assertion:

$$(**) \qquad \forall\, x \in K, \qquad x \in 2K \iff e(x,y) = 1, \text{ all } y \in K \text{ such that } 2y = 0.$$

Since $\mathcal{G}(L^8)$ is a non-degenerate extension, e is a non-degenerate skew-symmetric form on K, and (**) is clearly true.

We can now apply the 3rd set of statements about non-degenerate extensions that we listed above. We deduce:

1) that each E_ℓ, F_ℓ, G_ℓ is non-empty,

2) that G_ℓ is an irreducible $\delta(\mathcal{G}(L^4))$-module.

The theorem now follows. By (1), choose $s \in E_\ell$, $t \in F_m$ with $s \neq 0$, $t \neq 0$. Then $\tau(s\otimes t)$ is the section $s\otimes t$ of $L^8 \otimes P_{\alpha+\beta}$, which is not zero. So the image of τ contains at least one non-zero element of G_ℓ, for each ℓ. But the image of τ is invariant under $\delta(\mathcal{G}(L^4))$, so by (2), it contains all of G_ℓ. Thus τ is surjective.

QED

Now consider the case char(k) = p ≠ 0. To make the proof work we must use the full group scheme $\mathcal{G}(L)$. First we need some theory about group schemes $\mathcal{G}$ which are central extensions of the type:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G} \longrightarrow K \longrightarrow 1$$

where K is a finite commutative group scheme. As before, we call $\mathcal{G}$ <u>non-degenerate</u> if $\mathbb{G}_m$ is the full scheme-theoretic center of $\mathcal{G}$ (i.e., ∀ S-valued points x of $\mathcal{G}$, if x commutes with all S'-valued points y of $\mathcal{G}$ for all S'/S, then x should be a point of $\mathbb{G}_m$). There is no simple structure theorem for such $\mathcal{G}$'s. However, they do satisfy:

(2') $\mathcal{G}$ has a unique irreducible representation V in which $\mathbb{G}_m$ acts by its natural character. All such representations are sums of V with itself.

(3') If $H \subset \mathcal{G}$ is an abelian subgroupscheme such that $H \cap \mathbb{G}_m = \{1\}$ scheme-theoretically, and if $R_H = \Gamma(\mathcal{O}_H)$ regarded as a representation of H (the "regular representation"), then $V \cong R_H^m$ for some m as an H-space. In particular, for all characters $\lambda: H \longrightarrow \mathbb{G}_m$, the eigenspace $V_\lambda \subset V$ for λ is non-empty. Moreover, if $\mathcal{G}'$ is the scheme-theoretic centralizer of H in $\mathcal{G}$, then

$$\mathcal{G}'/\{\lambda(x)^{-1}\cdot x \mid x \in H\}$$

acts on V_λ, is again a non-degenerate extension, and V_λ is its irreducible representation.

Note that in (3') $V \supset \oplus V_\lambda$, but if $\mathrm{char}(k) \mid \mathrm{order}\ (H)$, it is possible that $V \supsetneqq \oplus V_\lambda$. In compensation, we have the extra fact, $V \cong R_H^m$.

Next, we still have:

<u>Theta-structure theorem</u>: If L is non-degenerate of index i, then $\mathcal{G}(L)$ is a non-degenerate extension and $H^i(L)$ is its unique irreducible representation, with $\mathbb{G}_m$ acting naturally.

The proofs of these facts are, unfortunately, not yet published. Now let's generalize the proof of Theorem 9 to char(p).

(I.) The lemma remains true. However to prove it, it is necessary to show that for all rings R/k, all R-valued points α of $\mathcal{G}(L\otimes M)$, the automorphism of the R-module $\Gamma(L\otimes M) \otimes_k R$ induced by α takes $W\otimes R$ into itself. This follows as before provided that we first prove the following:

(*) For all R-valued points α of $\hat{X}$, if P_α is the invertible sheaf $(1\times\alpha)^*P$ on $X\times\mathrm{Spec}(R)$, then the image of the map:

$$\Gamma(p_1^*L\otimes P_\alpha) \otimes \Gamma(p_1^*M\otimes P_{-\alpha}) \longrightarrow \Gamma(p_1^*(L\otimes M)) = \Gamma(L\otimes M)\otimes_k R$$

is contained in $W \otimes_k R$.

First if R is a finitely generated integral domain over k, then the intersection of the maximal ideals in R is (0): so to prove that an element $x \in \Gamma(L\otimes M) \otimes_k R$ is in $W \otimes_k R$ for such an R, it suffices to show that for all homomorphisms $\phi: R \longrightarrow k$, the image $1 \otimes \phi(x) \in \Gamma(L \otimes M)$ is in W. And this is just a case of (*) for a k-valued point of X, i.e., it is part of the hypothesis. But since X is an integral scheme of finite type over k, for any R, and any R-valued point α of X, α is induced by an R'-valued point β of X via a homomorphism $R' \longrightarrow R$, with R' an integral domain finitely generated over k. And if (*) is true for β, it follows immediately for α. This proves (*) in general.

(II.) Once the lemma is proven, Theorem 9 is reduced to the case $n = m = 4$ exactly as before.

(III.) Next, isomorphisms $\mathcal{G}(L) \cong \mathcal{G}(L \otimes P_\alpha)$, $\alpha \in \hat{X}$, L non-degenerate, can be set up exactly as before. We need only the well-known lemma:

<u>Lemma</u>: If $0 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G} \longrightarrow K \longrightarrow 0$ is an abelian extension, and K is a finite group scheme, then $\mathcal{G} \cong \mathbb{G}_m \times K$.

Moreover, we get a homomorphism of group schemes $\delta: \mathcal{G}(L^4) \longrightarrow \mathcal{G}(L^8)$ exactly as before, and τ turns out again to be $\mathcal{G}(L^4)$-linear.

(IV.) Now, if $\mathrm{char}(k) \neq 2$, the rest of the proof works over k without alteration: $\mathcal{G}(L^4)$ splits over X_2, the vector spaces $\Gamma(L^4 \otimes P_\alpha)$, $\Gamma(L^4 \otimes P_\beta)$, $\Gamma(L^8 \otimes P_{\alpha+\beta})$ split into eigenspaces, and we apply statement (3') about the group theory of non-degenerate $\mathcal{G}$'s. <u>However</u>, if char(k) = 2, X_2, the kernel of multiplication by 2, is never a reduced group scheme. We can still split $\mathcal{G}(L^4)$ over X_2, and $\delta(\mathcal{G}(L^4))$ is still the centralizer of $\delta(\rho(X_2))$ in $\mathcal{G}(L^8)$, <u>but</u> since the representations of X_2 are not completely reducible,

$$\Gamma(L^4 \otimes P_\alpha) \underset{\neq}{\supset} \bigoplus_{\ell \in \hat{X}_2} E_\ell \,, \quad \text{etc.}$$

We must finish the proof in a new way. Let W = image of τ. Let $W^\perp \subset \Gamma(L^8 \otimes P_{\alpha+\beta})^*$ be the space of linear maps that kill W. Assume $W \underset{\neq}{\subset} \Gamma(L^8 \otimes P_{\alpha+\beta})$, hence $W^\perp \neq (0)$ Now W and hence $W^\perp$ is invariant under the action of $\mathcal{G}(L^4)$, hence of the action of $\rho(X_2)$. Therefore $W^\perp$ contains an eigenvector for at least one character $\ell \in \hat{X}_2$. Let $G_\ell^* \subset \Gamma(L^8 \otimes P_{\alpha+\beta})^*$ be the eigenspace for the character ℓ. Now $\Gamma(L^8 \otimes P_{\alpha+\beta})^*$ is an irreducible representation space for the opposed group to $\mathcal{G}(L^8)$, i.e., with multiplication reversed, and in this representation $\mathbb{G}_m$ acts by its natural character. Therefore applying statement (3') to this opposed group, it follows that G_ℓ^* is $\mathcal{G}(L^4)$-irreducible. Therefore $W^\perp \supset G_\ell^*$.

Now we must construct something inside W. By (3') for $\mathcal{G}(L^4)$, $\Gamma(L^4 \otimes P_\beta)$ contains a non-zero $\rho(X_2)$-invariant t. For all $s \in \Gamma(L^4 \otimes P_\alpha)$, $s \neq 0$, the element $\tau(s \otimes t) \in \Gamma(L^8 \otimes P_{\alpha+\beta})$ is not zero, so τ defines an isomorphism of $\Gamma(L^4 \otimes P_\alpha) \otimes s$ with a subspace $W_o \subset W$. As a representation space for $\rho(X_2)$, W_o is therefore isomorphic to $\Gamma(L^4 \otimes P_\alpha)$, hence to R_2^m, where R_2 denotes the regular representation of X_2. Since R_2 is an injective object in the category of representations of X_2, it follows that:

$$\Gamma(L^8 \otimes P_{\alpha+\beta}) \cong W_o \oplus \tilde{W}$$

where $\tilde{W}$ is also X_2-invariant. Now the dual space to R_2 contains eigenvectors for every character of X_2: so there is an element $x \in W_o^*$ which is an eigenvector for the character ℓ. Extend x to a linear map on $\Gamma(L^8 \otimes P_{\alpha+\beta})$ that is zero on $\widetilde{W}$. Then $x \in G_\ell^*$, but $x \neq 0$ on W, i.e., $x \notin W^\perp$. This is a contradiction.

QED

§-. Abelian varieties: the method of the variable pencil.

First of all, we need some more results about the index of invertible sheaves. For proofs of these results, see my book on Abelian Varieties and the appendix to this paper by George Kempf.

Definition: Let L be a degenerate invertible sheaf on an abelian variety X. Let $K = K(L)^{\circ}$, the connected component of K(L), Y = X/K, and $\pi: X \longrightarrow Y$ the canonical map. Then there is a non-degenerate sheaf M on Y such that $L \cong P_{\alpha} \otimes \pi^*M$, some $\alpha \in \hat{X}$ (cf. appendix). We define index (L) to be the interval:

$$[\text{index }(M), \text{ index }(M) + \dim K].$$

The following result is proven in the appendix:

Proposition: If $i \notin$ index (L), then $H^i(L) = (0)$.

Now suppose L and M are 2 invertible sheaves on X, and L is ample. Consider the collection of sheaves $L^p \otimes M^q$ and the polynomial:

$$P(p,q) = \chi(L^p \otimes M^q).$$

The following theorem is proven in §16 of my book and in the appendix to this paper:

Theorem: If g = dim X, then there are $\alpha_1, \cdots, \alpha_g \in \mathbb{R}$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_g$ such that

$$P(x,y) = \prod_{i=1}^{g} (x - \alpha_i y).$$

Moreover, for all $p,q \in \mathbb{Z}$, $q > 0$, if $\alpha_{i+1} = \cdots = \alpha_{i+k} = \frac{p}{q}$,

$\alpha_i > \frac{p}{q} > \alpha_{i+k+1}$, then

$$\text{index } (L^p \otimes M^q) = [i, i+k].$$

The precise result that we need is slightly stronger. I want to assume only that $\Gamma(L) \neq (0)$ (i.e., $L = \pi^* L_o$ for some $\pi: X \longrightarrow X/K$, L_o ample on X/K). In this case, I claim:

Theorem: Suppose L and M are 2 invertible sheaves, $\Gamma(L) \neq (0)$ and M non-degenerate. Then

$$P(x,y) = \prod_{i=1}^{r} (x - \alpha_i y) \cdot y^{g-r}$$

for some r and some $\alpha_i \in \mathbb{R}$ with $\alpha_1 \geq \cdots \geq \alpha_r$. For $N >> 0$, let $i_o = \text{index } (L^N \otimes M)$. Then for all p,q, $q > 0$, if $\alpha_{i+1} = \cdots = \alpha_{i+k} = p/q$, $\alpha_i > \frac{p}{q} > \alpha_{i+k+1}$, then

$$\text{index } (L^p \otimes M^q) = [i + i_o, i + k + i_o].$$

This theorem is deduced easily from the 1st one, by introducing an ample L_1 and considering all the sheaves $L^p \otimes L_1^{p_1} \otimes M^q$ and the polynomial: $P(p, p_1, q) = \chi(L^p \otimes L_1^{p_1} \otimes M^q)$.
We omit this step.

The purpose of this section is to prove:

Theorem 10: Let X be an abelian variety, L an ample invertible sheaf and $n \geq 4$ an integer. L^n defines an immersion:

$$\phi_{L^n}: X \longrightarrow \mathbb{P}(\Gamma(L^n)) .$$

Then $\phi_{L^n}(X)$ is ideal-theoretically an intersection of quadrics of rank ≤ 4.

Proof: First, let's construct a set of quadrics containing $\phi_{L^n}(X)$. Once and for all, fix p and q, $p,q \geq 2$, such that $n = p+q$. Consider the map:

$$\Gamma(L^p \otimes P_\alpha) \otimes \Gamma(L^q \otimes P_{-\alpha}) \longrightarrow \Gamma(L^n).$$

If $s \in \Gamma(L^p \otimes P_\alpha)$, $t \in \Gamma(L^q \otimes P_{-\alpha})$, let the induced section $s \otimes t$ of $\Gamma(L^n)$ be denoted $\langle s,t \rangle$ to prevent a confusion of notation. Then for all $s_1, s_2 \in \Gamma(L^p \otimes P_\alpha)$, $t_1, t_2 \in \Gamma(L^q \otimes P_{-\alpha})$, we get 4 sections of L^n: $\langle s_i, t_j \rangle$, $i,j = 1$ and 2. In $\Gamma(L^{2n})$, we get an identity:

$$\langle s_1, t_1 \rangle \otimes \langle s_2, t_2 \rangle = \langle s_1, t_2 \rangle \otimes \langle s_2, t_1 \rangle.$$

Therefore

$$q_{s_1,t_1,s_2,t_2} = \langle s_1,t_1 \rangle \otimes \langle s_2,t_2 \rangle - \langle s_1,t_2 \rangle \otimes \langle s_2,t_1 \rangle \in \mathcal{R}(L^n, L^n).$$

If Q_{s_1,t_1,s_2,t_2} is the quadric in $\mathbb{P}(\Gamma(L^n))$ defining by $q_{s_1,t_1,s_2,t_2} = 0$, then we will actually prove:

$$(*) \quad \left\{ \begin{array}{l} \phi_{L^n}(X) \text{ is the ideal-theoretic intersection of the quadrics} \\ \qquad Q_{s_1,t_1,s_2,t_2} \text{ for all } \alpha, s_i, t_i. \end{array} \right.$$

For most of this proof, we will deal with the fact that $\phi_{L^n}(X)$ is the set-theoretic intersection of these quadrics. At the end, we will indicate the easy extension of the method to proof that $\phi_{L^n}(X)$ is also an ideal-theoretic intersection.

The first step is to translate (*) into an assertion on X itself, not involving $\mathbb{P}(\Gamma(L^n))$. The points of $\mathbb{P}(\Gamma(L^n))$ correspond to non-zero linear maps $\ell: \Gamma(L^n) \longrightarrow k$, modulo scalars. Fix one such ℓ. Then it is easy to see that the point defined by ℓ lies on Q_{s_1,t_1,s_2,t_2} if and only if

$$\ell(\langle s_1,t_1\rangle)\cdot\ell(\langle s_2,t_2\rangle) = \ell(\langle s_1,t_2\rangle)\cdot\ell(\langle s_2,t_1\rangle).$$

Moreover, it is elementary linear algebra that this holds for all s_1,t_1,s_2,t_2 if and only if there are linear maps:

$$m_\alpha: \Gamma(L^p \otimes P_\alpha) \longrightarrow k$$
$$n_\alpha: \Gamma(L^q \otimes P_{-\alpha}) \longrightarrow k$$

such that:

$$\ell(\langle s,t\rangle) = m_\alpha(s)\cdot n_\alpha(t), \qquad \text{all } s \in \Gamma(L^p \otimes P_\alpha)$$
$$t \in \Gamma(L^q \otimes P_{-\alpha}).$$

On the other hand, what does it mean to say that the "point" ℓ is in $\phi_{L^n}(X)$? This means that there is a point $x \in X$, and an isomorphism $L^n \cong \mathcal{O}_X$ near x, such that, evaluating sections by this isomorphism:

$$\ell(s) = s(x), \quad \text{all } s \in \Gamma(L^n).$$

Thus (*) comes down to the assertion:

(**) If $\ell: \Gamma(L^n) \longrightarrow k$ is a non-zero linear map, such that for all $\alpha \in \hat{X}$, there exist linear maps $m_\alpha: \Gamma(L^p \otimes P_\alpha) \longrightarrow k$, $n_\alpha: \Gamma(L^q \otimes P_{-\alpha}) \longrightarrow k$ for which $\ell(\langle s,t\rangle) = m_\alpha(s). n_\alpha(t)$, then for some $x \in X$, $\ell(s) = s(x)$ all $s \in \Gamma(L^n)$.

In order to prove (**), the basic idea is to treat all α simultaneously, i.e., to put the m_α's and n_α's together into a single homomorphism. In fact, consider the invertible sheaves:

$$p_1^*L^p \otimes P \quad \text{and} \quad p_1^*L^q \otimes P^{-1} \quad \text{on} \quad X \times \hat{X}.$$

These have the property:

$$p_1^*L^p \otimes P\big|_{X\times(\alpha)} \cong L^p\otimes P_\alpha; \qquad p_1^*L^q\otimes P^{-1}\big|_{X\times(\alpha)} \cong L^q \otimes P_{-\alpha}.$$

Define

$$E_p = p_{2,*}(p_1^*L^p\otimes P)$$

$$F_q = p_{2,*}(p_1^*L^q\otimes P^{-1}).$$

Since the higher cohomology groups of $L^p\otimes P_\alpha$, $L^q\otimes P_{-\alpha}$ are zero, E_p and F_q are locally free sheaves on S such that

$$E_p\otimes k(\alpha) \cong \Gamma(L^p\otimes P_\alpha); \quad F_q\otimes k(\alpha) \cong \Gamma(L^q\otimes P_{-\alpha}).$$

There is a natural pairing:

$$E_p \otimes_{\mathcal{O}_{\hat{X}}} F_q \longrightarrow p_{2,*}(p_1^*L^n) \cong \Gamma(L^n) \otimes_k \mathcal{O}_{\hat{X}}.$$

This is the globalized form of the individual pairings $\Gamma(L^p\otimes P_\alpha) \otimes \Gamma(L^q\otimes P_{-\alpha}) \longrightarrow \Gamma(L^n)$. In order to go further, we need:

Lemma 1: If ℓ: $\Gamma(L^n) \longrightarrow k$ satisfies the condition of (**), then for all α, ℓ does not vanish identically on the image of $\Gamma(L^p\otimes P_\alpha) \otimes \Gamma(L^q\otimes P_{-\alpha})$ in $\Gamma(L^n)$.

We will prove the lemma later. Assuming this, we next globalize the m_α and n_α as follows: I claim there is an invertible sheaf K on $\hat{X}$ and surjective homomorphisms:

$$m: E_p \longrightarrow K$$
$$n: F_q \longrightarrow K^{-1}$$

such that the diagram:

$$\begin{array}{ccc} E_p \otimes F_q & \longrightarrow & \Gamma(L^n) \otimes_k \mathcal{O}_{\hat{X}} \\ \downarrow{\scriptstyle m\otimes n} & & \downarrow{\scriptstyle \ell\otimes 1} \\ K \otimes K^{-1} & \xrightarrow{\approx} & \mathcal{O}_{\hat{X}} \end{array}$$

commutes. To see this, consider the composite map:

$$E_p \otimes F_q \longrightarrow \Gamma(L^n) \otimes \mathcal{O}_{\hat{X}} \xrightarrow{\ell\otimes 1} \mathcal{O}_{\hat{X}}.$$

It induces a map of locally free sheaves:

$$m': E_p \longrightarrow \underline{\mathrm{Hom}}\ (F_q, \mathcal{O}_{\hat{X}}) \ .$$

By the hypothesis in (**), this map, after taking $\otimes k(\alpha)$, is always of rank 0 or 1; by lemma 1, it never has rank 0. Therefore, its image is an invertible subsheaf K of $\underline{\mathrm{Hom}}(F_q, \mathcal{O}_{\hat{X}})$ which is locally a direct summand. m gives a surjective homomorphism $m: E_p \longrightarrow K$. On the other hand, the inclusion of K in $\underline{\mathrm{Hom}}(F_q, \mathcal{O}_{\hat{X}})$ induces a surjection:

$$n: F_q = \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(F_q, \mathcal{O}_{\hat{X}}), \mathcal{O}_{\hat{X}}) \longrightarrow \underline{\mathrm{Hom}}(K, \mathcal{O}_{\hat{X}}) = K^{-1}.$$

It is clear that the sheaf K and the homomorphisms m,n make the diagram above commute.

To motivate the next steps, let's imagine that (**) is true and see what K, m, and n ought to turn out to be. For all $x \in X$, let $Q_x = P|_{\{x\}\times\hat{X}}$. Then Q_x is an invertible sheaf on $\hat{X}$ and, if we pick an isomorphism $L^p \xrightarrow{\sim} \mathcal{O}_X$ in a neighborhood of x, then there is a natural restriction map:

$$p_1^*L^p \otimes P \longrightarrow p_1^*L^p \otimes P|_{\{x\}\times\hat{X}} \cong P|_{\{x\}\times\hat{X}}.$$

This induces a map of locally free sheaves on $\hat{X}$:

$$r_x: E_p \longrightarrow Q_x$$

which is a global form of the linear maps:

$$E_p \otimes k(\alpha) \cong \Gamma(L^p \otimes P_\alpha) \xrightarrow{\text{"evaluation at x"}} k.$$

Similarly there is a map:

$$s_x: F_q \longrightarrow Q_x^{-1}$$

which is a global form of the linear maps:

$$F_q \otimes k(\alpha) \cong \Gamma(L^q \otimes P_{-\alpha}) \xrightarrow{\text{"evaluation at x"}} k.$$

Therefore, what we want to prove is:

$$(***)\quad \begin{cases} K \cong Q_x, \text{ for some } x \in X \\ \text{and } m \text{ is a multiple of } r_x, \quad n \text{ of } s_x. \end{cases}$$

If we prove (***), then it follows immediately that ℓ, as a point of $\mathbb{P}(\Gamma(L^n))$, equals $\phi_{L^n}(x)$. In fact, choosing an isomorphism of L^n and $\mathcal{O}_X$ near x, let ℓ': $\Gamma(L^n) \longrightarrow k$ by the evaluation map $s \longmapsto s(x)$.

Then what (***) asserts is that the 2 composite homomorphisms:

$$\begin{matrix} L \\ L'' \end{matrix}: E_p \otimes F_q \longrightarrow \Gamma(L^n) \underset{k}{\otimes} \mathcal{O}_{\hat{X}} \underset{\ell'\otimes 1}{\overset{\ell\otimes 1}{\rightrightarrows}} \mathcal{O}_{\hat{X}}$$

differ by a scalar. Say $L = \lambda.L'$. Then on the image of each map

$$\Gamma(L^p \otimes P_\alpha) \otimes \Gamma(L^q \otimes P_{-\alpha}) \longrightarrow \Gamma(L^n),$$

$\ell = \lambda.\ell'$. By the lemma of §3, these images generate $\Gamma(L^n)$, so $\ell = \lambda.\ell'$ on all of $\Gamma(L^n)$ and the Theorem is proven.

To prove (***), we proceed as follows. First apply Serre duality to the morphism $p_2: X\times\hat{X} \longrightarrow \hat{X}$:

$$\begin{aligned}\Gamma(\hat{X}, \underline{\mathrm{Hom}}(E_p, K)) &= \Gamma(\hat{X}, \underline{\mathrm{Hom}}(p_{2,*}(p_1^* L^p \otimes P), K)) \\ &\cong \Gamma(\hat{X}, R^g_{p_{2,*}}(p_1^* L^{-p} \otimes P^{-1} \otimes p_2^* K))\end{aligned}$$

Since all the cohomology groups of the restriction $p_1^* L^{-p} \otimes P^{-1} \otimes p_2^* K|_{X\times\{\alpha\}} \cong L^{-p} \otimes P_{-\alpha}$ are zero, except for the g^{th} group,

$$R^i p_{2,*}(p_1^* L^{-p} \otimes P^{-1} \otimes p_2^* K) = (0), \quad i \neq g.$$

Therefore, we conclude by the Leray spectral sequence that:

$$\Gamma(\hat{X}, \underline{\mathrm{Hom}}(E_p, K)) \cong H^g(X\times\hat{X}, p_1^* L^{-p} \otimes P^{-1} \otimes p_2^* K).$$

Similarly:

$$\Gamma(\hat{X}, \underline{\mathrm{Hom}}(F_q, K^{-1})) \cong H^g(X\times\hat{X}, p_1^* L^{-q} \otimes P \otimes p_2^* K^{-1})$$

hence by Serre duality on $X\times\hat{X}$:

$$\Gamma(\hat{X}, \underline{Hom}(F_q, K^{-1}))^* \cong H^g(X\times\hat{X}, p_1^*L^q \otimes P^{-1} \otimes p_2^*K).$$

Therefore, we have at our disposal the 2 apparently meagre bits of information:

$$H^g(X\times\hat{X}, p_1^*L^m \otimes P^{-1} \otimes p_2^*K) \neq (0), \quad \text{for} \quad m = -p \text{ and } q.$$

But, amazingly, these facts turn out to trigger a Rube Goldberg-like set of cohomological implications that we will describe later. We summarize this part of the proof for now in:

Lemma 2: Let L be ample on X, K any invertible sheaf on $\hat{X}$. If there exist integers $a,b \geq 2$ such that

$$H^g(X\times\hat{X}, p_1^*L^m \otimes P^{-1} \otimes p_2^*K) \neq (0)$$

for $m = -a$ and b, then, in fact, for all m:

i) $p_1^*L^m \otimes P^{-1} \otimes p_2^*K$ is non-degenerate of index g,

ii) $\dim H^g(p_1^*L^m \otimes P^{-1} \otimes p_2^*K) = 1$,

iii) $K \in Pic^o(\hat{X})$.

But by the theorem of biduality, the invertible sheaf P on $X\times\hat{X}$ makes X into the dual $\hat{\hat{X}}$ of $\hat{X}$ with Poincaré sheaf still P. Therefore, all sheaves in $Pic^o(\hat{X})$ are isomorphic to Q_x, some $x \in X$, hence $K \cong Q_x$, some $x \in X$.

Finally to show that m is a multiple of r_x, and n is a multiple of s_x, it suffices to prove that

$$\dim \Gamma(\hat{X}, \underline{Hom}(E_p, K)) = 1$$
$$\dim \Gamma(\hat{X}, \underline{Hom}(F_q, K^{-1})) = 1.$$

But we saw above that these dimensions equal

$$\dim H^g(X\times\hat{X},\ p_1^*L^{-p}\otimes P^{-1}\otimes p_2^*K)$$

and

$$\dim H^g(X\times\hat{X},\ p_1^*L^{q}\otimes P^{-1}\otimes p_2^*K)$$

and these are both 1 by lemma 2. This proves (***)!

We now go on to the lemmas:

Proof of lemma 1: Suppose $\ell \equiv 0$ on the image of $\Gamma(L^p\otimes P_\alpha)\otimes\Gamma(L^q\otimes P_{-\alpha})$. Since ℓ is not zero everywhere, and since $\Gamma(L^p\otimes P_\beta)\otimes\Gamma(L^q\otimes P_{-\beta})$ generate $\Gamma(L^n)$ as β varies, choose a point $\gamma\in\hat{X}$ such that

$$\ell \not\equiv 0 \text{ on } \Gamma(L^p\otimes P_{\alpha+\gamma})\otimes\Gamma(L^q\otimes P_{-\alpha-\gamma})$$

By the hypothesis on ℓ, ℓ on this last space is of the form $m\otimes n$, where $m\not\equiv 0$ and $n\not\equiv 0$. By the same reasoning, for almost all $\delta\in\hat{X}$,

$$m\not\equiv 0 \quad\text{on}\quad \Gamma(L^{p-1}\otimes P_{\alpha+\gamma+\delta})\otimes\Gamma(L\otimes P_{-\delta}),$$

and again for almost all $\delta\in\hat{X}$

$$n\not\equiv 0 \quad\text{on}\quad \Gamma(L^{q-1}\otimes P_{-\alpha+\delta})\otimes\Gamma(L\otimes P_{-\gamma-\delta}).$$

Choose a δ for which $m\not\equiv 0$ and $n\not\equiv 0$. Then it follows that $\ell\not\equiv 0$ on the image in $\Gamma(L^n)$ of:

$$[\Gamma(L^{p-1}\otimes P_{\alpha+\gamma+\delta})\otimes\Gamma(L\otimes P_{-\delta})]\otimes[\Gamma(L^{q-1}\otimes P_{-\alpha+\delta})\otimes\Gamma(L\otimes P_{-\gamma-\delta})].$$

But by interchanging the 2nd and 4th factors, this image is the same as the image in $\Gamma(L^n)$ of:

$$[\Gamma(L^{p-1}\otimes P_{\alpha+\gamma+\delta}) \otimes \Gamma(L\otimes P_{-\gamma-\delta})] \otimes [\Gamma(L^{q-1}\otimes P_{-\alpha+\delta}) \otimes \Gamma(L\otimes P_{-\delta})].$$

The map of this 4-way tensor product into $\Gamma(L^n)$ factors through $\Gamma(L^p\otimes P_\alpha) \otimes \Gamma(L^q\otimes P_{-\alpha})$, so this contradicts the assumption that $\ell \equiv 0$ on the image of this space in $\Gamma(L^n)$.

QED

Proof of lemma 2: This is where we will use the theorem quoted in the beginning of this section. First we compute $H^i(X\times\hat{X}, P \otimes p_2^*K^{-1})$. Apply the Leray spectral sequence:

$$H^i(\hat{X}, R^j p_{2,*}(P) \otimes K^{-1}) \Longrightarrow H^{i+j}(X\times\hat{X}, P \otimes p_2^*K^{-1}).$$

But, as is shown in my book, §13:

$$R^j p_{2,*}(P) = (0), \quad i < g$$

$$R^g p_{2,*}(P) = k(0).$$

Therefore:

$$H^{i+g}(X\times\hat{X}, P\otimes p_2^*K^{-1}) = \begin{cases} (0), & i \neq 0 \\ H^i(\hat{X}, k(0) \otimes K^{-1}), & i = 0. \end{cases}$$

Hence $H^i(X\times X, P\otimes p_2^*K^{-1}) = (0)$ if $i \neq g$, and is 1-dimensional if $i = g$.

By Serre duality, the same is true of $P^{-1} \otimes p_2^*K$. Now consider the family of sheaves:

$$M_{p,q} = p_1^*L^p \otimes (P^{-1} \otimes p_2^*K)^q$$

and their Euler characteristics:

$$P(p,q) = \chi(M_{p,q}) .$$

We know by the above computation and by our hypothesis that:

$$
\text{(I)} \quad \begin{cases} M_{0,1} \text{ non-degenerate, index} = g \\ g \in \text{index } (M_{b,1}) \\ g \in \text{index } (M_{-a,1}) \end{cases}
$$

It follows from the Theorem that $P(x,1)$ has no zeroes in the open interval $-a < x < b$. But now $P(x,1)$ is a real polynomial of x such that

i) P has only real zeroes,

ii) $P(0) = (-1)^g$,

iii) P has no zeroes with $-a < x < b$.

iv) $P(n) \in \mathbb{Z}$, for all $n \in \mathbb{Z}$.

But (i) implies that P has a unique local maximum or minimum between any 2 zeroes: let $-\alpha < 0 < \beta$ $(\alpha, \beta \in \mathbb{R}^+)$ be its zeroes of smallest absolute value. Since $-\alpha < 1 < \beta$, and $|P(1)| \geq 1 = |P(0)|$, P must have a local maximum or minimum between 0 and β; since $-\alpha < -1 < \beta$, and $|P(-1)| \geq 1 = |P(0)|$, P must also have a local maximum or minimum between $-\alpha$ and 0. This is a contradiction — <u>unless</u> P <u>is constant</u>.

Applying the theorem again, it follows that

$$
\text{(II)} \quad \begin{cases} M_{p,q} \text{ non-degenerate} \\ \text{index } (M_{p,q}) = g \\ \dim H^g(M_{p,q}) = 1 \end{cases}
$$

for all $p,q \in \mathbb{Z}$, $q \neq 0$. This proves (i) and (ii) of the lemma. To prove (iii), apply the Leray spectral sequence:

If $\mathfrak{J}_i = R^i p_{1,*}(P^{-1} \otimes p_2^* K)$, then

$$E_2^{i,j} = H^i(X, L^m \otimes \mathfrak{J}_j) \Longrightarrow H^{i+j}(X \times \hat{X}, p_1^* L^m \otimes P^{-1} \otimes p_2^* K) .$$

In particular, since L is ample, $E_2^{ij} = (0)$ if $i > 0$, $m >> 0$, hence the spectral sequence reduces to :

$$H^0(X, L^m \otimes \mathfrak{J}_j) \cong H^j(X \times \hat{X}, p_1^* L^m \otimes P^{-1} \otimes p_2^* K), \quad \text{if } m >> 0.$$

Therefore because of (II) the whole sheaf $\mathfrak{J}_j$ must be zero if $j < g$. The spectral sequence now reduces to

$$H^i(X, L^m \otimes \mathfrak{J}_g) \cong H^{i+g}(X \times \hat{X}, p_1^* L^m \otimes P^{-1} \otimes p_2^* K), \quad \text{all } m ,$$

hence

$$\chi(L^m \otimes \mathfrak{J}_g) = 1, \quad \text{all } m.$$

This shows first that $\mathrm{Supp}(\mathfrak{J}_g)$ is 0-dimensional, since its Hilbert polynomial is a constant; and second, that $\dim H^0(\mathfrak{J}_g) = 1$, hence $\mathfrak{J}_g \cong k(x)$, some $x \in X$.

Now recall from EGA, Ch. 3, §7 that the cohomology of $P^{-1} \otimes p_2^* K$ along the fibre $[x] \times \hat{X}$ of p_1 is computed from the higher direct images by a spectral sequence:

$$\mathrm{Tor}_{-i}^{\mathcal{O}_X}(k(x), R^j_{p_{1,*}}(P^{-1} \otimes p_2^* K)) \Longrightarrow H^{i+j}(P^{-1} \otimes p_2^* K \big|_{[x] \times \hat{X}}) .$$

Since $P^{-1} \otimes p_2^* K \big|_{\{x\}\times\hat{X}} \cong Q_x^{-1} \otimes K$, and since $\mathcal{J}_j = (0)$, $j < g$, we find

$$H^{g-i}(Q_x^{-1} \otimes K) \cong \mathrm{Tor}_i^{\mathcal{O}_X}(k(x),k(x)).$$

Thus $H^i(Q_x^{-1} \otimes K) \neq (0)$, for all i. For $i = 0$, this gives $\Gamma(Q_x^{-1} \otimes K) \neq (0)$, and for $i = g$, this gives (by Serre duality) $\Gamma(Q_x \otimes K^{-1}) \neq (0)$. Therefore $K \cong Q_x$ hence $K \in \mathrm{Pic}^0(\hat{X})$. QED

This completes the proof that $\phi_{L^n}(X)$ is the set-theoretic intersection of the quadrics Q_{s_1,t_1,s_2,t_2}. To prove that it is also ideal-theoretically equal to this intersection, it is enough, as we remarked in the introduction, to prove that for all $x \in X$, the tangent space to $\phi_{L^n}(X)$ at x is the intersection of the tangent spaces to the quadrics Q_{s_1,t_1,s_2,t_2} at x. Equivalently, let $R = k[\epsilon]/(\epsilon^2)$: then we must prove that for all R-valued points x of $\mathbb{P}(\Gamma(L^n))$, x is in $\phi_{L^n}(X)$ if and only if x is in all the quadrics. But such a point x is defined by a k-linear map $\lambda: \Gamma(L^n) \longrightarrow R$ such that $\mathrm{Image}(\lambda) \not\subset k\cdot\epsilon$. Translating suitably the conditions that x is in $\phi_{L^n}(X)$ and in the quadrics, we find that the assertion to be proven comes out as:

(**) If ℓ: $\Gamma(L^n) \longrightarrow R$ is a k-linear map with $\mathrm{Im}(\ell) \not\subset k\cdot\epsilon$, such that for all $\alpha \in \hat{X}$, there exist linear maps $m_\alpha: \Gamma(L^p\otimes P_\alpha) \longrightarrow R$ and $n_\alpha: \Gamma(L^q\otimes P_{-\alpha}) \longrightarrow R$ for which $\ell(\langle s,t\rangle) = m_\alpha(s).n_\alpha(t)$, then for some R-valued point x of X, $\ell(s) = s(x)$, all $s \in \Gamma(L^n)$.

This is proven by a straightforward generalization of our proof for k-valued points. Lemma 1 is unchanged and one finds first an invertible sheaf K on $\hat{X}\times\mathrm{Spec}(R)$ and surjective homomorphisms:

$$m: E_p \otimes_k R \longrightarrow K$$

$$n: F_q \otimes_k R \longrightarrow K^{-1}$$

on $\hat{X} \times \mathrm{Spec}(R)$ which globalize m_α and n_α. For all R-valued points $x: \mathrm{Spec}(R) \longrightarrow X$ of X, define Q_x on $\hat{X}\times\mathrm{Spec}(R)$ to be the pull-back of P by $x \times 1_{\hat{X}}$. We get restriction maps $r_x: E_p \otimes_k R \longrightarrow Q_x$, $s_x: F_q \otimes_k R \longrightarrow Q_x^{-1}$ as before, and (**) reduces as before to:

(***) $K \cong Q_x$, for some R-valued point x of X, and $m = \mu\cdot r_x$, $n = \nu\cdot s_x$ for some units $\mu,\nu \in R$.

But by our proof for k-valued points, we know already that $K|_X \cong Q_{x_o}$ for some k-valued point x_o of X. Therefore, since Pic^o is an "open" subfunctor of Pic, and since X is the dual of $\hat{X}$, it follows immediately that $K \cong Q_x$ for some R-valued point x of X. To prove the rest of (***), it is only necessary to check that

$$\Gamma(\hat{X} \times \mathrm{Spec}(R), \underline{\mathrm{Hom}}\ (E_p, K)) \cong R$$

$$\Gamma(\hat{X} \times \mathrm{Spec}(R), \underline{\mathrm{Hom}}\ (F_q, K^{-1})) \cong R\ .$$

Then since the restriction m_o of m to X is a non-zero multiple of r_{x_o}, m must be a unit times r_x; and similarly for n.

As before, we compute:

$$\Gamma(\hat{X} \times \mathrm{Spec}(R), \underline{\mathrm{Hom}}(E_p, k)) \cong H^g(X \times \hat{X} \times \mathrm{Spec}(R), p_1^* L^{-p} \otimes p_{12}^* P^{-1} \otimes p_{23}^* K)\ .$$

We can then apply the remark:

If L is an invertible sheaf on $Z \times \mathrm{Spec}(R)$ such that $H^i(L|_Z) = (0)$, $i \neq i_o$, then $H^i(L) = (0)$ if $i \neq i_o$ and $H^{i_o}(L)$ is a free R-module.

This completes the proof of Theorem 10.

Appendix*, by George Kempf

Let X be an abelian variety, L an invertible sheaf on X, Y = connected component of $K(L)$, and $p: X \longrightarrow X/Y$ the canonical map.

Theorem 1: (i) If $L|_Y$ is non-trivial, then $H^i(X,L) = (0)$, all i.

(ii) If $L|_Y$ is trivial, there exists a non-degenerate invertible sheaf M on X/Y with $L = p^*M$, and if i_o = index(M):

$$H^i(X,L) \cong H^{i_o}(X,M) \otimes H^{i-i_o}(Y,\mathcal{O}_Y), \quad \text{all } i.$$

Proof: The theorem follows from:

Lemma 1: $T_x^*L|_Y \cong L|_Y$ for all $x \in X$, and

Lemma 2: Let $P \xrightarrow{f} Z$ be a principal homogeneous space (in the flat topology) with structure group Y, an abelian variety. Then

$$R^if_*(\mathcal{O}_P) \cong H^i(Y,\mathcal{O}_Y) \otimes \mathcal{O}_Z.$$

By Lemma 1, we see that $L|_Y \in \mathrm{Pic}^o(Y)$ and also that $L|_{x+Y} = L|_{p^{-1}(p(x))}$ is isomorphic to $L|_Y$. Now if $L|_Y$ is non-trivial, then

$$(0) = H^i(Y, L|_Y) = H^i(p^{-1}(p(x)), L|_{p^{-1}(p(x))})$$

for all i (see Mumford, Abelian Varieties, §13). By the theorems on cohomology and base extension, $R^ip_*(L) = (0)$ for all i. The Leray spectral sequence then implies that $H^i(X,L) = (0)$ for all i.

---

* The results in this appendix were independently discovered by C. P. Ramanujam.

If $L|_Y$ is trivial, hence $L|_{p^{-1}(p(x))}$ is trivial, the see-saw principle shows that if $M = p_*(L)$, then M is an invertible sheaf such that $L = p^*(M)$. This M is clearly non-degenerate. Note that:

$$\begin{aligned} R^i p_*(L) &= R^i p_*(p^* M) \\ &\cong R^i p_*(\mathcal{O}_X) \otimes_{\mathcal{O}_{X/Y}} M \\ &\cong H^i(Y, \mathcal{O}_Y) \otimes_k M \qquad \text{(by lemma 2).} \end{aligned}$$

Therefore

$$H^j(X/Y,\ R^i p_*(L)) \cong H^i(Y, \mathcal{O}_Y) \otimes_k H^j(X/Y,\ M),$$

and this is zero unless $j = i_o$, the index of M. Thus the Leray spectral sequence shows:

$$\begin{aligned} H^i(X,L) &\cong H^{i_o}(X/Y,\ R^{i-i_o} p_*(L)) \\ &\cong H^{i-i_o}(Y, \mathcal{O}_Y) \otimes_k H^i(X/Y,\ M). \end{aligned}$$

Proof of Lemma 1: If $m: X \times X \longrightarrow X$ is the addition morphism, we know that $m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ on $X \times X$ is trivial when restricted to $Y \times X$. Define $s: Y \longrightarrow Y \times X$ by $s(y) = (y,x)$. Then

$$\begin{aligned} & s^*(m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}|_{Y \times X}) \\ & \quad \cong T_x^*L|_Y \otimes L|_Y^{-1} \otimes \mathcal{O}_Y \end{aligned}$$

is also trivial. QED

Proof of lemma 2: Since $P \times_Z P \cong Y \times P$, it will suffice to prove the stronger:

<u>Sublemma</u>: Given $f: X \longrightarrow S$ a morphism of schemes $/k$ such that there exists $\pi: S' \longrightarrow S$ where π is faithfully flat and $\pi_X(\mathcal{O}_{S'}) \cong \mathcal{O}_S$ with the property:

$\exists$ φ, Y and a diagram

$$\begin{array}{ccc} X \underset{S}{\times} S' & \xrightarrow[\sim]{\varphi} & Y \underset{k}{\times} S' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle p_2} \\ S' & = & S' \end{array}$$

where Y is proper over k. Then we have an isomorphism $R^i f_*(\mathcal{O}_X) \cong H^i(Y, \mathcal{O}_Y) \underset{k}{\otimes} \mathcal{O}_S$.

<u>Proof</u>: $\pi^*(R^i f_*(\mathcal{O}_X)) \cong R^i f'_*(\mathcal{O}_{X \underset{S}{\times} S'})$ since $S' \longrightarrow S$ is a flat base extension and $R^i f'_*(\mathcal{O}_{X \underset{S}{\times} S'}) \cong H^i(Y, O_Y) \underset{k}{\otimes} \mathcal{O}_{S'}$ because of the existence of φ. Because $H^i(Y, \mathcal{O}_Y)$ is finite-dimensional, $R^i f'_*(\mathcal{O}_{X \underset{S}{\times} S'})$ is a vector bundle. Hence $R^i f_*(\mathcal{O}_X)$ is a vector bundle because π is faithfully flat. Now we define an isomorphism

$$\begin{aligned} R^i f_*(\mathcal{O}_X) &\xrightarrow{\sim} \pi_* \pi^* R^i f_*(\mathcal{O}_X) && (\text{since } \pi_* \mathcal{O}_{S'} = \mathcal{O}_S) \\ &\cong \pi_*[H^i(Y, \mathcal{O}_Y) \otimes_k \mathcal{O}_{S'}] \\ &\cong H^i(Y, \mathcal{O}_Y) \otimes_k \mathcal{O}_S && (\quad " \quad) \end{aligned}$$

<u>QED</u>

<u>Theorem 2</u>: Let L and M be invertible sheaves on an abelian variety X, with L ample. Let

$$P_{L,M}(n) = \chi(L^n \otimes M).$$

Then (i) all the roots of $P_{L,M}$ are real and dim K(M) is the multiplicity of 0 as a root,

(ii) Counting roots with multiplicities:

$$H^k(X,M) = (0), \text{ if } 0 \leq k < \text{number of positive roots}$$

$$H^{g-k}(X,M) = (0), \text{ if } 0 \leq k < \text{number of negative roots.}$$

<u>Proof</u>: The theorem is proven in Mumford, Abelian varieties, §16, for M non-degenerate. It is obvious when $M \in \mathrm{Pic}^0(X)$ because in this case

$$P_{L,M}(n) = \chi(L^n) = \frac{(L^g)\cdot n^g}{g!}$$

and X = K(M). Now suppose $X = X_1 \times X_2$, $L = p_1^* L_1 \otimes p_2^* L_2$ and $M = p_1^* M_1 \otimes p_2^* M_2$ where $M_1 \in \mathrm{Pic}^0(X_1)$, M_2 is non-degenerate on X_2 and L_i is ample on X_i. Then by the Künneth formula,

$$(1) \qquad P_{L,M}(n) = P_{L_1,M_1}(n) \cdot P_{L_2,M_2}(n)$$

and $K(M) = K(M_1) \times K(M_2)$. So in this case the theorem follows from the above special cases and the Künneth formula.

We shall reduce the theorem to this case. Suppose $f: Y \longrightarrow X$ is an isogeny. Then

(2) $$P_{f^*L,f^*M}(n) = \deg f \cdot P_{L,M}(n)$$

by the Riemann-Roch theorem, and $\dim K(f^*M) = \dim K(M)$. Therefore assertion (i) is invariant under an isogeny. Let Y be the identity component of K(M) and let Z be a complementary subvariety for Y in X We have an isogeny $f: Y \times Z \longrightarrow X$. Now $Y \subset K(f^*M)$ and if $M_1 = f^*(M)|_Y$, then as in the proof of Theorem 1, $M_1 \in \mathrm{Pic}^0(Y)$ and M is of the form $p_1^*M_1 \otimes p_2^*M_2$ where M_2 is a non-degenerate invertible sheaf on Z. The next problem is to see that the theorem does not depend on the ample L. Then we can replace f^*L by $p_1^*L_1 \otimes p_2^*L_2$ and we have reduced the proof of (i) to a case where (i) has been proven.

<u>Claim: $P_{L,M}$ and $P_{L',M}$ have the same number of positive, zero, and negative roots (counted with multiplicity).</u>

Let δ (resp. δ') be the smallest positive root of $P_{L,M}$ (resp. $P_{L',M}$). Let a (resp. a') be the number of positive roots of $P_{L,M}$ (resp. $P_{L',M}$). Then

a = number of positive roots of $P_{L,M}(t+\epsilon)$, if $0 < \epsilon < \delta$

a' = " " " $P_{L',M}(t+\epsilon')$, if $0 < \epsilon' < \delta'$.

But
$$\begin{aligned} s^g P_{L,M}(n+\tfrac{r}{s}) &= P_{L^s,M^s}(n+\tfrac{r}{s}) \\ &= \chi(L^{ns+r} \otimes M^s) \\ &= \chi(L^{sn} \otimes (L^r \otimes M^s)) \\ &= P_{L,L^r\otimes M^s}(sn) \\ &= s^g P_{L,L^r\otimes M^s}(n). \end{aligned}$$

So if $0 < \frac{r}{s} < \delta$, then $L^r \otimes M^s$ is non-degenerate and

$$a = \text{number of positive roots of } P_{L, L^r \otimes M^s}$$
$$= \text{index } (L^r \otimes M^s).$$

Now let N be large enough so that $(L')^N \otimes L^{-1}$ is ample and choose r and s so that $0 < r/s < \delta$, $0 < \frac{Nr}{s} < \delta'$. Then

$$a = \text{index } (L^r \otimes M^s)$$
$$\geq \text{index } (((L')^N \otimes L^{-1})^r \otimes L^r \otimes M^s) \text{ (by Th. for non deg. M)}$$
$$= \text{index } ((L')^{Nr} \otimes M^s)$$
$$= a'.$$

By symmetry, it follows that $a = a'$. The claim is proven similarly for the multiplicity of 0 and the number of negative roots.

To prove (ii), we may assume that $M = p^*N$ for a non-degenerate N on X/Y, since otherwise M has no cohomology at all. We have the commutative diagram:

$$\begin{array}{ccc} Y \times Z & \xrightarrow{\quad f \quad} & X \\ \big\downarrow & & \big\downarrow p \\ Z & \xrightarrow{\quad g \quad} & X/Y \end{array}$$

for some isogeny g. Then g^*N is non-degenerate and index $(g^*N) =$ index(N). So:

number of pos. rts of $P_{L,M}$ = number of pos. rts of P_{f^*L, f^*M} (by formula 2)

= number of pos. rts of $P_{L|_Z, g^*N}$ (by formula 1)

= index g^*N (Th in non-deg. case)

= index N.

Now (ii) follows from Thèorem 1.

QED