Bhagawandas

Abstract Theta Functions

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1. Introduction: Let A be an abelian variety defined over k, an algebraically closed field complete with respect to a real absolute value. Let R be the ring of integers in k, and $\bar{k}$ the residue field; suppose char $\bar{k} \neq 2$. Our aim is to show that A has a "good reduction" over R: i.e., that there is a fibre product diagram:

$$\begin{array}{ccccc} A & \hookrightarrow & \mathcal{A} & \hookleftarrow & \bar{A} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} R & \hookleftarrow & \operatorname{Spec} \bar{k} \end{array}$$

where $\mathcal{A}$ is a group scheme over R, and $\bar{A}$ is an extension of $\mathcal{G}_m^M$ by an abelian variety. The existence of such reductions provides an abstract analogue of the existence of the Satake compactification of the moduli scheme of A.

If $\bar{A} = (\mathcal{G}_m)^n$, we will say that A has totally-degenerate reduction: in this case one can get a p-adic analytic uniformization $\pi: V \longrightarrow A$, and hence the Tate-Morikawa-McCabe Theory.

We will use an abstract theory of theta-functions to perform the reduction; and we begin by sketching such a theory.

2. Abstract Theta-functions: Classically, the theta-functions associated to an abelian variety A arise in the following way. Let A be defined over k, and suppose there is a surjective homomorphism $\pi: V \longrightarrow A$; then $A \cong V/_{\operatorname{Ker} \pi}$. For example,

if $k = \mathbb{C}$, $V = \mathbb{C}^g$ ($g = \dim A$) and Ker π is a lattice; or in the Tate-Morikawa-McCabe Theory, k is a local field, $V = (k^*)^g$ and Ker π is a "multiplicative (annular) lattice". Then the theta-functions on V associated to A are holomorphic functions on V which satisfy a certain functional equation with respect to Ker π.

Quite generally, suppose we have a homomorphism $\pi: V \longrightarrow A$; and an ample, invertible sheaf L on A so that if $\mathbb{L}$ is the induced line bundle, then $\pi^* \mathbb{L} \cong \mathbb{I}$ (the trivial line-bundle on V - i.e., induced from $\mathcal{O}_V$). Then sections $S \in \Gamma(A, \mathbb{L})$ pull back to sections $\pi^*(S) \in \Gamma(V, \mathbb{I})$, and these are naturally interpreted as k-valued functions on V which can be called theta-functions. These functions satisfy a kind of periodicty with respect to Ker π, as the following argument shows. Let $\gamma \in$ Ker π, and interpret it as a translation map on V; then $\pi \cdot \gamma = \pi$ so $\gamma^* \pi^* = \pi^*$. Therefore we have a commutative diagram

$$\begin{array}{ccc} \gamma^* \pi^* \mathbb{L} & = & \pi^* \mathbb{L} \\ \wr\wr & & \wr\wr \\ \gamma^* \mathbb{I} & \xrightarrow[\text{mult. by } c_\gamma]{} & \mathbb{I} \end{array}$$

where c_γ is a suitable nowhere-zero function on V. Thus, if f is the k-valued function $\pi^*(s)$, then we have

$$f(\gamma z) = c_\gamma(z) f(z) \ , \ z \in V.$$

We apply this formulation as follows. Let p be a prime, A_{p^∞} = pts of order p^n in A (some n) and $V_p(A) = \varprojlim_p A_{p^\infty}$, the Tate-module of A at p. Then there is an exact sequence

$$0 \longrightarrow \Lambda_p \longrightarrow V_p(A) \xrightarrow{\pi} A_{p^\infty} \longrightarrow 0$$

where π is given by $\pi(a_0, a_1 \ldots) = a_0$. Recall that there are isomorphisms $V_p(A) \cong (\mathbb{Q}_p)^{2g}$ and $\Lambda_p \cong (\mathbb{Z}_p)^{2g}$ ($g = \dim A$). Thus we can discuss "local" theta-functions corresponding to the uniformization $\pi: V_p(A) \longrightarrow A_{p^\infty}$. There is an analogous theory of global theta-functions in which $V_p(A)$ is replaced by the adèle group $\prod_p V_p(A)$. However, there seem to be difficulties in the local case for $p \neq 2$; hence we shall restrict our attention, from now on, to the case $p = 2$.

3. Construction of Theta-functions.

Let L be an ample invertible sheaf on A, $\ell: \mathbb{L} \longrightarrow A$ the corresponding line bundle. For $y \in A$, let $\mathbb{L}_y = \ell^{-1}(y)$, the fibre over y. Assume that

i) There is an isomorphism $\rho: i^* \mathbb{L} \longrightarrow \mathbb{L}$ where $i: A \longrightarrow A$ is the map $i(x) = -x$; i.e., that $\mathbb{L}$ is symmetric.

ii) We are given a specific isomorphism $\varphi_0: \mathbb{L}_0 \xrightarrow{\sim} k$.

Now we can trivialize $\pi^* \mathbb{L}$ if we can find isomorphisms $\mathbb{L}_x \xrightarrow{\sim} k$ for all $x \in A_{2^\infty}$. We proceed to do so as follows.

Let $t_x: A \longrightarrow A$ be the translation $t_x(y) = x+y$, and suppose that for some particular x we are given an isomorphism $\tau_x: t_x^* \mathbb{L} \xrightarrow{\sim} \mathbb{L}$. Consider the diagram:

$$
\begin{array}{ccc}
i^* t_x^* \mathbb{L} & \xrightarrow{\ i^*\tau_x\ } & i^* \mathbb{L} \\
\| & & \Big\downarrow \rho \\
(t_x \circ i)^* \mathbb{L} & & \\
\| & & \mathbb{L} \\
(i \circ t_{-x})^* \mathbb{L} & & \Big\downarrow t_{-x}^* \tau_x \\
\| & & \\
t_{-x}^* i^* \mathbb{L} & \xrightarrow{\ t_{-x}^* \rho\ } & t_{-x}^* \mathbb{L}
\end{array}
$$

As it stands, there is no reason to expect this diagram to commute. However, if we modify τ_x by a suitable automorphism of $\mathbb{L}$ (which is just an element of k^* - since $\mathbb{L}$ is a line bundle over a projective variety) we can force the diagram to commute. Now suppose that $\alpha, \beta \in k^*$ are automorphisms such that $\alpha \cdot \tau_x$ and $\beta \cdot \tau_x$ make the diagram commutative, then an easy chase shows that $\alpha^2 = \beta^2$, or $\alpha = \pm \beta$. Thus $\alpha \cdot \tau_x$ and $-\alpha \cdot \tau_x$ are the only isomorphisms of ℓ^*_x with $\mathbb{L}$ which make the diagram commutative, and if we stipulate that this should be so, then an isomorphism $\tau_x' : t_x^* \mathbb{L} \xrightarrow{\sim} \mathbb{L}$ is determined canonically up to ± 1.

We can define a <u>completely</u> canonical isomorphism σ_{2x} from $t_{2x}^* \mathbb{L}$ to $\mathbb{L}$ as follows,

$$\sigma_{2x} : t_{2x}^* \mathbb{L} = t_x^*(t_x^* \mathbb{L}) \xrightarrow{\ t_x^* \tau_x'\ } t_x^* \mathbb{L} \xrightarrow{\ \tau'_x\ } \mathbb{L}$$

since $-\tau_x'$ and τ_x' give the same σ_{2x}. Thus we can get canonical isomorphisms

$$(\sigma_{2x})_0 : (t_{2x}^* \mathbb{L})_0 \longrightarrow \mathbb{L}_0$$ and from this

$\varphi_o \circ (\sigma_{2x})_o : (t_{2x}^* \mathcal{L})_o \longrightarrow k$. But $(t_{2x}^* \mathcal{L})_o = \mathcal{L}_{2x}$.
Therefore we have a canonical isomorphism

$$\varphi_o \circ (\sigma_{2x})_o : \mathcal{L}_{2x} \longrightarrow k \quad .$$

Therefore we can trivialize $\pi^* \mathcal{L}$ along those fibers $\mathcal{L}_x$ such that $t_y^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$ for some y solving $2y = x$. But this isomorphism exists for only a few points in A. We use the following lemma to enable us to obtain isomorphisms $\mathcal{L}_x \xrightarrow{\sim} k$ for all $x \varepsilon A_{2^\infty}$. Put $H(L) = \{x\varepsilon A: t_x^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}\}$

<u>Lemma</u>: $(n^2 > n)$. Let $n\delta: A \longrightarrow A$ be the isogeny $n\delta(x) = nx$. For all $x \varepsilon A$ of finite order, $(n\delta)^* t_x^* \mathcal{L}$ is isomorphic to $(n\delta)^* \mathcal{L}$ for some n (n = order of x will do).

(The proof is easy: let $x = ny$, so y has order n^2. Then $(n\delta)^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{n^2}$. But for any $\mathcal{M}$ and k, $H(\mathcal{M}^k) \supseteq A_k$. So $(n\delta)^* t_x^* \mathcal{L} \xrightarrow{\sim} t_y^* (n\delta)^* \mathcal{L} \longrightarrow t_y^* \mathcal{L}^{n^2} \simeq (n\delta)^* \mathcal{L})$

Now let $x_o \varepsilon A_{2^\infty}$ be some fixed but arbitrary element. Then x_o sits in at least one sequence $(x_o, x_1, \ldots,) \varepsilon V_2(A)$. We will not in general have an isomorphism $t_{x_1}^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$; however, for large enough m, $(2^m\delta)^*(t_{x_1}^* \mathcal{L}) \xrightarrow{\sim} (2^m\delta)^* \mathcal{L}$ by the lemma. Since $(2^m\delta)^* t_{x_1}^* \mathcal{L} \longrightarrow t_{x_{m+1}}^* (2^m\delta)^* \mathcal{L}$ we get a canonical (up to $\pm$ 1) isomorphism

$\tau'_{x_{m+1}} : t_{x_{m+1}}^* (2^m\delta)^* \mathcal{L} \longrightarrow (2^m\delta)^* \mathcal{L}$ and thus a completely canonical $\sigma_{x_{\hat{m}}}$, and therefore isomorphisms

$\varphi_o \circ (\sigma_{x_m})_o : ((2^m\delta)^* \mathcal{L})_{x_n} \longrightarrow k$. But $((2^n\delta)^* \mathcal{L})_{x_n} = \mathcal{L}_{x_o}$.

Thus we get the desired isomorphism of the fiber $\mathcal{L}_{x_0}$ with k.

Glossing over the development, the final theory comes out something like this. We begin with a symmetric, ample, invertible sheaf L on A of degree 1, as above. L determines

i) a bimultiplicative, skew-symmetric form

$e: V_2(A) \times V_2(A) \longrightarrow \{2^n \text{ th. roots of } 1 \text{ in } k\}$

ii) a "quadratic character" $e_*: (1/2\, \Lambda)/\Lambda \longrightarrow \{\pm 1\}$

satisfying $e(\alpha,\beta)^2 = e_*(\bar\alpha+\bar\beta)\, e_*(\bar\alpha)\, e_*(\bar\beta)$ for $\alpha,\beta \in \frac{1}{2}\Lambda$.

We can assume that the Arf invariant of e_* is zero by replacing L by some $t_x^* L$, $x \in A_2$, if necessary. (The quadratic form e is classical: see for example Lang, Abelian Varieties).

In terms of this data we obtain theta-functions $\theta_{[s]}: V_2(A) \longrightarrow k$ for all $s \in \Gamma(A, \mathcal{L}^n)$, satisfying

$$(*)\ \theta_{[s]}(\alpha+\beta) = [e_*(\beta/2)\cdot e(\beta/2,\alpha)]^n\, \theta_{[s]}(\alpha) \text{ for all } \alpha \in V_2(A),\ \beta \in \Lambda.$$

This gives a homomorphism of k-algebras

$$\theta: \bigoplus_{n=1}^{\infty} \Gamma(A, \mathcal{L}^n) \longrightarrow \{\text{k-valued functions on } V_2(A) \text{ satisfying } (*)\}$$

where multiplication of sections s_1, s_2 is given by $s_1 \otimes s_2$. Further, θ is injective; in fact, if $a = (a_0, a_1 \ldots) \in V_2(A)$, $\theta_{[t]}(a) = 0$ if and only if $\Longleftrightarrow t(a_0) = 0$. If s_0 denotes the canonical section of $\mathcal{L}$ then we put $\theta_{[s_0]} = \Theta$ the Riemann theta-function. It satisfies

$$(**)\quad \Theta(-\alpha) = \Theta(\alpha)$$

$$(***)\quad \prod_{i=1}^{4} \Theta(\alpha_i) = 2^{-g} \sum_{\eta\in\frac{1}{2}\Lambda} e(\gamma,\eta) \prod_{i=1}^{4} \Theta(\alpha_i+\gamma+\eta)$$

$$\text{where } \gamma = -\tfrac{1}{2}\sum \alpha_i$$

$(****)$ for every $\alpha \in V_2(A)$ $\exists\, \beta \in \frac{1}{2}\Lambda$ so that $\Theta(\alpha+\beta) \neq 0$.

What is remarkable about these theta-functions is that beginning with just Θ we can recover the pair $(A, \mathcal{L})$. Suppose we start with a vector space V isomorphic to $(\mathbb{Q}_2)^{2g}$, Λ a maximal isotropic lattice in V, the form e and the quadratic character e_*. Then we can define a theta-function Θ on V as a k-valued function satisfying $(**)$, $(***)$ and $(****)$. We then put

M = k-vector space spanned by $e(\alpha,\beta)\,\Theta(2\alpha-\beta)$ where $\alpha \in V$, $\beta \in \frac{1}{2}\Lambda$. (This will equal the space of $\theta_{[s]}$'s, $s \in \Gamma(A, \mathcal{L}^4)$).

$S_0(M) = k$

$S_1(M) = M$

$S_n(M)$ = space spanned by n-fold products of elements from M.

Then $A = \mathrm{Proj}\,(\oplus\, S_n(M))$ is the abelian variety sought for, and $\mathcal{L}$ is easily recovered from A and $\oplus\, S_n(M)$.

Finally, let us note an important correspondence between theta-functions on V and finitely-additive measures on a certain subspace of V. These measures arise from the Fourier transforms of the theta-functions, and are examples of Schwartz-Bruhat distributions. Explicitly, we can describe them as follows:

decompose V as $V_1 \oplus V_2$ where V_1, V_2 are isotropic with respect to the pairing e, $\Lambda = (\Lambda \cap V_1) + (\Lambda \cap V_2)$ and $e_*(\alpha) = 1$ for $\alpha \in \frac{1}{2}\Lambda \cap V_i$. A finitely-additive measure μ on the Boolean ring of compact open subsets of V_1 is called Gaussian if and only if

i) $\mu(U) = \mu(-U)$

ii) Given the map $\xi\colon V_1 \times V_1 \longrightarrow V_1 \times V_1$

$\xi(x, y) = (x+y, x-y)$, then

$(\mu \times \mu)(\xi U) = (\nu \times \nu)(U)$ where ν is some other measure on the same ring.

The correspondence between theta-functions on V and Gaussian measures μ on V_1 is given thus:

$$\mu(\alpha_1 + 2^n \Lambda_1) = 2^{-ng} \sum_{\alpha_2 \in 2^{-n} \Lambda_2/\Lambda_2} e(\alpha_1, {}^{\alpha_2}/_2)\, \theta(\alpha_1 + \alpha_2)$$

and

$$\theta(\alpha_1 + \alpha_2) = e(\alpha_1, {}^{\alpha_2}/_2) \int_{\alpha_1 + \Lambda_1} e(\alpha_2, \beta)\, d\mu(\beta)$$

where $\Lambda_i = \Lambda \cap V_i$ $(i = 1,2)$ and $\alpha_i \in \Lambda_i$.

4. The Reduction of A over R

We now analyze the relation of Θ to the integers R in k (R,k as in section 1). Let $|\ |$ denote the real absolute value of k; $V_2(A)$, Λ and Θ as before.

Proposition. $\max_{\alpha\varepsilon V} |\Theta(\alpha)|$ is finite and is taken on for some $\alpha \varepsilon \frac{1}{2} \Lambda$.

Proof. The Riemann theta-relation

$$\prod_{i=1}^{4} \Theta(\alpha_i) = 2^{-g} \sum_{\eta\varepsilon\frac{1}{2}\Lambda} e(\gamma,\eta) \prod_{i=1}^{4} \Theta(\alpha_i+\gamma+\eta)$$

gives
$$\prod_{i=1}^{4} |\Theta(\alpha_i)| \leq \max_{\eta\varepsilon\frac{1}{2}\Lambda/\Lambda} \prod_{i=1}^{4} |\Theta(\alpha_i+\gamma+\eta)|$$

since $|\Theta(\alpha)|$ is constant on cosets of Λ.

If we put $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = -\alpha_4$, then

$$(\dagger) \quad |\Theta(\alpha)|^4 \leq \max_{\eta\varepsilon\frac{1}{2}\Lambda/\Lambda} |\Theta(\eta)|^3 \, |\Theta(2\alpha - \eta)|$$

since $(\dagger)$ is valid for all $\alpha \varepsilon V$, applying it successively to $2\alpha-\eta$, $4\alpha-3\eta$, ..., $2^n\alpha-(2^n-1)\eta$, ... and substituting back in $(\dagger)$ we get

$$|\Theta(\alpha)|^4 \leq \max_{\eta\varepsilon\frac{1}{2}\Lambda/\Lambda} |\Theta(\eta)|^{r_n} \cdot |\Theta(2^n\alpha-(2^n-1)_\eta)|^{s_n}$$

where $r_n = \sum_{i=0}^{n} 3/4^i$, $\quad s_n = \frac{1}{2n}$

Now in the 2-adic topology $2^n\alpha - (2^n-1)\eta$ converges to η. We know r_n converges to 4. Therefore we get

$$|\Theta(\alpha)|^4 \leq \max_{\eta=\frac{1}{2}\Lambda/\Lambda} |\Theta(\eta)|^4$$

whence

$$\max_{\alpha\varepsilon V} |\Theta(\alpha)| \leq \max_{\eta\varepsilon\frac{1}{2}\Lambda/\Lambda} |\Theta(\eta)|^4 \leq \max_{\alpha\varepsilon V} |\Theta(\alpha)|^4$$

which yields the result.

Using this proposition we can normalize Θ so that its values lie in R, but not all lie in the meximal ideal M; that is, if $\bar{\Theta}$ denotes the induced function to $\bar{k}$, $\bar{\Theta}(\alpha) \neq 0$ for some $\alpha \in V$.

We now invoke the main result used for the Satake compactification. (see [1])

<u>Theorem</u>: For every theta-function θ on V, (i.e., a function satisfying (*),(**),(***) but <u>not</u> necessarily (****)) there is a subspace $W \subseteq V$ with $W^{\perp} \subseteq W$ ($\perp$ with respect to e) and a non-degenerate theta-function Φ on $W/_{W^{\perp}}$ such that

$$\operatorname{supp} \theta \subseteq W + \Lambda + \eta_0 \,, \quad \eta_0 \in \tfrac{1}{2} \Lambda$$

and

$$\theta(\eta_0+\eta_1+\alpha) = e_*(\eta_1/2)\; e(\eta_1/2, \alpha)\; e\left(\frac{\eta_0+\eta_1}{2}, \alpha\right)\; \Phi(\bar{\alpha})$$

$(\eta_1 \in \Lambda, \ \alpha \in W)$

The theta-function Φ is used to construct an abelian variety $B/_{\bar{k}}$ of dimension h(where $\dim W/_{W^{\perp}} = 2h$) in the same way that A was constructed from Θ (see section 3). Then the special fibre $\bar{A}$ of the sought-for group scheme $\mathcal{A}$ should be in an extension

$$0 \longrightarrow \mathbb{G}_m^h \longrightarrow \bar{A} \longrightarrow B \longrightarrow 0 \,.$$

Notice that if the reduction is to be totally-degenerate, then we must have $W = W^{\perp}$, and hence $B = \{0\}$.

To construct $\mathcal{A}$ however, we must first study how many R-valued theta-functions come from a given $\bar{k}$-valued non-degenerate theta-function Φ on a vector space $W/_{W^{\perp}}$ of smaller dimension. For this question the measure-theoretic point of view (outlined at the end of section 3) is much better. For ease of exposition we confine ourselves to the case of totally-degenerate reduction, i.e., $W = W^{\perp}$. In this case the R-valued measure μ corresponding to Θ reduces to a $\bar{k}$-valued measure $\bar{\mu}$, where $\bar{\mu}$ is just the point mass at 0, δ_o (the so-called "Dirac delta-function"). The main result concerning these measures is this.

<u>Theorem</u>: Let μ be a non-degenerate R-valued Gaussian measure on $\mathbb{Q}_2^g$ such that $\bar{\mu} = \delta_o$. Then there is a unique subgroup M' in $\mathbb{Q}_2^g$ isomorphic to $Z[\frac{1}{2}]^g$ (and equal to it after a suitable change of co-ordinates) and a unique quadratic character $c': M' \longrightarrow R-\{0\}$ such that

$$\mu = \sum_{x \in M'} c'(x)\ \delta_x$$

Moreover, if we tensor M' with $\mathbb{R}$, then there is a positive-definite quadratic form $Q: M' \otimes \mathbb{R} \longrightarrow \mathbb{R}$ so that $|c'(x)| = e^{-Q(x)}$

Ideally, at this point we should write down $\mathcal{A}$ explicitly in terms of Θ. However, this presents certain complications, and it is faster to construct $\mathcal{A}$ by means of the theory of the Néron - model and to check that its special fibre $\bar{A}$ is $\mathbb{G}_m^h$ by means of Galois Theory (following a suggestion of Grothendieck).

Choose a subfield $k_o \subseteq k$ with a discrete absolute value so that A is defined over k_o, and let $\mathcal{A}_o$ be the Néron model of A over R_o = integers in k_o. Let $\mathcal{A} = \mathcal{A}_o$ (minus the components of its special fibre not containing zero) then $G = \text{Gal}(k/k_o)$ acts on $V_2(A)$ preserving Λ, e, W, M', and Q. On $V/_W$ the action of G is determined by its action on $M' \cap (\Lambda/\Lambda \cap W) \cong \mathbb{Z}^g$. Hence we have a representation

$$G \longrightarrow O(Q)_{\mathbb{Z}}$$

into an integral orthogonal group (corresponding to the quadratic form (Q) on $\mathbb{Q}_2^{2g}$. But this group is finite! Hence replacing k_0 by a finite extension k, if necessary we see that $G_1 = \mathrm{Gal}\ (k/_{k_1})$ acts trivially on $V/_W$. Since G_1 preserves the actión of e, it acts trivially on W too; thus the representation takes the form

$$\sigma \longrightarrow \left(\begin{array}{c|c} I & * \\ \hline 0 & I \end{array}\right)$$

Thus A_{2^∞} contains a subgroup H which is k_1-rational and is a maximal isotropic subgroup of points of order 2^n, isomorphic to $(^{Q_2}/_{\mathbb{Z}_2})^g$. Now by one of the key properties of Néron models, all k_1-rational points of A extend to R_1-rational points of $\mathcal{a}_1$($\mathcal{a}_1$ = Néron model of A over R_1). Since H is divisible, all points of H give R_1-rational points of $\mathcal{a}_1$ hence H induces a subgroup $\bar{H} \subseteq \bar{A}$, isomorphic to $(\mathbb{Q}_2/\mathbb{Z}_2)^g$.

Now from quite general structure theorems on group schemes, we have an exact sequence

$$0 \longrightarrow L \longrightarrow \bar{A} \xrightarrow{\pi} B \longrightarrow 0$$

when L is a linear group of dimension r, and B is an abelian variety. It can be shown that $\pi(\bar{H})$ is still isotropic in B and since $\underline{B}$ has dimension g-r, $\pi(\bar{H}) \cong (Q_2/\mathbb{Z}_2)^k$, $k \leq g-r$. Therefore $\bar{H} \cap L$ has a subgroup $(Q_2/\mathbb{Z}_2)^r$, whence $L = \mathbb{G}_m^r$. Using the total-degeneracy of the theta function we can then show that $\bar{A} = \mathbb{G}_m^g$ - i.e., that $B = \{0\}$.

5. Analytic Theta-Functions.

In this section we will show how our theta-functions with totally-degenerate reduction are essentially equal to suitable holomorphic theta-functions of Tate-Morikawa-McCabe, and hence that the abelian varieties uniformized by the Tate theory are exactly those with totally-degenerate reduction.

In the algebraic theory we have outlined, the exact sequence

$$0 \longrightarrow \Lambda \longrightarrow V_2(A) \longrightarrow A_{2^\infty} \longrightarrow 0$$

is the analogue of the sequence

$$0 \longrightarrow M \xrightarrow{q} V(M) \longrightarrow A(k) \xrightarrow{?} 0$$

(where V(M) is the g-dimensional torus with character group M) of the holomorphic theory. See Tate's Bowdoin Colloquium talks for details. Now using the theorem of the previous section we can express every theta-function with totally degenerate reduction, Θ_a (the subscript $\underline{a}$ emphasizes that it is the algebraic theta-function) as

$$\Theta_a(\alpha+\beta) = e(\beta/2,\alpha) \sum_{x \in M''} e(\beta,x)\ c'(\alpha+x)$$

where $V = V_1 \oplus V_2$ is a suitable decomposition, $\alpha \in V_1$, $\beta \in V_2$, $M'' = M' \cap \Lambda$, $M' \subset V_1$, V_2 is the W of the previous section, c' and M' as in Theorem 2.

In the holomorphic theory there is a quadratic character $c: M \longrightarrow R$ which determines q via the identity

$$(q^x)^y = \frac{c(x+y)}{c(x)c(y)}$$

The unique holomorphic theta-function Θ_h is equal to

$$\Theta_h(u) = \sum_{x \in M} c(x) u^x .$$

It is now easy to relate Θ_h and Θ_a. Explicitly, we construct a map

$$\begin{array}{ccc} f: \; M' \oplus ({}^{V_2}/_{V_2 \cap \Lambda}) & \longrightarrow & \{x \in V(M) : x^{2^m} \in q^M\} \\ \cap \downarrow & & \cap \downarrow \\ V_2(A)/_{V_2 \cap \Lambda} & & V(M) \end{array}$$

so that if $\alpha \in V_2$, $x \in M$

$$f(\alpha)^x = e(\alpha, x)$$

and if $\alpha \in M'$, $x \in M$ then

$$f(\alpha)^x = \frac{c(\alpha+x)}{c(\alpha)c(x)}$$

Note that $f({}^{V_2}/V_2 \cap \Lambda) \xrightarrow{\sim} \{\text{points of order } 2^n \text{ in } V(M)\}$ and $f(M')$ is a "2-divisible hull" of q^M in $V(M)$.

Define: $\gamma: M' \oplus ({}^{V_2}/_{V_2 \cap \Lambda}) \longrightarrow R$ by

$$\gamma(\alpha+\beta) = c(\alpha)^{-1} e(\alpha, {}^{\beta}/2)$$

Then a simple verification yields

$$\Theta_h(f(x)) = \gamma(x) \cdot \Theta_a(x)$$

Since we have essentially the same theta functions in the algebraic and holomorphic cases, it is easy to deduce that the two theories provide uniformizations of the same abelian variety.