EXISTENCE OF THE MODULI SCHEME
FOR CURVES OF ANY GENUS

A thesis presented

by

David Bryant Mumford

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Introduction

The purpose of this thesis is the construction of "Arithmetic Moduli" of curves of any genus greater than one. This object, first of all, is to be a normal model M in the sense of Nagata over $\mathbb{Z}$; or a normal integral scheme, dominating and of finite type over Spec $(\mathbb{Z})$[1] in the sense of Grothendieck. Its fundamental property is to classify curves of genus g over every algebraically closed field Ω. In fact, the bi-regular equivalence classes of such curves should be isomorphic to the closed points of $M \times_{Z} \text{Spec } \Omega$, or equivalently to

$$\text{Hom}(\text{Spec } \Omega, M)$$

where Hom (X, Y) stands for the set of morphisms or regular maps from X to Y.

But what does the word "isomorphic" mean? The isomorphism must certainly be natural, and not merely set-theoretic. The requirement roughly is to be "compatible with specialization", or "compatible with algebraic families". More precisely, the first statement means: for all valuation rings R, and specializations of algebraic curves $C_1 \longrightarrow C_2$ over R, there should be given a morphism $\phi: \text{Spec } R \longrightarrow M$, such that if Ω_1, Ω_2 are algebraically closed fields containing the

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[1] In the following, where no confusion seems likely, we shall write R for Spec R.

quotient field and residue field of R respectively, i. e. fields of definition of C_1, C_2, then the obvious composite maps

$$\mathrm{Spec}\ \Omega_i \longrightarrow \mathrm{Spec}\ R \longrightarrow M$$

should be the morphisms corresponding to the curves C_i. Similarly, the second statement means: for all normal varietes A, V over an algebraically closed field Ω, and projective morphisms $f : A \longrightarrow V$ such that $f^{-1}(x)$ is a non-singular curve with multiplicity 1 for every x, there should be given a morphism

$$\phi : V \longrightarrow M$$

which maps every closed point $x \in V$ to the point of $M \underset{\underset{\sim}{Z}}{\times} \Omega$ corresponding to the curve $f^{-1}(x)$. The most complete way to combine both requirements, and even allow consideration of infinitesimal families of curves seems to be this:

> (*) For all schemes X, and all proper, simple[2] morphisms $A \xrightarrow{f} X$ all of whose geometric fibres[2] are curves of genus g, there should be given a morphism $\phi(f) : X \longrightarrow M$.

$\phi(f)$ is, so-to-speak, the generalized "j-morphism". Compatibility is ensured by:

> (**) For all morphisms $Y \xrightarrow{s} X$, if $B = A \underset{X}{\times} Y$ and the projection is $B \xrightarrow{g} Y$, then $\phi(g) = \phi(f) \circ s$.

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[2] For definitions of this terminology see Grothendieck's notes [7], parts 2 and 4.

Any such pair (M, ϕ) will be called an *invariant scheme* (for curves of genus g). Since for any morphism $M \xrightarrow{f} N$, $(N, f \circ \phi)$ is also an invariant scheme, we can consider the Universal Mapping Property:

(***) For all invariant schemes (N, ψ), there is a unique morphism $M \xrightarrow{f} N$ such that $\psi = f \circ \phi$. An invariant scheme satisfying this will be called a *pre-moduli scheme* (for curves of genus g). Finally the first requirements we considered were:

(****) M is a normal model over $\mathbb{Z}$, and for all algebraically closed fields Ω, ϕ sets up an isomorphism between biregular equivalence classes of curves C of genus g defined over Ω, and geometric points of M in Ω, i. e. closed points of $M \times_{\mathbb{Z}} \Omega$, via

$$C/\Omega \longrightarrow P, \quad \text{where} \quad f : C \longrightarrow \operatorname{Spec} \Omega .$$

and

$$P = \operatorname{Im}\left[\Omega \xrightarrow{\phi(f) \times id} M \times_{\mathbb{Z}} \Omega\right]$$

If (M, ϕ) satisfies all these properties, we shall call it a *moduli scheme*. Its existence (uniqueness is immediate by (***)) is the main theorem of this thesis.

I shall follow this terminology:

i) a <u>scheme</u> : as in Grothendieck

ii) a <u>model over R</u>, R a noetherian integral domain: an integral scheme of finite type over R, dominating R, ([16]).

iii) a <u>variety over k</u> : a model over a field k.

iv) a <u>morphism</u>: as usual. I prefer this to regular map since it is more concise; "map" alone is ambiguous.

v) $\underset{\sim}{P}^n$: the scheme over $\underset{\sim}{Z}$, $\mathrm{Proj}\{Z[X_0, \cdots, X_n]\}$,[3] or what might have been awkwardly called in pre-Grothendieck terminology, the union over all p of the loci of specializations of $\underset{\sim}{P}^n/Q \xrightarrow{\underset{\sim}{Z}(p)} \underset{\sim}{P}^n/[\underset{\sim}{Z}/(p)]$. Write also $\underset{\sim}{P}^n_R$ for $\underset{\sim}{P}^n \underset{\underset{\sim}{Z}}{\times} \mathrm{Spec}\, R$.

vi) If X and Y are schemes, $X \times Y$ means product over $\underset{\sim}{Z}$ unless otherwize qualified and π_1, π_2 will be its canonical projections. If, on the other hand, we consider $X \underset{S}{\times} Y$ and the morphism $f : X \longrightarrow S$ or $f : Y \longrightarrow S$ is not clear, I shall write $(X \underset{S}{\times} Y)_f$.

vii) We shall follow Grothendieck's useful distruction between <u>point</u>, <u>geometric point and rational point over k</u>. Given a scheme X over a scheme S, a point is simply a point of the underlying topological space; a geometric point, however, is a closed point of the space of $X \underset{S}{\times} \mathrm{Spec}(\Omega)$ for some algebraically closed field Ω, (and morphism $\mathrm{Spec}\ \Omega \longrightarrow S$); a rational point over k is a morphism $\mathrm{Spec}(k) \longrightarrow X$ (which is equivalent to a point of $X \underset{S}{\times} \mathrm{Spec}(k)$ with residue field k). The residue field of a point $x \in X$

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[3] See Grothendieck's Elements, Chapter II, p. 25, [5] .

will be denoted $\kappa(x)$.

It seems to me valuable to preserve the words "model" and "variety" rather than adopting "scheme such that ..... " not only for brevity and for historical reasons, but because the two concepts of scheme and normal projective variety particularly call for a totally different set of technical tools, neither including the other, and imply very different pictures of their geometry — dominated one by the functor K, the other by the Chow ring A, [1]. For example, flat morphisms are often called for in dealing with schemes, where open morphisms suffice in dealing with normal varieties; and Cartier divisors on schemes may be replaced by sums of prime divisors on normal varieties.

In the first part of this thesis, the construction of the moduli scheme will be reduced to a problem of finding a quotient by the action of a group. In the second, some general results on such quotients are presented. In the third, the quotient in question is shown to exist, once a key construction, presented in section four, is carried out. One appendix deals with some tools for studying the behaviour of cohomology in a base extension; another with the specialization of Jacobians.

I would like to express my indebtedness to Professors Tate and Zariski for their encouragement; to T. Matsusaka for the crucial suggestion that projective families of abelian varieties may be simpler

in structure than projective families of space curves; and to Grothendieck and Hilbert without whose work on schemes and invariant theory respectively I would never have thought this problem approachable.

Section 1. Pluri-Canonical Embeddings of Curves

For simplicity, we shall say:

DEFINITION. X a scheme, $A \longrightarrow X$ a proper simple morphism all of whose geometric fibres are curves of genus g, then A is an algebraic family of curves of genus g.

We shall sometimes drop "of curves of genus g" if no confusion is possible. Because the following is not entirely well-known, recall:

DEFINITION. A scheme G over a scheme S is a relative group scheme over S if S-morphisms

i) $G \times_S G \longrightarrow S$

ii) $G \longrightarrow G$

iii) $S \longrightarrow G$

are given satisfying the usual identities for multiplication inverse and identity respectively; if $S = Z$, G is an absolute group scheme.

DEFINITION. A relative group scheme G/S is said to operate on a scheme X/S if an S-morphism

$$G \times_S X \xrightarrow{\sigma} X$$

is given satisfying the usual identities. The letter σ will always be used for such morphisms below.

The central definition of this section is:

DEFINITION. An algebraic family $A \xrightarrow{\varpi} K$, and an operation of an absolute group scheme G on K is a <u>canonical</u> <u>triple</u> (A, K, G) if both of the following hold:

1) for all algebraic families $B \xrightarrow{f} Y$, there is an open covering $\{U_i\}$ of Y and morphisms $f_i : U_i \longrightarrow K$ and U_i-isomorphisms:

$$f^{-1}(U_i) \simeq (A \times_K U_i)_{f_i}$$

2) given $f_1, f_2 : Y \longrightarrow K$, then there is a Y-isomorphism

$$(A \times_K Y)_{f_1} \simeq (A \times_K Y)_{f_2}$$

if and only if there is a morphism $g : Y \longrightarrow G$ such that

(1)
$$\begin{tikzcd} Y \arrow[r, "g \times f_1"] \arrow[dr, "f_2"'] & G \times K \arrow[d, "\sigma"] \\ & K \end{tikzcd}$$

commutes.

The following consequence is the motivation of this definition:

PROPOSITION 1. *If* (A, K, G) *is a canonical triple, then there is a* 1-1 *correspondence between invariant schemes* $(M, \emptyset)$ *and morphisms* $K \xrightarrow{f} M$ *such that*

$$\begin{tikzcd} G \times K \arrow[r, "\sigma"] \arrow[d, "\pi_2"'] & K \arrow[d, "f"] \\ K \arrow[r, "f"] & M \end{tikzcd} \tag{2}$$

commutes.

Proof: Let $\varpi : A \longrightarrow K$ be the given morphism, then to an invariant scheme $(M, \emptyset)$, associate $f = \emptyset(\varpi) : K \longrightarrow M$. To see the commutativity of (2), apply the 2^{nd} property of a canonical triple to $f_1 = \sigma$, $f_2 = \pi_2 : G \times K \longrightarrow K$. It follows that there is a $G \times K$-isomorphism

$$[A \underset{K}{\times} (G \times K)]_{\sigma} \simeq [A \underset{K}{\times} (G \times K)]_{\pi_2} .$$

Let the projections be called

$$\varpi_1 : [A \underset{K}{\times} (G \times K)]_{\sigma} \longrightarrow G \times K$$

$$\varpi_2 : [A \underset{K}{\times} (G \times K)]_{\pi_2} \longrightarrow G \times K \ .$$

Then $f \circ \pi_2 = \phi(\varpi) \circ \pi_2 = \phi(\varpi_2) = \phi(\varpi_1) = \phi(\varpi) \circ \sigma = f \circ \sigma$.

Conversely, say $K \xrightarrow{f} M$ is given, and let us construct ϕ. For all algebraic families $B \xrightarrow{g} Y$, let $\{U_i\}$ be the open covering of Y, and $f_i : U_i \longrightarrow K$ the morphisms given by the 1st property of canonical triples. Define $\phi(g) \mid U_i$ by $f \circ f_i$. We must show (a) On $U_i \cap U_j$, $f \circ f_i = f \circ f_j$, and (b) the compatibility property (**). As for (a), note that there are $U_i \cap U_j$-isomorphisms

$$[A \underset{K}{\times} (U_i \cap U_j)]_{f_i} \simeq g^{-1}(U_i \cap U_j) \simeq [A \underset{K}{\times} (U_i \cap U_j)]_{f_j}$$

by the choice of f_i and f_j; hence by the 2nd property of canonical triples, there is a morphism

$$g_{ij} : U_i \cap U_j \longrightarrow G$$

such that the left triangle below commutes:

(3)
$$\begin{tikzcd}
U_i \cap U_j \arrow[r, "g_{ij} \times f_i"] \arrow[dr, "f_j"] & G \times K \arrow[r, "\pi_2"] \arrow[d, "\sigma"] & K \arrow[d, "f"] \\
 & K \arrow[r] & M
\end{tikzcd}$$

Since, by assumption, the square commutes, the outer triangle commutes, hence

$$f \cdot f_j = f \cdot \pi_2 \cdot (g_{ij} \times f_i) = f \cdot f_i \quad .$$

(b) is proven by a similar argument. Q. E. D.

The usefulness of a canonical triple is therefore clear. To construct one, we shall use the Hilbert schemes which Grothendieck has recently defined. These schemes are a variant of Chow varieties, and, in fact the part of the Hilbert scheme we need would seem to be isomorphic to part of a "Chow model", that is to say, the closure in integral projective space $\underset{\sim}{P}^N$ of a Chow variety in $\underset{\sim}{P}^N \times_{\underset{\sim}{Z}} \underset{\sim}{Q}$, (or better the locus of solutions of the canonical integral equations which define the Chow variety — which in general define an ideal such that primes p are in some of its isolated prime ideals). However, in his report [6], Part IV, Grothendieck has sketched the proof of very strong theorems about his Hilbert schemes — hence in this case about Chow models too — which in particular give one a description of its infinitesimal properties. These are necessary for proving the defining property of a canonical triple in the case the Y of the definition is not reduced (for example, if $Y = \text{Spec } A$, A an Artin local ring); and for proving that the K we obtain is normal (in fact, simple over $\underset{\sim}{Z}$). To the best of my knowledge, no analogous results are known for Chow coordinates. In fact Grothendieck shows (and he even takes it as his defining property) that, for every n, polynomial $\underline{P}$ in $\underline{\overline{X}}$, and noetherian schemes X

> (#) there is a 1-1 correspondence between closed sub-schemes $Y \subset X \times \underset{\sim}{P}^n$ such that $\mathcal{O}_Y$ is flat over X, and for one and hence all $x \in X$, the Hilbert characteristic polynomial of $Y \underset{X}{\times} \kappa(x) \subset \underset{\sim}{P}^n_{\kappa(x)}$ is $P(\underline{\overline{X}})$, and morphisms $\phi : X \longrightarrow \underline{\text{Hilb}}_n^{P(\underline{\overline{X}})}$, $\underline{\text{Hilb}}_n^{P(\underline{\overline{X}})}$ being the Hilbert scheme.

The analogous classical result is, for all n, m, d and normal varieties V over a field k, of characteristic P,

> (##) there is a 1-1 correspondence between positive cycles Z on $V \times \underset{\sim}{P}^n_k$ such that if $\text{supp } Z = \cup Z_i$, then $\pi_1 : Z_i \longrightarrow V$ is an open morphism, and for one, and hence all simple points $x \in V$ the cycle

$$\pi_2[Z \cdot (x \times \underset{\sim}{P}^n_k)]$$

on $\underset{\sim}{P}^n_{k(x)}$ is purely of dimension m, and degree d, and has no component with multiplicity divisible by p; and morphisms $\phi : V \longrightarrow \underline{\text{Chow}}^{m,d}_n$, where $\underline{\text{Chow}}^{m,d}_n$ is a suitable open subset of a Chow Variety over k.

This can probably be generalized to "Chow models" over $\underset{\sim}{Z}$, and normal models V instead of varieties; and in that case it seems clear that except for the two points noted above, Chow coordinates plus very different classical techniques could be used in this section.

Fix any $n \geq 3$, and let $\mathcal{H}_n = \underline{\text{Hilb}}^{(g-1)(2nX-1)}_{(g-1)(2n-1)-1}$.

PROPOSITION-DEFINITION 2. *There exists a unique reduced sub-scheme* $K_n \subset \mathcal{H}_n$ *such that the geometric points of* K_n *with values in* Ω *are precisely the sub-schemes* $C \subset \underset{\sim}{P}^{(g-1)(2n-1)-1}$, C *irreducible and simple over* Ω, $C \not\subset$ *any hyperplane*, C *a curve of genus* g, *and such that the line bundle on* C *induced by this embedding is isomorphic to* $\mathcal{O}_C(nK)$ (i.e. $[\Omega^1_{C/\Omega}]^{\otimes n}$).

Proof: Write $\underset{\sim}{P}^{(g-1)(2n-1)-1} = \underset{\sim}{P}$ for simplicity. By the remark of p. 99 of [6], Part IV, there is an open subset $U_1 \subset \mathcal{H}_n$ parametrizing the sub-schemes $C \subset X \times \underset{\sim}{P}$ such that C is simple and absolutely irreducible over X. Since the Hilbert characteristic polynomial of the geometric fibres is $(g-1)(2n\underline{\overline{X}}-1)$, it follows that these fibres are indeed curves of genus g. Let $A_1 \xrightarrow{\varpi_1} U_1$ be the algebraic family of curves induced, via #, from the identity $U_1 \xrightarrow{\sim} U_1$; we are also given $A_1 \subset U_1 \times \underset{\sim}{P}$. Let $\underset{\sim}{P}^*$ be the dual projective space to $\underset{\sim}{P}$, and $I \subset \underset{\sim}{P} \times \underset{\sim}{P}^*$ the divisor of incident points and hyperplanes. Set $\underline{P}_0(\underline{\overline{X}}) = (g-1)(2n\underline{\overline{X}}-1)$, then in Grothendieck's terminology, there is a natural morphism:

$$\underline{\mathrm{Hilb}}^{\underline{P}_0}_{I/\underset{\sim}{P}^*} \xrightarrow{f} \mathrm{Hilb}^{\underline{P}_0}_{\underset{\sim}{P} \times \underset{\sim}{P}^* / \underset{\sim}{P}^*} \simeq \mathcal{H}_n \times \underset{\sim}{P}^* \,.$$

Let $E = \mathrm{Projection}_{U_1}\,[\mathrm{Supp}\,(U_1 \times \underset{\sim}{P}^*) \cap \mathrm{Supp}\,(\mathrm{Image}\ f)]$. E is clearly a closed subset of U_1; let $U_2 = U_1 - E$, and $A_2 = \varpi^{-1}(U_2)$. It is immediate that the geometric points of U_2 correspond precisely to the curves in A_1 not contained in any hyperplane. Finally, let L be the line bundle induced on A_2 by hyperplanes of $\underset{\sim}{P}$ via $A_2 \hookrightarrow U_2 \times \underset{\sim}{P} \xrightarrow{\pi_2} \underset{\sim}{P}$. Look at:

$$\mathcal{F} = (R^1\varpi)\left(L \otimes \left[\Omega^1_{A_2/U_2}\right]^{-(n-1)}\right).$$ [4]

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[4] The definition of the sheaf $\Omega^1_{X/Y}$ for any simple morphism $X \longrightarrow Y$ can be found in [7], p. 1, Part 1.

Define $K_n \subset U_2$ to be the reduced sub-scheme with support = supp $\mathcal{F}$. To prove that this is precisely the locus of geometric points with the last property required, note that if C_Ω is a geometric fibre of ϖ lying over the ordinary fibre C_x, for $x \in U_2$, and L_Ω, L_x stand for the line bundles induced on these curves by L, then noting that $\deg(c_1(L)) = n(2g-2)$ because of the coefficient of $\mathfrak{X}$ in P_0,

$$
\begin{aligned}
& & L_\Omega &\simeq \left[\Omega^1_{C_\Omega/\Omega}\right]^{\otimes n} \\
&\Longleftrightarrow & L_\Omega \otimes \left[\Omega^1_{C_\Omega/\Omega}\right]^{-(n-1)} &\simeq \Omega^1_{C_\Omega/\Omega} \\
&\Longleftrightarrow & H^1\left(L_\Omega \otimes \left[\Omega^1_{C_\Omega/\Omega}\right]^{-(n-1)}\right) &\neq (0) \\
&\Longleftrightarrow & H^1\left(L_x \otimes \left[\Omega^1_{C_x/k(x)}\right]^{-(n-1)}\right) &\neq (0) ,
\end{aligned}
$$

and by Proposition 7 of Appendix 1, this last cohomology group is

$$\simeq (R^1\varpi)\left(L \otimes \left[\Omega^1_{A_2/U_2}\right]^{-(n-1)}\right) \otimes k(x) ,$$

hence:

$$\Longleftrightarrow \quad \mathcal{F}_x \neq (0) .$$

Q. E. D.

Let A_n be the algebraic family induced over K_n by $K_n \hookrightarrow \mathcal{H}_n$, and let $\pi_n : A_n \longrightarrow K_n$. Then

PROPOSITION 3. Let L be the line bundle induced on A_n by hyperplane sections via $A_n \hookrightarrow K_n \times \underset{\sim}{P} \xrightarrow{\pi_2} \underset{\sim}{P}$. There exists a line bundle L_0 on K_n such that

$$L \simeq (\pi_n^* L_0) \otimes \left[\Omega^1_{A_n/K_n}\right]^n .$$

Proof: This is essentially an application of the "see-saw" principle. Let

$$E = L \otimes \left(\Omega^1_{A_n/K_n}\right)^{-n} , \quad L_0 = (R^0 \pi_n)(E) .$$

I claim L_0 is locally free, rank 1, and for all $y \in K_n$, $L_0 \otimes \kappa(y) \simeq H^0(E \otimes \kappa(y))$, which implies the result — since there is a canonical homomorphism

$$(\pi_n^* L_0) \otimes \left[\Omega^1_{A_n/K_n}\right]^n \longrightarrow L ,$$

which restricted to each fibre C_y is

$$H^0\left(L_y \otimes \left[\Omega^1_{c_y/\kappa(y)}\right]^{-n}\right) \otimes \left[\Omega^1_{c_y/\kappa(y)}\right]^{n} \longrightarrow L_y$$

— an isomorphism since

$$\left[\Omega^1_{c_y/\kappa(y)}\right]^{n} \simeq L_y \quad .$$

But by Proposition 8 of Appendix 1, as K_n is reduced, and $E_y = E \otimes \kappa(y) \simeq \mathcal{O}_{c_y}$ for every $y \in K_n$, it follows that L_0 is indeed locally free, rank 1 such that $L_0 \otimes \kappa(y) \simeq H^0(E_y)$. Q.E.D.

COROLLARY.

$$P\left(R^0\pi_n\left(\left(\Omega^1_{A_n/K_n}\right)^n\right)\right) \simeq P(R^0\pi_n(L)) \simeq K_n \times \underset{\sim}{P} \quad .^{5}$$

Now let PGL(N) be the absolute group scheme of projective transformations of $\underset{\sim}{P}^N$: in fact,

$$PGL(N) = \text{open subset of } \operatorname{Proj}\{\underset{\sim}{Z}[U_{00}, \cdots, U_{0N}, \cdots, U_{N0}, \cdots, U_{NN}]\}$$

where $\det(U_{ij}) \neq 0$. Its group structure is induced by the Hopf

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[5] The symbol $P(E)$, E a locally free sheaf is defined on p. 71, of [5], Chapter II.

algebra structure on its homogeneous coordinate ring:

$$U_{ij} \longrightarrow \sum_{k=0}^{N} U'_{ik} \otimes U''_{kj}$$

giving a homomorphism

$$\underset{\sim}{Z}[U_{ij}] \longrightarrow \underset{\sim}{Z}[U'_{ij}] \otimes \underset{\sim}{Z}[U''_{ij}] \quad .$$

Let $G_n = PGL((g-1)(2n-1)-1)$. Clearly G_n operates on $\mathcal{H}_n$, and by the invariance of its defining property also on K_n.

THEOREM 1. (A_n, K_n, G_n) <u>is a canonical triple for any</u> $n \geq 3$.

<u>Proof:</u> Start with an algebraic family $B \xrightarrow{\pi} X$. We seek an open covering $\{U_i\}$ of X and morphisms $f_i : U_i \longrightarrow K_n$ such that there is a U_i-isomorphism

$$\pi^{-1}(U_i) \xrightarrow{\sim} \left(A_n \underset{K_n}{\times} U_i\right)_{f_i} \quad .$$

Consider $E = (R^0\pi)((\Omega^1_{B/X})^n)$. By Proposition 7 of Appendix 1, it follows E is locally free, and for all $y \in X$,

$$E\otimes\kappa(y)\simeq H^0\left(\left(\Omega^1_{B_y/\kappa(y)}\right)^n\right) .$$

Pick U_i so that $E | U_i$ is free, and for simplicity write $X = U_i$, and assume E is free. Choose a basis of E, hence an identification $P(E) \simeq X \times \underset{\sim}{P}$, where $\underset{\sim}{P} = \underset{\sim}{P}^{(g-1)(2n-1)-1}$ again. Consider the usual rational map

$$\begin{tikzcd}
B \arrow[rr] \arrow[dr] & & X \times \underset{\sim}{P} \arrow[dl] \\
& X &
\end{tikzcd} \tag{4}$$

I claim that if $n \geq 3$, this is a closed immersion, hence (after trivially checking the Hilbert characteristic function) defines a morphism $f : X \longrightarrow \mathcal{H}_n$, inducing the family B. But it is a well known classical fact that for all points $y \in B$ the induced map $B_y \longrightarrow \underset{\sim}{P}_{\kappa(y)}$ is a closed immersion. Therefore $B \longrightarrow X \times \underset{\sim}{P}$ is well defined, i.e. a morphism, and is 1-1, (to be well defined means for any point x of B there is a local section of E not zero at x; but if there is such a section in

$$H^0\left(\left(\Omega^1_{B_{\pi(x)}/\kappa(\pi(x))}\right)^n\right),$$

it lifts to a local section in E). By the Stein factorization $B \to X \times \underset{\sim}{\mathbf{P}}$ is a finite morphism. Let $x \in B$ and $y \in X \times \underset{\sim}{\mathbf{P}}$ its image. We still must show $\mathcal{O}_x = \mathcal{O}_y/\mathfrak{a}$ where $0 \to \mathfrak{a} \to \mathcal{O}_y \to \mathcal{O}_x$. But if $\mathfrak{b} \subset \mathcal{O}_y$ is the ideal generated by the maximum ideal of $\mathcal{O}_{f(x), X}$, then we know $\mathcal{O}_x/\mathfrak{b}\,\mathcal{O}_x = \mathcal{O}_y/\mathfrak{a} + \mathfrak{b}$. Therefore $\mathcal{O}_x/\mathcal{O}_y$ is a finite $\mathcal{O}_y$-module such that $\mathfrak{b}\cdot(\mathcal{O}_x/\mathcal{O}_y) = \mathcal{O}_x/\mathcal{O}_y$, hence by Nakayama's lemma, $\mathcal{O}_x/\mathcal{O}_y = (0)$, i.e. $\mathcal{O}_x = \mathcal{O}_y/\mathfrak{a}$.

We still must show that the morphism f factors: $X \to K_n \hookrightarrow \mathcal{H}_n$. Clearly $\mathrm{Supp}(\mathrm{Image}\ f) \subset \mathrm{Supp}(K_n)$, hence f factors if X is reduced. To see this in general, it suffices to prove that for all Artin local rings A_0, and morphisms $A_0 \xrightarrow{g} X$, then $f \circ g$ factors through K_n. But given any such A_0, by a result of I. Cohen [2], Corollary 2, p. 89, there is a complete regular local ring A_1 such that $A_0 = A_1/\mathfrak{a}$, some primary $\mathfrak{a}$. Consider the induced algebraic family B_0:

$$\begin{tikzcd} B_1 \arrow[d, "\pi_1"] & B_0 \arrow[l] \arrow[r] \arrow[d, "\pi_0"] & B \arrow[d, "\pi"] \\ \mathrm{Spec}\ A_1 & \mathrm{Spec}\ A_0 \arrow[l] \arrow[r] & X \end{tikzcd} \tag{5}$$

By a result of Grothendieck, [9], Theorem 9, p. 13, there exists an algebraic family B_1 over A_1 also inducing B_0. The free sheaf

$$E_1 = (R^0 \pi_1)\left(\left(\Omega^1_{B_1/A_1}\right)^n\right)$$

is such that

$$E_1 \underset{A_1}{\otimes} A_0 \simeq E_0 = (R^0 \pi_0)\left(\left(\Omega^1_{B_0/A_0}\right)^n\right) \simeq E \underset{\mathcal{O}_X}{\otimes} A_0$$

(Proposition 7 of Appendix 1). The chosen basis of E induces one of E_0, and this may be lifted to a basis of E_1. We obtain

(6)
$$\begin{tikzcd}
 & B_1 \arrow[dl] \arrow[dd] & & B_0 \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dd] & & B \arrow[dl] \arrow[dd] \\
\operatorname{Spec} A_1 & & \operatorname{Spec} A_0 \arrow[ll] \arrow[rr] & & X & \\
 & P(E_1) \arrow[ul] \arrow[d, "s"'] & & P(E_0) \arrow[ll] \arrow[rr] \arrow[ul] \arrow[d, "s"'] & & P(E) \arrow[ul] \arrow[d, "s"'] \\
 & \underset{\sim}{P}_{A_1} & & \underset{\sim}{P}_{A_0} \arrow[ll] \arrow[rr] & & X \times \underset{\sim}{P}
\end{tikzcd}$$

which commutes. As above $B_1 \hookrightarrow \underset{\sim}{P}_{A_1}$ is a closed immersion. Hence we obtain

(7)
$$\begin{tikzcd}
A_0 \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\
A_1 \arrow[r] \arrow[dr, dashed] & \mathcal{H}_n \\
& K_n \arrow[u, hook]
\end{tikzcd}$$

Since A_1 is reduced, the factorisation along the dotted line exists, hence also $f \circ g$ factors through K_n. This completes the proof of the first property of a canonical triple.

Secondly, suppose $f_1, f_2 : X \longrightarrow K_n$ are given and a X-isomorphism

$$\begin{array}{ccc}
\left(A_n \underset{K_n}{\times} X\right)_{f_1} & \simeq & \left(A_n \underset{K_n}{\times} X\right)_{f_2} \\
\| & & \| \\
B_1 & & B_2
\end{array}$$

There exist embeddings

$$j_i : B_i \hookrightarrow X \times \underset{\sim}{P} , \qquad i = 1, 2 ,$$

inducing back the morphisms $f_i : X \longrightarrow K_n$, $i = 1, 2$, according to the definition of Hilbert schemes. Let $\pi_i : B_i \longrightarrow X$, and let

$$E_i = (R^0\pi_i)\left(\left(\Omega^1_{B_i/X}\right)^n\right) .$$

By the Corollary of Proposition 3, there are canonical X-isomorphisms

$$\begin{tikzcd}
X \times \underset{\sim}{P} \arrow[rr, "\simeq"] \arrow[dr] & & P(E_i) \arrow[dl] \\
& X &
\end{tikzcd}$$

and the isomorphism of the B_i's induces an X-isomorphism $P(E_1) \simeq P(E_2)$. Composing, let

$$\begin{tikzcd}
\phi : X \times \underset{\sim}{P} \arrow[rr, "\simeq"] \arrow[drrr] & & P(E_1) \arrow[rr, "\simeq"] \arrow[dr] & & P(E_2) \arrow[dl] & & X \times \underset{\sim}{P} \arrow[ll, "\simeq"'] \arrow[dlll] \\
& & & X & & &
\end{tikzcd}$$

Now the group scheme $PGL((g-1)(2n-1)-1) = G_n$ has the fundamental property:

$$\mathrm{Hom}(X, G_n) \simeq \mathrm{Aut}_X(X \times \underset{\sim}{P}) .$$

(See [6], IV, p. 19 and II, p. 13). Thus ϕ induces $g : X \longrightarrow G_n$. It remains to verify that

(8)
$$\begin{tikzcd}
X \arrow[r, "g \times f_1"] \arrow[dr, "f_2"'] & G_n \times K_n \arrow[d, "\sigma"] \\
& K_n
\end{tikzcd}$$
commutes.

But consider:

(9)
$$\begin{tikzcd}
B_1 \arrow[r, hook] \arrow[d, "\wr"] & P(E_1) \arrow[d, "\wr"] & X \times \underset{\sim}{P} \arrow[l, "\sim"'] \arrow[d, "\phi"] \\
B_2 \arrow[r, hook] & P(E_2) & X \times \underset{\sim}{P} \arrow[l, "\sim"']
\end{tikzcd}$$

By construction, the inner squares commute, hence the rectangle does also. But by the relation of ϕ and g, it follows that:

(10)
$$\begin{tikzcd}
B_1 \arrow[rr, "(g \circ \pi_1) \times j_2"] \arrow[d, "\wr"] \arrow[ddr] & & G_n \times (X \times \underset{\sim}{P}) \arrow[d, "\sigma"] \arrow[ddl] \\
B_2 \arrow[rr, "j_2"] \arrow[dr] & & X \times \underset{\sim}{P} \arrow[dl] \\
& X &
\end{tikzcd}$$
commutes.

Therefore (8) commutes (by definition of the induced operation of G_n on $\mathcal{H}_n$).

The converse of the second property of canonical triples is immediate. Q. E. D.

COROLLARY OF PROOF. K_n is simple over $\mathbb{Z}$.

Proof: By Grothendieck's criterion, Theorem 3.1, Part 3, [7], this is equivalent to: for all Artin local rings A_1, and $A_0 = A_1/\mathfrak{a}$, and morphisms

$$\mathrm{Spec}\ A_0 \longrightarrow K_n$$

there is a factorization

$$\mathrm{Spec}\ A_0 \longrightarrow \mathrm{Spec}\ A_1 \longrightarrow K_n \quad .$$

But, in effect, this is exactly what we showed above, except that there A_1 was a complete, regular local ring, in the part of the proof using diagrams (5), (6) and (7). The argument there applies verbatim.

Q. E. D.

Section 2. Generalities on Quotients by Algebraic Groups.

Suppose an absolute group scheme G, a scheme K, and an operation of G on K are given.

DEFINITION: A scheme Q and a morphism $\phi : K \longrightarrow Q$ will be called a quotient pair (Q, ϕ) if

(1)
$$\begin{tikzcd}
G \times K \arrow[r, "\sigma"] \arrow[d, "\pi_2"'] & K \arrow[d, "\phi"] \\
K \arrow[r, "\phi"] & Q
\end{tikzcd}$$
commutes,

(2) for all algebraically closed fields Ω, ϕ set up an isomorphism between the orbits $\mathrm{Hom}(\Omega, K)/\mathrm{Hom}(\Omega, G)$ and the geometric points $\mathrm{Hom}(\Omega, Q)$,

(3) ϕ is an open morphism,

(4) the sheaf $\mathcal{O}_Q$ consists of the invariant functions of $\mathcal{O}_K$ in the following sense: $\forall U \subset Q$ open, $\Gamma(\mathcal{O}_Q, U) \simeq \{f \in \Gamma(\mathcal{O}_K, \phi^{-1}(U)) \mid$ if f defines $f' : \phi^{-1}(U) \longrightarrow \underset{\sim}{A}^1$, then

$$\begin{tikzcd}
G \times \phi^{-1}(U) \arrow[r, "\sigma"] \arrow[d, "\pi_2"'] & \phi^{-1}(U) \arrow[d, "f'"] \\
\phi^{-1}(U) \arrow[r, "f'"'] & A^1
\end{tikzcd}$$
commutes$\}$.

The uniqueness of such a pair follows from:

PROPOSITION 4. *Given any scheme* P, *and morphism* $\psi : K \longrightarrow P$ *such that*

$$\begin{tikzcd} G \times K \arrow[r, "\sigma"] \arrow[d, "\pi_2"] & K \arrow[d, "\psi"] \\ K \arrow[r, "\psi"'] & P \end{tikzcd}$$

commutes,

there is a unique morphism $f : Q \longrightarrow P$ *such that* $\psi = f \circ \phi$.

Proof: Let $x \in Q$. By (2), the set $\psi(\phi^{-1}(x))$ consists of exactly one point $y \in P$. Let $V \subset P$ be open and affine, $y \in V$. By (3), $\phi(\psi^{-1}(V)) \subset Q$ is open. Let $U \subset \phi(\psi^{-1}(V))$ be open and affine, $x \in U$. We shall define f locally by a homomorphism

$$\Gamma(\mathcal{O}_P, V) \longrightarrow \Gamma(\mathcal{O}_Q, U) .$$

Namely, if $g \in \Gamma(\mathcal{O}_P, V)$, g defines $g' : V \longrightarrow \underset{\sim}{A}^1$, hence $g'' : \phi^{-1}(U) \longrightarrow \underset{\sim}{A}^1$, and it is easy to verify the condition of (4). Hence g'' defines $g''' \in \Gamma(\mathcal{O}_Q, U)$ by (4). This map is obviously a homomorphism; that these local f's agree, and that the final f is unique follows clearly. Q.E.D.

The concept of a quotient becomes somewhat more managable in light of:

THEOREM 2. *If* G, K *are models over* $\underset{\sim}{Z}$, K *normal and* G *absolutely irreducible*, *then given* $\phi : K \longrightarrow Q$, (Q, ϕ) *is a quotient pair if and only if* (1), (2) *hold and* (4*) Q *is a normal model over* $\underset{\sim}{Z}$.

Proof: Let us first prove the sufficiency of these new conditions. The first step is to show $\emptyset$ is equidimensional. We need:

LEMMA 1. $X \xrightarrow{f} Y$ *of finite type*, Y *a noetherian scheme*, X *irreducible*, *and* f *dominating*. *Then*

$$X \times_Y X \xrightarrow{\pi_1} X \text{ equidimensional} \Longrightarrow X \xrightarrow{f} Y \text{ equidimensional.}$$

Proof: Note that the fibres of π_1 are merely ground *field* extensions of the fibres of f. Q. E. D.

LEMMA 2. $X \xrightarrow{f} Y \xrightarrow{g} Z$ *of finite type*, Z *a noetherian scheme*, X *irreduicible*, *and* f *and* g *surjective, then*

$$g \circ f \text{ equidimensional} \Longrightarrow g \text{ equidimensional} .$$

Proof: f and g surjective implies Y and Z are irreducible. Let z_1, z_2 be the generic and an arbitrary point of Z respectively, and let y_1, y_2 be the generic point of Y, and the generic point of any component C of $g^{-1}(z_2)$ respectively. Also, let D be any component of $f^{-1}(y_2)$. Then we know

$$\dim C \geq \dim g^{-1}(z_1)$$

$$\dim D \geq \dim f^{-1}(y_1)$$

hence

$$\begin{aligned}\dim(g \circ f)^{-1}(z_1) &= \dim g^{-1}(z_1) + \dim f^{-1}(y_1)\\ &\leq \dim C + \dim D\\ &= \dim\,[\text{one component of } (g \circ f)^{-1}(z_2)]\\ &= \dim\,(g \circ f)^{-1}(z_1)\ .\end{aligned}$$

Therefore $\dim C = \dim g^{-1}(z_1)$. Q. E. D.

Now G is irreducible and dominates $\underset{\sim}{Z}$, and consequently is equidimensional over $\underset{\sim}{Z}$.[6] Therefore $K \times G \xrightarrow{\pi_1} K$ is equidimensional, and factoring this by $K \times G \xrightarrow{\pi_1 \times \sigma} K \underset{Q}{\times} K \xrightarrow{\pi_1} K$, it follows by Lemma 2 that $K \underset{Q}{\times} K \xrightarrow{\pi_1} K$ is equidimensional. Therefore by Lemma 1, $K \xrightarrow{\phi} Q$ is equidimensional. Now by Theorem 5, Section 4, Chapter III of [5], since Q is normal, it follows $K \xrightarrow{\phi} Q$ is an open morphism, ie. (3). Note that in this argument we have not used the full strength of (2). In fact, we have used precisely:

(2_0) ϕ is dominating, and the map of the orbits

$$\mathrm{Hom}(\Omega\,, K)/\mathrm{Hom}(\Omega\,, G) \longrightarrow \mathrm{Hom}(\Omega\,, Q) \quad \text{is} \quad 1-1\ .$$

To prove (4), let $f : \phi^{-1}(U) \longrightarrow A^1$ be given satisfying the condition of (4).

---

[6] For example, note that if $G \xrightarrow{\pi} \underset{\sim}{Z}$, $U \subset G$ open, then πU is constructible, and $\dim \underset{\sim}{Z} = 1$, hence πU is open or closed. G irred. $\Longrightarrow$ U not closed, hence π is an open morphism. Hence by Theorem 5; Section 4, Chapter III of [5], π is equidimensional.

Let F be the reduced subscheme of $A^1 \times U$ with support the closure of the image of $f \times \phi$. F is a model over $\underset{\sim}{Z}$, hence it has a normalization F_{nor}. Moreover the dominating morphism $\phi^{-1}(U) \longrightarrow F$ has a unique factorization:

$$\phi^{-1}(U) \xrightarrow{\psi} F_{nor} \xrightarrow{\varpi} F$$

since K is normal. I claim that the pair (F_{nor}, ϕ) satisfies (1), (2_0) and (4^*) for the operation of G on $\phi^{-1}(U)$. For, first of all, consider:

(11)
$$\begin{tikzcd}
G \times \phi^{-1}(U) \arrow[r, "\sigma"] \arrow[d, "\pi_2"] & \phi^{-1}(U) \arrow[d, "\psi"'] \arrow[ddr, "\phi"] & \\
\phi^{-1}(U) \arrow[r, "\psi"] \arrow[drr, "\phi"'] & F_{nor} \arrow[dr, "\varpi"'] & \\
& & F
\end{tikzcd}$$

Since $\phi \circ \pi_2 = \phi \circ \sigma$, hence $\varpi \circ (\psi \circ \pi_2) = \varpi \circ (\psi \circ \sigma)$, and since $G \times \phi^{-1}(U)$ is integral, $\phi \circ \sigma = \phi \circ \pi_2$ is dominating, and ϖ birational, it must follow that $\psi \circ \pi_2$ and $\psi \circ \sigma$ agree on the generic point of $G \times \phi^{-1}(U)$, and hence agree. This proves (1) for (F_{nor}, ψ); (2_0) follows since the map $Hom(\Omega, K)/Hom(\Omega, G) \longrightarrow Hom(\Omega, Q)$ factors thru $Hom(\Omega, F_{nor})$ by (1), and hence $Hom(\Omega, K)/Hom(\Omega, G) \longrightarrow Hom(\Omega, F_{nor})$ must be 1-1. Therefore by the first part of the proof, it follows $\psi : \phi^{-1}(U) \longrightarrow F_{nor}$ is an open morphism, and in particular, $\psi(\phi^{-1}(U)) = (F_{nor})_0$ is open in F_{nor}, hence is a model itself. Consider the morphism $(F_{nor})_0 \xrightarrow{\rho} U$. By what we have shown ρ induces an isomorphism of geometric points, hence since the characteristic is 0, ρ is birational; hence since U is normal, is an isomorphism. Therefore we may define the composite

$$U \xrightarrow{\rho^{-1}} (F_{nor})_0 \xrightarrow{\sigma} F \hookrightarrow \underset{\sim}{A}^1 \times U \xrightarrow{\pi_1} \underset{\sim}{A}^1$$

which defines $f' \in \Gamma(\mathcal{O}_Q, U)$. This f' clearly induces $f \in \Gamma(\mathcal{O}_K, \phi^{-1}(U))$. It remains still to note that $\Gamma(\mathcal{O}_Q, U) \longrightarrow \Gamma(\mathcal{O}_K, \phi^{-1}(U))$ is 1-1. This follows since $\phi : \phi^{-1}(U) \longrightarrow U$ is surjective, and U is reduced.

Let us now prove the converse. Assume (Q, ϕ) a quotient pair. Then by (4), the local rings of Q are subrings of those of K, hence Q is locally integral. By (2), ϕ is surjective, hence Q is actually integral. To show that Q is a normal model, let $U \subset Q$ be affine and open, $R = \Gamma(\mathcal{O}_Q, U)$. First of all, let $f \in$ quotient field of R, and integrally dependent on R. Then $f \in \Gamma(\mathcal{O}_K, \phi^{-1}(U))$ since K is normal, and consider the diagram:

(12)

$$\begin{tikzcd} G \times \phi^{-1}(U) \arrow[r, "\sigma"] \arrow[d, "\pi_2"] & \phi^{-1}(U) \arrow[d, "f"] \\ \phi^{-1}(U) \arrow[r, "f"'] & \underset{\sim}{A}^1 \end{tikzcd}$$

.

Since $f = g/h$, g, $h \in R$, therefore for $x \in G \times \phi^{-1}(U)$ generic,

$$\begin{aligned} f(\pi_2(x)) &= g(\pi_2(x))/h(\pi_2(x)) \\ &= g(\sigma(x))/h(\sigma(x)) \\ &= f(\sigma(x)) \end{aligned}$$

hence the diagram commutes. Therefore by (4), $f \in R$. It remains to show Q of finite type over $\underset{\sim}{Z}$. Now R is a countable set : enumerate its elements $f_1, f_2, f_3, \cdots$, and set

$$R_n = \underset{\sim}{Z}[f_1, f_2, \cdots, f_n] \subset R \quad .$$

Let $U_n = \operatorname{Spec} R_n$. There are canonical morphisms

$$\phi^{-1}(U) \longrightarrow U \longrightarrow \cdots \longrightarrow U_3 \longrightarrow U_2 \longrightarrow U_1 \longrightarrow \underset{\sim}{Z} \, .$$

Let $\quad \Delta_n = \phi^{-1}(U) \underset{U_n}{\times} \phi^{-1}(U) \subset \phi^{-1}(U) \times \phi^{-1}(U)$

and $\quad \Delta = \phi^{-1}(U) \underset{U}{\times} \phi^{-1}(U) \subset \phi^{-1}(U) \times \phi^{-1}(U)$

then $\quad \Delta \subset \cdots \subset \Delta_3 \subset \Delta_2 \subset \Delta_1 \subset \phi^{-1}(U) \times \phi^{-1}(U) \quad .$

By the descending chain condition on sub-schemes of a noetherian scheme, there exists an n_0

$$\Delta_{n_0} = \Delta_{n_0+1} = \cdots = \Delta_n = \cdots \quad ,$$

(any $n \geq n_0$), hence $\Delta_{n_0} = \Delta$ in fact. Then let $\widetilde{U}$ = normalization of the closure of the image of U in U_{n_0} . Since U is a normal model, there is a unique factorization

$$\phi^{-1}(U) \longrightarrow U \longrightarrow \widetilde{U} \longrightarrow U_{n_0} \quad .$$

Let $\widetilde{\phi} : \phi^{-1}(U) \longrightarrow \widetilde{U}$. Then (1), (2_0) and (4^*) hold for the pair $(\widetilde{U}, \widetilde{\phi})$, hence by the first part of this proof, $\widetilde{\phi}$ is an open morphism. In particular $\widetilde{U}_0 = \widetilde{\phi}(\phi^{-1}(U))$ is also a normal model, and $(\widetilde{U}_0, \widetilde{\phi})$ satisfies (1), (2) and (4^*). Hence by the first half of this theorem, $(\widetilde{U}_0, \widetilde{\phi})$ is a quotient pair for the action of G on $\phi^{-1}(U)$. By Proposition 4 , quotient pairs are unique and $U \xrightarrow{\sim} \widetilde{U}_0$. Hence U is a model too. Q. E. D.

In fact, we will not use the converse of this theorem. It is interesting, however, insofar as it sheds some light on why Nagata [17] was able to obtain his counter-example to Hilbert's 14th problem. He had a group $G/\underset{\sim}{Q}$ operating on $\underset{\sim}{A}{}^{N}_{\underset{\sim}{Q}}$, some N, linearly. Even projectivizing, his action is extremely non-hausdorff, and (2_0) is far from holding. The proof above breaks down at the step where you wish to conclude that the image of U in U_n is open.

The following corollary is the precise result we will use in the construction of moduli :

COROLLARY: *If* (A, K, G) *are a canonical triple,* K, G *models over* $\underset{\sim}{Z}$, K *normal,* *and* G *absolutely irreducible, and for every* p *there exists a morphism*[*]

$$K_{(p)} \xrightarrow{\phi_p} Q_p$$

such that (1) *and* (2_0) *hold, and* Q_p *is a model over* $\underset{\sim}{Z}$, *then the moduli scheme exists.*

Proof: By Proposition 1 and Proposition 4, it suffices to find a quotient pair $K \xrightarrow{\phi} Q$ for the operation of G on K. By the uniqueness of quotients, it suffices to find one locally, e.g. for every p, $K_{(p)} \xrightarrow{\phi_p} Q_p$. But given (Q_p, ϕ_p) satisfying (1) and (2_0), first replace Q_p by its normalization $Q_{p'}$. Then, just as in the proof of Theorem 2, $\phi'_p : K_{(p)} \longrightarrow Q'_p$ is an open morphism; set Q''_p = image of ϕ'_p . Then (Q''_p, ϕ'_p) satisfies (1), (2) and (4*), hence is a quotient pair by Theorem 2. Q. E. D.

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[*] If X is any model of $\underset{\sim}{Z}$, then $X_{(p)} = X \times_{\underset{\sim}{Z}} \underset{\sim}{Z}_{(p)}$.

Section 3. The Construction of the Moduli Scheme

In this section, we shall construct the "quotient" Q_p with the properties of the last Corollary using any of the canonical triples (A_n, K_n, G_n). This construction does not depend on any general principles for the existence of such quotients (which would be more satisfactory), but on a reduction to a very simple situation where the quotient may be explicitly constructed. This explicit work will be carried out in Section 4, but we state the results here:

DEFINITION. Let $(\underset{\sim}{P}^n)^m_0 \subset (\underset{\sim}{P}^n)^m = \overbrace{\underset{\sim}{P}^n \times \cdots \times \underset{\sim}{P}^n}^{m}$ be the open sub-scheme whose geometric points are those m-tuples $(x_1, x_2, \cdots, x_m)$, $x_i \in \underset{\sim}{P}^n$, such that, for all hyperplanes $H \subset \underset{\sim}{P}^n$, if m_0 points x_i are $\in H$, then

$$\frac{m_0}{m} < \frac{1}{n+1} \quad .$$

DEFINITION. Let $(\underset{\sim}{P}^n)^m_s = (\underset{\sim}{P}^n)^m / \mathcal{S}_m$, [7] where $\mathcal{S}_m$ = symmetric group on m letters, operating on $(\underset{\sim}{P}^n)^m$ by permuting the factors. This quotient exists by a standard result (See [7], Part 5,

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[7] X/G means the quotient of X by the operation of G: if it exists.

Section 1) depending on $\mathcal{S}_m$ being finite and $(\mathbf{P}^n)^m$ being projective over $\mathbf{Z}$.

DEFINITION. Let $[(\mathbf{P}^n)^m_s]_0 \subset (\mathbf{P}^n)^m_s$ be the open subset which is the image of $(\mathbf{P}^n)^m_0$.

THEOREM 3. $(\mathbf{P}^n)^m_0$ *is a principal fibre bundle over a quasi-projective normal model* Q *over* Z, *with group* $PGL(n)$.

The proof will be given in Section 4.

COROLLARY. $[(\mathbf{P}^n)^m_s]_0/PGL(n)$ *exists, and is a quasi-projective normal model* $M_{n,m}$ *over* $\mathbf{Z}$.

Proof of Corollary: Let $Q = (\mathbf{P}^n)^m_0/PGL(n)$ as in Theorem 3. By uniqueness of Q, the operation of $\mathcal{S}_m$ on $(\mathbf{P}^n)^m_0$ induces an operation of $\mathcal{S}_m$ on Q (the operation of $\mathcal{S}_m$ and that of $PGL(n)$ commute with each other). Since Q is quasi-projective over $\mathbf{Z}$, $Q/\mathcal{S}_m = M_{n,m}$ exists and is a quasi-projective normal model over $\mathbf{Z}$. By the universal mapping property of $(\mathbf{P}^n)^m_s = (\mathbf{P}^n)^m/\mathcal{S}_m$, there is a unique ϕ :

(13)
$$\begin{tikzcd}
(\underset{\sim}{P}^n)^m_0 \arrow[r] \arrow[d] & {[(\underset{\sim}{P}^n)^m_s]_0} \arrow[d, "\phi_{n,m}"] \\
Q \arrow[r] & M_{n,m}
\end{tikzcd}$$

such that this commutes. It follows immediately that $(M_{n,m}, \phi_{n,m})$ is a quotient pair for the action of $PGL(n)$ on $[(\underset{\sim}{P}^n)^m_s]_0$. Q. E. D.

Using this, we can construct Q_p. Fix $n \geq 3$, and p a prime, and also an integer ν such that $p^{\nu} > 2^{(2g-2)\cdot g}\sqrt{g!}$. In the first part of the argument, we shall outline a procedure, symbolically referred to as (Θ), for associating to any curve C, simple over a field k of characteristic $\neq p$ (and absolutely irreducible), a rational point over k in $M_{\mu-1, m^{2g}}$ where $m = p^{\nu}$, $\mu = 2^{(2g-2)\cdot g}$.

<u>Procedure</u> (Θ): First, let J be the Jacobian variety of C, also defined over k. Now, unfortunately the Θ-divisor of J is not necessarily defined over k; however, the divisor class $2^{2g-2}\cdot\Theta$ is so. In fact, if J is constructed by Chow's method via the cycles of degree $N\cdot(2g-2)$ for example, with the cycles in $|NK|$ acting as origin, then the cycles of the linear systems

$$|(N-1)K + 2(P_1 + \cdots + P_{g-1})|$$

where $P_1, \cdots, P_{g-1}$ are variable points of C, span a prime divisor Θ' in J rational over k. Over an algebraically closed field $\Omega \supset k$, it is immediate that Θ' is the image of Θ by $J \xrightarrow{\lambda} J$, $\lambda(x) = 2x$, and therefore Θ' is algebraically equivalent to $2^{2g-2} \cdot \Theta$, and

$$(\Theta'^g) = 2^{(2g-2)g} \cdot (\Theta^g)$$

$$= \mu \cdot g!$$

By the Riemann-Roch theorem for abelian varieties, [20],

$$\dim|\Theta'| = \frac{(\Theta'^g)}{g!} - 1$$

$$= \mu - 1$$

(and, in fact, $H^i(\mathcal{O}_J(\Theta')) = (0)$, $i > 0$). Embed $J \subset \underset{\sim}{P}_k^{\mu-1}$ by $|\Theta'|$: this is an embedding since $2^{2g-2} \geq 3$, and [18].

Consider the diagram

(14)
$$\begin{tikzcd}
J \arrow[dr] & & k \arrow[dl, "e"'] \\
& J \arrow[d] & \\
& k &
\end{tikzcd}$$

where $\lambda(x) = p^{\nu} \cdot x = m \cdot x$, and Image (e) = identity of J.

Let $F_\nu = (J \underset{J}{\times} k)_{\lambda, e}$, a sub-scheme of J since k is a sub-scheme of J via e, and immersions are stable under products. Consider F_ν as a sub-scheme of $\underset{\sim}{P}_k^{\mu-1}$. Now F_ν is a reduced sub-scheme since $p \times \text{char } J$: in fact, $\deg(\lambda) = m^{2g}$, hence $\Gamma(\mathcal{O}_{F_\nu})$ is an algebra over k of dimension m^{2g}; and since $\text{supp}[F_\nu \underset{k}{\times} \Omega]$ consists of m^{2g} distinct points, $\Gamma(\mathcal{O}_{F_\nu})$ must be a separable algebra over k, and in particular has no nilpotent elements. Therefore F_ν defines a 0-cycle on $\underset{\sim}{P}_k^{\mu-1}$, and it has a Chow point:

$$a_\nu \in \left(\underset{\sim}{P}_k^{\mu-1}\right)_s^{m^{2g}}, \qquad \text{rational over } k \quad .$$

Together with the canonical morphism, this defines

$$a'_{y} : \operatorname{Spec} k \longrightarrow (\underset{\sim}{P}^{\mu-1}_{k})^{m^{2g}}_{s} \longrightarrow (\underset{\sim}{P}^{\mu-1})^{m^{2g}}_{s} \quad .$$

a rational point over k in $(\underset{\sim}{P}^{\mu-1})^{m^{2g}}_{s}$. At this point we require:

LEMMA 1. $A \subset \underset{\sim}{P}^{n}$ *an abelian variety of degree* d, *dimension* g *defined over an algebraically closed field* Ω, $H \subset \underset{\sim}{P}^{n}$ *a hyperplane not containing* A. *Then if* N_0 *of the* $N = m^{2g}$ *points of order* m *on* A (*characteristic* $\Omega \nmid m$), *are in* H, *then*

$$N_0/N \leq d/m^2 \quad .$$

Proof: Consider $\lambda : A \longrightarrow A$, $\lambda(x) = m \cdot x$. Let γ be a curve through the identity on A which is the intersection of A with a linear space of appropriate dimension. Let $h = (A \cdot H)$, then $\lambda^{-1}(\gamma)$ passes through every point of order m, hence:

$$\begin{aligned} N_0 &\leq (h \cdot \lambda^{-1}(\gamma))_A \\ &= (\lambda(h) \cdot \gamma)_A \\ &= m^{2g-2}(h \cdot \gamma)_A \end{aligned}$$

by Proposition 2, p. 92, and Theorem 6, p. 109 of [13]. Therefore

$$N_0/N \leq (h \cdot \gamma)/m^2 = d/m^2 \quad . \qquad \text{Q.E.D.}$$

Hence, in our case $A = J$, if we choose ν so that $m = p^{\nu} > \mu\sqrt{g!}$, then

$$N_0/N \leq \mu \cdot g!/m^2 < \mu \cdot g!/\mu^2 \cdot g! = 1/\mu \quad ,$$

consequently

$$\text{Image}\,(\alpha_{\nu}') \in \left[\left(P^{\mu-1}\right)_s^{m^{2g}}\right]_0$$

in fact. Therefore we may define a rational point over k in $M_{\mu-1,\, m^{2g}}$, α_{ν}'', as the composition of α_{ν}' and $\phi_{\mu-1,\, m^{2g}}$. Then $C \longrightarrow \alpha_{\nu}''$ is the result of Procedure $(\mathcal{P})$.

Set $Q_{p,\nu} = M_{\mu-1,\, m^{2g}}$, and write $K_{(p)} = (K_n)_{(p)}$, $A_{(p)} = (A_n)_{(p)}$ for convenience for the members of the canonical triple localized outside p. We shall define a morphism $\psi_{p,\nu} : K_{(p)} \longrightarrow Q_{p,\nu}$: let $x \in K_{(p)}$ be its generic point, and let $C_x / K(x)$ be the curve in $A_{(p)}$ over x. Then let $(\mathcal{P})$ associate to C_x, the morphism $\psi_0 : \text{Spec}\ K(x) \longrightarrow Q_{p,\nu}$. Then $\psi_{p,\nu}$, as a <u>rational</u> map, is defined simply as the closure of ψ_0.

PROPOSITION 5.

i) $\psi_{p,\nu}$ *is a morphism*

ii) *if* $x' \in K_{(p)}$, $y' = \psi_{p,\nu}(x')$, *then* y' *is the image of the rational point associated by* $(\mathcal{P})$ *to the curve* $C_{x'}$ *in* $A_{(p)}$ *which is the fibre over* x'.

Proof: Let $\Gamma_{p,\nu} \subset K_{(p)} \times Q_{p,\nu}$ be the closed sub-scheme which is the graph of $\psi_{p,\nu}$. To prove (i), by Zariski's main theorem (see Corollary 5, Theorem 2, Chapter III of [5]), since $K_{(p)}$ is a normal model, it is enough to show that $\Gamma_{p,\nu} \longrightarrow K_{(p)}$ gives an isomorphism of the points of $\Gamma_{p,\nu}$ with those of $K_{(p)}$. Therefore (i) *and* (ii) follow if we show that, given any $x' \in K_{(p)}$, if $y'' : \mathcal{K}(x') \longrightarrow Q_{p,\nu}$ is the rational point associated by $(\mathcal{P})$ to $C_{x'}$, and if $x'' : \mathcal{K}(x') \longrightarrow K_{(p)}$ is the rational point determined by x', then the image of

$$x'' \times y'' : \mathcal{K}(x') \longrightarrow K_{(p)} \times Q_{p,\nu}$$

is *the* point of $\Gamma_{p,\nu}$ above x'. It is enough, therefore, to show that if R is any discrete, rank 1, valuation ring, and Spec R $\xrightarrow{f}$ $K_{(p)}$ is a dominating morphism which maps the closed point of Spec (R) to x' , then f can be lifted to a morphism h

(15)
$$\begin{tikzcd} & \Gamma_{p,\nu} \arrow[d, "\pi_1"] \\ \text{Spec (R)} \arrow[ur, "h"] \arrow[r, "f"'] & K_{(p)} \end{tikzcd}$$

(h is unique since R is a valuation ring!) mapping the closed point of Spec (R) to the image of $x'' \times y''$.

To show this, let

$$A_R = \left(\text{Spec R} \underset{K_{(p)}}{\times} A_{(p)}\right)_f$$

be the algebraic family induced over R. By Appendix 2, A_R/R has a Jacobian, or more simply, there is a relative group scheme J_R, proper over R, whose generic and closed fibres are precisely the Jacobians of the generic and closed fibres of A_R/R. Moreover, by Appendix 2, there is also an integral closed sub-scheme $\Theta'_R \subset J_R$ which intersects the generic and closed fibres of J_R in precisely the

Θ'-divisors defined in $(\mathcal{P})$. Note that the local rings of J_R are regular, since R itself is a regular local ring, and $J_R \xrightarrow{\pi_R} R$ is a simple morphism ([7], Part 2, Proposition 3.1). Therefore $\mathcal{O}_{J_R}(\Theta'_R) = L_R$ is a line bundle. Also L_R induces on the generic and closed fibres of J_R the line bundles corresponding to the Θ'-divisors: this is clear by what we claim about Θ'_R, except that L_R might possibly induce on the closed fibre the line bundle corresponding to a *multiple* of Θ'. But as $(\Theta'^g) = \mu \cdot g!$ on both fibres, and as intersection multiplicity is preserved under specialization, this cannot happen. By Proposition 7 of the first appendix, $(R^i\pi_R)(L_R) = (0)$, $i > 0$, and $(R^0\pi_R)(L_R) = E_R$ is a free R-module reducing to the cohomology groups $H^0(\mathcal{O}(\Theta'))$ at the generic and closed points of R, i.e. tensored with the quotient and residue field of R. Choosing a basis of E_R, we can define an embedding

$$J_R \subset \mathbb{P}_R^{\mu-1}$$

which reduces to ones *projectively equivalent* to the embeddings of the generic and closed fibres obtained in the course of $(\mathcal{P})$; one cannot be sure it reduces to the same embedding on the generic fibre since the chosen basis of the tensor product of E_R with the quotient field of R may not come from a basis of E_R. The closed sub-scheme

$F_{\nu,R}$ is defined exactly as in $(\mathcal{P})$, hence defines

$$g : \mathrm{Spec}\ R \longrightarrow \left[\left(\underset{\sim}{P}_R^{\mu-1}\right)_s^{m^{2g}}\right]_0 \longrightarrow \left[\left(\underset{\sim}{P}^{\mu-1}\right)_s^{m^{2g}}\right]_0 \longrightarrow Q_{p,\nu}$$

exactly as in $(\mathcal{P})$. By construction, it is clear that the images of the generic and closed points of Spec (R) are the points associated by $(\mathcal{P})$ to the corresponding fibres of A_R. Consequently,

$$\mathrm{Spec}\ (R) \xrightarrow{f \times g} K_{(p)} \times Q_{p,\nu}$$

(a) maps the generic point to the generic point of $\Gamma_{p,\nu}$;

(b) maps the closed point to the image of $x'' \times y''$.

By (a), it factors through $\Gamma_{p,\nu}$. By (b), the required result is proven. Q. E. D.

In order to complete the construction of moduli, we must construct $\phi_p : K_{(p)} \longrightarrow Q_p$ satisfying the requirements of the Corollary to Theorem 2. We shall take as ϕ_p :

$$\phi_{p,N} = \psi_{p,\nu_0} \times \psi_{p,\nu_0+1} \times \cdots \times \psi_{p,N}$$

where $N >> 0$, ν_0 = least n such that $p^n > 2^{(2g-2)\cdot g}\sqrt{g!}$, and as Q_p :

$$Q_{p,N} = \text{Closure of Image of } K_{(p)} \text{ in } Q_{p,\nu_0} \times \cdots \times Q_{p,N} .$$

By Proposition 5, $(Q_{p,N}, \phi_{p,N})$ satisfies condition (1) of the Corollary, and clearly $\phi_{p,N}$ is dominating. It remains to examine

$$\Delta_N = \left(K_{(p)} \underset{Q_{p,N}}{\times} K_{(p)}\right)_{\phi_{p,N}, \phi_{p,N}} .$$

Note that $\Delta_{\nu_0} \supset \Delta_{\nu_0+1} \supset \cdots \supset \Delta_N \supset \cdots\cdots$. By the descending chain condition on closed sub-schemes of the noetherian scheme $K_{(p)} \times K_{(p)}$, there exists an N_0 such that $\Delta_{N_0} = \Delta_{N_0+1} = \cdots\cdots$. We must show that supp (Δ_{N_0}) = the image of $G_{(p)} \times K_{(p)}$ under $\sigma \times \pi_2$, and then we are finished. We require:

LEMMA 2. $A_1, A_2 \subset \underset{\sim}{P}^n$ <u>abelian varieties over an algebraically closed field</u> Ω, <u>of characteristic</u> $\neq p$. <u>If for all</u> $\ell \geq \nu_0$, *there exists a transformation* $T_\ell \in PGL(n) \times \Omega$ *such that* $T_\ell(A_1)$ *and* A_2 *have the same points of order* p^ℓ, *then there exists a transformation* $T_\infty \in PGL(n) \times \Omega$ *such that* $T_\infty(A_1) = A_2$.

Proof: Let $G_k \subset PGL(n) \times \Omega$ be the closed reduced sub-schemes whose points are the transformations T such that all points of order p^k on A_2 are contained in $T(A_1)$. Then $G_k \supset G_{k+1} \supset \cdots\cdots$. Since $G_k \neq 0$ for any k, and since, by the descending chain condition on closed sub-schemes, there exists an N such that $G_N = G_{N+1} = \cdots\cdots$, it follows that some T is in $\bigcap_1^\infty G_k \subset PGL(n) \times \Omega$. Then every point of order p^N, any N, of A_2 is in $T(A_1)$. Since p characteristic (Ω), the union of all points of order p^N, some N, are Zariski-dense in A_2,[8] hence $A_2 = T(A_1)$. Q. E. D.

Finally, if x_1, x_2 are two geometric points of $K_{(p)}$ such that x_1 x_2 is in Δ_N, every N, then consider the curves C_{x_1} over x_1 in $A_{(p)}$, and their Jacobians $J_{x_1} \subset \underset{\sim}{P}_\Omega^{\mu-1}$. Since $x_1 \times x_2 \in \Delta_N$, it follows that there is a projective transformation T_N of $\underset{\sim}{P}_\Omega^{\mu-1}$ such that the points of order p^N on J_{x_2} are the same as those of $T_N(J_{x_1})$.

---

[8] Let $W \subset A_2$ be the least closed subvariety of A_2 containing all points of order p^N, every N. W is clearly invariant under translations by the points of a closed subvariety containing the points of order p^N, any N. Hence W is a subgroup in fact. But W and A_2 have the same number of points of order p, hence $\dim W = \dim A_2$, and $W = A_2$.

By the lemma, J_{x_1} and J_{x_2} are projectively equivalent, hence isomorphic as polarized Jacobians. By Torelli's theorem [14], C_{x_1} and C_{x_2} are isomorphic, and by the properties of a canonical triple, there is a geometric point σ of $G_{(p)}$ such that $x_1^{\sigma} = x_2$, i.e. $x_1 \times x_2 \in$ image of $G_{(p)} \times K_{(p)}$ under $\sigma \times \pi_2$. The requirements of the Corollary to Theorem 2 are therefore completely satisfied, and moduli exist.

Section 4. *The Operation of* PGL(n) *on* $(\underset{\sim}{P}^n)^m$

In this section, we shall prove Theorem 3. The methods are those of classical (i.e. Hilbert and before) invariant theory. Let the i^{th} factor in $(\underset{\sim}{P}^n)^m$ have homogeneous coordinates $X_0^{(i)}, X_1^{(i)}, \cdots, X_n^{(i)}$. For all (n + 1)-triples $i_0, i_1, \cdots, i_n$ of integers between 1 and m, let

$$D_{i_0, i_1, \cdots, i_m} = \operatorname*{Det}_{0 \le k, \ell \le n} \left(X_\ell^{(i_k)} \right) .$$

The first step is a purely combinatorial result stating that suitably many of these D's are not zero at a geometric point of $(\underset{\sim}{P}^n)_0^m$:

PROPOSITION 6. *Let* $P = (P_1, \cdots, P_m)$ *be a geometric point of* $(\underset{\sim}{P}^n)_0^m$. *Then*

(a) *there exists* N *and* $N_0 > 0$, *and for every* i *between* 1 *and* m, *a monomial* $\prod^{(i)}$ *in the* D's *such that* $\prod^{(i)}(P) \neq 0$, *and the degree of* $\prod^{(i)}$ *in* $X_*^{(j)}$ *is* N, *if* $j \neq i$, *and* $N - N_0$, *if* $j = i$.

(b) *there exists* $N > 0$, *and for every* $i \neq j$ *integers between* 1 *and* m, *a monomial* $\prod^{(i,j)}$ *in the* D's *such that* $\prod^{(i,j)}(P) \neq 0$, *and the degree of* $\prod^{(i,j)}$ *in* $X_*^{(k)}$ *is* N, *if* $k \neq i, j$, *and* $N + 1$, *if* $k = i$, *and* $N - 1$, *if* $k = j$.

Proof: Let E be the real vector space of dimension m, and let H be the convex cone spanned by the points $P_{i_0, \cdots, i_n} = (x_1, x_2, \cdots, x_m)$ where $x_i = 0$ if $i \neq$ any i_k, $x_i = 1$ if $i =$ some i_k, and where $(i_0, \cdots, i_n)$ range over the $(n+1)$-tuples such that $D_{i_0, \cdots, i_n}(P) \neq 0$. Then there is a 1-1 correspondence between monomials π in D's such that $\pi(P) \neq 0$, and positive integral sums of the points P (taking the exponent of a given D to the coefficient of the corresponding P). Then I claim (a) follows from the assertion $(1, 1, \cdots, 1) \in \text{Int}(H)$. For this implies that every point with rational coordinates sufficiently near $(1, 1, \cdots, 1)$ is a positive rational sum of the P's. Hence there is an N_1 such that all points

$$\left(1, 1, \cdots, 1, 1 - \frac{1}{N_1}, 1, \cdots, 1\right)$$

are such rational sums. Therefore, for $N = N_1 N_0$ large enough, $(N, N, \cdots, N, N - N_0, N, \cdots, N)$ are positive integral sums of the P's, hence (a). Now if $(1, 1, \cdots, 1) \in \text{Int}(H)$, there is a linear functional on E, zero at $(1, 1, \cdots, 1)$ and non-positive on H, i.e. $\exists \alpha_1, \alpha_2, \cdots, \alpha_m$ such that

$$\text{(i)} \qquad \sum_{k=0}^{n} \alpha_{i_k} \leq 0 \qquad \text{if} \qquad D_{i_0, \cdots, i_k}(P) \neq 0 \quad ,$$

$$\text{(ii)} \quad \sum_{i=1}^{m} a_i = 0 \quad .$$

Say without loss of generality that $a_1 \geq a_2 \geq \cdots \geq a_m$. Pick the sequence $i_0, i_1, \cdots, i_n$ as follows: $i_0 = 1$; i_1 = smallest i such that $P_i \neq P_{i_0}$; i_2 = smallest i such that $P_i \notin$ line spanned by P_{i_0}, P_{i_1} in $\underset{\sim}{P}^n$; i_3 = smallest i such that $P_i \notin$ plane spanned by $P_{i_0}, P_{i_1}, P_{i_2}$ in $\underset{\sim}{P}^n$; etc. It follows from the hypothesis $P \in (\underset{\sim}{P}^n)_0^m$ that $i_n - 1 < (m/n+1)$(since $P_1, P_2, P_3, \cdots, P_{i_n-1}$ all lie in a hyper-plane). Therefore, if $\mu = [m/n+1]$, $i_n \leq \mu + 1$, hence

$$\begin{aligned} 0 &\geq \mu \cdot \sum_{k=0}^{n} a_{i_k} \\ &\geq \mu \cdot a_1 + \mu \cdot n a_{i_n} \\ &\geq \mu \cdot a_1 + \mu \cdot n \cdot a_{\mu+1} \quad . \end{aligned}$$

But therefore $a_{\mu+1} \leq 0$, hence

$$\begin{aligned} \mu \cdot a_1 + \mu \cdot n a_{\mu+1} &\geq \mu \cdot a_1 + (m - \mu) a_{\mu+1} \\ &\geq \sum_{i=1} a_1 + \sum_{i=\mu+1}^{m} a_i = 0 \quad . \end{aligned}$$

Therefore all inequalities are equalities, and by the last one, $a_1 = a_2 = \cdots = a_\mu$, and $a_{\mu+1} = a_{\mu+2} = \cdots = a_m$ and by the first one, $a_{i_1} = a_{i_2} = \cdots = a_{i_n}$, and by the second, $a_{i_n} = a_{\mu+1}$. Consequently, $a_1 = a_2 = \cdots = a_m = 0$. Therefore (a) is proven.

To prove (b), for note that for $i \neq j$ between 1 and m, there exists $i_1, \cdots, i_n$ such that

$$D_{i,i_1,\cdots,i_n}(P) \neq 0$$

$$D_{j,i_1,\cdots,i_n}(P) \neq 0 \quad .$$

For if this were not so, one sees easily that there would be two hyperplanes, H_1 through P_i, and H_2 through P_j such that every point P_k was on H_1 or H_2. But $< m/2$ points (P_i) are on any one hyperplane, so this is impossible. Then set

$$\Pi^{(i,j)} = D_{i,i_1,\cdots,i_n} \cdot (D_{j,i_1,\cdots,i_n})^{N_0-1}$$

$$\cdot \, \Pi^{(i_1)}, \cdots, \Pi^{(i_n)} \cdot \Pi^{(j)} \quad .$$

Q. E. D.

Before proving Theorem 3, let me make precise the meaning of principle fibre bundle:

DEFINITION. X is said to be a *principal fibre bundle* over Y, with group G, where X, Y, and G are schemes over S, if (a) there is an S-operation $G \underset{S}{\times} X \xrightarrow{\sigma} X$ of G on X, (b) there is a S-morphism $\phi : X \longrightarrow Y$, such that

(1)
$$\begin{tikzcd}
G \underset{S}{\times} X \arrow[r, "\sigma"] \arrow[d, "\pi_2"] & X \arrow[d, "\phi"] \\
X \arrow[r, "\phi"'] & Y
\end{tikzcd}$$
commutes.

(2) There is an open covering $\{U_i\}$ of Y, and S-morphisms $s_i : U_i \longrightarrow X$ such that $\phi \circ s_i$ = identity, and $G \underset{S}{\times} U_i \xrightarrow{f_i} \phi^{-1}(U_i)$ given by $G \underset{S}{\times} U_i \xrightarrow{1_G \times s_i} G \underset{S}{\times} \phi^{-1}(U_i) \xrightarrow{\sigma} \phi^{-1}(U_i)$ is an isomorphism.

THEOREM 3. $(\underset{\sim}{P}^n)_0^m$ *is a principal fibre bundle over a quasi-projective normal model* Q, *over* $\underset{\sim}{Z}$, *with group* PGL(n).

Proof: Let $S = \underset{\sim}{Z}[\cdots, X_j^{(i)}, \cdots]$, $R \subset S$ the sub-ring generated by monomials of equal degree in each set of variables $X_*^{(i)}$, i.e. generated by $X_{i_1}^{(1)}, X_{i_2}^{(2)}, \cdots, X_{i_m}^{(m)}$. Let

$S_0 = \underset{\sim}{Z}[\cdots, D_{i_0,\ldots,i_n}, \cdots]$, and $R_0 = S_0 \cap R$.

(A) R_0 is finitely generated.

Proof of (A): Notice that R_0 is generated by monomials in the D's which are of equal degree in each set of variables $X_*^{(i)}$. If we list the D's as $D_1, D_2, \cdots, D_N$, then R_0 is generated by the monomials $D_1^{r_1} \cdot D_2^{r_2}, \cdots, D_N^{r_N}$ such that

$$\sum_{\substack{X_*^{(1)} \text{ occurs} \\ \text{in } D_i}} r_i = \sum_{\substack{X_*^{(2)} \text{ occurs} \\ \text{in } D_i}} r_i = \cdots = \sum_{\substack{X_*^{(m)} \text{ occurs} \\ \text{in } D_i}} r_i \quad .$$

But Gordon ([4], p. 199) has proved the lemma:

LEMMA 1. Given a finite set of homogeneous linear equations with integral coefficients, there is a finite set of positive integral solutions such that every other positive integral solution is a positive integral combination of the given solutions.

This implies that there is a finite set of such monomials so that every other is a monomial in these monomials. Q. E. D.

Recall the lemma, [15]:

LEMMA 2. $R = \sum_{n=0}^{\infty} R_n$ a graded ring, finitely generated over R_0. There exists an N such that $R(N) = \sum_{n=0}^{\infty} R_{nN}$ is finitely generated by elements of $R(N)_1 = R_N$ over R_0.

Apply this lemma to R_0, and consider the inclusion of graded rings $R_0(N) \subset R(N)$. Now R is the homogeneous coordinate ring of the well-known Segre model for $(\underset{\sim}{P}^n)^m$. Therefore Proj $R(N) \simeq (\underset{\sim}{P}^n)^m$ too. Let $\overline{Q}$ = Proj $R_0(N)$, a projective model over Z. The inclusion $R_0(N) \subset R(N)$ induces a rational map $\overline{\Psi} : (\underset{\sim}{P}^n)^m \longrightarrow \overline{Q}$.

(B) $\overline{\Psi}$ restricted to $(\underset{\sim}{P}^n)^m_0$ is a morphism.

Proof of (B): As usual, this is equivalent to saying, if P is a geometric point of $(\underset{\sim}{P}^n)^m_0$, then there exists an $\alpha \in R_0(N)_1$ such that $\alpha(P) \neq 0$. But by part (a) of Proposition 6, $\prod^{(1)}, \cdots, \prod^{(m)}$ is a homogeneous element of $R_0(N)$, not vanishing at P. Since $R_0(N)$ is generated over $\underset{\sim}{Z}$ by $R_0(N)_1$, some α as required must exist. Q.E.D.

Let $\Psi = \overline{\Psi} \restriction (\underset{\sim}{P}^n)^m_0$. It is immediate by the elementary properties of determinants that

$$(16)\qquad \begin{tikzcd} PGL(n) \times (\underset{\sim}{P}^n)^m_0 \arrow[r, "\sigma"] \arrow[d, "\pi_2"] & (\underset{\sim}{P}^n)^m_0 \arrow[d] \\ (\underset{\sim}{P}^n)^m_0 \arrow[r, "\psi"] & \overline{Q} \end{tikzcd}$$

commutes.

Now let a geometric point $P \in (\underset{\sim}{P}^n)^m_0$ be fixed. We wish to show that there is an open $U \subset \overline{Q}$, $\psi(P) \in U$, and a morphism $S : U \longrightarrow (\underset{\sim}{P}^n)^m_0$ such that:

(i) $\psi \circ S$ = identity

(ii) $U \times PGL(n) \xrightarrow{f} \psi^{-1}(U)$ given by $\sigma \circ (S \times \text{identity})$ is an isomorphism.

When this is proven, it follows that $\psi((\underset{\sim}{P}^n)^m_0) = Q$ is open in $\overline{Q}$, hence Q is a model over $\underset{\sim}{Z}$, over which $(\underset{\sim}{P}^n)^m_0$ is a principal fibre bundle. It follows immediately that Q is the quotient of $(\underset{\sim}{P}^n)^m_0$ by $PGL(n)$, hence is normal by Theorem 2. Therefore the theorem will be completely proven.

To construct S, we use a so-called Typische Darstellung. Namely, seek:

(1) $I_{k,j} \in (R_0)_{N\ell}$, $0 \leq k \leq n$, $1 \leq j \leq m$

(2) $a_j \in S_0$, $1 \leq j \leq m$, homogeneous in $X_*^{(k)}$, all k

(3) $\sigma_{i,k} \in S$, $0 \leq i, k \leq n$, homogeneous in $X_*^{(\ell)}$, all ℓ with degree independent of i, k (but possibly dependent on ℓ) ,

such that:

$$(*) \qquad a_j X_i^{(j)} = \sum_{k=0}^{n} \sigma_{i,k} I_{k,j} \quad , \qquad 0 \leq i \leq n , 1 \leq j \leq m \quad ,$$

$$(\#) \qquad a_j(P) \neq 0 \quad , \quad 1 \leq j \leq m \quad .$$

Their existence follows by Proposition 6, if we fix any subscript a , and also one $(n+1)$-tuple $i_0, \cdots, i_n$ such that $D_{i_0,\cdots,i_n}(P) \neq 0$, and then set:

$$(1') \qquad I_{k,j} = (D_{i_0,i_1,\cdots,i_n})^{N_0-1} \cdot D_{i_0,\cdots,\widehat{i_k},j,\cdots,i_n} \cdot \Pi^{(a,j)}$$

$$\cdot \Pi^{(i_0,a)}, \cdots, \Pi^{(i_n,a)} \cdot \Pi^{(i_0)}, \cdots, \Pi^{(i_n)}$$

$$(2') \qquad \alpha_j = (D_{i_0,i_1,\dots,i_n})^{N_0}\cdot \prod{}^{(\alpha,j)} \cdot \prod{}^{(i_0,\alpha)} ,$$

$$\dots \cdot \prod{}^{(i_n,\alpha)} \cdot \prod{}^{(i_0)} , \dots , \prod{}^{(i_n)} .$$

$$(3') \qquad \sigma_{i,k} = \prod{}^{(i_0,\alpha)} , \dots , \widehat{\prod{}^{(i_k,\alpha)}} , \dots , \prod{}^{(i_n,\alpha)} \cdot X_i^{(i_k)} .$$

Then (*) follows by means of the standard identity:

$$(*') \qquad D_{i_0,\dots,i_n} X_i^{(j)} = \sum_{k=0}^{n} D_{i_0,\dots,\widehat{i_k},j,\dots,i_n} \cdot X_i^{(i_k)} .$$

What does this apparently complicated formula mean? First of all, $\{\sigma_{i,j}\}$ define a rational map $\phi : (\underset{\sim}{P}^n)^m \longrightarrow PGL(n)$, while for each j, $\{l_{k,j}\}$ define a rational map $\overline{Q} \longrightarrow \underset{\sim}{P}^n$, hence all together define a rational map $s : \overline{Q} \longrightarrow (\underset{\sim}{P}^n)^m$. Define $U \subset Q$ to be the set of points x such that $\alpha_j(x) \neq 0$, all j. Then (*) implies that s is a morphism on U, and ϕ is a morphism on $\psi^{-1}(U)$; and that the following composed morphism is the identity:

$$\psi^{-1}(U) \xrightarrow{\phi \times (s \circ \psi)} PGL(n) \times (\underset{\sim}{P}^n)^m \xrightarrow{\sigma} (\underset{\sim}{P}^n)^m \quad .$$

Then (i) follows immediately, and to show (ii), define $g = \psi \times \phi : \psi^{-1}(U) \Longrightarrow U \times PGL(n)$. Then $f \circ g$ = identity follows from this identity. To see that $g \circ f$ = identity, note first that no m-triple $P = (P_1, \cdots, P_m) \in (\underset{\sim}{P}^n)^m_0 \times \Omega$ has any stabilizer, i.e. if $\sigma(T \times P) = P$, then $T = I$, where $T \in PGL(n) \times \Omega$. For, suppose that $D_{i_0, \cdots, i_n}(P) \neq 0$. Since $< (m/n+1)$ of the P_i lie in any one hyperplane, it follows that $< m$ of the P_i lie in one of the hyperplanes H_k, H_k spanned by $P_{i_0}, \cdots, \widehat{P_{i_k}}, \cdots, P_{i_n}$. Therefore, there is a $P_{i_{n+1}}$ such that $P_{i_0}, \cdots, P_{i_n}, P_{i_{n+1}}$ form a projective base of $\underset{\sim}{P}^n$. Consequently, if T leaves every P_i fixed, T is the identity. Therefore, it follows that $g \circ f$ is the identity on geometric points, and since $U \times PGL(n)$ is reduced, $g \circ f$ is the identity. Q. E. D.

Appendix 1. Cohomology and Base Extension

Fix a flat, proper morphism $f : X \longrightarrow Y$, and a locally free sheaf E on X. Fix $y \in Y$, and let $Xy = X \underset{Y}{\times} \kappa(y)$ be the fibre over y, $Ey = E \otimes \mathcal{O}_{Xy}$ be the sheaf induced on Xy by E. The problem is to compare $(R^q f)(E)$ with $H^q(Ey)$. This question being local on Y, assume $Y = \mathrm{Spec}\,(A)$ is affine, and assume A is noetherian. Let $U_1, \cdots, U_n$ be a finite covering of Xy by affine sets. Form the Čech cochain complex with respect to the covering $\{U_i\}$, with sections of E:

$$\underline{C} : 0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow C^3 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0 \quad .$$

This is a complex of A-modules, and

$$H^i(\underline{C}) = H^i(E, X) = \Gamma((R^i f)(E), Y)$$

since Y is affine. On the other hand, U_i induces an affine covering of Xy, for which the Čech cochain complex, with sections of Ey is:

$$\underline{C} \otimes \kappa(y) : 0 \longrightarrow C^1 \otimes \kappa(y) \longrightarrow C^2 \otimes \kappa(y) \longrightarrow \cdots \longrightarrow C^n \otimes \kappa(y) \longrightarrow 0 .$$

Therefore $H^i(\underline{C} \otimes \kappa(y)) = H^i(Ey, Xy)$. To relate $H^i(\underline{C} \otimes \kappa(y))$ $= H^i(\mathrm{tor}^0(\underline{C}, \kappa(y)))$ with $H^i(\underline{C}) \otimes \kappa(y) = \mathrm{tor}^0(H^i(\underline{C}), \kappa(y))$, we use the Spectral Sequences of Hyperhomology (see p. 146, formula (2. 4. 2) of [8]). In our case, E being locally free, and f being flat, it follows that $\underline{C}$ is flat over A, i. e. $\mathrm{tor}^i(\underline{C}, \kappa(y)) = (0)$, $i > 0$, and one of the two sequences degenerates. Consequently, noting that tor is <u>right</u> exact, we have, in the usual notation,

$$\mathrm{tor}_A^{-p}(H^q(\underline{C}), \kappa(y)) \xRightarrow[p]{} H^n(\underline{C} \otimes \kappa(y)) \quad .$$

Notice that unlike most spectral sequences, this one is in the upper <u>left</u> hand quadrant:

(1 7)

$$\begin{array}{cccc|c}
 & 0 & 0 & 0 & \\
\cdots\cdots & \mathrm{tor}^2(H^n(\underline{C}), \kappa(y)) & \mathrm{tor}^1(H^n(C), \kappa(y)) & H^n(C) \otimes \kappa(y) & \\
 & \vdots & \vdots & \vdots & 0 \\
\cdots\cdots & \mathrm{tor}^2(H^3(C), \kappa(y)) & \mathrm{tor}^1(H^3(C), \kappa(y)) & H^3(C) \otimes \kappa(y) & \\
\cdots\cdots & \mathrm{tor}^2(H^2(C), \kappa(y)) & \mathrm{tor}^1(H^1(C), \kappa(y)) & H^2(C) \otimes \kappa(y) & \\
\cdots\cdots & \mathrm{tor}^2(H^1(C), \kappa(y)) & \mathrm{tor}^1(H^1(C), \kappa(y)) & H^1(C) \otimes \kappa(y) & \\
\cdots\cdots & \mathrm{tor}^2(H^0(C), \kappa(y)) & \mathrm{tor}^1(H^0(C), \kappa(y)) & H^0(C) \otimes \kappa(y) & \\
\hline
 & & 0 & & 0
\end{array}$$

The dotted line represents a typical d_2. Consequently, we have:

$$\operatorname{tor}^{-p}_{\mathcal{O}_y}((R^q f)(E)_y, \kappa(y)) \xRightarrow[p']{} H^n(Ey, Xy) \quad .$$

From this spectral sequence, we need only a few results:

PROPOSITION 7. *If for any* q_0, $H^q(Ey, Xy) = (0)$ *for* $q > q_0$, *then* $(R^q f)(E)$ *is* (0) *near* y *for* $q > q_0$, *and* $H^{q_0}(Ey, Xy) = (R^{q_0} f)(E) \otimes \kappa(y)$. If $q_0 \neq 0$, *then it also follows that* $(R^{q_0} f)(E)$ *is locally free near* y.

Proof: Use decreasing induction on q. Say we know $(R^{q'} f)(E) = (0)$ near y for $q' > q$. Then by the spectral sequence $H^q(Ey, Ey) = (R^q f)(E) \otimes \kappa(y)$, and if $q > q_0$ also, $(R^q f)(E) \otimes \kappa(y) = (0)$, hence by Nakayama's lemma, $(R^q f)(E) = (0)$ near y. If $q_0 = 0$, the spectral sequence consists of one horizontal line, hence

$$\operatorname{tor}^{-p'}((R^0 f)(E), \kappa(y)) \simeq H^n(Ey, Xy) \quad .$$

Therefore $\operatorname{tor}^1((R^0 f)E, \kappa(y)) = (0)$, which implies $(R^0 f)(E)_y$ is a flat $\mathcal{O}_y$-module (see [7], part 4).

But, as is well known, $(R^0f)(E)_y$ is therefore a *free* $\mathcal{O}_y$-module.

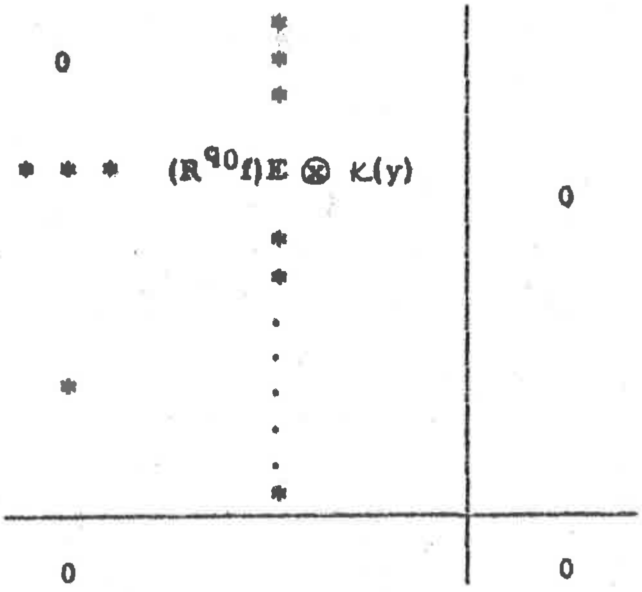
Q. E. D.

Slightly stronger is:

PROPOSITION 8. *If* Y *is reduced, and for* $q > q_0$, $\dim_{\kappa(y)} H^q(Ey, Xy)$ *is independent of* y, *then* $H^{q_0}(Ey, Xy) = (R^{q_0}f)(E) \otimes \kappa(y)$. *If* $\dim_{\kappa(y)} H^{q_0}(Ey, Xy)$ *is also independent of* y, *then* $(R^{q_0}f)(E)$ *is a locally free sheaf.*

Proof: Use decreasing induction on q_0. Say the proposition is proven for $q_0' > q_0$. Then we know $H^q(Ey, Xy)$ is locally free for $q > q_0$, hence the spectral sequence has the form:

(18)



Hence clearly $H^{q0}(Ey, Xy) \simeq (R^{q0}f)(E) \otimes \kappa(y)$. Now suppose also that $\dim_{\kappa(y)} H^{q0}(Ey, Xy)$ is independent of y. The proposition follows from:

LEMMA. $\mathcal{J}$ *a coherent sheaf on a reduced noetherian scheme* Y. *If* $\dim_{\kappa(y)} \mathcal{J} \otimes \kappa(y)$ *is independent of* $y \in Y$, *then* $\mathcal{J}$ *is a locally free sheaf.*

Proof: Fix $y \in Y$, and let $f_1, \cdots, f_m \in \mathcal{J}_y$ induce a basis of $\mathcal{J}_y \otimes \kappa(y)$. By Nakayama's lemma, $f_1, \cdots, f_m$ span $\mathcal{J}_y$ as an $\mathcal{O}_y$-module. I claim there are no relations. But suppose $\Sigma a_i f_i = 0$, $a_i \in \mathcal{O}_y$. Let $(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$ in $\mathcal{O}_y$ be the decomposition of the (0) ideal. Let K_i = quotient field of $\mathcal{O}_y / \mathfrak{p}_i$. Then by assumption, the f_i are independent in $\mathcal{J}_y \times K_i$, hence $\{a_j\}$ are all zero in $\mathcal{O}_y / \mathfrak{p}_i$. Therefore the a_i are in $\cap \mathfrak{p}_i = (0)$. Q. E. D.

Appendix 2. The Specialisation of Jacobians

Grothendieck has recently proved sweeping results on the compatibility of specialization of generalized Picard varieties, as a corollary to the existence of Picard schemes attached to broad classes of morphisms ([6]). For the case of specialization of non-singular curves over rank 1, discrete valuation rings, however, it is hardly necessary to refer to these new techniques. The work of Chow, Igusa, Hironaka, and Shimura leaves only a few details to complete, ([3], [10], [11], [12], and [19]). In fact, Igusa in his first paper on fibre systems of elliptic curves proves that the Jacobian of a non-singular curve specializes compatibly over an equi-characteristic specialization. His proof, in view of Hironaka's and Shimura's work, goes over without modification to the general case. For the sake of completeness, however, let me sketch how the problem of setting up the Jacobian scheme of any algebraic family A/X of curves is attacked taking Chow's method, but the present terminology and Grothendieck's techniques.

One first considers:

$$A(d) = \overset{d \times}{\overline{\left(A \times_X A \times_X \cdots \times_X A\right)}}_{s} = \mathrm{Hilb}^{d}_{A/X}$$

a scheme simple over X , and $\mathcal{H} = \mathrm{Hilb}_{A(d)/X}$. Then we can set up a procedure ($\mathcal{J}$) , assigning to a point $x \in A(d)$, a rational point over $\kappa(x)$ of $\mathcal{H}$, as follows: given x, there is a canonical $\kappa(x) \longrightarrow \mathrm{Hilb}^d_{A/X}$, hence a canonical $D \subset A_x$ if $A_x = A \underset{X}{\times} \kappa(x)$. D defines the line bundle $\mathcal{O}_{A_x}(D) = L_x$. Let $\underset{\sim}{P} = P(H^0(L_x, A_x))^*$, a projective space over $\kappa(x)$, (* standing for the dual projective space). Then the canonical map

$$\begin{tikzcd}
A_x \arrow[rr] \arrow[dr] & & P(H^0(L_x, A_x)) \arrow[dl] \\
& \kappa(x) &
\end{tikzcd}$$

defines a section of $\pi_1 : A_x \times \underset{\sim}{P}^* \longrightarrow A_x$, hence dually a divisor $H \subset A_x \times \underset{\sim}{P}$. H , in turn, defines

$$\underset{\sim}{P} \longrightarrow \mathrm{Hilb}^d_{A_x/\kappa(x)} \overset{\sim}{\longrightarrow} \left(\mathrm{Hilb}^d_{A/X}\right) \underset{X}{\times} \kappa(x) \quad .$$

One can prove (a) H flat over $\underset{\sim}{P}$, (b) $\underset{\sim}{P} \longrightarrow \mathrm{Hilb}^d_{A_x/\kappa(x)}$ an immersion. Then $\underset{\sim}{P} \subset A(d) \underset{X}{\times} \kappa(x)$ defines $\kappa(x) \longrightarrow \mathcal{H}$, the sought-for rational point.

Then, if X is integral, we can define the Jacobian scheme J/X of A/X , as the closure of the image of the rational point corresponding to the generic point of $A(d)$. An analog of Proposition 5 can then be proven, by step-by-step tracing the compatibility of $(\mathcal{g})$ with specialisation. Finally, picking $d = (2g - 2) \cdot N$, a relative group structure is set up in the usual way, using as identity the following morphism $X \longrightarrow J$: Let $\underset{\sim}{P} = P((R^0\pi)((\Omega^1_{A/X})^N))^*$, where $\pi : A \longrightarrow X$ and $*$ stands for taking the dual projective spaces. Then the usual

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & P\left((R^0\pi)\left(\left(\Omega^1_{A/X}\right)^N\right)\right) \arrow[dl] \\ & X & \end{tikzcd}$$

defines dually, as in $(\mathcal{g})$, a divisor

$$H \subset P \underset{X}{\times} A \quad ,$$

hence a map (which is a morphism), $\underset{\sim}{P} \longrightarrow \mathrm{Hilb}_{A/X}$, which is, in fact, an immersion $\underset{\sim}{P} \subset A(d)$. This defines $X \longrightarrow \mathrm{Hilb}_{A(d)/X}$, which factors through J and defines the identity.

Finally, let us note how the relative Θ'-divisor is seen to exist. Whether the reader prefers Grothendieck's construction, Igusa's construction, or the one just outlined, there is in all cases a relative group scheme J/X (or $J/\mathrm{Spec}\, R$, if the reader prefers to look at that case), <u>and</u> an X-morphism

$$\phi_N : A(N\cdot(2g-2)) \longrightarrow J \quad .$$

This last comes from Chow's method, and also, if Grothendieck's approach is followed, from this alternative consideration: J is canonically the identity component of the larger relative group scheme classifying divisor classes of all degrees. If J_n/X is the component classifying divisor classes of degree n , then

i) $J \simeq J_0$

ii) there is a canonical X-morphism

$$\phi : A \longrightarrow J_1$$

iii) there is a canonical section

$$K : X \longrightarrow J_{2g-2} \quad .$$

Here ϕ is, of course, the Albanese morphism, and K comes, as usual, from the canonical line bundle $\Omega^1_{A/K}$. In any case, these define

I) $\phi^{(i)} : A(i) \longrightarrow J_i$ by factoring through $A(i)$, the composite

$$\overbrace{A \underset{X}{\times} \cdots \underset{X}{\times} A}^{ix} \xrightarrow{\phi \times \cdots \times \phi} \overbrace{J_1 \underset{X}{\times} \cdots \underset{X}{\times} J_1}^{ix} \xrightarrow{\text{addition}} J_i \quad .$$

II) $K^{(N)} : J_{N(2g-2)} \xrightarrow{\simeq} J_0$, since the existence of a section of the principal homogeneous space J_{2g-2} over J_0 shows that it, and its multiples, are trivial, (as principal homogeneous spaces).

III) $\phi_N = K^{(N)} \circ \phi^{(N\cdot(2g-2))} : A(N\cdot(2g-2)) \longrightarrow J_0 \simeq J$.

In any case, assuming this last morphism defined, the Θ'-divisor can also be easily defined. Passing to an open subset of X if necessary, there will be some rational $(N - 1)$-times pluri-canonical divisor. Let it be represented by the section

$$K' : X \longrightarrow A((N - 1) \cdot (2g - 2)) \quad .$$

Consider the composite:

$$\overset{(g-1)X}{\overline{A \underset{X}{\times} A \underset{X}{\times} \cdots \underset{X}{\times} A}} \xrightarrow{K' \times \text{diagonal}} A((N-1)(2g-2)) \overset{(2g-2)X}{\overline{\underset{X}{\times} A \underset{X}{\times} \cdots \underset{X}{\times} A}}$$

$$\longrightarrow A(N \cdot (2g-2)) \xrightarrow{\phi_N} J \quad .$$

Then Θ' is the reduced sub-scheme with the image of this morphism as support.

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