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How to get middle school students to love a formula

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I believe there is a way to present algebra to middle schoolers that breaks the log jam of 'what the hell are *x* and *y*?' To make middle and high school math work it is essential to get students (or most of them anyway) to see how formulas are useful and intuitive ways to see how numbers in their real lives are connected to each other. It is pointless to drill students for three or four years in something most of them will forget as soon as they have taken their SATs. The blog below was addressed to NYTimes readers in the hope that the Times OpEd department might print it and some readers might loose some of their 'fear and loathing' of algebra. Other articles in this vein are in the education page.

In a nutshell the reason for the usefulness of algebra is this: life is full of situations where several numbers are needed to describe a situation, these numbers vary from one situation to another but in each case the numbers have a fixed arithmetical relationship to each other that doesn't vary. Writing this relationship as an equation gives you a clearer grasp of all these situations, much as having the right word in your vocabulary can help you grasp immediately new situations described by this word: in both cases, your mind learns a structure that will fit many situations in the future. An equation can be thought of as a quantitative metaphor. Those who never internalize this equation are condemned to dredge up isolated rules every time similar situations come their way.

The simplest case is that in any trip, distance travelled is the product of the time the trip takes by the speed of travel. Going by plane, 3000 miles from NYC to SF equals 6 hours times 500 miles per hour; a 2 mile walk is 40 minutes (2/3 of an hour) times a typical walking pace of 3 miles per hour. We write this:

$$d = s.t$$

using simple abbreviations for distance, speed and time. Clearly, if s and t are known, the formula tells us what d is. But algebra tells us that we can also play the game using:

$$s = d/t \text{ or } t = d/s$$

so that if we know d and t, we get out s, etc. The rules of algebra show how a numerical relationship of one kind can be used in multiple ways. Once you get the hang of thinking in terms of a formula, the formula becomes a much clearer way of describing a situation than an awkward long sentence. It becomes the natural way of grasping how numbers *fit together*. But before this happens, you need to see a lot of meaningful instances and schools, all too often, just drill the student in abstract formulas with no real world meaning.

It is in financial matters that most of us need to grasp numerical relationships more clearly and where formulas can help a lot and give us the power not to have to accept blindly everything told to us by 'experts' (who are usually salesmen). A spreadsheet is a terrific stepping-stone for some: to use these efficiently, you enter formulas into cells that calculate a new value from values in other cells. The spreadsheet is not merely a set of numbers but a whole web of numerical relationships.

A major high school topic, pretty much always taught without showing any relevance to the real world, is the theory of polynomials. This is sad because they are relevant in understanding paying off loans. Your average student wants a car and may be able to get one on time. But, for instance, if they charge a bad credit risk teen-ager 1.33% interest per month (16% APR) on a 5 year loan, he would do well to know that his total cost works out to be about 50% more for the car as he would pay if he had the cash.

His high school class can give him the confidence to "do the math" himself and not rely on others with their own agendas. The first step is to assign abbreviations to the numbers involved. Use C for the cost, P for the monthly payment, r for the rate of interest. Then one month's interest increases the loan from C to $C.(1+r)$ and one payment decreases it to $C.(1+r)-P$. Repeating this for the second month, the balance owed becomes $(C.(1+r)-p).(1+r)-p$. Seems like a mess only a math nerd would love. But use the rules of algebra and it becomes a quadratic polynomial in the number (1+r):

$$C.(1+r)^2 - P.(1+r) - P.$$

If you go on for, say 4 months, the balance owed will be this polynomial:

$$C.(1+r)^4 - P.\left((1+r)^3 + (1+r)^2 + (1+r) + 1\right).$$

We're not giving a lecture here, just hoping to show how algebra can be useful. So let's just say -- if you use the stuff taught in every Algebra II class and pursue what we have started, you'll wind up easily seeing that, if you need to pay off the loan in 5 years (60 months), your payment P will be

$$P = C.\frac{r}{1-(1+r)^{-60}}.$$

In the example above, make r = 0.0133 and work out his total cost, 60P, on a hand held calculator, and you get about 1.5 times the cost C of the car.

The formula above, though it might show up in a New Yorker cartoon with white-coated scientists, when you play with it reveals an

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essentially simple relationship between interest rates, loans and payments. The majority of real life mathematicians work on real problems like this and not on abstract stuff in ivory towers.

A nation-wide discussion, verging on a political fight, is going on right now pro and con the Common Core State Standards in Math (CCSS-M) and the involvement of the Department of Education. As we see it, the CCSS-M have considerably upped the ante in abstract math but have also opened the option of introducing 'modeling', a code word for math that might relate to the real world as students know it. All K-12 math can be enlivened and made relevant, exciting even, to students by dipping into the vast array of applications that math has to real life. Our message: math, properly taught, need not turn you off.

For a long time, I thought algebra was a natural language for everyone who had had a decent middle or high school math teacher and I thought that probably around half of all high school graduates had the gist of it down. Then Dave Wright, Caroline Series and I wrote *Indra's Pearls* and I gave copies to quite a few non-mathematical friends. Essentially all of them loved the pictures but, when reading the text, got stuck on Chapter One where we tried to write a gentle introduction to the arithmetic of complex numbers. I began to understand why no general circulation magazine or newspaper will ever print a formula.

Where did math class loose this large group of students? An image that has gone viral shows a blackboard on which a student has written "Dear Algebra, Please stop asking us to find your X. She's never coming back and don't ask Y." Sigh.

Comments:

Nov.2: In response to this post, William McCallum, Director of the Institute for Mathematics and Education at the University of Arizona and a major author of the Common Core State Standards in Mathematics, wrote to me:

> Very nice blog post ... you describe where we want to arrive in mathematics education very well with the idea of an equation as a "quantitative metaphor," a "natural way of grasping how numbers fit together." The question is how to get there.
>
> An important part of the problem that interests me (by no means the entire solution) is grasping the complex relationship between fluency and conceptual understanding. One of the great divides in mathematics education is between those who pay lip service to one or the other of these. Those of us who love algebra bring to it a certain native fluency that we are sometimes not even aware of; it frees us to look at complex algebraic expressions with confidence and equanimity. What do we do with students who don't have that confidence? Some dry practice is necessary, but it should be practice in simple things deeply

understood, like the relationship between the three equations resulting from $d = rt$.

This starts in elementary school, with the relationship between multiplication and division facts. A first step would be a truce between the warring camps there: yes, kids need to know their facts cold, but they don't need to memorize every single one as a separate fact: if you know 3 × 5 = 15, then you know 15 ÷ 5 = 3 and 15 ÷ 3 = 5, not to mention 5 × 3 = 15.

Arithmetic is a great seed bed for algebra. This was brought home to me forcefully when I watched one of the Tea Party anti Common Core videos complaining about the complicated way an elementary classroom was dealing with 9 + 6 = 15. The teacher decomposed the 6 as 1 + 5, then put the 1 with the 9 to make a 10, then added the 5 to get 15. Why don't they just do it the old-fashioned way? scoffed the narrator: just memorize the facts. No sense of irony here: the narrator clearly had no idea that if you understand why 9 + 6 = 15, then you understand why 19 + 6 = 25, 29 + 6 = 35, etc. And also why 8+6 = 14, 7 + 6 = 13. It struck me like a lightning bolt that the narrator actually *did* memorize all these as separate facts. But if you start kids out with a flexible understanding of arithmetic, then they are more likely to appreciate your formula for the total amount paid on a loan (by the way, you ignored the time value of money there, but never mind).

On the other side, we have people

> expressing horror at the very idea of memorizing anything at all. And yet, which one of us has not experienced the pleasure of memorizing a poem, a formula, how a spectral sequence works? Why deny kids that pleasure?
>
> Procedural fluency is very much informed by understanding (as in the 9+6 example), and understanding is very much informed by procedural fluency (as in the ability to see $r = t/d$ right away from $d = rt$); they are deeply intertwined in ways I have never quite figured out how to articulate.

I agree Bill, procedural fluency (sometimes through memorization) and understanding have to grow side by side. The part that I feel is often overlooked, though, is that using letters for numbers is a strange and challenging idea for many and that concrete examples using abbreviations are a natural stepping stone to the doubly abstract x and y. It would seem natural to me to introduce very simple formulas with abbreviations at least a year before x and y. I hope there will be good sources for concrete examples available to teachers. Zalman Usiskin sent me a preprint of a paper of his forthcoming in *Mathematics Teaching in the Middle School*

Nov.3: I also received an extended reply from Al Cuoco, a major author and advisor at the Educational Development Corporation:

> I agree that the post is a very nice

description of why equations (and functions and expressions) are so useful. And I can't stress enough how much I agree with "The question is how to get there."

How to get there is not easy. As Bill's examples show, these mathematical ways of thinking start early in arithmetic and require nurturing throughout middle and high school.

Your example of monthly payments on a loan is a perfect example of just how subtle the ideas are and how they need to be built up over time. You claim that "The first step is to assign abbreviations to the numbers involved." My colleagues at EDC and I have used this example for decades in our own CME high school curriculum in our high school teaching before that (I know that you're familiar with all this history), and the step of writing down the relationships in precise algebraic language is somewhere near the midpoint of a long development that is preceded by carefully orchestrated numerical calculations, an introduction to functions and recursively defined functions, and experiments with a spreadsheet and later with a CAS. Once the basic algebraic relationships are in place, there are a host of other sophisticated ideas that need to be in place before one can get the closed form for the monthly payment.

A more detailed description of how all this might be developed is in chapter 2 of the NCTM monograph "Reasoning and Sense

Making in Algebra." The PARCC Content Frameworks uses the example to describe this habit of using precise language as an (essential, in my experience) intermediate step between numerical examples and algebraic generalization:

"Capturing a situation with precise language can be a critical step toward modeling that situation mathematically. For example, when investigating loan payments, if students can articulate something like, ?What you owe at the end of a month is what you owed at the start of the month, plus 1/12 of the yearly interest on that amount, minus the monthly payment,? they are well along a path that will let them construct a recursively defined function for calculating loan payments.'"

Our curriculum develops the idea over three courses, introducing refinements as the kids develop more sophisticated tools. I took some time this morning to cull out the relevant lessons, attached here.

In elementary algebra, students learn to define functions recursively and then to experiment with the monthly payment context using a spreadsheet.

In advanced algebra, they build a recurrence to calculate the balance on a loan, month by month, and they analyze the computational complexity of the resulting function (modeled in a CAS), using simple algebra to make it more efficient.

> Later, they learn several techniques for resolving recurrences of different types and hence derive a formula for the monthly payment on a loan (using, for example, geometric series or affine transformations). A real source of excitement here is that they finally get to understand some things they noticed empirically before, like the fact that, interest and term held constant, the monthly payment is a linear function of the cost of the car.
>
> So, many of us agree with you that this kind of investigation is exciting to many students, at many levels of sophistication. I've used the monthly payment investigation with high school juniors and seniors who got by their previous courses by the skins of their teeth.
>
> But I'm convinced that it's not the context that's the source of excitement. It's the challenge, the ability to see into the structure of a problem, and the chance to use mathematics to exploit that structure. I've seen HS kids get just as excited trying to figure out which integers can be written as a sum of two squares.

Although I agree with much that Al says, I diverge from him where he says "the step of writing down the relationships in precise algebraic language is somewhere near the midpoint of a long development that is preceded by carefully orchestrated numerical calculations, an introduction to functions and recursively defined functions and experiments with a spreadsheet and later with a CAS (?)". Numerical calculations and spreadsheets sure but why teach the general concept of function and recursion first? This seems to me the point of view of a pure mathematician -- that you cannot understand an idea until you have a general definition for it. I would put it backwards: you cannot understand the general idea of a recursive function until you have seen some motivating examples. After working with numbers in spreadsheets, a recursive formula with abbreviations is not a big step. I want to stick to my guns: show *real* examples first, trusting that the concrete context allows the teacher to explain easily the arithmetic in the formula. After enough examples are seen, then one might introduce general functions and general recursion rules. A confession: this is how my mind works and Al's approach is a stumbling block for me reading many math books.

Here is the attachment he sent me: CME Pages, three sections from three books I guess, leading up to the formula for the monthly payment in my blog. It is hard to judge the pedagogy from such fragments, BUT If you have a look at the very beginning, note that the general idea of a function, using the letter *f*, has been introduced earlier and that an unmotivated linear recursive function is recalled (I hope it was motivated by approximating data). I suspect that this section is probably the first place where the student sees that a recursion is actually needed. My interest in this example was that here is a financial situation which might motivate studying polynomials.

Nov.13. I received the following from Ulf Persson, a student of mine many decades ago who teaches math at Chalmers University in Sweden and often writes me long emails. I think he typifies how most mathematicians react to discussions of curricula:

> It is true when it comes to counting, until recently at least, counting meant for most people counting money. Even uneducated people in a store usually could count in their head quickly and accurately, this is no longer the case I suspect as there is no longer the need, you only push in the figures on the keys and out it comes. I guess most people think of this as progress. When I attended elementary school there were a lot of word problems to the effect that you buy so and so many shirts at such and such a price and add a percentage or what, and with other costs you compute the profit (or as we say in Swedish 'vinst' (gain, from winning) as 'profit' is a dirty word indeed). I found it extremely boring. You saw through the structure once and after that it was just mindless repetition. Such real life connections are supposed to motivate people, or at least give to the exercises a certain meaning. I doubt whether it works at all. When it comes to say celestial mechanics, which of course can be thought of applied mathematics, it was different. The idea that you could compute the paths of planets, the flattening of the earth (which is very hard as I realized when I tried it myself, I wonder how Newton did it), the height of tides

(likewise quite difficult, the standard explanations are not quantitative) or the period of the precession, is very exciting. Here it helps that it has 'real life' applications, just as in optics, when the fact that the lines obeying certain reflection and refraction laws are supposed to be light rays not just formal laws for the behavior of certain lines, stimulate your imagination. When I was considering the problem of measuring heights of lunar mountains from their shadows cast on a perspectively distorted sphere, I began to feel the need for systematic trigonometry. It is not just a question of pure versus applied, some applications stimulate your imagination, others simply kill it.

I must say that I am a bit skeptical about stimulating the imagination of mathematically recalcitrant children by giving them something from which they may tangibly gain such as their personal financial well-being. It certainly would not have worked with me, this does not mean that it would not work with others, we are all different supposedly. On the other hand we are less unique than we vainly believe, and many of our deficiencies are shared with substantial fractions of the population, much to our consolation. If it is just the case of one single formula their response would just be, why not simply have it implemented in some application so you can just plug in the values. If they are truly made to love formulas, it involves them setting up formulas for themselves for slight variations of the problem. But to do so I suspect there must be some additional intellectual stimulation beyond finding facts relevant to their financial situation. It is like 'like meeting like'. Problems in astronomy, mechanics, optics etc very much have the same flavor as problems in mathematics making for the possibility of transform, which is the ultimate goal of education. It is also why a physical intuition is useful mathematically and the other way round. Financial accounting is something different, it does not even have to do with making money. I recall a story which was spread around in my childhood about some indifferent student who was doing very well being asked why. 'I am buying the stuff for a dollar a piece and selling it for four, and on those four percent I am doing quite well' was his reply. My point is that there has to be something else to kick in for people to love a formula. Of course I have no empirical experience to draw on, but am resorting to the cheap and flexible and thus time-honored method of introspection.

How did I learn to count? I remember the occasion very vividly. Unfortunately I do not remember my precise age at the time. I was north at my grandparents farm and helping, no doubt very ineptly, to put hay on some stacks along with my father. I asked or was told that trettio and trettio was sextio, and knowing beforehand for some reason that tre and tre was sex, everything fell into place, and I recall thinking to myself that things hang together. Obviously it made a deep

> impression on me, and I have often afterwards recalled the episode in my mind, with the danger of elaborating on it. As I have grown older and wiser I have been able to make more sophisticated interpretations of it. Now I think that it was then I realized that you could count not only hay stacks and cows, but numbers themselves. This is indeed a very potent idea which unfolds an entire new world, which I instinctively embraced (and became subsequently for my age quite adept at doing arithmetical operations in my head, impressing my mother of the clever method I multiplied by splitting up in factors and rearranging them, in short I fell in love with numbers). This anecdote also shows the advantage of the illustrative example rather than the general theory. It is much more instructive to be led to a flight of fancy, than to mechanically decode and reduce. A case of which we seem to be in total agreement. Learning to count you never thought of the commutative and associative laws, they were internalized. Thus mathematics is not a question of following rules, as it is often presented as.

What is clear in Ulf's story is that he is a born mathematician: bored with any example worked more than once, wanting really challenging real world problems and picking up on the idea that numbers have a life of their own not merely as numbers of cows. This is, of course, why teaching math is so frustrating -- bore one student, confuse others. Placing students in different tracks is great but not always possible. My emphasis on finance and accounting is due to this being a topic that most kids from 7th grade on *really want to master*. His friend who profited 4% (or 300%) certainly illustrates this. Of course you need to present multiple variations of each class of problems and teaching that you can transform a formula from one setting to another is a central goal but not an easy one for the weaker students.