David Mumford

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Professor Emeritus
Brown and Harvard Universities
David_Mumford@brown.edu

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Can one explain schemes to biologists

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John Tate and I were asked by Nature magazine to write an obituary for Alexander Grothendieck. Now he is a hero of mine, the person that I met most deserving of the adjective "genius". I got to know him when he visited Harvard and John, Shurik (as he was known) and I ran a seminar on "Existence theorems". His devotion to math, his disdain for formality and convention, his openness and what John and others call his naiveté struck a chord with me.

So John and I agreed and wrote the obituary below. Since the readership of Nature were more or less entirely made up of non-mathematicians, it seemed as though our challenge was to try to make some key parts of Grothendieck's work accessible to such an audience. Obviously the very definition of a scheme is central to nearly all his work, and we also wanted to say something genuine about categories and cohomology. Here's what we

came up with:

Alexander Grothendieck
David Mumford and John Tate

Although mathematics became more and more abstract and general throughout the 20th century, it was Alexander Grothendieck who was the greatest master of this trend. His unique skill was to eliminate all unnecessary hypotheses and burrow into an area so deeply that its inner patterns on the most abstract level revealed themselves -- and then, like a magician, show how the solution of old problems fell out in straightforward ways now that their real nature had been revealed. His strength and intensity were legendary. He worked long hours, transforming totally the field of algebraic geometry and its connections with algebraic number theory. He was considered by many the greatest mathematician of the 20th century.

Grothendieck was born in Berlin on March 28, 1928 to an anarchist, politically activist couple -- a Russian Jewish father, Alexander Shapiro, and a German Protestant mother Johanna (Hanka) Grothendieck, and had a turbulent childhood in Germany and France, evading the holocaust in the French village of Le Chambon, known for protecting refugees. It was here in the midst of the war, at the (secondary school) Collège Cévenol, that he seems to have first developed his fascination for mathematics. He lived as an adult in France but remained stateless (on a "Nansen passport") his whole life, doing most of his revolutionary work in the period 1956 - 1970, at the Institut des

Hautes Études Scientifique (IHES) in a suburb of Paris after it was founded in 1958. He received the Fields Medal in 1966.

His first work, stimulated by Laurent Schwartz and Jean Dieudonné, added major ideas to the theory of function spaces, but he came into his own when he took up algebraic geometry. This is the field where one studies the locus of solutions of sets of polynomial equations by combining the algebraic properties of the rings of polynomials with the geometric properties of this locus, known as a *variety*. Traditionally, this had meant complex solutions of polynomials with complex coefficients but just prior to Grothendieck's work, Andre Weil and Oscar Zariski had realized that much more scope and insight was gained by considering solutions and polynomials over arbitrary fields, e.g. finite fields or algebraic number fields.

The proper foundations of the enlarged view of algebraic geometry were, however, unclear and this is how Grothendieck made his first, hugely significant, innovation: he invented a class of geometric structures generalizing varieties that he called *schemes*. In simplest terms, he proposed attaching to *any* commutative ring (any set of things for which addition, subtraction and a commutative multiplication are defined, like the set of integers, or the set of polynomials in variables *x,y,z* with complex number coefficients) a geometric object, called the *Spec* of the ring (short for spectrum) or an affine scheme, and patching or gluing together these objects to form the scheme. The ring is to be thought of as the set of functions on its affine scheme.

To illustrate how revolutionary this was, a ring can be formed by starting with a field, say the field of real numbers, and adjoining a quantity ε satisfying $\varepsilon^2 = 0$. Think of ε this way: your instruments might allow you to measure a small number such as $\varepsilon = 0.001$ but then $\varepsilon^2 = 0.000001$ might be too small to measure, so there's no harm if we set it equal to zero. The numbers in this ring are $a + b \cdot \varepsilon$ with real a,b. The geometric object to which this ring corresponds is an infinitesimal vector, a point which can move infinitesimally but to second order only. In effect, he is going back to Leibniz and making infinitesimals into actual objects that can be manipulated. A related idea has recently been used in physics, for superstrings. To connect schemes to number theory, one takes the ring of integers. The corresponding Spec has one point for each prime, at which functions have values in the finite field of integers mod p and one classical point where functions have rational number values and that is 'fatter', having all the others in its closure. Once the machinery became familiar, very few doubted that he had found the right framework for algebraic geometry and it is now universally accepted.

Going further in abstraction, Grothendieck used the web of associated maps -- called morphisms -- from a variable scheme to a fixed one to describe schemes as *functors* and noted that many functors that were not obviously schemes at all arose in algebraic geometry. This is similar in science to having many experiments measuring some object from which the unknown real thing is pieced together or even finding something unexpected from its influence on known things. He applied

this to construct new schemes, leading to new types of objects called *stacks* whose functors were precisely characterized later by Michael Artin.

His best known work is his attack on the geometry of schemes and varieties by finding ways to compute their most important topological invariant, their cohomology. A simple example is the topology of a plane minus its origin. Using complex coordinates (z, w), a plane has four real dimensions and taking out a point, what's left is topologically a three dimensional sphere. Following the inspired suggestions of Grothendieck, Artin was able to show how with algebra alone that a suitably defined third cohomology group of this space has one generator, that is the sphere lives algebraically too. Together they developed what is called *étale cohomology* at a famous IHES seminar. Grothendieck went on to solve various deep conjectures of Weil, develop *crystalline cohomology* and a meta-theory of cohomologies called *motives* with a brilliant group of collaborators whom he drew in at this time.

In 1969, for reasons not entirely clear to anyone, he left the IHES where he had done all this work and plunged into an ecological/political campaign that he called *Survivre*. With a breathtakingly naive spririt (that had served him well doing math) he believed he could start a movement that would change the world. But when he saw this was not succeeding, he returned to math, teaching at the University of Montpellier. There he formulated remarkable visions of yet deeper structures connecting algebra and geometry,

e.g. the symmetry group of the set of all algebraic numbers (known as its Galois group Gal ($\overline{\mathbb{Q}}/\mathbb{Q}$)) and graphs drawn on compact surfaces that he called 'dessin d'enfants'. Despite his writing thousand page treatises on this, still unpublished, his research program was only meagerly funded by the CNRS (Centre Nationale de Recherche Scientifique) and he accused the math world of being totally corrupt. For the last two decades of his life he broke with the whole world and sought total solitude in the small village of Lasserre in the foothills of the Pyrenees. Here he lived alone in his own mental and spiritual world, writing remarkable self-analytic works. He died nearby on Nov. 13, 2014.

As a friend, Grothendieck could be very warm, yet the nightmares of his childhood had left him a very complex person. He was unique in almost every way. His intensity and naivety enabled him to recast the foundations of large parts of 21st century math using unique insights that still amaze today. The power and beauty of Grothendieck's work on schemes, functors, cohomology, etc. is such that these concepts have come to be the basis of much of math today. The dreams of his later work still stand as challenges to his successors.

The sad thing is that this was rejected as much too technical for their readership. Their editor wrote me that 'higher degree polynomials', 'infinitesimal vectors' and 'complex space' (even complex numbers) were things at least half their readership had never come across. The gap between the world I have lived in and that *even of scientists* has never seemed larger.

I am prepared for lawyers and business people to say they hated math and not to remember any math beyond arithmetic, but this!? Nature is read only by people belonging to the acronym 'STEM' (= Science, Technology, Engineering and Mathematics) and in the Common Core Standards, all such people are expected to learn a hell of a lot of math. Very depressing.

Added on Dec. 28

Well, Nature magazine really wanted to publish *some* obit on Grothendieck and wore us out until we agreed with a severely stripped down re-edit. The obit is coming out, I believe, in the Jan.15 issue, and copyright prevents me from putting it here. The whole issue of trying to bridge the gap between the mathematician's world and that of other scientists or that of lay people is a serious one and I believe mathematicians could try harder to find bridges. An example is Gower's work on bases in Banach spaces: when he received the Fields Medal, no one to my knowledge used the example of musical notes to explain Fourier series and thus bases of function spaces to the general public.

In the case of our obit, I had hoped that the inclusion of the unit 3-sphere in $\mathbb{C}^2 - (0,0)$ would be fairly clear to most scientists and so could be used to explain the Mike Artin's breakthrough that $H^3_{etale}(\mathbb{A}^2 - (0,0)) \neq (0)$. No: excised by Nature. I had hoped that the "web of maps" was an excellent metaphor for the functor represented by an object in a category and gave one the gist. No: excised by

Nature. I had hoped that the "symmetry group of the set of all algebraic numbers" might pass muster to define this Galois group. No: excised by Nature. To be fair, they did need to cut down the length and they didn't want to omit the personal details.

The essential minimum I thought for a Grothendieck obit was to make some attempt to explain schemes and say something about cohomology. To be honest, the central stumbling block for explaining schemes was the word "ring". If you haven't taken an intro to abstract algebra, where to begin? The final draft settled on mentioning in passing three examples -- polynomials (leaving out the frightening phrase "higher degree"), the dual numbers and finite fields. We batted about Spec of the dual numbers until something approaching an honest description came out, using "very small" and "infinitesimal distance". As for finite fields, in spite of John's discomfort, I thought the numbers on a clock made a decent first exposure. OK, $\mathbb{Z}/12\mathbb{Z}$ is not a field but what faster way to introduce finite rings than saying "a type of number that are added like the hours on a clock -- 7 hours after 9 o'clock is not 16 o'clock, but 4 o'clock"? We then describe characteristic p as a "discrete" world, in contrast to the characteristic 0 classical/continuous world. In another direction, we also added the clause "inspired by the ideas of the French mathematician Jean-Pierre Serre", an acknowledgement of their extraordinary collaboration.

The whole thing is a compromise and I don't want to say Nature is foolish or stupid not to

allow more math. The real problem is that such a huge and painful gap has opened up between mathematicians and the rest of the world. I think that Middle and High School math curricula are one large cause of this. If math was introduced as connected to the rest of the world instead of being an isolated exercise, if it was shown to connect to money, to measuring the real world, to physics, chemistry and biology, to optimizing decisions and to writing computer code, fewer students would be turned off. In fact, why not drop separate High School math classes and teach the math as needed in science, civics and business classes? If you think about it, I think you'll agree that this is not such a crazy idea.

Comments

Dec.31: My old friend at UCLA, David Gieseker, wrote to me about what is happening there:

> We've been having a lot of trouble with scientists, in particular life scientists. They are teaching calculus by radically dumbing it down. E.g. no trig, a half page on the chain rule, ... and very weak exams. This is being pushed by the Dean of LS, ostensibly so that math phobic students are not turned off science. The people in charge seem to be ecologists and they don't believe in any math that's not what they use. I suspect these students will be in real trouble when they take physics. I also suspect the readers of Nature think they know all important math and get upset if it's hinted that there's important math they haven't even heard of.

A sad story. How much math do biologists need? I would argue first of all that oscillations are central part of *every* science plus engineering/economics/business (arguably excluding computer science) and one needs the basic tools for describing them -- sines and cosines, all of trig of course, Euler's formula $e^{ix} = \cos(x) + i.\sin(x)$ and especially Fourier series. And, of course, modeling a system by the path of a state vector in some $\mathbb{R}^n$, often with a PDE, is also ubiquitous. For example, surely all ecologists have studied the Lotka-Volterra equation (wolf and rabbit population cycles). Algebra is more of a mixed bag. Splines are much more useful than polynomials for engineers, finite fields arise mostly in coding applications and I doubt that the abstract idea of a *ring* is ever needed. But polynomials *and* varieties have been used in Sturmfels' algebraic statistics and, as Lior Pachter noted

(see below), is very effectively used in modeling genome mutation. But evolutionary genomics is one community within biology and John and I figured we needed to throw into the obit a rough definition of a ring.

Jan.2: I received email from Steven Salzberg about the challenge of bridging the gap between math and biology, including a link to a fascinating blog on this gap by Lior Pachter. Pachter details how varieties arise as sets of probabilities consistent with a class of models, an application I was only dimly aware of when writing the obit with John Tate. He then elaborates at length on the many ways in which the culture of mathematicians and of biologists differ, cultures that he straddles at UC Berkeley. As he goes on to say, "The extent to which the two cultures have drifted apart is astonishing" and worse, both sides seem happy to ignore each other. To illustrate this, he cites another side to the situation at UCLA mentioned by Gieseker -- that the math dept is not one of 15 partner departments to UCLA's new "Institute for Quantitative and Computational Biosciences". This split is to their joint detriment and as he says:

> The laundry list of differences between biology and math that I aired above can be overwhelming. Real contact between the subjects will be difficult to foster, and it should be acknowledged that it is neither necessary nor sufficient for the science to progress. But wouldn't it be better if mathematicians proved they are serious about biology and biologists truly

experimented with mathematics?

Jan.3: I received the following in an email from Eric Zaslow in Northeastern University:

David Mumford asks on his blog, "Can one explain schemes to biologists?" in the context of his and John Tate's obituary for Grothendieck being prepared for publication in Nature. He offers a first draft obit which was rejected as too technical, along with a lament about the chasm between math and other scientific fields. Their draft introduces Grothendieck's field of algebraic geometry as follows: "This is the field where one studies the locus of solutions of sets of polynomial equations by combining the algebraic properties of the rings of polynomials with the geometric properties of this locus, known as a variety."

I find it surprising how someone who has worked at the interface of mathematics, applied math and biology for so long was surprised at Nature's reception. Of course this is too technical!

So, I wanted to try to take up Mumford's challenge.

Here's my first draft. Comments welcome.

Algebraic geometry is about solving equations. Not fancy equations involving trigonometric functions and exponentials, but ordinary, garden-variety equations

involving x, x^2, x^3 and so on. Here's one:
$3x^2 + 4x = 5x^3 + 6$. Moving everything
to the left side, we can write this as
$-6 + 4x + 3x^2 - 5x^3 = 0$. The thing on
the left is a polynomial, that is a sum of
terms, each one a multiple of some pure
power of x. Let us call the polynomial $f(x)$.
So, deciding that we'll move everything to
the left side, we study equations like
$f(x) = 0$. We can also study polynomials
in two variables, such as
$g(x, y) = x^2 - y^2$. In this case, the
equation $g(x, y) = 0$ can be solved by
factoring: $x^2 - y^2 = 0$ means
$(x + y)(x - y) = 0$, so the solutions are
either $y = x$ or $y = -x$. That is, any point
that lies on either the line $y = x$ or $y = -x$
(or both) gives a solution: indeed the point
(4,-4) is solution since $4^2 - (-4)^2 = 0$.
The set of solutions looks like a giant,
infinite X shape. Some polynomials
cannot be factored. For example, if we put
$h(x, y) = x^2 + y^2 - 2$ then $h(x, y) = 0$
means $x^2 + y^2 = 2$, and the set of
solutions looks like a circle. Note that the
giant, infinite X had two pieces,
corresponding to the factors of $f(x, y)$,
while the circle has one piece,
corresponding to the fact that we could
not factor $h(x, y)$. We could also consider
solving more than one equation
simultaneously. For example, if we try to
solve $g(x, y) = 0$ and $h(x, y) = 0$, this
means that we need to find a point that
both lies on the giant, infinite X and the
circle $h(x, y) = 0$. In total, there are four
such points: (1,1), (1,-1), (-1,1), and (-1,-1).
So algebraic geometry tries to characterize

what the set of solutions to some number
of polynomial equations looks like.

Polynomials are interesting mathematical
objects because, like numbers, you can
multiply them. Think of a polynomial j
with one variable, say, as a rule which
transforms a number x into a new number
$j(x)$. Then given two polynomials, j and k,
we can define the product jk by the rule
which transforms x into the number
$j(x)k(x)$, i.e. the product of $j(x)$ and $k(x)$.
In mathematical language, we say that
they form a ring. You can also add them:
$j + k$ transforms x into the sum of $j(x)$ and
$k(x)$. And like with numbers, you get
distributivity and other nice properties. In
fact, above we found that certain
polynomials could be factored, while
others could not. This is analogous to the
fact that certain whole numbers can be
factored, e.g. $6 = 2$x3, while other "prime"
numbers such as 5 cannot. This algebraic
property (being prime or composite) is
reflected in the geometry of the solution
space: the prime polynomial $h(x, y)$ had
one component in the geometry of its
solution set (a circle) while the composite
polynomial $g(x, y)$ had two (the two lines
which cross). Algebraic geometry is the
study of this interplay. For example, note
here that both g and h were "degree-two"
polynomials, since terms like x^2 or y^2
involve the multiplication of two things,
like an x with an x or a y with a y, and two
is the maximum number required by any
term in the polynomial. When we
considered the simultaneous set of
solutions to g and h, we found four points.

Here we meet a demonstration of a
mathematical theorem in algebraic
geometry: Bezout's theorem says that the
number of points equals the product of the
degrees, and indeed here we have 4 = 2x2.

A function is a machine which takes as
input a point in some space and has as
output a number. For example, our
polynomials g and h are functions on the
plane, since the inputs (x, y) are points in
the plane. The output, such as
$g(2, 3) = 2^2 - 3^2 = -5$, is always a
number. Recall that the set of points to
which g assigns the number zero formed a
giant X. What if we wanted to talk about
functions on that X itself? That is, what if
we were interested in assigning a number
to each point on that X? In algebraic
geometry, we often want to do such a
thing. In order to study a space, you might
study how it appears inside other spaces
(such as the X in the plane) and you might
study how other spaces appear inside it
(such as the four points inside the X). Now
here is one way to consider a function on
X. Start with a function on the plane and
restrict your inputs to points which lie on
X. For example, we could apply the
function $h(x, y)$ to points (x, y) that lie on
X (which is to say, points with $g(x, y) = 0$
). That's fine, but then you soon realize
that sometimes two different functions on
the plane restrict to the same function on
X. For instance, if we compare h and
$h + g$, then on the plane they are different
but on X they are the same, since $h + g$
equals $h + 0$ because on X, g is zero, and
$h + 0$ equals h). After we impose this

notion of sameness, we get a new "ring" of functions, and in general these rings can have interesting properties. For example, consider the function $j(x, y) = x + y$ as a function on X. Note j is equal to zero along the line from northwest to southeast, but j is nonzero on the other line. Therefore j is not the "zero" function which assigns zero to every point. Likewise, the function $k(x, y) = x - y$ is nonzero, but is equal to zero along the line from southwest to northeast. Now note that together on X we have $jk = 0$. The product of these two nonzero functions is zero when considered as functions on X. This is a very different phenomenon from what we are used to with numbers. With numbers, if the product of two numbers is zero, then one of them must be zero (possibly both). The lesson is that functions can be multiplied just like polynomials. Sometimes, the ring that they define can be interesting in novel ways, such as having the product of two nonzero objects being zero.

Recap: we can learn about the geometry of the space of solutions of some polynomial equations by studying their algebraic properties. The relationship between factoring and having multiple components was one example. Bezout's theorem was another. Functions on a space organize into an algebraic structure called a ring, since you can multiply them, and these rings can be more exotic than the rings formed by numbers or by plain old polynomials.

Now here is the crucial insight: we free

ourselves from geometry and simply describe a space with its ring of functions. The plane would be described by the polynomials in two variables (omitting, always, the fancy trigonometric functions and such, in the land of algebra). The X would be described by that ring but with h and $h + g$ thought of as the same, i.e. with g being identified with the zero function. The one-dimensional line would be described by polynomials in one variable. Even a single point can be described in this way! A function on a point must assign to that point a number, so the ring of functions is the ordinary ring of numbers, where multiplication is the usual multiplication. So we may think of each space as providing a generalization of the algebraic structure of ordinary numbers: each space is defined by (or defines) a ring of functions. This construction gives many interesting objects -- the so-called "affine schemes," but algebraic geometry contains yet more.

A scheme is a space described locally by a ring of functions. To give a flavor for what this might mean -- particularly the word "locally" -- consider a space which looks more like a Q than an X. Near where the tail of the Q meets the circle, there is a crossing which looks like a miniature X. What that means is that we should be able to zoom in our perspective and describe the points near the crossing as we would describe the X itself. Now in truth giving a notion of "nearness" can be subtle. Up until this point, we haven't relied on distances. For instance, we could have

made all the same essential conclusions above using $x^2 + y^2 - 50$ instead of $x^2 + y^2 - 2$ and the circle h described would have been five times as large. Thus a scheme is a set of points equipped with a notion of nearness, such that on each "small" region a ring of functions is given. Further, these rings of functions must be compatible when considering the overlap of two regions. What this means is that if a small region A is contained in both region B and in region C, then the functions on A can be considered as restrictions of functions on B or as restrictions of functions on C.

That's about it. You take your geometric object (if you have a notion of geometry), look at a "small region" and describe the object by some equations. These equations tell you what the space of functions on the object is (e.g. which polynomials to consider "the same"). And you do this on enough small regions so that the whole object is described. If you want to free yourself from geometry entirely, you must provide a set with a notion of "nearby points," and give a ring (of functions) for each such neighborhood.

In fact, the only thing left to specify here is what we meant at the start by a "number." That is, we have to decide on the set of functions on a point! This choice determines which "numbers" we are allowed to use in our polynomial expressions. We might have meant the real numbers, the rational numbers, complex numbers, the whole numbers, or

-- and here it gets deep very fast --
something more exotic. The only thing we
really require is that whatever we decide a
"number" is, we ensure that multiplication
is associative and commutes, like for
ordinary numbers.

Why do all this? Having "algebraicized"
the problem completely, the power of this
approach emerges when geometry breaks
down. For example, if you plot the
solutions to the equation $y^2 - x^3 = 0$ you
see a pointy object which doesn't have a
tangent line at the origin (0,0). So certain
geometric constructions are off limits.
However, this kind of space poses no
problems in scheme theory. The ring of
functions is simply obtained: for example,
you need to set two polynomials equal to
each other if they differ by $y^2 - x^3$, since
that is zero on the space.

Obviously, Tate and Mumford were not
afforded this much space by Nature, and
just as obviously, without constraints they
could communicate these ideas, too -- and
better. Whatever its origin, the challenge
was a good one. Did I meet it?

Eric, this is certainly a simple introduction to
some of the ideas needed to explain schemes.
But I think that it also illustrates why
mathematicians are often unsuccessful in
explaining their ideas to other scientists. The
reason is that it seems to me to suffer from the
mathematicians compulsion to always be
100% precise and complete, defining every
concept used. All scientists know what a

function is already and the idea of restricting a function to some smaller set does not need to be spelled out in such detail. The second issue is that mathematicians, when giving examples, tend to start with trivial examples instead of going for an example that illustrates best the core idea. In your case, I think the equation $x^2 - y^2 = 0$ is just too simple and emphasizing reducible varieties seems to me just distracting. In the version Nature accepted, John and I use your third example, the circle -- a variety certainly known to all scientists -- and say "Algebraic geometry is the field that studies the solutions of sets of polynomial equations by looking at their geometric properties. For instance a circle is the set of solutions of $x^2 + y^2 = 1$ and in general such a set of points is called a variety." I think the trick is to bootstrap the math on things scientists know, simplifying definitions (Stewart's maxim "Lie a little") and getting to some core non-trivial motivating example if possible.

Jan.5. I received from Jean-Michel Kantor a three part obituary by Michel de Pracontal containing some excellent efforts at explaining Grothendieck's work to lay people. I quote some of his article here. First a quote from Michel Demazure:

> "Expliquer les maths de Grothendieck en termes du langage quotidien? Ça me semble difficile ... " Michel Demazure, lui-même mathématicien et élève d'Alexandre Grothendieck dans les années 1960, sait de quoi il parle. "D'une part, il y a une

histoire mathématique dans laquelle il s'inscrit totalement, de l'autre son approche est très personnelle et d'une certaine manière unique. Grothendieck a reconstruit la géométrie algébrique, mais il n'a jamais écrit une équation algébrique. Il ne regardait jamais un objet particulier, le cadre était plus important pour lui que les objets qui s'y trouvaient. Tous les matheux ont des objets en tête, mais il n'avait pas les mêmes que les autres? "

But frankly, I was quite disappointed by their struggle to say something meaningful about what schemes and functors are. They start, as John and I finally did, with a circle but discussing how one can look at the integer and rational solutions of the equation of a circle as well as real and complex solutions. This leads them to the following passage where schemes and functors are strangely conflated. I'm not sure why they say a set of equations could have no solutions -- what happened to the *nullstellensatz*? I guess they meant the variety has no points rational over the ground field.

Or, la mathématique étant le pays de la liberté, il n'y a aucune raison de ne pas considérer les solutions d'une équation, ou d'un système d'équations, pour n'importe laquelle des espèces de nombres évoqués ci-dessus. Ce qui enrichit encore considérablement la variété des variétés ...

Et c'est là qu'intervient Grothendieck. Rappelons-nous qu'une variété est un objet géométrique, qui représente les

solutions d'un système d'équations. Mais il y a des cas où le système n'a pas de solution, de sorte que la variété correspondante n'a pas de points. On ne peut pas la dessiner comme une figure géométrique. Mais peut-on quand même l'étudier ? L'idée de Grothendieck est de généraliser la notion de variété, en passant par les propriétés algébriques, et en "ignorant" les points : "Grothendieck ne se préoccupe pas des points, il les oublie délibérément, explique le mathématicien français Jean-Michel Kantor. Son raisonnement revient à dire : même si j'ai une équation sans solution, je veux pouvoir étudier cet objet ; donc je vais rassembler toute une série de variétés, sans savoir s'il y a des points, et je vais construire un objet plus général, qui inclut tous les cas possibles."

Cet objet plus général s'appelle un "schéma". L'intérêt des schémas est qu'ils élargissent le cadre de l'algèbre, tout en conservant les propriétés les plus importantes. Les schémas permettent de traiter dans le même cadre le monde des nombres entiers et celui des grandeurs continues, répondant aux questions soulevées par Diophante il y a 1 800 ans. Ainsi, avec les schémas, notre cercle peut être étudié aussi bien en considérant les nombres entiers que les réels ou un autre type de nombres.