THE COMPLEX BURGERS EQUATION, THE HCIZ INTEGRAL AND THE CALOGERO-MOSER SYSTEM

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Abstract. This article is the text of a talk at the meeting on Random Matrix Theory at CMSA (Harvard) in Jan 2017. The goal of this talk is to present a new approach to the limit PDE obtained by Matytsin [13] in his analysis of the asymptotics of the HCIZ integral. The main new idea is to treat this PDE as a zero dispersion continuum limit of the Calogero-Moser system.

Contents

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1. Introduction

1.1. **The mass transportation problem.** My most direct goal is to gain a better understanding of the following boundary value problem. Consider a compressible fluid with density $\rho(x,t)$ and velocity field $v(x,t)$ satisfying Euler's equations

$$\partial_t \rho + \partial_x(\rho v) = 0, \quad \partial_t v + v\partial_x v - \pi^2 \rho \partial_x \rho = 0. \tag{1.1}$$

Here $x \in \mathbb{R}$, $t \in (0,1)$, $\rho(x,t) \geq 0$, and the boundary conditions are

$$\rho(x,t)|_{t=0} = \rho_A(x), \quad \rho(x,t)|_{t=1} = \rho_B(x), \tag{1.2}$$

where ρ_A and ρ_B are given density fields. Our goal is to obtain explicit understanding of how the probability measure ρ_A is transported to the probability measure ρ_B according to the flow of (1.1).

The system (1.1) may be rewritten as follows. Define the complex field [1]

$$f(x,t) = v(x,t) + i\pi\rho(x,t), \tag{1.3}$$

and observe that the system (1.1) is equivalent to the single equation

$$\partial_t f + f\partial_x f = 0. \tag{1.4}$$

This looks like Burgers equation, and it is tempting to think of (1.4) in terms of the (implicit) solution formula

$$f(x,t) = f_0(x - tf(x,t), \quad f_0(x) := f(x,0). \tag{1.5}$$

However, since $f(x,t)$ is complex at each point where $\rho(x,t) > 0$, we must treat the equation more carefully. To see what the issue is, recall that the *linear* partial differential equation

$$\partial_t f + c\partial_x f = 0, \tag{1.6}$$

is the transport equation when c is real. In this case the solution to (1.6) is simply $f(x,t) = f_0(x - ct)$. Further, this solution formula makes sense even when f_0 is a function with minimal smoothnes. On the other when $c = i$, f is necessarily complex, and equation (1.6) is the Cauchy-Riemann equations in disguise. More generally, when the imaginary part of c is non-zero, f is complex, and (1.6) is an *elliptic* equation and the formal solution formula $f(x,t) = f_0(x - ct)$ is meaningful only if the domain of f_0 can be extended into a complex neighborhood of the real line. Thus, it can be solved only for analytic initial conditions.

Returning to the complex Burgers equation (1.4), we see that whether it is hyperbolic or elliptic at a point (x,t) depends on whether $\rho(x,t) > 0$ (hyperbolic) or $\rho(x,t) = 0$ (elliptic). The fact that (1.4) is elliptic also explains the fact that we prescribe boundary conditions on only 'half the variables' (i.e. only on ρ, not on v). The problem is over-determined if both ρ and v are prescribed at $t = 0$ and $t = 1$.

1.2. **The asymptotics of the HCIZ integral.** The above boundary value problem arises in a central problem in random matrix theory– the asymptotic analysis of matrix integrals. The integral of interest here is the HCIZ integral defined as follows. Assume given two strictly increasing sequences of real numbers, denoted $a_1 < a_2 < \ldots < a_N$ and $b_1 < b_2 < \ldots < b_N$. Form the diagonal matrices

$$A_N = \mathrm{diag}(a_1, a_2, \ldots, a_N), \quad B_N = \mathrm{diag}(b_1, b_2, \ldots, b_N), \tag{1.7}$$

[1] As usual, i denotes $\sqrt{-1}$.

and consider the matrix integral over the unitary group $U(N)$ with normalized Haar measure

$$I_N(A_N, B_N) = \int_{U(N)} e^{N\mathrm{tr}(A_N U B_N U^*)}\, dU. \tag{1.8}$$

Harish-Chandra (HC) discovered [10], and Itzykson and Zuber (IZ) re-discovered [11], the beautiful exact formula

$$I_N(A_N, B_N) = \frac{1}{\omega_n} \frac{\det\left(e^{N a_j b_k}\right)_{1\le j,k\le N}}{V_N(A_N) V_N(B_N)}, \tag{1.9}$$

where $V_N(A_N)$ denotes the Vandermonde determinant

$$V_N(A_N) := \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_N \\ a_1^2 & a_2^2 & \dots & a_N^2 \\ \vdots & & & \vdots \\ a_1^{N-1} & a_2^{N-1} & \dots & a_N^{N-1} \end{vmatrix} = \prod_{j<k} (a_k - a_j), \tag{1.10}$$

$V_N(B_N)$ is defined in an identical manner, and ω_N is the volume of $U(N)$, given by

$$\omega_N = \prod_{j=1}^{N-1} j!. \tag{1.11}$$

It would take me too far afield to derive this formula, since there are at least three different approaches [18]. The most intuitive (to me) uses the fundamental solution to the heat equation in the space of Hermitian matrices with metric $ds^2 = \mathrm{tr}(dM)^2$. A second proof uses the character expansions for $U(N)$. Finally, a third proof is based on the Duistermaat-Heckman theorem from symplectic geometry. The last of these proofs has a rather striking consequence – since our interest is in large N, if one naively applies the saddle-point method to the right hand side of (1.8), we find that it yields (1.9) *exactly*!

The asymptotic problem is as follows. We fix spectral measures,

$$\mu_A(dx) = \rho_A(x)\, dx, \quad \mu_B(dx) = \rho_B(x)\, dx, \tag{1.12}$$

and consider sequences of diagonal matrices $A_N = \mathrm{diag}(a_1^{(N)}, \dots, a_N^{(N)})$ and $B_N = \mathrm{diag}(b_1^{(N)}, \dots, b_N^{(N)})$ whose spectral measures

$$\mu_{A_N} = \frac{1}{N} \sum_{j=1} \delta_{a_j^{(N)}}, \quad \mu_{B_N} = \frac{1}{N} \sum_{j=1} \delta_{b_j^{(N)}}, \tag{1.13}$$

converge weakly to μ_A and μ_B respectively. [2] In a beautiful paper, Matytsin [13] discovered that in this regime

$$\lim_{N\to\infty} \frac{1}{N^2} \log I_N(A_N, B_N) = F_0(\rho_A, \rho_B), \tag{1.14}$$

where

$$F_0(\rho_A, \rho_B) = S(\rho_A, \rho_B) + \frac{1}{2}\left(m_2(\rho_A) + m_2(\rho_B)\right) - \frac{1}{2}\left(\Sigma(\rho_A) + \Sigma(\rho_B)\right). \tag{1.15}$$

[2] Since we assume that we have limit densities ρ_A and ρ_B, this simply means that the distribution functions $\mu_{A_N}(x)$ and $\mu_{B_N}(x)$ converge pointwise to $\mu_A(x)$ and $\mu_B(x)$ at each x.

The key term here is the coupling $S(\rho_A, \rho_B)$. It turns out that it is the classical action

$$S(\rho_A, \rho_B) = \int_0^1 \int_{\mathbb{R}} \left(\frac{1}{2}\rho v^2 + \frac{\pi^2}{6}\rho^3 \right) dxdt, \tag{1.16}$$

evaluated on the (unique) solution to (1.1) with the boundary condition (1.2). Each of the remaining terms in (1.15) depends on either ρ_A or ρ_B, but not both of them. For instance,

$$m_2(\rho_A) := \int_{\mathbb{R}} x^2 \rho_A(x)\, dx, \quad \Sigma(\rho_A) := \int_{\mathbb{R}} \int_{\mathbb{R}} \log|x-y| \rho_A(x)\rho_A(y)\, dx\, dy. \tag{1.17}$$

A rigorous formulation of Matytsin's work has been provided by Guionnet and Zeitouni [8, 9]. Further, it is of considerable interest to understand higher-order terms, and to rigorously establish an asymptotic expansion of the form

$$\frac{1}{N^2} \log I_N(A_N, B_N) = F_0(\rho_A, \rho_B) + \frac{1}{N^2} F_1(\rho_A, \rho_B) + \frac{1}{N^4} F_2(\rho_A, \rho_B) + \dots \tag{1.18}$$

since the terms in the expansion have deep significance in enumerative geometry [7].

The purpose of a rigorous analysis of (1.4) is now clear. I expect it to complement the above rigorous results by yielding new exact solutions, numerical schemes, and perhaps even simpler proofs of some known theorems. The reason is that the complex Burgers equation is a continuum limit of a fundamental integrable system – the classical Calogero-Moser system. We may therefore apply several methods that have been developed for the Calogero-Moser systems to the analysis of (1.4). The work presented here is a preliminary attempt. There is a great deal more that one can do [3].

1.3. **The Calogero-Moser system.** The (classical) Calogero-Moser (CM) is a system of N identical particles on the line interacting through an inverse square potential. The coordinates of the particles are $x_1 < x_2 < \dots < x_N$ and their momenta are denoted p_1,p_2, $\dots p_N$. The phase space of the system is $\mathcal{W}_N \times \mathbb{R}^n$, where $\mathcal{W}_N$ denotes the Weyl chamber

$$\mathcal{W}_N = \{ x \in \mathbb{R}^N \,|\, x_1 < x_2 < \dots < x_N \}. \tag{1.19}$$

The potential energy and Hamiltonian of the system are respectively

$$V(x) = \frac{g^2}{2} \sum_{k \neq j} \frac{1}{(x_j - x_k)^2}, \quad H(x,p) = \frac{1}{2}|p|^2 + V(x). \tag{1.20}$$

The equations of motion are given by

$$\dot{x}_j = \partial_{p_j} H(x,p) = p_j, \quad \dot{p}_j = -\partial_{x_j} H(x,p) = 2g^2 \sum_{k \neq j} \frac{1}{(x_j - x_k)^3}, \quad 1 \leq j \leq N. \tag{1.21}$$

The parameter g measures the strength of the interaction. When g is real, the particles repel one another; however, when g is imaginary, the particles are attracted to one another, and coalesce in finite time. Most of the literature on the CM system assumes that g is real. In this case, equation 1.20 shows that the Hamiltonian is

[3] This connection appears as a footnote in [13]. Continuum limits of the Calogero-Moser system have been studied by several physicists, especially Jevicki, but the version studied here is different, because it is a zero dispersion limit.

necessarily positive. If $H(x(0), p(0)) = E > 0$, then the particles can never get too close to one another because of the uniform bound

$$0 < \frac{g^2}{2} \sum_{k \neq j} \frac{1}{(x_j - x_k)^2} \leq E. \tag{1.22}$$

Before turning to the exact solvability of the CM system, let us build some intuition by considering the simplest case, which is $N = 2$. Without loss of generality, we may assume that the center of mass and mean momentum are 0. Therefore, using equation (1.21) we find that $s = x_2 - x_1$ satisfies the second-order differential equation

$$\ddot{s} = \frac{4g^2}{s^3}. \tag{1.23}$$

This equation may be integrated using conservation of energy. We find that

$$\dot{s}^2 = 2E - \frac{2g^2}{s^2}, \quad \text{or} \quad \dot{s} = \pm\sqrt{2E - \frac{2g^2}{s^2}}. \tag{1.24}$$

The phase portraits for this system when $g = 1$ (repulsive) and $g = i$ (attractive) are shown in Figure 1. When $g = i$, for each $E < 0$, there is a unique, positive solution on a maximal interval $(-T(E), T(E)$ with $\lim_{t \to \pm T(E)} s(t) = 0$. This solution has a square-root singularity $s(t) \sim (T(E) - t)^{1/2}$ and $s(t) \sim (t + T(E))^{1/2}$ as $t \to \pm T(E)$, since $\dot{s} \sim \pm s^{-1}$ as $s \to 0_+$. This critical solution is of interest to us for the following reason. It gives rise to an exact solution to the boundary value problem (1.1)–(1.2).

1.4. **An exact solution to complex Burgers.** Bun, Bouchaud, Majumdar and Potters [5] [4] observed that (1.1)–(1.2) may be solved exactly by substituting the ansatz

$$f(x,t) = b(t)x + i\frac{\sqrt{4\sigma(t)^2 - x^2}}{2\sigma(t)}, \quad |x| \leq 2\sigma(t), \tag{1.25}$$

in equation (1.4), and solving the resulting ordinary differential equations for $b(t)$ and $\sigma(t)$. The exact form of this solution may be found in their work; what I would like to point out here is that $\sigma(t)$ and $b(t)$ are related as follows:

$$\ddot{\sigma} = -\frac{4}{\sigma^3}, \quad b(t) = \frac{\dot{\sigma}}{\sigma}. \tag{1.26}$$

That is, the $N = 2$ CM system is embedded within this exact solution, and b is obtained by the Riccati transformation of $\sigma(t)$. The solution is symmetric about $t = 1/2$ and may be extended to a maximal time interval corresponding to the critical solution to the CM system with $N = 2$. This calculation is reminescent of the construction of solitons and breathers via a pole ansatz, and I suspect that several other exact solutions may be constructed in this manner.

[4] Matytsin makes a similar ansatz to solve induced QCD [13, Sec. 4].

2. Large deviations for Dyson Brownian motion and the CM system

2.1. **The principle of least action.** The CM system arises naturally in the study of the HCIZ integral through a somewhat unexpected confluence of two different variational problems. Recall that the equations of classical mechanics may be formulated using the principle of least action. The Lagrangian for the CM system is the function

$$L(x,y) = \frac{1}{2}|y|^2 - V(x), \quad x \in \mathcal{W}_N, y \in \mathbb{R}^N. \tag{2.1}$$

We consider C^1 paths $\gamma : [0,1] \to \mathcal{W}_N$ with boundary conditions $\gamma(0) = (a_1, a_2, \ldots, a_N)$ and $\gamma(1) = (b_1, b_2, \ldots, b_N)$ and we associate to each path the *action*

$$S[\gamma(\cdot)] = \int_0^1 L(\gamma(s), \dot{\gamma}(s))\, ds. \tag{2.2}$$

The principle of least action asserts that the 'true' path is a minimizer for the above variational principle. In this case, the Euler-Lagrange equations are precisely the equations of motion for the CM system (we're using $p = \dot{x}$). This is a formal principle and it is not always true that there is a minimizing path for the repulsive CM system. Surprisingly, the attractive CM system is better behaved.

Lemma 2.1. *Assume $g = i\kappa$ for some $\kappa \in R$. Assume given any two points a and b in the Weyl chamber $\mathcal{W}_N$ and any $T > 0$. Then there is a unique C^1 path $\gamma : [0,T] \to \mathcal{W}_N$ with $\gamma(0) = a$ and $\gamma(T) = b$ that minimizes the action. In particular, for every $a, b \in \mathcal{W}_N$ and $T > 0$, there exists a unique solution to the CM system defined on $[0,T]$ such that $\gamma(0) = a$ and $\gamma(T) = b$.*

Sketch of the proof. The main point is that when g is imaginary, the Lagrangian $L(x,y)$ of the CM system is strictly convex in both x and y, since

$$L(x,y) = \frac{1}{2}|y|^2 + \frac{\kappa^2}{2}\sum_{k\neq j} \frac{1}{(x_j - x_k)^2}, \quad x \in \mathcal{W}_N, y \in \mathbb{R}^N. \tag{2.3}$$

Further, the Weyl chamber $\mathcal{W}_N$ is a convex set, since it may be written as an intersection of half-spaces. A standard argument yields existence, uniqueness and smoothness of minimizers. □

2.2. **Large deviations of Dyson Brownian motion.** Let us now indicate how an *a priori* different variational principle is connected to this approach. The starting point for Guionnet and Zeitouni, as well as Bun *et al*, is the large deviations of Dyson Brownian motion. For each $x \in \mathcal{W}_N$ we define the Coulomb energy

$$U(x) = \sum_{j<k} \log|x_j - x_k|, \tag{2.4}$$

and recall that Dyson Brownian motion originating at a point $a \in \mathcal{W}_N$ is the unique solution to the SDE

$$d\gamma_l = -\partial_{x_l} U(\gamma)\, dt + \varepsilon dB_l, \quad \gamma(0) = a, \tag{2.5}$$

where dB_l, $l = 1, \ldots, N$ are independent standard Brownian motions. Let us focus on Dyson Brownian bridges γ with $\gamma(0) = a$ and $\gamma(1) = b$. Ignoring technicalities, as $\varepsilon \downarrow 0$, such a bridge concentrates at the path that minimizes the *Wiener action*

$$W[\gamma(\cdot)] = \frac{1}{2}\int_0^1 |\dot{\gamma} + \nabla U(\gamma)|^2\, ds, \tag{2.6}$$

where we again minimize over all C^1 paths $\gamma : [0,T] \to \mathcal{W}_N$ with $\gamma(0) = a$ and $\gamma(1) = b$.

There is a lovely identity that connects the variational principle for the Wiener action with the principle of least action for the CM system. Since $U(x)$ is the Coulomb energy, we differentiate to find

$$\partial_{x_l} U(x) = \sum_{j \neq l} \frac{1}{x_l - x_j}, \quad l = 1, \dots, N. \tag{2.7}$$

Therefore, observing that the cross-terms cancel, we find

$$\begin{aligned} \frac{1}{2}|\nabla U(x)|^2 &= \frac{1}{2}\sum_{l=1}^{N} \left(\partial_{x_l} U(x)\right)^2 \\ &= \frac{1}{2}\sum_{l=1}^{N}\sum_{j\neq l}\sum_{k\neq l} \frac{1}{(x_l - x_j)(x_l - x_k)} = \frac{1}{2}\sum_{k\neq j} \frac{1}{(x_j - x_k)^2} = \frac{1}{g^2}V(x), \end{aligned} \tag{2.8}$$

where $V(x)$ is the potential energy for the CM system defined in (1.20).

We apply this identity to the Wiener action, to obtain

$$W[\gamma(\cdot)] = \int_0^1 \left(\frac{1}{2}|\dot\gamma|^2 + \frac{1}{g^2}V(\gamma)\right) ds - \int_0^1 \dot\gamma \cdot \nabla U(\gamma)\, ds. \tag{2.9}$$

The last term is a total differential (a *null Lagrangian* in the terminology of the calculus of variations), since

$$\int_0^1 \dot\gamma \cdot \nabla U(\gamma)\ ds = \int_0^1 \frac{d}{ds} U(\gamma(s))\, ds = U(b) - U(a). \tag{2.10}$$

To summarize, we have obtained the following identity for the Wiener action

$$W[\gamma(\cdot)] = \int_0^1 \left(\frac{1}{2}|\dot\gamma|^2 + \frac{1}{g^2}V(\gamma)\right) ds - \left(U(b) - U(a)\right). \tag{2.11}$$

In particular, when $g = i$, the classical action $S[\gamma()]$ and the Wiener action $W[\gamma()]$ are minimized on the same path. Further, the minimum values are related by the difference in the Coulomb energy

$$\mathrm{argmin} W[\gamma(\cdot)] = \mathrm{argmin} S[\gamma(\cdot)] - \left(U(b) - U(a)\right). \tag{2.12}$$

2.3. **The Hamilton-Jacobi equation and Matytsin's approach.** Let us now connect the principle of least action to the Hamilon-Jacobi theory. Following Arnol'd [3, §46C], we define the *action function*

$$S(x,t) = \int_0^t L(\gamma(s), \dot\gamma(s))\, ds, \tag{2.13}$$

where $\gamma(t)$ is the extremal path connecting $\gamma(0) = a$ to an arbitrary point $x \in \mathcal{W}_N$ at time t. Any initial condition would do, we have chosen $\gamma(0) = a$, only to be concrete. By Lemma 2.1, the action function is well-defined when g is purely imaginary. It then follows that $S(x,t)$ solves the Hamilton-Jacobi equation

$$\partial_t S = H(x, \partial_x S), \quad \text{i.e.} \quad \partial_t S = \frac{1}{2}\sum_{j=1}^{N} \left(\partial_{x_j} S\right)^2 + \frac{g^2}{2}\sum_{k\neq j} \frac{1}{(x_j - x_k)^2}. \tag{2.14}$$

The reason for introducing this formalism here is to contrast Matytsin's approach to the HCIZ integral with Guionnet's. Matytsin's approach is not explicitly

probabilistic. More precisely, he uses the heat equation in the space of Hermitian matrices, but not Dyson Brownian motion. His starting point is the fact that a time-dependent version of the HCIZ integral solves a heat equation in $\mathcal{W}_N$. He then makes a WKB (or inverse Cole-Hopf) transformation to convert the linear heat equation to a nonlinear Hamilton-Jacobi equation, obtaining after some calculations a Hamilton-Jacobi equation with diffusion

$$\partial_t \tilde{S} = \frac{1}{2N}\sum_{j=1}^{N} \partial_{x_j}^2 \tilde{S} + \frac{N}{2}\sum_{j=1}^{N}\left(\partial_{x_j}\tilde{S}\right)^2 - \frac{1}{2N^3}\sum_{k\neq j}\frac{1}{(x_j-x_k)^2}. \tag{2.15}$$

At this stage, everything is exact. His key assumption is then to neglect the diffusion term, obtaining the Hamilton-Jacobi equation [13, Eqn (2.7)]

$$\partial_t \tilde{S} = \frac{N}{2}\sum_{j=1}^{N}\left(\partial_{x_j}\tilde{S}\right)^2 - \frac{1}{2N^3}\sum_{k\neq j}\frac{1}{(x_j-x_k)^2}. \tag{2.16}$$

We may write this equation in the form

$$\partial_t \tilde{S} = \tilde{H}_N(x,\tilde{S}), \quad \tilde{H}_N(x,p) = \frac{N}{2}|p|^2 - \frac{1}{2N^3}\sum_{k\neq j}\frac{1}{(x_j-x_k)^2}, \tag{2.17}$$

Aside from scaling factors, equation (2.16) is identical to (2.14). Thus, implicit in Matytsin's calculation is a reduction of the HCIZ integral to the CM system. (This rescaling is discussed in Section 5.)

The corresponding Lagrangian of the rescaled system is

$$\tilde{L}(x,\dot{x}) = \frac{1}{2N}\sum_{j=1}^{N}\dot{x}_j^2 + \frac{1}{2N^3}\sum_{k\neq j}\frac{1}{(x_j-x_k)^2}. \tag{2.18}$$

As explained in Section 5 when $N\to\infty$, the above Lagrangian has the the continuum limit

$$\tilde{L}(\rho,v) = \int_{\mathbb{R}} \rho(x,t)\left(\frac{1}{2}|v(x,t)^2 + \frac{\pi^2}{3}\rho^3(x,t)\right), dx, \tag{2.19}$$

Continuum limits of the CM system have been considered in the physics literature [1, 17]. In these papers, $N\to\infty$, but g does not depend on N. In our work, we must also take $g\to 0$ at the rate $1/N$. This changes the character of the problem; we now have a zero-dispersion continuum limit, much like the celebrated Lax-Levermore-Venakides theory.

3. Solving the Calogero-Moser system

There are many distinct solution techniques for the CM system. Two of these are considered here: (i) the projection method of Olshanetsky and Perelomov; and (ii) the use of pole dynamics and the doubled Benjamin-Ono equation (2BO). There is also a third approach that I have not worked out. This involves the use of τ-functions and the KP hierarchy. I remark on this briefly at the end of this note.

In this section N is held fixed, and we drop the subscript N in the notation. We assume given $a, b \in \mathcal{W}_N$ and we write $A = \mathrm{diag}(a)$, $B = \mathrm{diag}(b)$. We set $T = 1$ and focus on solving the boundary value problem for the CM system corresponding to the principle of least action for paths $\gamma : [0,1] \to \mathcal{W}_N$ with $\gamma(0) = a$, $\gamma(1) = b$. This is equivalent to determining the most likely path for the Dyson Brownian bridge connecting a and b as the noise vanishes when g is purely imaginary. In order to explain the solution to this boundary value problem, it is important to first recall the solution to the *initial* value problem for the CM system.

3.1. Moser's matrices and the projection method.

Consider the CM system

$$\dot{x}_j = y_j, \quad \dot{y}_j = 2g^2 \sum_{k \neq j} \frac{1}{(x_j - x_k)^3}, \tag{3.1}$$

with the initial conditions $x(0) = a \in \mathcal{W}_N$, $y(0) = v \in \mathcal{W}_N$. Moser showed that this problem is completely integrable [5] by introducing the following matrices. Given (x, y), introduce the matrices $P(x, y)$ and $Q(x, y)$ with entries

$$P_{jj} = y_j, \qquad P_{jk} = \frac{ig}{(x_j - x_k)}, \quad j \neq k, \tag{3.2}$$

$$Q_{jj} = -\sum_{l \neq j} \frac{ig}{(x_j - x_k)^2}, \qquad Q_{jk} = \frac{ig}{(x_j - x_k)^2}. \tag{3.3}$$

For comparison with continuum results with the Hilbert transform of the density, we also introduce the discrete matrix

$$H_{jj} = 0, \quad H_{jk}(x, y) = \frac{1}{x_j - x_k}, \quad j \neq k. \tag{3.4}$$

While it requires considerable ingenuity to discover these matrices, a direct computation shows that (3.1) is equivalent to the Lax equation

$$\dot{P} = [P, Q]. \tag{3.5}$$

As a consequence, the eigenvalues of P are constants of motion for (3.1). These can be shown to be in involution and it follows that (3.1) is an integrable system.

Given a matrix H with real, distint eigenvalues, let $\mathrm{eig}(H) \in \mathcal{W}_N$ denote the eigenvalues listed in increasing order. We will use the following refinement of Moser's method introduced by Olshanetsky and Perelomov [15] to directly solve the initial value problem for (3.1). Given $(a, v) \in \mathcal{W}_N \times \mathbb{R}^N$, define the matrix $P(a, v)$ as in (3.2). Then as long as the solution to (3.1) exists, it is given by the simple formula

$$x(t) = \mathrm{eig}(A + tP(a, v)). \tag{3.6}$$

There is an important difference between the repulsive case (g real) and the attractive case (g imaginary). When g is real, the matrix $H(t) := A + tP(a, v)$ is

[5] He implicitly assumes g is real, but this is not necessary.

Hermitian, the eigenvalues are real for all t, and the energy bound (1.22) ensures that the eigenvalues never collide. Thus, for each $(a, v) \in \mathcal{W}_N \times \mathbb{R}^N$ there is a solution to the CM system for every $t \in \mathbb{R}$.

This is no longer true when $g = i$. The initial value problem is not defined for all time, and the eigenvalues collide. In fact, since $A+tP(a, v)$ is a real matrix that is not symmetric, there is no *a priori* reason to expect that it has real eigenvalues (though this follows from (3.1), and remains true until the first collision of eigenvalues). The above solution formula is called the projection method, since the CM flow is obtained by projecting the linear flow $H(t) = A + tP(a, v)$ in $\mathrm{Her}(n)$ (when g is real) or $\mathrm{GL}(n; \mathbb{R})$ (when g is imaginary) onto $\mathcal{W}_N$ via the map $H \mapsto \mathrm{eig}(H)$.

Nevertheless, Lemma 2.1 always allows us to solve the *boundary value problem*. More precisely, when $g = i$, given $a, b \in \mathcal{W}_N$, there is always a unique path with $\gamma(0) = a$ and $\gamma(1) = b$ that minimizes the action. This path is necessarily smooth, and in particular, if we set $v = \dot{\gamma}(0)$, we find that

$$b = \mathrm{eig}(A + P(a, v)). \tag{3.7}$$

Observe that b is given and v is determined implicitly through (3.7). This solution formula yields a shooting method to solve the boundary value problem.This yields the initial condition v for the CM flow connecting a to b. The entire flow is then given by (3.6).

This solution formula allows us to approximate and visualize the transport map from ρ_A to ρ_B that seems so mysterious when written in the form (1.1)–(1.2). Roughly, all we have to do is to approximate the spectral measures μ_A and μ_B with N suitably chosen point masses, choose $g = i/N$, and find the CM solution that connects a and b. For instance, given ρ_A and ρ_B and N, let us define the vectors $a, b \in \mathcal{W}_N$ by setting

$$a_j = \mu_A^{-1}(\frac{j}{N}), \quad b_j = \mu_B^{-1}(\frac{j}{N}), \quad 1 \le j \le N, \tag{3.8}$$

where $\mu_A(x) := \int_{-\infty}^{x} \rho_A(s)\, ds$ is the distribution function of μ_A, and we define the inverse as $\mu_A^{-1}(\alpha) = \inf_x\{\mu_A(x) \ge \alpha\}$. We then apply the numerical scheme below.

3.2. **A naive numerical scheme.** In order to solve the discrete transport map numerically, it is only necessary to determine the initial velocity v that solves equation (3.7). We use an eigenvalue solver and a Newton scheme as follows. Define the nonlinear map

$$v \mapsto \lambda(v) := \mathrm{eig}(A + P(a, v)), \tag{3.9}$$

and consider the quadratic cost function

$$h(v) = \frac{1}{2}|\lambda(v) - b|^2. \tag{3.10}$$

The Newton-Raphson scheme to solve $h(v) = 0$ consists of an iterative sequence $v^{(n)} \in \mathbb{R}^N$ defined by

$$v_j^{(n+1)} = v_j^{(n)} - \frac{h(v^{(n)})}{\partial_{v_j} h(v^{(n)})}, \quad j = 1, \dots, N. \tag{3.11}$$

By the product rule, the gradient of h is

$$\partial_{v_j} h(v) = (\lambda(v) - b) \cdot \left(\partial_{v_j} \lambda(v)\right) = \sum_{k=1}^{N} (\lambda_k(v) - b_k) \left(\partial_{v_j} \lambda_k(v)\right). \tag{3.12}$$

Therefore, our problem reduces to standard perturbation theory for eigenvalues. Let us recall the general facts, and then apply them in our context.

Consider a smooth curve $(-1,1) \ni \tau \mapsto C(\tau) \in \mathrm{GL}(N,\mathbb{R})$ in the space of matrices, such that $\mathrm{eig}(C(\tau)) \in \mathcal{W}_N$, $\tau \in (-1,1)$. Let $U(\tau)$ denote the matrix of eigenvectors and $\Lambda(\tau) = \mathrm{diag}(\mathrm{eig}(C(\tau))$ denote the diagonal matrix of eigenvalues. We differentiate the equation

$$C(\tau) = U(\tau)\Lambda(\tau)U^{-1}(\tau), \tag{3.13}$$

and rearrange terms to obtain

$$U^{-1}\dot{C}U = [U^{-1}\dot{U}, \Lambda] + \dot{\Lambda}. \tag{3.14}$$

The diagonal entries of the commutator $[U^{-1}\dot{U}, \Lambda]$ vanish. Thus,

$$\dot{\Lambda} = \mathrm{diag}\left(U^{-1}\dot{C}U\right). \tag{3.15}$$

We apply the general formula(3.15) to our situation as follows. In order to compute the derivative

$$\partial_{v_j}\lambda(v)\big|_{v=v^{(n)}} \tag{3.16}$$

we consider a curve $C(\tau)$ with

$$C(\tau) = A + igH(a) + ig\,\mathrm{diag}(v(\tau)), \quad v(\tau) = v^{(n)} + \tau e_j \tag{3.17}$$

where $e_j \in \mathbb{R}^n$ is the j-th standard basic vector. In this case,

$$C(0) = A + igH(a) + ig\,\mathrm{diag}(v^{(n)}) = A + P(a, v^{(n)}), \quad \text{and} \quad \dot{C}(0) = \mathrm{diag}\,(e_j). \tag{3.18}$$

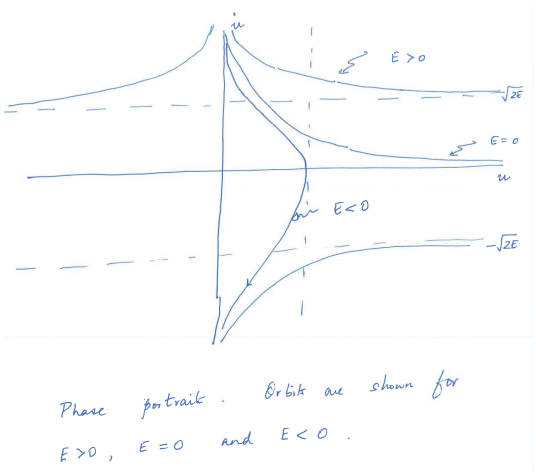
We then have

$$\partial_{v_j}\lambda(v)\big|_{v=v^{(n)}} = ig\,\mathrm{diag}\left(U^{-1}\,\mathrm{diag}(e_j)U\right), \tag{3.19}$$

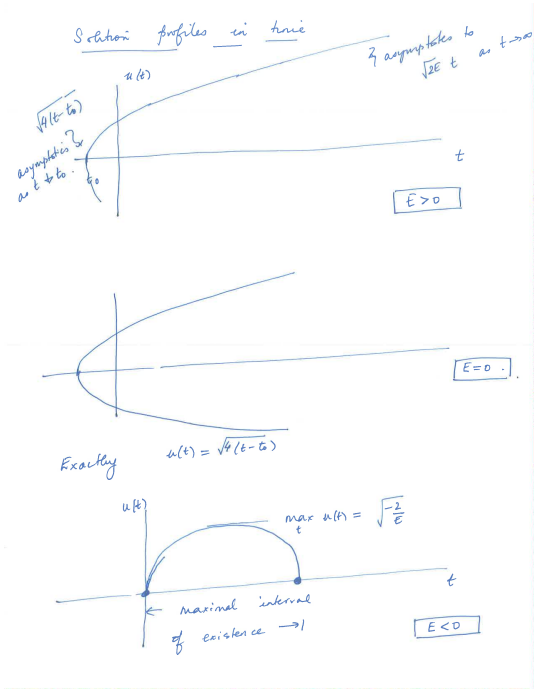
where $U = U(v^{(n)})$ is the matrix of eigenvectors of $A + P(a, v^{(n)})$.

In summary, the numerical scheme is as follows. Given $v^{(n)}$, we compute the eigenvalues $\lambda^{(n)}$ and eigenvector matrix $U^{(n)}$ of $A + P(a, v^{(n)})$ using an eigenvalue solver for non-symmetric matrices. We then compute the cost function $h(v^{(n)})$ and its derivative using the formulas (3.10), (3.12) and (3.19). These yield the next iterate $v^{(n+1)}$ in the Newton-Rapshon scheme. The scheme requires a careful choice of initial conditions. [6] In particular, since the matrix $A + P(a, v^{(n)})$ is not-symmetric, we must stop the iteration if it has complex eigenvalues.

[6] As of Feb. 2016, I have not been able to use this scheme to solve the BVP! It appears that what one needs is a multiple shooting method.



(a)



(b)

Figure 1: **Phase portrait for attractive CM system when $N = 2$.**

4. The complex Burgers equation

4.1. **Matytsin's functional equations.** In this section, I will explain how the CM system provides the correct interpretation for the complex Burgers equation. In order to explain the main issues, let us return to the PDE (1.4). Since this

equation is elliptic at all points where $\rho(x,t) > 0$, we cannot use the solution formula (1.5) without assuming that f_0 is analytic in a complex neighborhood of the x-axis. However, f_0 cannot be analytic for any ρ_A with compact support, not even for the exact solution (1.25)!

Matytsin does not let such mathematical niceties stop him, and he solves induced QCD by the method of characteristics as follows. Let f_0 and f_1 denote $f(x,0)$ and $f(x,1)$ respectively. Define the forward characteristic map $G_+ : x \mapsto x + f_0(x)$ and the backward characteristic map $G_- : x \mapsto x - f_1(x)$, and (formally) use (1.4) to obtain the identities

$$G_-(G_+(x)) = x, \quad G_+(G_-(x)), \quad x \in \mathbb{R}, \tag{4.1}$$

Matytstin formally solves these functional equations with boundary conditions

$$\mathrm{Im}(G_+(x)) = \pi\rho_A(x), \quad \mathrm{Im}(G_-(x)) = \pi\rho_B(x), \quad x \in \mathbb{R}, \tag{4.2}$$

by making a clever ansatz.

In my view, the use of this functional equations does not actually shed any light on the problem, since it is an easy consequence of the (unjustified) use of the method of characteristics. Further, when one applies this method to the study of the Kosterlitz-Thouless phase transitions as in [14], the functional equation leads to a problem with small divisors, a sure sign that there is something rather subtle in the background. Therefore, it strikes me as important to nail down the method of characteristics in this problem. In particular, this requires a more careful specification of the domain of the problem (Is x real or complex? How is f_0 defined in the complex plane, when we only prescribe ρ_A and ρ_B on the line? etc.).

4.2. **The Hilbert and Cauchy transforms.** Given a complex function $u(x)$, $x \in \mathbb{R}$, its Hilbert transform is defined by

$$\mathcal{H}u(x) = \frac{\text{p.v.}}{\pi} \int_{\mathbb{R}} \frac{1}{\lambda - x} u(\lambda)\, d\lambda = \quad x \in \mathbb{R} \tag{4.3}$$

The principal value integral is defined by

$$\mathcal{H}u(x) = \frac{\text{p.v.}}{\pi} \int_{\mathbb{R}} \frac{u(x-s)}{s}\, ds = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|s|>\varepsilon} \frac{u(x-s)}{s}\, ds. \tag{4.4}$$

The Hilbert transform is a bounded operator on $L^p(\mathbb{R})$, $1 < p < \infty$. When the Fourier transform of u is defined by

$$\hat{u}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} u(x)\, dx, \tag{4.5}$$

the Hilbert transform has the multiplier [7]

$$\widehat{(\mathcal{H}u)}(\xi) = -i\mathrm{sgn}(\xi)\, \hat{u}(\xi), \quad \xi \in \mathbb{R}. \tag{4.6}$$

When u is real, the Hilbert transform provides the boundary values of the complementary harmonic function for the harmonic extension of u. Here is what this means. Assume u is real and consider its Cauchy transform

$$\mathcal{C}u(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} u(\lambda)\, d\lambda, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{4.7}$$

[7] We use Stein's convention to define the Hilbert transform, but the modern sign convention for the Fourier transform. This is why the multiplier is $-i\,\mathrm{sgn}(\xi)$, as compared with $i\,\mathrm{sgn}(\xi)$ [16, p.55].

The function $\mathcal{C}u$ is analytic in $\mathbb{C}\backslash\mathbb{R}$. When u is a positive density it is a Herglotz function.[8] Let $(\mathcal{C}u)_\pm(x)$ denote its boundary values as $z \to x \in \mathbb{R}$ from the upper and lower-half plane respectively. Then

$$\mathcal{C}u_\pm(x) = \pi\mathcal{H}u(x) \pm i\pi u(x), \quad x \in \mathbb{R}. \tag{4.8}$$

4.3. **Free convolution with the semicircle law.** To build some intuition for complex Burgers equation, let us first recall the following fact from free probability theory. Let ν_t denote the semicircle law with width $2t$

$$d\nu_t(x) = \frac{1}{2\pi t}\sqrt{4t - x^2}, \quad |x| \le 2t. \tag{4.9}$$

Assume given a measure μ_A with density ρ_A, and let $\mu_t = \mu_A \boxplus \nu_t$ denote the free convolution of μ_A with the semicircle law ν_t. Let ρ_t denote the density of μ_t and (abusing notation) let us set $f(z,t) = \mathcal{C}_{\mu_t}(z) = \mathcal{C}_{\rho_t}(z)$. Then a basic fact in free probability theory is that f solves the complex Burgers equation

$$\partial_t f - f\partial_z f = 0, \quad t > 0, z \in \mathbb{C}_+. \tag{4.10}$$

The difference with (1.4) is that we are not explicit about the domain of our function, and the use of complex characteristics is completely justified. The characteristics of (4.10) solve the ordinary differential equation

$$\dot{z} = -f(z,t), \quad z \in \mathbb{C}_+. \tag{4.11}$$

On the real axis, these ordinary differential equations become

$$\dot{x} = \pi\mathcal{H}\rho_t(x), \quad \dot{y} = -\pi\rho_t(x), \quad x \in \mathbb{R}, t > 0. \tag{4.12}$$

Since $\rho_t \ge 0$, the characteristics always flow out of $\mathbb{C}_+$. Thus, the domain $\mathbb{C}_+$ is positively invariant under the flow (4.11). This property is what allows us to use the method of characteristics in this problem. In summary, in the absence of a boundary condition at $t = 1$, equation (1.4) is well-posed on the domain $\mathbb{C}_+ \times (0,\infty)$ by extending $\rho(\cdot,t)$ to the analytic function $f(z,t)$ using the Cauchy transform. It may also be checked that for each $t > 0$ the PDE continues to hold on the x-axis. With a little work, one can now establish regularity for ρ_t (Biane does not explicitly use the method of characteristics, but it is implicit in his proof[4]).

The complex Burgers equation (4.10) should be thought of as the analogue of the heat equation in free probability. The solution admits the following direct stochastic interpretation. Let S_t denote a standard free Brownian motion, and let A be an operator with spectral measure ρ_A that is free with respect to the process S_t. Then ρ_t is the spectrum of $A + S_t$. The appearance of complex Burgers equation in

[8]There is a rather annoying difference in sign conventions between the use of Cauchy transforms in the theory of Herglotz function and in free probability theory. Herglotz functions are analytic function $f : \mathbb{C}_+ \to \mathbb{C}_+$. A classical representation theorem asserts that each Herglotz function is the potential of a positive measure, that is $f = \mathcal{C}\mu$ where μ is a positive measure on $\mathbb{R}$. This sign convention ensures that Herglotz functions are closed under composition and that they are matrix monotone [6]. In the literature on free probability, the Cauchy transform is normalized as $G_\mu(z) := -\mathcal{C}\mu(z)$. The transform G_μ is an anti-Herglotz function, that is $G_\mu : \mathbb{C}_+ \to \mathbb{C}_-$. This convention has the unfortunate effect of flipping the sign in the inversion formula. For simplicity, assume μ has a density ρ. Then

$$\lim_{y\downarrow 0} \operatorname{Re} G_\mu(x+iy) = \pi\mathcal{H}\rho(x), \quad \lim_{y\downarrow 0} \operatorname{Im} G_\mu(x+iy = -\pi\rho(x),$$

in comparison with (4.8). Since Matytsin's starting point is the function $f = v + i\pi\rho$, we have preferred to use the convention on Herglotz functions, rather than the convention of free probability.

Matytsin's work also has a free probability interpretation. However, now we must consider a free Brownian bridge, not a free Brownian motion. Let S be a standard semicircular operator, and let A and B be operators with spectral measures ρ_A and ρ_B respectively, and suppose A,B and S are free with respect to one another. Then the density $\rho(\cdot, t)$ corresponding to the solution to (1.4) is the law of the free variable [9]

$$X_t = (1-t)A + tB + \sqrt{t(1-t)}\, S, \quad t \in [0,1]. \tag{4.13}$$

4.4. **The domain for (1.4).** In order to solve (1.4) by the method of characteristics, we must extend the spatial domain from x in the support of ρ, to a suitable subset of the complex plane. Classical potential theory and the calculation above suggests that the natural extension of ρ to the complex plane is provided by its Cauchy transform. Further, the calculations above show that each positive density ρ induces a velocity field $\pi\mathcal{H}(\rho)(x)$. However, when solving the boundary value problem (1.2)–(1.4), it is clear that the 'shooting velocity' from ρ_A cannot simply be $\pi\mathcal{H}(\rho_A)$, since this would imply $\rho_t = \rho_A \boxplus \nu_t$. Thus, we are finally led to separate the velocity field into two parts: a self evolution corresponding to the Hilbert transform and free convolution, and a second part, which 'shoots' ρ_A at $t = 0$ to ρ_B at $t = 1$. [10]

This idea is implemented as follows. For each $\delta > 0$ we define the open strip $\mathcal{S}_\delta = \{z \,|\, |\mathrm{Im}(z)| < \delta\}$. Given ρ_A and ρ_B we seek $\delta > 0$ and a function $f = f_+ + f_-$ such that f_- is analytic in $\mathcal{S}_\delta$ and

$$f_+(z,t) = \mathcal{C}\rho(\cdot, t), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{4.14}$$

Observe that this separates the velocity field $v(x,t)$, $x \in \mathbb{R}$ into two parts:

$$v_+(x,t) = \pi\mathcal{H}(\rho(\cdot,t))(x) = \mathrm{Re}\, f_+(x,t), \quad v_-(x,t) = \mathrm{Re}\, f_-(x,t), \quad x \in \mathbb{R}. \tag{4.15}$$

The first part corresponds to the 'self-velocity' of free convolution. The second part is the 'driving' velocity that steers $\rho(\cdot, t)$ from ρ_A to ρ_B. The point is that the solution formula (1.5) may be rigorously applied for $z \in S_\delta$.

The difference between the velocity $v(x,t)$ and $\mathcal{H}\rho(\cdot, t)$ may be computed for the exact solution (1.25) constructed in [5]. In this example, we find that

$$f_-(z,t) = \left(b(t) + \frac{1}{2\sigma^2(t)} \right) z. \tag{4.16}$$

In this case, $S_\delta = \mathbb{C}$ and $f_-(z,t)$ is an entire function of z. Matytsin always assumes that the 'driving' velocity field is a polynomial, which is what allows him to use the method of characteristics. However, this situation is atypical, and for many other interesting solutions, f_- is analytic only in a strip of finite width.

4.5. **A generalized Hilbert transform.** Aficionados of integrable systems are aware of many unexpected connections between distinct models. For instance, it has been known since the work of Airault, McKean and Moser that the CM system describes the evolution of rational solutions to the KdV hierarchy [2]. Roughly, the poles of certain rational solutions to KdV evolve according to the CM system. Such connections were treated systematically for the KP hierarchy by Krichever [12]. The

[9] This statement follows Guionnet [8]. But surely one should be able to use a 'true' free Brownian bridge (i.e. a process S_t), rather than just scaling a fixed free variable S.

[10] In the discrete CM system, this splitting correspond to the off-diagonal and diagonal terms of the matrix P defined in (3.2).

version of this set of ideas that is most useful for us is a connection between CM solutions and the doubled Benjamin-Ono equation studied by Abanov, Bettelheim and Wiegmann [1] (see also [17]).

In order to introduce these equations, we will modify the definition of the Hilbert transform as follows. Consider a simple closed curve Γ that separates the complex plane into exterior and interior domains, denoted $\Omega_\pm$ respectively. Given a sufficiently regular function $\varphi : \Gamma \to \mathbb{C}$ we extend it to the analytic functions

$$\varphi_\pm(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{\varphi(s)}{s-z}\, ds, \quad z \in \Omega_\pm. \tag{4.17}$$

The Plemelj formulas provide jump and continuity conditions on the curve $z \in \Gamma$

$$\varphi_-(z) - \varphi_+(z) = \varphi(z), \quad \varphi_-(z) + \varphi_+(z) = \frac{\text{p.v.}}{\pi i} \oint_\Gamma \frac{\varphi(s)}{s-z}\, ds. \tag{4.18}$$

Therefore, we may define the Hilbert transform with respect to Γ, as

$$\mathcal{H}_\Gamma \varphi(z) = \frac{\text{p.v.}}{\pi} \oint_\Gamma \frac{\varphi(s)}{s-z}\, ds = i\left(\varphi_-(z) + \varphi_+(z)\right). \tag{4.19}$$

Of particular importance for us are the following eigenfunctions of the operator $\mathcal{H}_\Gamma$. Using the above formulas, we find that when

$$\varphi(z) = \frac{1}{z-a}, \quad a \in \Omega_+ \cup \Omega_-, \tag{4.20}$$

$$\mathcal{H}_\Gamma \varphi(z) = -i\varphi(z), \quad a \in \Omega_+, \quad \mathcal{H}_\Gamma \varphi(z) = +i\varphi(z), \quad a \in \Omega_-. \tag{4.21}$$

These eigenfunctions are computed as follows. To be concrete, we assume $a \in \Omega_+$. Then $\varphi(z)$ is analytic in a sufficiently small neighborhood of Γ, and the integrand in (4.17) may be written as follows:

$$\frac{\varphi(s)}{s-z} = \frac{1}{z-a}\left(\frac{1}{s-a} - \frac{1}{s-z}\right). \tag{4.22}$$

Substituting in (4.17), we find that

$$\varphi_-(z) = \frac{-1}{z-a}, \quad \varphi_+(z) = 0. \tag{4.23}$$

Similarly, when $a \in \Omega_-$ we find

$$\varphi_-(z) = 0, \quad \varphi_+(z) = \frac{1}{z-a}. \tag{4.24}$$

We substitute (4.23)–(4.24) in (4.19) to obtain (4.21).

4.6. **The doubled Benjamin-Ono equation.** The Benjamin-Ono equation with respect to the curve Γ is the PDE

$$\partial_t f + f\partial_z f = \frac{g}{2}\partial_z^2 \mathcal{H}_\Gamma f, \quad z \in \Gamma, \quad t > 0. \tag{4.25}$$

This PDE is connected to the CM system via the following ansatz. Assume Γ is a sufficiently curve that circles a sufficiently large interval I on the x-axis. Assume N and M are integers, and let $x_1(t) < x_2 < \ldots < x_N(t)$ be points on I, and let $\{w_k(t)\}_{k=1}^M$ be points in $\Omega_+ \subset \mathbb{C}$. We substitute the ansatz

$$f(z,t) = \sum_{j=1}^N \frac{ig}{z - x_j(t)} - \sum_{k=1}^M \frac{ig}{z - w_k(t)}, \tag{4.26}$$

into (4.25), use (4.20)–(4.21), and the identity

$$\frac{1}{z-\alpha}\frac{1}{(z-\beta)^2}+\frac{1}{(z-\alpha)^2}\frac{1}{z-\beta}=\frac{1}{\alpha-\beta}\left(\frac{1}{(z-\alpha)^2}-\frac{1}{(z-\beta)^2}\right), \tag{4.27}$$

to obtain the ordinary differential equations

$$\dot{x}_j = \sum_{l=1,l\neq j}^{N}\frac{ig}{x_l-x_j}-\sum_{m=1}^{M}\frac{ig}{w_m-x_j}, \quad j=1,\dots,N, \tag{4.28}$$

$$\dot{w}_k = -\sum_{l=1}^{N}\frac{ig}{w_k-x_l}+\sum_{m=1,m\neq k}^{M}\frac{ig}{w_k-w_m}, \quad k=1,\dots,M. \tag{4.29}$$

Thus, we obtain a closed system of $N+M$ equations for the evolution of the poles. Now it turns out that one may further differentiate these equations, and use (4.28) and (4.29) to eliminate $\dot{x}_j$ and $\dot{w}_k$ to find that

$$\ddot{x_j}=\sum_{l\neq j}\frac{2g^2}{(x_j-x_l)^3}, \quad j=1,\dots,N. \tag{4.30}$$

Thus, the poles x_j satisfy the CM system (1.21). In order to ensure that these solutions stay real, we must ensure that the initial w's are such that $\dot{x}_j(0)$ is real for each j. Equation (4.28) shows that this always holds provided M is even and the poles w are pairs of complex conjugates. The w's satisfy a complementary CM system. Except for the coupling through the initial condition, these two systems evolve independently (despite what (4.28)–(4.29) suggests at first sight).

4.7. τ **functions and the zero dispersion continuum limit.** Though the details are not presented here, it is a short step from (ii) to the use of τ-functions to construct multi-soliton solutions to (1.1). Roughly, exact solutions to the CM system may be expressed as determinants using τ-functions for the KP hierarchy. Further, the $N\to\infty$, $g=O(1)$ continuum limit of these solutions has already been worked out by Abanov *et al*. However, it does remain to see how this approach yields solutions to the complex Burgers equation when we also take $g\to 0$. If we recall that the zero-dispersion limit of (real valued) KdV is Burgers equation – this is the core observation of Lax and Levermore – we see exactly how the $N\to\infty$ limit studied by Matytsin is an analogous zero-dispersion limit. In order to explain this point, let us examine the scaling limit more carefully.

5. The scaling limit

Our starting point is the Calogero-Moser system with the rescaled Hamiltonian

$$H(x,p) = \frac{N}{2}|p|^2 - \frac{\kappa^2}{2N^3}\sum_{k\neq j}\frac{1}{(x_j - x_k)^2}. \tag{5.1}$$

In the notation of equation (1.20), we have chosen

$$g = \frac{i\kappa}{N}, \tag{5.2}$$

where κ is a fixed parameter, independent of N. In most of what follows $\kappa = 1$. This parameter is included to provide a unified treatment of the attractive and repulsive CM system.

Recall that fluid equations such as (1.1) may be written in either the Eulerian or Lagrangian formulation. We first write the equations of motion for the particle system with Hamiltonian (5.1) in a way that is suggestive of the continuum limit in the Lagrangian formulation. To this end, let $\alpha \in [0,1]$ denote the material coordinate; the position of the particle $x(\alpha,t)$ is given implicitly through the relationship

$$\int_{-\infty}^{x(\alpha,t)} \rho(s,t)\,ds = \alpha, \tag{5.3}$$

and the map $\alpha \mapsto x(\alpha,t)$ is an increasing function. At points where $\rho(x,t) > 0$, we may differentiate (5.3) to obtain the relationship

$$x'(\alpha,t) := \partial_\alpha x(\alpha,t) = \frac{1}{\rho(x,t)}. \tag{5.4}$$

The Lagrangian form of equation (1.1) is the system

$$\partial_t x(\alpha,t) = u(\alpha,t), \tag{5.5}$$

$$\partial_t u(\alpha,t) = -\kappa^2\pi^2\frac{x''}{(x')^4}, \tag{5.6}$$

where the Eulerian and Lagrangian velocity fields are related via

$$u(\alpha,t) = v(x(\alpha,t),t) = v(x,t). \tag{5.7}$$

The equivalence between the two systems of equations *at points where* $\rho(x,t)$ is strictly positive and differentiable may be established by differentiating (5.5) and using (5.3), (5.4) and (5.7).

We now explain how to obtain (5.5) as the $N \to \infty$ limit of the CM system with Hamiltonian (5.1). For each N, let us denote the vector $x \in \mathbb{R}^N$ (resp. v, p), as a function $x^{(N)}(\alpha,t)$ defined at each lattice point $\alpha_k = k/N$, $1 \leq k \leq N$ by the relation

$$x^{(N)}(\alpha_k,t) = x_k(t), \quad p^{(N)}(\alpha_k,t) = p_k(t), \quad u^{(N)}(\alpha_k,t) = u_k(t). \tag{5.8}$$

In these variables the equations of motion for the Hamiltonian (5.1) take the form

$$\partial_t x^{(N)}(\alpha,t) = Np^{(N)}(\alpha,t), \tag{5.9}$$

$$\partial_t p^{(N)}(\alpha,t) = \frac{2\kappa^2}{N^3}\sum_{\beta\neq\alpha}\frac{1}{\left(x^{(N)}(\beta,t) - x^{(N)}(\alpha,t)\right)^3}, \tag{5.10}$$

where the Lagrangian coordinates α and β range over the index set k/N, $1 \le k \le N$. In the limit, we expect the particle velocities to be well-defined, and working with $v^{(N)}$ instead of $p^{(N)}$, we find

$$\partial_t x^{(N)}(\alpha,t) = u^{(N)}(\alpha,t), \tag{5.11}$$

$$\partial_t u^{(N)}(\alpha,t) = \frac{2\kappa^2}{N^2}\sum_{\beta\neq\alpha}\frac{1}{\left(x^{(N)}(\beta,t)-x^{(N)}(\alpha,t)\right)^3}. \tag{5.12}$$

It is easy to see how (5.11) converges to (5.5), when we scale the momentum as in (5.1). Indeed, with this scaling, the kinetic energy in the Hamiltonian is

$$\frac{N}{2}\sum_{k=1}^{N} p_k^2 = \frac{1}{2N}\sum_{k=1}^{N} u_k^2 = \frac{1}{2N}\sum_{k=1}^{N}\left|u^{(n)}(\alpha_k,t)\right|^2 \approx \frac{1}{2}\int_0^1 |u(\alpha,t)|^2\, d\alpha, \tag{5.13}$$

using the Riemann sum to approximate the integral under the assumption that

$$u^{(n)}(\alpha_k,t) \approx u(\alpha_k,t), \quad 1 \le k \le N. \tag{5.14}$$

However, it is necessary to treat the singular terms systematically in order to obtain the continuum limit (5.6) from (5.12) under the scaling (5.2). We will explain a general procedure to approximate such singular sums.

5.1. **Approximations with the Euler-MacLaurin formula.** Assume $f:[0,1]\to\mathbb{R}$ is a C^∞ function. The Euler-MacLaurin formula

$$\int_0^1 f(\alpha)\,d\alpha = \frac{1}{N}\left(\frac{1}{2}f(0)+\sum_{k=1}^{N_1} f(\frac{k}{N})+\frac{1}{2}f(1)\right) + \frac{1}{12N^2}\left(f'(0)-f'(1)\right) - \frac{1}{720N^4}\left(f'''(0)-f'''(1)\right)+\dots \tag{5.15}$$

provides an asymptotic expansion for the Riemann integral of f. Observe that the error terms are controlled by the derivatives of f at the endpoints 0 and 1.

Our interest lies in the leading order asymptotics of sums of the form

$$\sum_{k=1,k\neq j}^{N}\frac{1}{(x_k-x_j)^p}, \tag{5.16}$$

when $j = N\alpha$ where $\alpha\in(0,1)$ is held fixed, and p is an integer ($p=1$, 2 and 3). In order to apply the Euler-MacLaurin formula to such functions, we will replace the summand with a suitably regularized function f_p, ($p=1$, 2 and 3) that regularizes the singularity near $k=j$. In what follows, we fix α, set $j=\alpha N$, and we use $k=\beta N$ to denote the indices being summed over.

5.1.1. *Case 1.* $\boxed{p=1}$. Fix $\varepsilon>0$ and consider separately the sum over β such that $|\alpha-\beta|\ge\varepsilon$ and $|\alpha-\beta|<\varepsilon$. In the region $|\alpha-\beta|\ge\varepsilon$, the summand is non-singular and we find

$$\lim_{N\to\infty}\frac{1}{N}\sum_{|j-k|<\varepsilon N}\frac{1}{x_k-x_j} = \int_{|\alpha-\beta|>\varepsilon}\frac{1}{x(\beta)-x(\alpha)}\,d\beta. \tag{5.17}$$

In order to treat the singular region, we consider the function

$$f_1(s) = \frac{1}{x(\alpha+s)-x(\alpha)} - \frac{1}{x'(\alpha)s}, \quad |s|<\varepsilon. \tag{5.18}$$

Under the assumption that $x(\beta)$ is C^2 at $\beta = \alpha$, we find that

$$\lim_{s\to 0} f_1(s) = \frac{1}{2}\frac{x''(\alpha)}{x'(\alpha)}. \tag{5.19}$$

We apply the Euler-MacLaurin approximation to f_1 on the interval $|s| < \varepsilon$ to obtain

$$\frac{1}{N}\sum_{|l|<\varepsilon N}\left(\frac{1}{x(\alpha+l/N)-x(\alpha)} - \frac{N}{x'(\alpha)l}\right) = \int_{|\alpha-\beta|<\varepsilon} f_1(\beta)\,d\beta + O(\frac{1}{N}). \tag{5.20}$$

On the other hand, we also have

$$\sum_{|l|<\varepsilon N,\, l\neq 0} \frac{1}{x'(\alpha)l} = 0. \tag{5.21}$$

Therefore, we can remove this term from the summand. Further, we can also remove the regularizing term in the integrand, provided we replace the divergent integral with its principal value. Combining these steps, we find that

$$\lim_{N\to\infty}\sum_{|l|<\varepsilon N} \frac{1}{x(\alpha+l/N)-x(\alpha)} = \text{p.v.}\int_{|\alpha-\beta|<\varepsilon} \frac{1}{x(\beta)-x(\alpha)}\,d\beta. \tag{5.22}$$

5.1.2. *Case 2.* $\boxed{p=2}$. As in Case 1, it is only necessary to consider the sum in the range $|\beta-\alpha| < \varepsilon$ for some fixed $\varepsilon > 0$. The regularizing function in this case takes the form

$$f_2(s) = \frac{1}{\left(x(\alpha+s)-x(\alpha)\right)^2} - \frac{c_0+c_1 s}{s^2}, \quad |s| < \varepsilon, \tag{5.23}$$

where the coefficients c_0 and c_1 are determined by the condition that $f_2(s)$ is continuous at $s = 0$. We find after some algebra (as explained below for the case $p = 3$) that

$$f_2(s) = \frac{1}{\left(x(\alpha+s)-x(\alpha)\right)^2} - \frac{1}{(x'(\alpha)^2 s^2} + \frac{x''(\alpha)}{x'(\alpha)s}, \quad |s| < \varepsilon, \tag{5.24}$$

Thus, adding the regularizing terms, and using the fact that $\sum_{|j-k|<\varepsilon, j\neq k}(k-j)^{-1} = 0$ and $\sum_{l=1}^{\infty} l^{-2} = \zeta(2) = \pi^2/6$, we obtain the approximation

$$\begin{aligned}\frac{1}{N}\sum_{|j-k|<\varepsilon N, j\neq k}\frac{1}{(x_k-x_j)^2} &= \frac{N\pi^2}{3x'(\alpha)^2}\\ &+\frac{1}{N}\left(\sum_{|j-k|<\varepsilon N, j\neq k}\frac{1}{(x_k-x_j)^2} - \frac{N^2}{(x'(\alpha)^2(j-k)^2} + \frac{Nx''(\alpha)}{x'(\alpha)(j-k)}\right) + O(\frac{1}{N}).\end{aligned}$$

By the Euler-MacLaurin expansion, the second term on the right hand side is $O(1)$, since it converges to the integral $\int_{|s|<\varepsilon} f_2(s)\,ds$. Thus,

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{|j-k|\leq\varepsilon N, j\neq k}\frac{1}{(x_k-x_j)^2} = \frac{\pi^2}{3x'(\alpha)^2}. \tag{5.25}$$

This also shows that the discrete CM potential energy has the scaling limit

$$\lim_{N\to\infty}\frac{-\kappa^2}{N^3}\sum_{k\neq j}\frac{1}{(x_k-x_j)^2} = -\kappa^2\int_0^1 \frac{\pi^2}{3x'(\alpha)^2}\,d\alpha. \tag{5.26}$$

The above convergence of the potential energy is enough to deduce that (5.6) is the continuum limit of the CM system, but let us also show that the same technique yields the convergence of (5.12) to (5.6).

5.1.3. *Case 3.* $\boxed{p=3}$. Observe that in Case 2, the leading order term is local (i.e. not a singular integral, unlike Case 1). This is typical for all $p \geq 2$. When $p = 3$, we proceed as for $p = 2$, seeking a function

$$f_3(s) = \frac{1}{\left(x(\alpha+s) - x(\alpha)\right)^3} - \frac{c_0 + c_1 s + c_2 s^2}{s^3}, \tag{5.27}$$

that is continuous as $s \to 0$. Assuming that we have found such an expansion, we fix an $\varepsilon > 0$, use the fact that

$$0 = \sum_{|j-k|<\varepsilon N, j\neq k} \frac{1}{(j-k)^3} = \sum_{|j-k|<\varepsilon N, j\neq k} \frac{1}{(j-k)} \tag{5.28}$$

and apply the Euler-MacLaurin expansion to f_3 in the interval $(\alpha-\varepsilon, \alpha+\varepsilon)$, to obtain the asymptotic expansion

$$\frac{1}{N^2} \sum \frac{1}{(x_k - x_j)^3} = c_1 \sum \frac{1}{(j-k)^2} + \frac{1}{N^2} \sum f_3(x_k - x_j), \tag{5.29}$$

where the sum is over the index k in the range $|k-j| < \varepsilon$, $k \neq j$.

The coefficients c_0, c_1 and c_2 are computed as follows. For brevity, let us denote the Taylor expansion

$$x(\alpha+s) - x(\alpha) = a_1 s + a_2 s^2 + a_3 s^3 + \dots \tag{5.30}$$

The continuity of f_3 at $s = 0$ imposes the requirement that

$$s^3 - (a_1 s + a_2 s^2 + a_3 s^3)^3 \left(c_0 + c_1 s + c_2 s^2\right) = O(s^6), \tag{5.31}$$

which yields the following set of polynomial equations for c_0, c_1 and c_2 in terms of a_1, a_2 and a_3:

$$\begin{aligned} O(1): &\qquad c_0 a_1^3 = 1, \\ O(s): &\qquad c_1 a_1^3 + 3 c_0 a_1^2 a_2 = 0, \\ O(s^2): &\qquad a_1^3 c_2 + 3 c_1 a_1^2 a_2 + 3 c_0 \left(a_1 a_2^2 + a_1^2 a_3\right) = 0, \end{aligned}$$

These equations have a unique solution when $a_1 \neq 0$. We find that

$$c_0 = \frac{1}{a_1^3}, \quad c_1 = -3\frac{a_2}{a_1^4}, \quad c_2 = 6\frac{a_2^2}{a_1^5} - 3\frac{a_3}{a_1^4}. \tag{5.32}$$

Equations (5.29) tells us that we only need c_1. Thus, using the fact that a_1 and a_2 are the first two terms in the Taylor expansion of $x(\alpha+s) - x(\alpha)$, we use (5.29) and (5.32) to find that for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{|j-k|<\varepsilon, j\neq k} \frac{1}{(x_k - x_j)^3} = -\frac{\pi^2}{3}\frac{3}{2}\frac{x''(\alpha)}{x'(\alpha)^4} = -\frac{\pi^2}{2}\frac{x''(\alpha)}{x'(\alpha)^4}. \tag{5.33}$$

We apply (5.33) to (5.12) to obtain equation (5.6), the continuum CM system in Lagrangian coordinates.

6. The Lax pair for the continuum limit

We have now found two distinct descriptions of the continuum limit of the CM system. The Eulerian formulation (1.1), and the Lagrangian formulation (5.5) and (5.6). It is easy to guess the form of the Lax pair for the Lagrangian formulation using the formulas (3.2)–(3.3), though these must be written in weak form.

Let $h : [0,1] \to \mathbb{C}$ be a test function that vanishes at all points where $x'(\alpha) = 0$, and define the singular integral operators

$$Ph(\alpha) = u(\alpha)h(\alpha) + \kappa \,\text{p.v.} \int_0^1 \frac{h(\beta)}{x(\beta) - x(\alpha)}\, d\beta \tag{6.1}$$

$$Qh(\alpha) = -\kappa \,\text{p.v.} \int_0^1 \frac{h(\beta) - h(\alpha)}{(x(\beta) - x(\alpha))^2}\, d\beta. \tag{6.2}$$

These operators are formal limits of the Lax pair (3.2)–(3.3), when $g = i\kappa/N$. The condition that the test function h vanishes at all points where $x'(\alpha) = 0$ is necessary to ensure that the integrals are meaningful. This condition is cumbersome and it is simpler to work with the operators written in Eulerian coordinates.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth test function and recall that $\rho(x,t)$ and $v(x,t)$ are the Eulerian density and velocity field for (1.1). We define the operators

$$L\varphi(x) = v(x,t)\varphi(x) + \kappa \,\text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{s - x}\, \rho(s,t)\, ds, \tag{6.3}$$

$$M\varphi(x) = -\kappa \,\text{p.v.} \int_{\mathbb{R}} \frac{f(s) - f(x)}{(s-x)^2}\, \rho(s,t)\, ds. \tag{6.4}$$

We claim that when $\kappa = 1$, (1.1) and (1.4) is equivalent to the Lax equation

$$\dot{L} = [L, M]. \tag{6.5}$$

While we have been led to this Lax pair by taking the continuum limit of the N particle CM system, it may be used directly for the analysis of (1.1). The verification that (6.5) is equivalent to (1.1) requires a careful computation, at the heart of which is the following commutator identity for the Hilbert transform. As with many Fourier identities, it is enough to check the lemma on functions in the Schwartz class $\mathcal{S}(\mathbb{R})$, and then to use standard density arguments to extend the identity to general function classes, say $L^p(\mathbb{R})$. Given a Schwartz function $n \in \mathcal{S}(\mathbb{R})$, we define the multiplication operator $\mathcal{M}_n : \mathcal{S} \to \mathcal{S}$ by $h \mapsto \mathcal{M}_n h$, $\mathcal{M}_n h(x) = n(x)h(x)$.

Lemma 6.1. *Suppose $p \in \mathcal{S}(\mathbb{R})$. Then we have the following commutator identity*

$$[\mathcal{H}, \mathcal{M}_{\mathcal{H}p}] = \mathcal{M}_p + \mathcal{H}\mathcal{M}_p\mathcal{H}. \tag{6.6}$$

The precise domain of definition of the above operators takes some care, since the Hilbert transform does not map $\mathcal{S}(\mathbb{R})$ into itself, but I'm going to ignore these technicalities for now. The main point is really that the above lemma provides the right way to think about the cancellation (4.28), which underlies the equivalence between the Lax equation (3.5) and the discrete CM system (3.1).

Proof. The lemma is best proved with Fourier multipliers. Let $\varphi \in \mathcal{S}(\mathbb{R})$ and consider

$$[\mathcal{H}, \mathcal{M}_{\mathcal{H}p}]\varphi = \mathcal{H}\mathcal{M}_n\varphi - \mathcal{M}_n\mathcal{H}\varphi.$$

Since $\mathcal{H}$ has multiplier $-i\mathrm{sgn}(\xi)$ and $\widehat{\mathcal{M}_n\varphi} = \hat{n} \star \hat{\varphi}$, we find that

$$\widehat{\mathcal{H}\mathcal{M}_{\mathcal{H}p}\varphi} = -i\mathrm{sgn}(\xi)\left(\hat{\mathcal{H}p} \star \hat{\varphi}\right)(\xi) = -\mathrm{sgn}(\xi)\int_{\mathbb{R}} \mathrm{sgn}(\eta)\hat{p}(\eta)\hat{\varphi}(\xi-\eta)\,d\eta. \tag{6.7}$$

A similar computation reveals that

$$\widehat{\mathcal{M}_{\mathcal{H}p}\mathcal{H}\varphi} = -\int_{\mathbb{R}} \mathrm{sgn}(\eta)\mathrm{sgn}(\xi-\eta)\hat{p}(\eta)\hat{\varphi}(\xi-\eta)\,d\eta. \tag{6.8}$$

Thus, we have found that the Fourier transform of $[\mathcal{H}, \mathcal{M}_{\mathcal{H}p}]\varphi$ is

$$\int_{\mathbb{R}} \mathrm{sgn}(\eta)\left(\mathrm{sgn}(\xi-\eta) - \mathrm{sgn}(\xi)\right)\hat{p}(\eta)\hat{\varphi}(\xi-\eta)\,d\eta. \tag{6.9}$$

Now an elementary computation shows that

$$\mathrm{sgn}(\eta)\left(\mathrm{sgn}(\xi-\eta) - \mathrm{sgn}(\xi)\right) = 1 - \mathrm{sgn}(\eta)\mathrm{sgn}(\xi-\eta), \quad \xi,\eta \in \mathbb{R}. \tag{6.10}$$

Therefore, the Fourier transform of $[\mathcal{H}, \mathcal{M}_{\mathcal{H}p}]\varphi$ is

$$\int_{\mathbb{R}} \hat{p}(\eta)\hat{\varphi}(\xi-\eta)\,d\eta - \int_{\mathbb{R}} \mathrm{sgn}(\eta)\mathrm{sgn}(\xi-\eta)\hat{p}(\eta)\hat{\varphi}(\xi-\eta)\,d\eta, \tag{6.11}$$

which is the Fourier transform of $(\mathcal{M}_p + \mathcal{H}\mathcal{M}_p\mathcal{H})\varphi$. □

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