Arch. Rational Mech. Anal. 162 (2002) 193–246
Digital Object Identifier (DOI) 10.1007/s002050200189

Gradient Systems with Wiggly Energies and Related Averaging Problems

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Dedicated to Prof. C. M. Dafermos on his sixtieth birthday

Communicated by F. Otto

Abstract

Gradient systems with wiggly energies of the form

$$\dot{x} = -\nabla\Big(F(x) + \varepsilon A\Big(\frac{x}{\varepsilon}\Big)\Big), \qquad x \in \mathbb{R}^d$$

and $A : \mathbb{T}^d \to \mathbb{R}$ were proposed by Abeyaratne, Chu & James [2] to study the kinetics of martensitic phase transitions. Their model may be recast in the framework of the theory of averaging as a dynamical system on $\mathbb{R}^d \times \mathbb{T}^d$, with the slow variable $x \in \mathbb{R}^d$ and fast variable $\theta \in \mathbb{T}^d$. However, this problem lies completely outside the classical theory of averaging, since the vertical flow on $\mathbb{T}^d$ is not ergodic for sets of positive measure, and we must interpret averages to mean weak limits.

We obtain rigorous averaging results for $d = 2$. We use Schwartz's generalization of the Poincaré-Bendixson theorem [37] to heuristically derive homogenized equations for the weak limits. These equations depend on the ω-limit sets for the vertical flow on fibres. When the vertical flow is structurally stable, we use the persistence of hyperbolic structures to prove that these are the correct equations. We combine these theorems with a study of two-parameter bifurcations of flows on $\mathbb{T}^2$ to characterize the weak limits. Our results may be interpreted as follows. The space $\mathbb{R}^2$ breaks into: $(\cdot 1)$ a bounded open set surrounding $\{\nabla F^{-1}(0)\}$ where there is only sticking, $(\cdot 2)$ a transition region outside this set, where the dynamics is a combination of sticking and slipping, and $(\cdot 3)$ the rest of the plane, which contains a countable number of resonance zones, with nonempty interior, and their nowhere dense complement. Inside a resonance zone the direction of the weak limits is given by the rotation number $\rho \in \mathbb{Q}$. The Cantor set structure of the resonance zones is described by well-known results of Arnol'd [7] and Herman [27] in the theory of circle diffeomorphisms. Consequently, the homogenized equations vary on all scales. We also study the linear transport equation associated with the wiggly gradient flow, and show that its homogenization limit is not well posed.

SMYSHLYAEV has studied this problem independently, and some of our results are similar [39].

1. Introduction

1.1. Motivation

Let $F : \mathbb{R}^d \to \mathbb{R}$ generate the gradient dynamical system

$$\dot{x} = -\nabla F(x). \tag{1}$$

Under natural conditions on F, namely that it is twice differentiable, and $F(x) \to \infty$ as $|x| \to \infty$, we know that almost all points $x \in \mathbb{R}^d$ are attracted to the wells of F. This is the invariance principle of LASALLE and BARBAŠIN & KRASOVSKIĬ [31, 9].

ABEYARATNE, CHU & JAMES [2] have considered a remarkable modification of (1) to explain their experiments. They observed hysteresis loops for the volume fraction (i.e., $d = 1$) in a martensitic phase transformation, and discovered that experimental observations were completely at odds with the solutions to (1) with periodic forcing. To resolve this, they considered a periodic function $A : \mathbb{R} \to \mathbb{R}$, and the kinetic law with a *wiggly energy*

$$\dot{x} = -\nabla\Big(F(x) + \varepsilon A\Big(\frac{x}{\varepsilon}\Big)\Big). \tag{2}$$

The physical insight underlying the modification is that the macroscopic dynamics may depend essentially on microstructural events like getting stuck in local minima. The goal is to derive an averaged equation for the macroscopic variable, x, that includes the effect of the microstructure, A. This was done by a weak convergence argument in [2], and the authors found excellent qualitative agreement between the macroscopic kinetic law and experimental measurements over a wide loading range.

The present article is a mathematical study of wiggly gradient systems for $d = 2$, especially the rigorous derivation of averaged or homogenized equations. It is important to mention that SMYSHLYAEV has studied this problem independently from a slightly different point of view, and there is some overlap in our conclusions [39].

We were motivated by some problems in mechanics with wiggly energies in addition to the experiments on martensites. Let us review these, and then comment on the physical meaning of multidimensional problems. ABEYARATNE [1] has derived the following examples based on Frenkel-Kontorova models: flow rules for plasticity based on motion of dislocations, and the adsorption of a gas onto a periodic substrate. WEISS & ELMER's [44] generalization of the Tomlinson model for dry friction is based on a wiggly energy (see also [30]). TRUSKINOVSKY & ZANZOTTO [41] have studied metastability in Ericksen's bar with a wiggly energy (though their model is quite different from the phenomenological model of ABEYARATNE *et al.* [2]).

A multi-dimensional wiggly energy does not correspond directly to any of these models. A plausible physical model in higher dimensions would be the evolution of a vector-valued order parameter. In two dimensions, we can think of the evolution of averages of a two-dimensional magnetic field $\mathbf{m} : \Omega \to S^1$ on some domain Ω. Then the mean magnetization $x = \int_\Omega \mathbf{m}/|\Omega| \in B(0, 1)$, and it could be interesting to study the reorientation of x under forcing. The evolution depends on fine details such as the domain and wall structure in the sample. But at a crude level, we may suppose that the detailed microstucture of the sample is modeled by some generic wiggly perturbation. This system has the feature that both components of x are comparable and correlated. However, we have not compared our analysis with any experiments on such systems yet. There have also been several articles in the condensed-matter physics literature on "landscape paradigms" (for example, the review [21]) that treat similar phenomenological models with less immediate physical interpretation. An amusing mental picture is to think of a light particle sliding down a rough slope. The particle takes a jerky path downhill, possibly getting stuck along the way.

There is also mathematical motivation to study this problem. The theory of averaging has a long history in dynamical systems, especially the results of the Soviet school, from the work of Bogoliubov, Krylov and Mitropolsky for single phase systems, to the definitive theorems of Anosov, Arnol'd and Neishtadt for multi-phase systems. We suggest the encylopaedic monongraph [8] as a review on averaging, and [33] for complete proofs. The averaging results that we derive are of a completely different nature, largely because of the differences in the source of the problem. The classical theory of averaging relates to perturbations of integrable Hamiltonian systems, and hypotheses of ergodicity or non-resonance play a crucial role. For gradient flows, as in wiggly energies, the hypothesis of ergodicity fails. Our response is to use weak limits to mean averages. In this sense our work is closer in spirit to homogenization of partial differential equations (see, e.g., Section 9), and one of the central questions (and the only one we tackle) is the derivation of averaged equations. Even this turns out to be surprisingly complicated, and the averaged equations are continuous, but not Lipshitz. In a different, but related vein, BORNEMANN has recently shown that weak convergence methods can be used to provide a unified view of some results in the classical theory of averaging [11].

1.2. The underlying bifurcation problem

We are interested in deriving a differential equation for a weak limit $x^0(t)$ with initial condition $x^0(0) = x_0 \in \mathbb{R}^2$. For $d \geqq 2$ the dynamics are remarkably rich in the generic case. This is because of an underlying bifurcation problem that determines the weak limits. The macroscopic scale acts as a bifurcation parameter, and as we move to different points in $\mathbb{R}^d$ there is an underlying toral flow that changes topologically. Bifurcations in the microscopic flow are seen macroscopically as the change in motion between slip, stick, and an intermittent combination of the two.

The connection is via the following blow-up transformation. For $\varepsilon > 0$ let

$$\alpha^\varepsilon = \frac{x_0}{\varepsilon} \mod \mathbb{Z}^d.$$

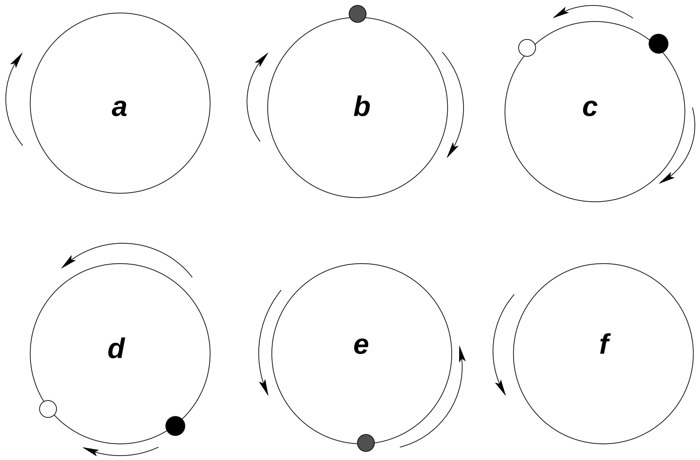

Fig. 1. (a) All orbits are periodic, $x_0 > 1$. (b) Birth of a semi-stable rest point in a saddle-node bifurcation for $x_0 = 1$. (c) Semi-stable rest point splits into a stable and unstable rest point, $-1 < x_0 < 1$. (d), (e), (f) The scenario unfolds in reverse.

We identify α^ε with a point in $[0, 1)^d$, and call it the phase of x_0/ε. The blow-up is

$$z = \frac{x - x_0}{\varepsilon} + \alpha^\varepsilon. \tag{3}$$

Since a is $\mathbb{Z}^d$-periodic we have

$$a\Big(\frac{x}{\varepsilon}\Big) = a\Big(\frac{x - x_0}{\varepsilon} + \alpha^\varepsilon\Big) = a(z).$$

Equation (2) may be rewritten in new coordinates as

$$\varepsilon\dot{z} = f(x_0 + \varepsilon(z - \alpha^\varepsilon)) + a(z).$$

Rescale time by setting $\tau = t/\varepsilon$. Then

$$\frac{dz}{d\tau} = f(x_0 + \varepsilon(z - \alpha^\varepsilon)) + a(z), \qquad z \in \mathbb{R}^d, \tag{4}$$

and in the limit $\varepsilon = 0$ we obtain the *microscopic vector field*

$$\frac{dz}{d\tau} = f(x_0) + a(z). \tag{5}$$

This is a vector field on the torus. The term x_0 is a parameter in (5), and the microscopic flow bifurcates as x_0 varies.

Let us illustrate this with a one-dimensional example. For simplicity suppose that $F(x) = x^2/2$, so that $f(x) = -x$ ranges from ∞ to $-\infty$, and $a(z) = \sin 2\pi z$. The qualitative change in the microscopic phase portrait is seen in Fig. 1. According to the macroscopic kinetic law derived in [2] and Section 4.2, the regions on the macroscale where all microscopic orbits are periodic ((a) and (f) in Fig. 1), are identical to the region where $\dot{x}^0 \neq 0$, and the regions where all orbits are trapped between rest points ((b)–(e) in Fig. 1), are identical to the region where $\dot{x}^0 = 0$. Thus we may distinguish between stick or slip on the macroscale by looking at the orbits on the microscale. The correspondence is quantitative: the effective equation is $\dot{x}^0 = 2\pi/T(x)$ where $T(x)$ is the (signed) period of oscillation, and $T(x) = \pm\infty$ when there is a rest point.

The interplay between two scales causes a fascinating bifurcation scenario. In most bifurcation problems, there is a control parameter λ, and variations in λ causes changes in the topological structure of the flow for the vector field $\dot{x} = f(x, \lambda)$. In this problem, all possible bifurcation scenarios are contained within *one* problem, since the bifurcation parameter *is* the macroscale.

1.3. Outline of results

Our analysis is based on classifying the microscopic flows, and using this information on the macroscale. This approach works for $d = 2$ because of topological restrictions. For example, we can combine several powerful results from the theory of circle maps, with simple geometric arguments (persistence of hyperbolic orbits) to derive homogenized equations. We outline our results here, and defer the precise statements of the theorems to later sections.

We show formally in Section 4 that there is a natural differential equation or inclusion for the weak limits for $d = 2$, which is valid for almost all points $x \in \mathbb{R}^2$, and each $A \in C^r(\mathbb{T}^2)$, $r > 3$. This is based on the generalized Poincaré-Bendixson theorem on compact two-manifolds of Schwartz [37].

It is far harder to rigorously justify these equations. What we prove is that in certain (large) regions of phase space the derivation of Section 4 is correct. We call these the regions of sticking and slipping (see Section 5 and Section 6). The region of sticking is an open neighborhood of the critical points of F. Here we prove that $\dot{x}^0 = 0$ (Theorem 2), hence the name "sticking". The region of slipping is where $\{\|\partial_{x_i} F\| > \|\partial_{x_i} a\|_\infty$ for some $i = 1, 2\}$ and the formal calculation of Section 4 is restricted to

$$\dot{x}^0 \in \left\{ \frac{1}{T(x,\theta)} \begin{pmatrix} 1 \\ \rho(x) \end{pmatrix} \right\}_{\theta \in \mathbb{T}^2}. \tag{6}$$

The direction of $\dot{x}^0$ is given by the rotation number ρ, and the magnitude by an appropriate time period of the microscopic flow. We are able to prove this rigorously for x_0 a.e. in the case when $\rho \in \mathbb{Q}$ (Theorems 3 and 4). We typically expect that the set where this assumption holds (a *resonance zone*) is a closed subset with nonempty interior that extends to ∞. We have not proved that (6) holds for $\rho \in \mathbb{R}\backslash\mathbb{Q}$. This remains the most important open problem, and a proof of this fact would result in a highly nontrivial improvement in the regularity of the weak limits following from a deep linearization theorem of Herman [27] on circle maps.

It is not obvious that the inclusion (6) is continuous at $\rho \in \mathbb{R}\backslash\mathbb{Q}$. We consider this question in Section 7. We prove a new result on weak continuity of invariant measures of circle maps that is of independent interest (Theorem 5), and obtain continuity as a corollary.

In Section 8 we give precise meaning to our statements about "typical" behavior, by studying generic bifurcations in the microscopic flows for C^∞ wiggles. Here we use the celebrated theorems of ARNOL'D [6] and HERMAN [27,26] on circle maps, along with some analysis of generic bifurcations to demonstrate the full Cantor set structure that is present for typical wiggly energies (see Lemma 17 and Theorem 6).

We interpret our theorems as follows. The macroscopic phase space breaks into three regions: ($\cdot$1) a bounded open set surrounding $\{\nabla F^{-1}(0)\}$ where there is only sticking, ($\cdot$2) a transition region outside this set, where the dynamics is a combination of sticking and slipping, and ($\cdot$3) the rest of the plane, which contains a countable number of closed domains called resonance zones, with nonempty interior, and their nowhere dense complement. Inside a resonance zone the limiting vector field has constant direction given by the rotation number of the microscopic flow. The direction varies continuously across the boundary of a resonance zone but not smoothly. A typical initial condition starting far outside $\{\nabla F^{-1}(0)\}$ would eventually be attracted to the sticky region surrounding $\{\nabla F^{-1}(0)\}$. However, the path it takes downhill is rough on all scales in the sense that the direction changes like a Cantor function. For example, in the physical context of a magnetization, the transformation would be observed to progress as a jump between states in which the two components are locked in a particular ratio. This idea is illustrated with an example in the next section.

Finally, we contrast our approach with previous work on homogenization of transport equations in Section 9. This section is largely independent of the rest of the article. However, it is an important digression since it identifies the need for a more careful analysis, and suggests some limitations of the homogenization method. We conclude in Section 10 with comments about models for dynamical systems that show rough behavior in time, and examples from physics that we believe make a mathematical study of such problems unavoidable.

Let us comment briefly on the assumptions we make. It suffices to assume that F is C^2. The assumptions on A are a lot more delicate. The smoothness assumptions are made in order to invoke either Sard's theorem, Schwartz's Poincaré-Bendixson theorem, or the linearization theorems of Herman. In all three cases, these theorems are sharp, and the requirement is not technical. In addition we make two global assumptions on the flow of $z' = -\nabla A$. Of these the Morse-Smale requirement is natural and generic (Assumption 1), but the second one is not (Assumption 2) and is undoubtedly technical. The reader uncomfortable with these comments, may safely assume throughout that A is C^∞ and belongs to an open subset of C^∞ defined by the functions with four non-degenerate critical points.

2. Description of an example

Let us now illustrate our results with an example (the details of this computation can be found in [35]):

$$F(x) = \frac{1}{2}\left(\lambda_1|x_1|^2 + \lambda_2|x_2|^2\right), 0 \neq \lambda_1 \neq \lambda_2 \neq 0.$$

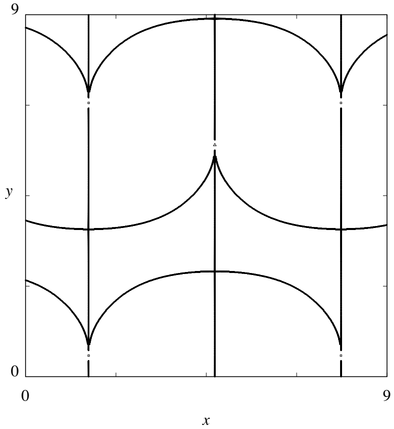

Fig. 2. The critical points and invariant manifolds of the saddle points for (8).

2.1. The wiggly perturbation

The choice of wiggly potential A is governed by the requirements of genericity and simplicity. A simple scalar function on the a torus embedded in $\mathbb{R}^3$ is the height function; however, it generates a flow with a saddle connection. This is non-generic, therefore we tilt the torus to obtain the potential (see [35])

$$A(y, z) = (R + r\cos z)\sin y\cos\beta + r\sin z\sin\beta. \tag{7}$$

The gradient vector field, $-\nabla A$, is

$$\begin{aligned} y' &= a(y, z) = -(R + r\cos z)\cos y\cos\beta, \\ z' &= b(y, z) = r\left(\cos\beta\sin y\sin z - \sin\beta\cos z\right). \end{aligned} \tag{8}$$

The equilibria of (8) in $[0, 2\pi)^2$ are the four points

$$\left(\frac{\pi}{2}, \beta\right), \quad \left(\frac{\pi}{2}, \pi + \beta\right), \quad \left(\frac{3\pi}{2}, 2\pi - \beta\right), \quad \left(\frac{3\pi}{2}, \pi - \beta\right).$$

These points and the invariant manifolds of the saddle points are shown in Fig. 2. In our calculation we have fixed $R = 2$, $r = 1$ and $\beta = \pi/3$.

2.2. Computation of ρ

Figure 3 shows the Cantor singular nature of ρ on the vertical line $\Omega_1 = 4$ ($r\sin\beta = 1/2$ in this picture). The pictures appear in order of increasing magnification. We can easily observe plateaus at "simple" rational values such as $0, 1, 1/2, 2 \ldots$. The width of the plateaus decreases rapidly with the order of the resonance, and they are soon invisible. Figure 4 shows the variation of ρ in a portion of the Ω plane. In this picture the resonance zones for $\rho = 1$ and $\rho = 2$ can be clearly distinguished, but the resolution is not good enough to distinguish other resonance zones.

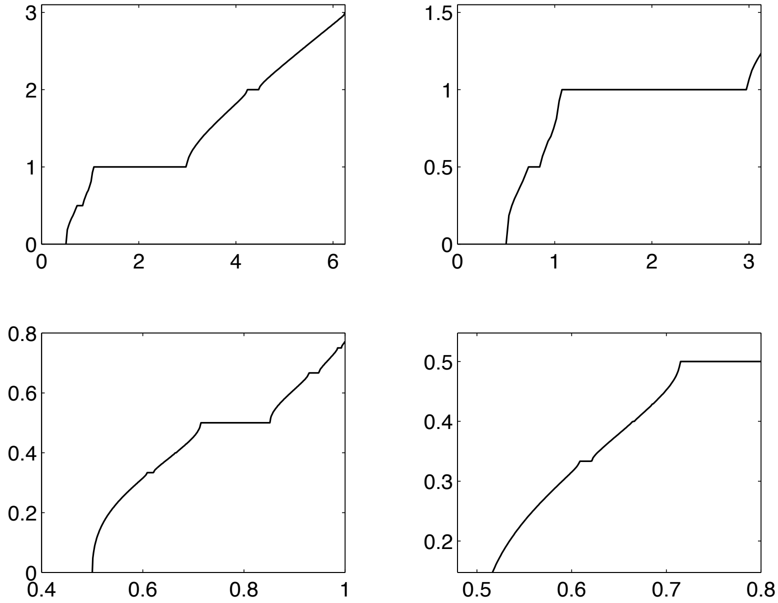

Fig. 3. The variation of ρ on the vertical line $\Omega_1 = 4$. $R = 2$, $r = 1$, and $\beta = \pi/3$ in this computation.

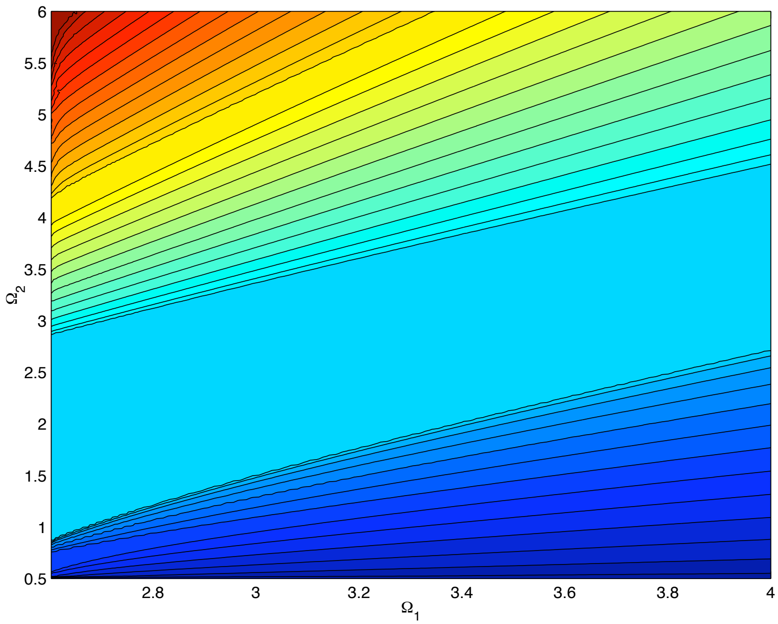

Fig. 4. Grey scale plot of a numerical computation of ρ with $R = 2$, $r = 1$, and $\beta = \pi/3$. The broad swathe is the region $\rho = 1$. The region $\rho = 2$ intersects the boundary at $\Omega_2 \approx 4$.

2.3. Characterization of weak limits

Figure 5 is a schematic picture of the weak limits. This is not a phase portrait since the time parametrization is typically not unique on one of the trajectories. The potential A satisfies Assumption 2, and it follows from Theorem 2 that there is an open region about the origin where the weak limits satisfy $\dot{x} = 0$. We do not know this region precisely, but we do know that it is open. The region $\rho^{-1}\{0\}$ is composed of the horizontal strips $\{|\Omega_1| > (R+r)\cos\beta,\ |\Omega_2| \leqq r\sin\beta\}$. The region $\rho^{-1}\{\pm\infty\}$ is the vertical strip $\{|\Omega_1| \leqq (R+r)\cos\beta,\ |\Omega_2| > r\sin\beta\}$. Outside the cross formed by these regions we have a countable number of resonance regions, $S_{p/q}$, with nonempty interior. In accordance with Theorem 3 and Lemma 17(iv), the direction of weak limits in this region is given by $\rho = p/q$, i.e., we have rectilinear motion in each strip. The trajectories can take only a finite number of speeds in this region given by Theorem 4. The intersection point of the boundaries of $\rho = 0$ and $\rho = \pm\infty$ at the points $(\lambda_1 x_1, \lambda_2 x_2) \equiv (\Omega_1, \Omega_2) = (\pm(R+r)\cos\beta, \pm r\sin\beta)$ are highly singular. For definiteness, consider the point in the fourth quadrant. On any horizontal segment to the left of the $\rho = \infty$ boundary, ρ increases sharply till it is ∞. Similarly, on any vertical segment below the $\rho = 0$ boundary, ρ decreases until it is zero. Therefore, in any open neighborhood of this point, the range of ρ is

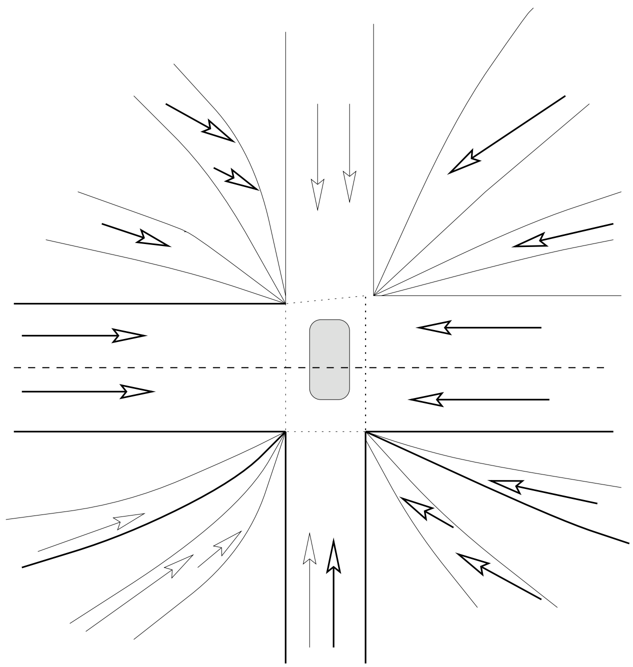

Fig. 5. Weak limits for $\dot{x} = -\nabla F(x) - \nabla A(\frac{x}{\varepsilon})$. The shaded region in the center corresponds to the "sticky" neighborhood of $x = 0$. Also shown are the resonance bands for $\rho = \{0, \pm\infty, \pm1, \pm2\}$. Within these bands the weak limits are straight lines with slope ρ.

$(0, \infty)$. We expect an infinite number of tongues, one for each $p/q \in \mathbb{Q} \bigcap (0, \infty)$, which taper in towards the point $\rho^{-1}\{0\} \cap \rho^{-1}\{\infty\}$.

We have drawn only a few resonance bands, but the reader may fill in a number of increasingly thinner bands in between these *ad infinitum* for other rationals. It is natural to extrapolate the picture to the region $\rho^{-1}\{\mathbb{R}\backslash\mathbb{Q}\}$. However, we emphasize that the theorems we prove utilize in an essential way the hyperbolicity of the microscopic flow, and this assumption is violated when ρ is irrational. We have also omitted to describe the dynamics in the region surrounding the region of sticking, but where $|\Omega_1| \leqq \|a\|_\infty$ or $|\Omega_2| \leqq \|b\|_\infty$. The ω-limit sets are typically either equilibria or periodic orbits in this region. Therefore, we expect that the weak limits are either stick, or slip, with the probability of the occurrence being governed by the sizes of the basins of attractions of equlibria and periodic orbits.

In conclusion, we see that the Cantor sets arising in the bifurcation of circle maps play an essential role in the homogenization. In addition to the convexification of the energy observed in one dimension, [2], in two dimensions we see the prevalence of preferred directions in the vicinity of equilibria. These correspond to the resonances of low order.

3. Weak convergence and averaging

3.1. Notation

Let us first fix our notation: $\mathbb{T}^d$ is defined as $\mathbb{R}^d/\mathbb{Z}^d$; m denotes Lebesgue measure on $\mathbb{R}^d$ and Haar measure on $\mathbb{T}^d$. There is a natural covering map $\Pi : \mathbb{R}^d \to \mathbb{T}^d$. If we let $[x]$ denote the integer part of x, then $\Pi(x)$ is identified with the point $x - [x] \in [0, 1)^d$. For $d = 1$, we use the notation S^1 instead of $\mathbb{T}$. For any integer $r \geqq 0$, $C^r(\mathbb{T}^d)$ denotes the space of r-times continuously differentiable functions, $f : \mathbb{T}^d \to \mathbb{R}$ which may be identified with the class of C^r functions from $\mathbb{R}^d$ to $\mathbb{R}$ that are 1-periodic in each coordinate. When r is not an integer, say $r = [r] + \beta$, then $C^r(\mathbb{T}^d)$ is the space of $C^{[r]}$ functions, with Hölder continuous derivative: $|D^{[r]}(y) - D^{[r]}(x)| \leqq C|x - y|^\beta$. We shall use the Sobolev space $W^{1,\infty}([0, T], \mathbb{R}^d)$ which can be identified with the space of Lipschitz functions from $[0, T] \to \mathbb{R}^d$. Finally, $D^r(S^1) \subset C^r(S^1)$ denotes the space of orientation preserving C^r diffeomorphisms of the circle.

3.2. Compactness in $W^{1,\infty}$

The problem may be stated as follows. We consider a C^r potential $A : \mathbb{T}^d \to \mathbb{R}$ and the kinetic law (2). The energies

$$G^\varepsilon(x) = F(x) + \varepsilon A\left(\frac{x}{\varepsilon}\right) \tag{9}$$

converge weakly to F in $W^{1,\infty}_{\mathrm{loc}}$. But G^ε does not converge strongly in this topology, since $\|\nabla G^\varepsilon(x) - \nabla F(x)\|_\infty = \|\nabla A\|_\infty > 0$.

If we fix $x_0 \in \mathbb{R}^d$, and consider a set of solutions $\{x^\varepsilon(t)\}, 0 < \varepsilon \leqq \varepsilon_0$, to equation (2) with $\lim_{\varepsilon \downarrow 0} x^\varepsilon(0) = x_0$, then

$$\sup_{\varepsilon} \sup_{t \geqq 0} |x^\varepsilon(t)| < \infty$$

since (2) is a gradient system, and the set $\{x | F(x) \leqq n\}$ is compact for all n. Since $x^\varepsilon(t)$ are restricted to a compact set, we also have uniform bounds on the speed,

$$\sup_{\varepsilon} \sup_{t \geqq 0} |\dot{x}^\varepsilon(t)| < \infty.$$

Thus the set of trajectories $\{x^\varepsilon\}$ is uniformly bounded in $W^{1,\infty}(\mathbb{R}_+; \mathbb{R}^d)$, and there exists a subsequence so that $x^{\varepsilon_n} \overset{*}{\rightharpoonup} x^0$ in $W^{1,\infty}(\mathbb{R}_+; \mathbb{R}^d)$. This means that, for any $0 < T < \infty$,

(·1) $x^{\varepsilon_n}(t) \to x^0(t)$ uniformly on $[0, T]$;
(·2) $\dot{x}^{\varepsilon_n}(t) \overset{*}{\rightharpoonup} \dot{x}^0(t)$ in $L^\infty([0, T])$.

The hypothesis that underlies this work is that all observable trajectories, are weak limits of solutions to (2). Thus the problem is to classify the weak limits, and to derive an averaged dynamical system, if it exists. Notice that there may be many weak limits through an initial point x_0. However, in one dimension the weak limit is unique. The existence of a unique weak limit is essential for the derivation of macroscopic kinetic laws. In higher dimensions there may be several distinct weak limits depending on the initial condition. Thus we cannot derive a deterministic macroscopic equation.

3.3. The method of averaging

We may recast (2) in a form suitable for averaging as follows. Define a phase variable

$$\theta = \frac{x}{\varepsilon} \mod 1. \tag{10}$$

The phase changes rapidly on the time scale, t. If we switch to the fast time scale $\tau = t/\varepsilon$, we may rewrite (2) as the system

$$\theta' = f(x) + a(\theta), \tag{11}$$
$$x' = \varepsilon(f(x) + a(\theta)), \tag{12}$$

where $f = -\nabla F$, $a = -\nabla A$, and $' = d/d\tau$. The phase space for this system is the fibre bundle $\mathbb{T}^d \times \mathbb{R}^d$: $\theta \in \mathbb{T}^d$ is the fibre variable, and $x \in \mathbb{R}^d$ is a point in the base space. In the limit $\varepsilon = 0$, the slow variable x is fixed, and the flow is said to be vertical.

The following standard multiple-scales argument (see e.g. [33]) suggests the form that the limiting equations must take. We make the ansatz

$$\begin{aligned} x^\varepsilon(t,\tau) &= x^0(t) + \varepsilon x^{(1)}(t,\tau) + \cdots \\ \theta^\varepsilon(t,\tau) &= \theta^0(\tau) + \varepsilon\theta^{(1)}(t,\tau) + \cdots . \end{aligned} \tag{13}$$

The goal is to compute $x^0(t)$. Let the initial conditions be (θ, x). Substituting the ansatz (13) in (11), and equating powers of ε we obtain the following equations for θ^0 and x^0:

$$\frac{d\theta^0}{d\tau} = f(x^0(t)) + a(\theta^0), \tag{14}$$

and

$$\frac{dx^0}{dt} + \frac{\partial x^{(1)}}{\partial \tau} = f(x^0(t)) + a(\theta^0(\tau)). \tag{15}$$

Let $\theta^0(\tau; \theta, x)$ denote the solution to (14) with initial condition (θ, x), i.e., the vertical flow on the fibre at x. Integrating (15) with respect to τ, we find that

$$x^{(1)}(t,\tau) - x^{(1)}(t,0) = \tau\left(-\frac{dx^0}{dt} + f(x^0(t)) + \frac{1}{\tau}\int_0^\tau a(\theta^0(s); \theta, x^0(t))\, ds\right).$$

In order to avoid secular terms in the expansion, the term on the right-hand side must be zero. This suggests that the limiting equation for $x^0(t)$ is

$$\frac{dx}{dt} = f(x) + \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau a(\theta^0(s); \theta, x)\, ds. \tag{16}$$

That is, the long-time behavior of the vertical flow on the fibres determines the homogenized equation. In order to obtain rigorous results we need to determine the limit in (16) (if it exists), and then prove that (16) is the right description of the average. The first question is considered in Section 4 where we derive averaged equations for $d = 1, 2$. A partial validation of these equations is in Sections 5 and 6.

The long-time limit is studied in ergodic theory and the method of Bogoliubov and Krylov (see [38, Lecture 2]) tells us that there is an invariant probability measure μ_x so that with probability 1

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau a(\theta^0(s; \theta, x))\, ds = \int_{\mathbb{T}^d} a(\theta) d\mu_x(\theta).$$

It is clear that there is a unique averaged equation only if the long-time limit in (16) exists, and is independent of the initial phase θ. The vertical flow is *ergodic* if μ_x is unique. Ergodicity was precisely linked to averaging by ANOSOV [5]. Under the hypothesis that the $\varepsilon = 0$ flow is ergodic on almost every fibre, ANOSOV showed, roughly speaking, that $x^\varepsilon(t)$ is close to $x^0(t)$, except for a set of initial conditions of small measure.

Equation (11) is an example of a system for which the limiting flow on fibres is *not* ergodic for sets in the base space with positive measure. For example, when $d = 2$ the flow is not ergodic on an open and dense subset of the base space $\mathbb{R}^2$. However, the existence of averages, in the sense of weak limits, is little more than an assertion of compactness, and is independent of assumptions of ergodicity. The cost of such generality is that the problem appears intractable for $d > 2$ for two reasons. Firstly, it is hard to characterize the invariant measures μ_x in higher dimensions. Secondly, it is harder to prove that (16) gives the correct averages in higher dimensions. Before considering these finer questions, we prove some basic results that rely only on *a priori* estimates and are independent of d.

3.4. *A priori estimates*

A natural question is whether the limiting behavior is gradient-like. Abeyaratne, Chu & James [2] found that the macroscopic kinetic law was a gradient flow: all points flow downhill into a flat valley of equilibria. In this section we use *a priori* estimates to show that there is a trapping region.

Suppose that $x^0(t)$ is a weak limit. Then for an appropriate subsequence $\{\varepsilon_n\}_{n=1}^{\infty}$, we have $x^{\varepsilon_n}(t) \overset{*}{\rightharpoonup} x^0(t)$. For any $t_1, t_2 \in [0, \infty)$ the differential equation (2) yields

$$\begin{aligned} F(x^{\varepsilon_n}(t_2)) - F(x^{\varepsilon_n}(t_1)) &= \int_{t_1}^{t_2} \nabla F(x^{\varepsilon_n}(s)) \cdot \frac{dx^{\varepsilon_n}(s)}{ds} \\ &= -\int_{t_1}^{t_2} |f(x^{\varepsilon_n}(s))|^2\, ds \\ &\quad - \int_{t_1}^{t_2} f(x^{\varepsilon_n}(s)) \cdot a\left(\frac{x^{\varepsilon_n}(s)}{\varepsilon_n}\right) ds. \end{aligned}$$

Since weak convergence in $W^{[1,\infty)}$ implies that $x^{\varepsilon_n}(t) \to x^0(t)$ uniformly on compact sets, and F is C^2, we have

$$\begin{aligned} &F(x^0(t_2)) - F(x^0(t_1)) \\ &\quad = -\int_{t_1}^{t_2} |f(x^0(s))|^2\, ds - \lim_{n\to\infty} \int_{t_1}^{t_2} f(x^{\varepsilon_n}(s)) \cdot a\left(\frac{x^{\varepsilon_n}(s)}{\varepsilon_n}\right) ds. \end{aligned} \tag{17}$$

The existence of the limit is part of the conclusion. There is another way we could calculate the change in energy. From (2) and (9)

$$\begin{aligned} G^{\varepsilon_n}(x^{\varepsilon_n}(t_2)) - G^{\varepsilon_n}(x^{\varepsilon_n}(t_1)) &= -\int_{t_1}^{t_2} |\nabla G^{\varepsilon_n}(x^{\varepsilon_n}(s))|^2\, ds \\ &= -\int_{t_1}^{t_2} |f(x^{\varepsilon_n}(s))|^2\, ds \\ &\quad - 2\int_{t_1}^{t_2} f(x^{\varepsilon_n}(s)) \cdot a\left(\frac{x^{\varepsilon_n}(s)}{\varepsilon_n}\right) ds \\ &\quad - \int_{t_1}^{t_2} \left|a\left(\frac{x^{\varepsilon_n}(s)}{\varepsilon_n}\right)\right|^2 ds. \end{aligned}$$

Letting $n \to \infty$ we find that

$$\begin{aligned} F(x^0(t_2)) - F(x^0(t_1)) &= -\int_{t_1}^{t_2} |f(x^0(s))|^2\, ds \\ &\quad - 2\lim_{n\to\infty} \int_{t_1}^{t_2} f(x^{\varepsilon_n}(s)) \cdot a\left(\frac{x^{\varepsilon_n}(s)}{\varepsilon_n}\right) ds \\ &\quad - \lim_{n\to\infty} \int_{t_1}^{t_2} \left|a\left(\frac{x^{\varepsilon_n}(s)}{\varepsilon_n}\right)\right|^2 ds. \end{aligned} \tag{18}$$

The existence of the second limit follows from the existence of the first. Comparing (17) and (18) we see that

$$F(x^0(t_2)) - F(x^0(t_1)) = -\int_{t_1}^{t_2} |f(x^0(s))|^2 ds + \lim_{n\to\infty} \int_{t_1}^{t_2} \left| a\left(\frac{x^{\varepsilon_n}(s)}{\varepsilon_n}\right)\right|^2 ds. \tag{19}$$

Since $\|a\|_\infty < \infty$, there is a subsequence, $\{\varepsilon_{n_k}\}_{k=1}^\infty$, and an associated Young measure, ν_t, such that

$$\lim_{k\to\infty} \int_{t_1}^{t_2} \left| a\left(\frac{x^{\varepsilon_{n_k}}(s)}{\varepsilon_{n_k}}\right)\right|^2 ds = \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\lambda|^2 d\nu_t(\lambda)\, dt = \int_{t_1}^{t_2} \operatorname{var}(\nu_t)\, dt.$$

But since the limit in (19) exists, we see that *all* Young measures associated with subsequences of ε_n have the same second moment (t a.e.). Furthermore, the contribution of the wiggles is always nonzero, unless the Young measures are Dirac masses. But $x^0(t)$ is differentiable almost everywhere. Hence

$$\frac{dF(x^0(t))}{dt} = -|\nabla F(x^0(t))|^2 + \operatorname{var}(\nu_t), \quad t \text{ a.e.} \tag{20}$$

Hence all weak limits enter wells around $\nabla F^{-1}\{0\}$, and these wells may have width that is at most $\|a\|_\infty$. The time taken to enter this trapping region is uniform on bounded sets since (20) provides a lower bound for the speed at which weak limits cross the contour lines of F. On the other hand, these *a priori* estimates do not provide any information about the averaged vector field, or the trajectories. This requires a more careful analysis, and we only have answers for $d = 1$ and $d = 2$.

4. Averaged equations for $d = 1, 2$

In this section $d = 1$ or $d = 2$. We derive averaged equations by calculating the limits

$$\langle a(\theta, x)\rangle_\pm \stackrel{\text{def}}{=} \lim_{\tau\to\pm\infty} \frac{1}{\tau}\int_0^\tau a(\theta^0(s;\theta,x))\, ds. \tag{21}$$

We expect the limits to depend only on the α- and ω-limit sets of θ and we may obtain different limits at $\pm\infty$. We begin with the following examples.

4.1. Examples

4.1.1. Critical points. Suppose $\omega(\theta) = \theta_*$, a critical point. Then

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau a(\theta^0(s;\theta,x))\, ds = \langle a(\theta, x)\rangle_+ = a(\theta_*). \tag{22}$$

Similarly, if $\alpha(\theta) = \theta_{**}$, then $\langle a(\theta, x)\rangle_- = a(\theta_{**})$. But $a(\theta_*) = a(\theta_{**}) = -f(x)$. It follows that the averaged vector field (16) is

$$\frac{dx^0}{dt} = f(x) + a(\theta_*) = 0. \tag{23}$$

4.1.2. Periodic orbits. Suppose $\omega(\theta)$ is a periodic oribit $\gamma^s : \mathbb{R} \to \mathbb{T}^d$, with period $T > 0$. In this case, a calculation similar to the one above shows that

$$\langle a(\theta, x)\rangle_+ = \frac{1}{T}\int_0^T a(\gamma^s(s))\, ds. \tag{24}$$

Clearly, if $\alpha(\theta)$ is a different periodic orbit, say $\gamma^u : \mathbb{R} \to \mathbb{T}^d$, then typically, $\langle a(\theta, x)\rangle_+ \neq \langle a(\theta, x)\rangle_-$.

4.1.3. Ergodic flows. Suppose $\alpha(\theta) = \omega(\theta) = \mathbb{T}^d$. Then the smallest minimal set is the entire fibre. In this case, the invariant probability measure, μ_x, obtained from the Bogoliubov-Krylov construction is unique, and the flow is ergodic with respect to this measure. The long-time limits $\langle a(\theta, x)\rangle_\pm$ exist for θ μ_x a.e., and are equal by the Birkhoff-Khinchine ergodic theorem [38].

4.2. Averaged equations for $d = 1$

There are only two possibilities. If there is at least one critical point, then $\omega(\theta)$ is a critical point for each $\theta \in S^1$. If there are no critical points, then $\omega(\theta)$ is S^1 for each θ. In the first case, the averaged equation is (23). In the second case, we can simplify (24) as follows. We invert (14) to obtain

$$\frac{d\tau}{d\theta} = \frac{1}{f(x) + a(\theta)},$$

so that the (signed) time period is

$$T(x) = \int_0^{2\pi} \frac{d\theta}{f(x) + a(\theta)}.$$

Thus, the long-time average of a may be written as

$$\begin{aligned}\langle a(\theta)\rangle &= \frac{1}{T}\int_0^T a(\theta^0(s; \theta, x))\, ds = \frac{1}{T}\int_0^{2\pi} \frac{a(\psi)}{f(x) + a(\psi)}\, d\psi \\ &= \frac{2\pi}{T} - \frac{f(x)}{T}\int_0^{2\pi} \frac{d\psi}{f(x) + a(\psi)}\, d\psi = \frac{2\pi}{T(x)} - f(x).\end{aligned}$$

Therefore, the averaged equation (16) is

$$\frac{dx}{dt} = \frac{2\pi}{T(x)}, \tag{25}$$

with the understanding that $T(x) = \pm\infty$ if there is a critical point. Of course, this is what we expect intuitively. Equations (23) and (25) are the macroscopic equations in one dimension that were derived in [2] by different methods.

4.3. *Averaged equations for $d = 2$*

In two dimensions the range of possible asymptotic behavior (for sufficiently smooth vector fields) is limited by Schwartz's generalization of the Poincaré-Bendixson theorem. We use the following version (the theorem is more general and applies to any compact two dimensional manifold).

Theorem 1 (SCHWARTZ [37]). *Suppose that φ_s, $s \in \mathbb{R}$ is a C^2 flow on $\mathbb{T}^2$, and M is a minimal set for φ_s. Then M must be one of the following:*

(·1) *A singleton consisting of a critical point.*
(·2) *A periodic orbit.*
(·3) *All of $\mathbb{T}^2$.*

The smoothness requirement is sharp. DENJOY showed that such an assertion is false for C^1 vector fields [16], and HERMAN has shown that for every $\beta \in (0, 1)$, there is a Denjoy counterexample that is $C^{2-\beta}$ [27].

We use Theorem 1 to characterize typical ω-limit sets. The vertical flow on a fibre satisfies an equation of the form

$$\dot{\theta} = \Omega + a(\theta), \quad \theta \in \mathbb{T}^2, \quad \Omega \in \mathbb{R}^2. \tag{26}$$

Equation (26) corresponds to the gradient system in $\mathbb{R}^2$,

$$\dot{z} = -\nabla V_\Omega(z) \stackrel{\text{def}}{=} -\nabla\left(A(z) - \Omega \cdot z\right). \tag{27}$$

We denote the flow of (26) by φ_s and the flow of (27) by $\tilde{\varphi}_s$. The flows of (26) and (27) are related by $\Pi(\tilde{\varphi}_t(z)) = \varphi_t(\Pi(z))$. The reason we consider the flow in $\mathbb{R}^2$ is that V_Ω is a Lyapunov function for (27) but not for (26). The following lemma is needed to classify the limit sets. The proof is standard, and may be found in [35].

Lemma 1. *Let $A \in C^r(\mathbb{R}^d)$, $r > 2d - 1$. Then for $\Omega \in \mathbb{R}^d$ in a residual set of full measure:*

(a) *All critical points of V_Ω are non-degenerate, and hence there are at most countably many of these.*
(b) *If z_1 and z_2 are distinct critical points, then $V_\Omega(z_1) \neq V_\Omega(z_2)$.*

The smoothness hypothesis $r > 2d-1$ is needed in an application of Sard's theorem to prove Lemma 1(b). We need Lemma 1(b) for the following Proposition.

Proposition 1. *Let Ω lie in the residual set of full measure of Lemma 1 and consider the flow generated by* (26). *Then for any $\theta \in \mathbb{T}^2$, the limit set $\omega(\theta)$ is one of the following:*

(·1) *A critical point.*
(·2) *A periodic orbit.*
(·3) *All of $\mathbb{T}^2$.*

Proof. Suppose that $\omega(\theta) \neq \mathbb{T}^2$. We show that we have (·1) or (·2). the limit set $\omega(\theta)$ is closed, invariant and nonempty. Hence, by Zorn's lemma, it must contain a minimal set. Applying Theorem 1, we see that either

Case 1: $\omega(\theta) \supset \gamma$, a periodic orbit, or
Case 2: $\omega(\theta) \supset \{\theta_*\}$, a critical point.

In Case 1, SCHWARTZ showed that $\omega(\theta) = \gamma$ [37, Corollary, p. 457]. Thus, the proposition reduces to proving that $\omega(\theta) = \{\theta_*\}$, and in order to show this we use the gradient structure.

Suppose that $\omega(\theta)\backslash\{\theta_*\}$ is nonempty. There are again two cases to consider.

Case 2(a): $\omega(\theta)\backslash\{\theta_*\}$ contains a minimal set.
Case 2(b): $\omega(\theta)\backslash\{\theta_*\}$ contains no minimal set.

Case 2(a): By Theorem 1 and Case 1 above, this means that there is a critical point $\theta_{**} \neq \theta_*$ in $\omega(\theta)$. Thus, there are real sequences $s_n, t_n \to \infty$, such that $\varphi_{s_n}(\theta) \to \theta_*$, and $\varphi_{t_n}(\theta) \to \theta_{**}$. Lifting the flow from $\mathbb{T}^2$ into $\mathbb{R}^2$ we find an initial point $z \in \Pi^{-1}\{\theta\}$, and distint critical points z_* and z_{**} such that $\tilde{\varphi}_{s_n}(z) \to z_*$ and $\tilde{\varphi}_{t_n}(z) \to z_{**}$. The function V_Ω is a Lyapunov function for the flow $\tilde{\varphi}_t$. In particular, it is monotonically decreasing and we find

$$\lim_{t\to\infty} V(\tilde{\varphi}_t(z)) = \lim_{n\to\infty} V(\tilde{\varphi}_{s_n}(z)) = V(z_*).$$

Similarly, we find $\lim_{t\to\infty} V(\tilde{\varphi}_t(z)) = V(z_{**})$. Thus, $V(z_*) = V(z_{**})$ which contradicts Lemma 1.

Case 2(b): In this case we claim that $\omega(\theta)$ contains an orbit homoclinic to θ_*. Let $\psi \in \omega(\theta)\backslash\{\theta_*\}$. Then $\alpha(\psi) \subset \omega(\theta)$ is nonempty, closed, and invariant. But the only closed invariant subsets of $\omega(\theta)$ are $\{\theta_*\}$ and $\omega(\theta)$. Thus, $\theta_* \in \alpha(\psi)$, and there is a sequence $s_n \to -\infty$ such that $\varphi_{s_n}(\psi) \to \theta_*$. Similarly, $\theta_* \in \omega(\psi)$, and there is a sequence $t_n \to \infty$ such that $\varphi_{t_n}(\psi) \to \theta_*$. But we know that $\{\theta_*\}$ is a non-degenerate critical point, so it is either a saddle, sink or a source. It cannot be either a source or a sink because both $\alpha(\psi)$ and $\omega(\psi)$ are θ_*. Thus, θ_* is a saddle point, and $\varphi_{s_n}(\psi) \in W^u_{\text{loc}}(\theta_*)$, $\varphi_{t_n}(\psi) \in W^s_{\text{loc}}(\theta_*)$, for large n. Since there are no other critical points in $\omega(\theta)$, the orbit through ψ, denoted γ, is homoclinic to θ_*.

We again draw a contradiction using the Lyapunov function V_Ω and the flow $\tilde{\varphi}$. There are two cases to consider. Let $z \in \Pi^{-1}\{\theta\}$.

Case (i): $\Pi^{-1}\{\gamma\} \subset \omega(z)$ contains a homoclinic orbit in the plane.
Case (ii): $\Pi^{-1}\{\gamma\} \subset \omega(z)$ contains a heteroclinic orbit in the plane.

In Case (i) let z_1 be a point in the orbit homoclinic to z_*. Then we have the contradiction

$$V(z_*) = \lim_{t\to-\infty} V(\tilde{\varphi}_t(z_1)) < V(z_1) < \lim_{t\to+\infty} V(\tilde{\varphi}_t(z_1)) = V(z_*).$$

In Case (ii), there are two distinct critical points z_* and z_{**} and sequences $s_n, t_n \to \infty$ such that $\tilde{\varphi}_{s_n}(z) \to z_*$ and $\tilde{\varphi}_{s_n}(z) \to z_{**}$. Since, $V(\tilde{\varphi}_t(z))$ is a decreasing function, and $V(z_*) \neq V(z_{**})$ this gives the contradiction of Case 2(a). □

Proposition 1 allows us to compute the limit $\langle a(\theta, x)\rangle_\pm$ for every θ. This is stronger than the μ_x-a.e. existence guaranteed by the Bogoliubov-Krylov construction. A few immediate consequences of this are:

1. There are typically as many limits $\langle a(\theta, x)\rangle_{\pm}$, as there are distinct α- and ω-limit sets. In the special circumstance where all limit sets are critical points, we obtain a unique limiting equation.
2. The averaging process has introduced an “arrow of time”. For example, if the flow decomposes into one attracting periodic orbit, γ^s, one repelling periodic orbit, γ^u, and their basins of attraction and repulsion, then $\langle a(\theta, x)\rangle_+$ is constant on the basin of attraction of γ^s (a set of full measure), and depends only on $a|_{\gamma^s}$. On the other hand, $\langle a(\theta, x)\rangle_-$ is constant on the basin of repulsion of γ^u, and is generically distinct from $\langle a(\theta, x)\rangle_+$.

Remark 1. The additional smoothness hypothesis of the proposition is not needed when $|f(x)| > \|a\|_\infty$. In this case we can reduce the problem to a study of circle maps as long as there is a well-defined flow (e.g., if f is Lipschitz). FURSTENBERG showed that the ergodic average for every $\theta \in S^1$ exists for any homeomorphsim of S^1 [22], and therefore for the flow of (26).

5. Persistence of gradient structure and sticking

In this section we characterize the weak limits in the vicinity of $\nabla F^{-1}\{0\}$ under some assumptions on the nature of the wiggles.

Theorem 2. *Suppose $x_* \in \nabla F^{-1}\{0\}$, and that $A \in C^2(\mathbb{T}^2)$ satisfies Properties 1 and 2. Then there exists $r > 0$ so that the homogenized equation in $B(x_*, r)$ is* $\dot{x}^0 = 0$.

Theorem 2 says that all the weak limits are constant near the critical points of F. Of our assumptions, one is generic (Property 1), and the other is technical (Property 2).

The proof relies on showing that if the lifts of microscopic orbits into $\mathbb{R}^d$ are bounded, then all orbits stick on the macroscale. Indeed, if we know that these bounds persist for some $\varepsilon > 0$, i.e., if we have

$$\sup_\tau |z^\varepsilon(\tau) - z^\varepsilon(0)| \leqq C \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0, \tag{28}$$

then since the blow-up transformation and change of time scale are invertible for $\varepsilon > 0$, we must have

$$\sup_t |x^\varepsilon(t) - x^\varepsilon(0)| \leqq C\varepsilon \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0. \tag{29}$$

Thus, if $x^\varepsilon(0)$ converges to x_0, we see that there is a unique weak limit through x_0, namely $x^0(t) = x_0$. Hence we see sticking on the macroscale. In fact, it is sufficient to obtain weaker estimates on the drift of trajectories in the microscale (for example an upper estimate of the form $C\varepsilon^{-\nu}$, $0 < \nu < 1$) to obtain the same conclusion.

5.1. Generic properties

The next step relies on a geometric analysis of a singular perturbation problem for vector fields in $\mathbb{R}^d$. Our arguments require structural stability of the microscopic flow in the vicinity of x_*. This is ensured by the generic hypothesis

Property 1. The flow for the vector field $a(\theta) = -\nabla_\theta A(\theta)$ is Morse-Smale.

Recall that for A in an open and dense set of $C^2(\mathbb{T}^2)$, the flow of

$$\dot{\theta} = a(\theta) = -\nabla A(\theta) \tag{30}$$

has the following properties.

(·1) There are at least four, and at most finitely many critical points. These are all non-degenerate, and are either sources, denoted $\{A_k\}_{k=1}^m$, sinks, $\{B_k\}_{k=1}^n$, or saddle points, $\{C_k\}_{k=1}^{m+n}$.
(·2) Each saddle point, C, has a one-dimensional stable manifold, $W^{\mathrm{s}}(C)$ and a one-dimensional unstable manifold $W^{\mathrm{u}}(C)$.
(·3) There are no saddle connections, i.e., the ω-limit set of any point on the unstable manifold, $W^{\mathrm{u}}(C)$ is a sink, and the α-limit set of any point on $W^{\mathrm{s}}(C)$ is a source.
(·4) The unstable (or stable) manifold $W^{\mathrm{u}}(C)$ (resp., $W^{\mathrm{s}}(C)$) consists of two disjoint C^1 curves imbedded in the torus.

Also, $m, n \geqq 1$ since the maximum and minimum of A are generically distinct. The number of saddle points must equal $m+n$ by the Poincaré-Hopf index theorem [24, p. 134] (·3) is a consequence of Peixoto's theorem [23, Section 1.9] and (·4) is seen as follows. In a neighborhood of C, $W^{\mathrm{u}}(C)$ is given as the graph of a C^1 map over the unstable subspace of the linearization at C. Hence, we can distinguish between a "left" and "right" branch of $W^{\mathrm{u}}(C)$ in the vicinity of C. Let D_{l} lie in the left branch, and D_{r} on the right branch. Then $W^{\mathrm{u}}(C) = \gamma(D_{\mathrm{l}}) \bigcup \gamma(D_{\mathrm{r}})$, where $\gamma(P)$ denotes the trajectory through the point P.

The ω-limit set of D_{l} is a sink, denoted by B. The closure of $W^{\mathrm{u}}(C)$ includes the endpoints B and C and it is a simple curve in the torus, for it can be written as the image of a continuous map $\tilde{\gamma} : [-1, 1] \to \mathbb{T}^2$ as follows. Let $h : [-1, 1] \to [-\infty, \infty]$ be defined by

$$h(s) = \frac{s}{1 - |s|},$$

and let $\tilde{\gamma}(-1) = C$, $\tilde{\gamma}(1) = B$, and

$$\tilde{\gamma}(s) = \varphi_{h(s)}(D_{\mathrm{l}}), \;\; s \in (-1, 1),$$

where φ_t, $t \in \mathbb{R}$ denotes the flow generated by (30). The orbit $\tilde{\gamma}$ is C^1 on $(-1, 1)$, and continuous at the endpoints.

Since the stable and unstable manifolds are simple curves, we shall say that a saddlepoint C is connected to a source A (or a sink B) if one branch of the stable (resp., unstable) manifolds of C terminates at A (resp., B).

5.2. Construction of barriers

At present, we are able to prove (28) only under the following Property on the flow of (30).

Property 2. There is at least one saddle point, say $C \in \mathbb{T}^2$, with the property that it connects to exactly one source A and one sink B. That is, both $W^{\mathrm{s}}(C) \bigcup \{C\} \bigcup \{A\}$ and $W^{\mathrm{u}}(C) \bigcup \{C\} \bigcup \{B\}$ form closed loops (see Fig. 6).

Notice that the simplest generic gradient fields on the torus have four critical points: one source, one sink and two saddles, and satisfy this hypothesis.

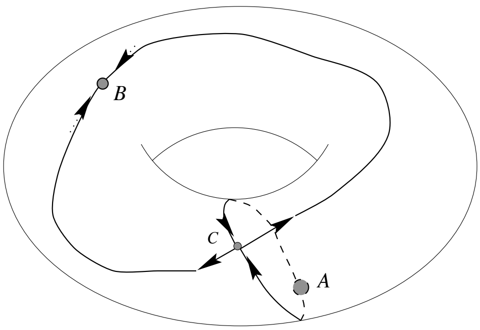

Fig. 6. Property 2.

We need to lift the properties of the gradient field on the torus into $\mathbb{R}^2$. For any point $P \in \mathbb{T}^2$ we use indexed lower-case letters to denote its inverse images $\Pi^{-1}\{P\} = \{p_n\}_{n \in \mathbb{Z}^2} \in \mathbb{R}^2$. Notice that every point $c_n \in \Pi^{-1}\{C\}$ is connected to one or two points in $\Pi^{-1}\{A\}$ because C is connected to A. In fact,

Lemma 2. *Each saddle point $c_n \in \Pi^{-1}\{C\}$ is connected to two distinct sources in $\Pi^{-1}\{A\}$, and two distint sinks in $\Pi^{-1}\{B\}$*

Proof. It is sufficient to prove the lemma for sources. Suppose both branches of the stable manifold of c were connected to the same source a. Then we would obtain a closed loop in the plane shown in Fig. 7. One branch of the unstable manifold of c must point into the loop, and thus there is a point $b \in \Pi^{-1}\{B\}$ that lies inside the loop. By Property 2, lifts into $\mathbb{R}^2$ of both branches of $W^{\mathrm{u}}(C)$ connect to b. One of these (say $W^{\mathrm{r}}(c)$) is shown in Fig. 7. The other branch, $W^{\mathrm{l}}(c)$, cannot connect to b without intersecting the loop aca. Thus, there is another point, say $c_2 \in \Pi^{-1}\{C\}$ that connects to b. The saddle point c_2 cannot lie within the loop, for the vector field is periodic, and thus a translate of the loop aca passing through c_2 would intersect aca. If c_2 lies outside the loop then the connection between b and c_2 intersects the loop aca. In either case we obtain a contradiction. □

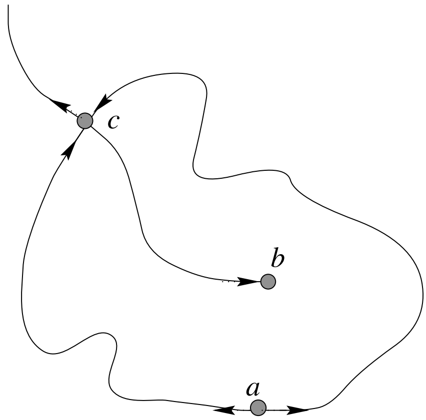

Fig. 7. Impossibility of closed loops in the plane.

Suppose c connects to two distinct sources a_1 and a_2. Then, $0 \neq a_1 - a_2 = k \in \mathbb{Z}^2$. Translating the arc $a_1 c a_2$ by nk for all integers n, we obtain a curve Γ^{s} that separates the plane. Similarly, considering the arc $b_1 c b_2$ obtained from the sinks b_1 and b_2 connected to c, we obtain a second curve, Γ^{u}. We call Γ^{u} and Γ^{s} *barriers*. The barriers Γ^{u} and Γ^{s} intersect transversely at c, and nowhere else. Considering integer translates of Γ^{s} and Γ^{u} we obtain a mesh of invariant cells in the plane separated by the barriers $\{\Gamma_n^s, \Gamma_n^u\}_{n\in\mathbb{Z}}$. The size of each cell depends only on the vector field a.

5.3. Perturbations of the microscopic flow

Now consider the microscopic flow

$$\dot{\theta} = f(x) + a(\theta), \quad \theta \in \mathbb{T}^2. \tag{31}$$

Lemma 3. *Suppose $f(x_*) = 0$. There is an $r > 0$ such that for $|x - x_*| < r$,*

(a) *there is a C^1 family of non-degenerate critical points $A(x)$, $B(x)$ and $C(x)$ in $\mathbb{T}^2$ of the same type as A, B and C respectively, and*
(b) *both branches of $W^{\mathrm{s}}(C(x))$ (and $W^{\mathrm{u}}(C(x))$) terminate at $A(x)$ (resp., $B(x)$).*
(c) *The barriers $\{\Gamma_n^{\mathrm{s}}, \Gamma_n^{\mathrm{u}}\}$ continue to a family $\{\Gamma_n^{\mathrm{s}}(x), \Gamma_n^{\mathrm{u}}(x)\}$. The variation is continuous and $\lim_{x\to x_*} \mathrm{dist}(\Gamma_n^{\mathrm{s}}(x), \Gamma_n^{\mathrm{s}}) = 0$, uniformly in n.*

The proof is almost the same (but simpler) than that of Lemma 5 below and is omitted (details may be found in [35]). It should be noted that in Lemma 3 (c), the variation in x may not be C^1 (for example, see [19]). Lemma 3 allows us to decompose $\mathbb{R}^2$ into a periodic mesh of invariant cells separated by barriers. This is shown in Fig. 8. We denote the diameter of the largest cell by $D(x_0)$.

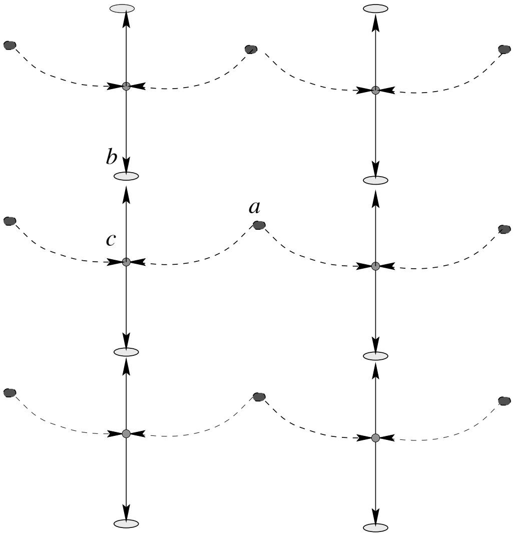

Fig. 8. Barriers Γ^{u} (solid line) and Γ^{s} (dashed line) in the plane.

5.4. Persistence for $\varepsilon > 0$

In order to prove the uniform estimate (28) we prove that the barriers continue for small $\varepsilon > 0$, and the size of the invariant cells is uniformly bounded.

To obtain an effective equation at x_0 it is sufficient to consider the weak limits in an arbitrarily small neighborhood of x_0. Let $\chi : \mathbb{R} \to \mathbb{R}$ be the C^∞ cut-off function

$$\chi(s) = \begin{cases} 1, & 0 \leqq s \leqq 1 \\ \exp\left(\dfrac{1-s}{2-s}\right), & 1 < s < 2 \\ 0, & 2 \leqq s < \infty \end{cases} \tag{32}$$
$$\chi(s) = \chi(-s), \qquad -\infty < s < 0.$$

It can be shown that $\max(\chi'(s)) < 2$. For any $\delta > 0$, define the cut-off function $\chi_\delta : \mathbb{R}^2 \to \mathbb{R}$

$$\chi_\delta(y, z) = \chi\left(\frac{y}{\delta}\right)\chi\left(\frac{z}{\delta}\right), \tag{33}$$

and consider the modified vector field in $\mathbb{R}^2$

$$z' = f(x_0 + \varepsilon(z - \alpha^\varepsilon)\chi_\delta(\varepsilon z)) + a(z) \stackrel{\text{def}}{=} f^\varepsilon(x_0, z) + a(z). \tag{34}$$

The modified vector field agrees with (4) in the region $\varepsilon|z| < \delta$, that is $|x - x_0| < \delta$. It is also uniformly close to the microscopic vector field (5) over the entire plane. Precisely,

Lemma 4. *The difference between the two vector fields is*

(a) $\sup_{z\in\mathbb{R}^2} |f^\varepsilon(x_0, z) - f(x_0)| \leqq C(\varepsilon + \delta)$,
(b) $\sup_{z\in\mathbb{R}^2} |D_z f^\varepsilon(x_0, z)| \leqq C\varepsilon$.

The constant C depends only the Lipschitz constant of f in an $O(\delta)$ neighborhood of x_0.

Proof. Let $\mathrm{Lip}(f) = \max_{|x-x_0|\leqq 4\delta} |Df(x)|$. From (34)

$$\begin{aligned}|f^\varepsilon(x_0, z) - f(x_0)| &\leqq \mathrm{Lip}(f)\varepsilon|z - \alpha^\varepsilon|\chi_\delta(\varepsilon z)\\ &\leqq \mathrm{Lip}(f)\big(\varepsilon|\alpha^\varepsilon| + \max_z \varepsilon|z|\chi_\delta(\varepsilon z)\big)\\ &\leqq \sqrt{2}\mathrm{Lip}(f)(\varepsilon + 2\delta).\end{aligned}$$

We used (32) and (33) in the last step. This proves (a). The gradient is estimated using the chain rule.

$$\begin{aligned}&|D_z f^\varepsilon(x_0, z)|\\ &\quad = |Df(x_0 + \varepsilon(z - \alpha^\varepsilon)\chi_\delta(\varepsilon z))\left(\varepsilon\chi_\delta(\varepsilon z) + \varepsilon(z - \alpha^\varepsilon) \otimes D_z\chi_\delta(\varepsilon z)\right)|\\ &\quad \leqq \mathrm{Lip}(f)(\varepsilon + \varepsilon \max_z |(z - \alpha^\varepsilon) \otimes D_z\chi_\delta(\varepsilon z)|) \leqq C(x_0, \delta)\varepsilon. \qquad \square\end{aligned}$$

The singular scaling in space prevents us from obtaining a better estimate in Lemma 4(a). In particular, the modified vector field is not $O(\varepsilon)$ close in the C^0 norm, and it is necessary to use the cut-off functions to control the difference.

We now show that the barriers persist for small $\varepsilon > 0$. To simplify notation we suppress the superscripts u, s and the dependence on x_0.

Lemma 5. *Let r be chosen as in Lemma 3 and suppose $|x_0 - x_*| < r$. Let $\eta > 0$ be given. Then there exist $\delta(x_0)$ and $\varepsilon_0(x_0, \delta)$ such that for $\varepsilon \in [0, \varepsilon_0]$ there is a family of barriers, $\{\Gamma_n^\varepsilon\}_{n\in\mathbb{Z}}$, for the flow of equation* (34) *and*

$$\sup_n \sup_\varepsilon \mathrm{dist}(\Gamma_n^\varepsilon, \Gamma_n) \leqq \eta.$$

Proof. It is sufficient to prove the persistence of an "unstable" barrier Γ_n^{u}: the proof for Γ_n^{s} is identical. We prove persistence of the barriers by proving persistence of the individual unstable manifolds $W^{\mathrm{u}}(c_k)$ uniformly in $k \in \mathbb{Z}^2$.

Choose $0 < \rho \leqq \eta$ so that $B(b_k, \rho)$ is strictly absorbing for all k. This is possible, since in the limit $\varepsilon = 0$ all the sinks, b_k, have the same linearization. Then by the implicit function theorem and Lemma 4, for small $\delta > 0$ and $\varepsilon_0 > 0$, each of these critical points persists for $\varepsilon > 0$, and the continuation $b_k^\varepsilon \in B(b_k, \rho)$. Similarly, the saddle points c_k continue to $c_k^\varepsilon \in B(c_k, \rho)$.

The periodicity of the limit also ensures that δ and ε may be chosen so that all the local stable and unstable manifolds of b_k and c_k persist. In particular, $B(b_k, \rho)$ remains a strictly absorbing ball and $\mathrm{dist}(W^{\mathrm{u}}(c_k), W^{\mathrm{u}}(c_k^\varepsilon)) \leqq C(\delta + \varepsilon)$ within $B(c_k, \rho)$. This follows from the persistence theorems of Fenichel [19].

A branch of the unstable manifold of c is shown in Fig. 9. Reducing ρ if necessary, $W^{\mathrm{u}}(c)$ is transverse to the circles $S(c, \rho)$ and $S(b, \rho)$. Thus, we can

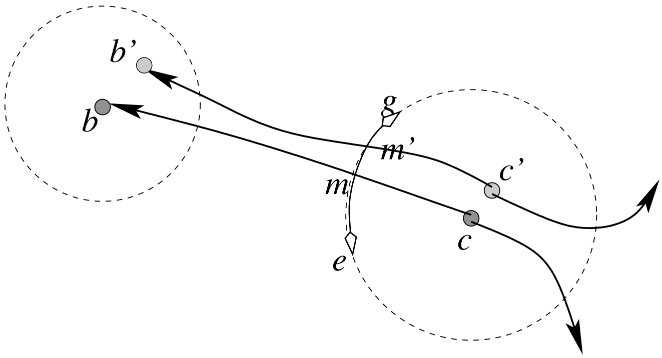

Fig. 9. Persistence of a branch of $W^{\mathrm{s}}(c)$.

define a Poincaré map $P^0 : (e, g) \to S(b, \rho)$ that maps an arc (e, g), with midpoint $m = W^u(c) \bigcap S(c, \rho)$, to $S(b, \rho)$. For small δ and ε_0, this continues to a Poincaré map P^ε. In particular, the map P^ε takes $m^\varepsilon = W^u(c^\varepsilon) \bigcap S(c, \rho)$ to $S(a, \rho)$. The time of flight between (e, g) and $S(a, \rho)$ is finite, thus the trajectories of m and m^ε can be made closer than η. Furthermore, since all the c_k are integer translates of c, and the vector fields (34) and (5) are uniformly close, δ and $\varepsilon_0 > 0$ can be chosen independent of k, so that the analogous map P_k^ε is defined on all integer translates of the arc (e, g).

Combining the above steps we see that the stable manifolds persist and

$$\operatorname{dist}(\operatorname{cl}(W^{\mathrm{u}}(c_k)), \operatorname{cl}(W^{\mathrm{u}}(c_k^\varepsilon))) \leqq \eta. \qquad \square$$

Proof of Theorem 2. Let $x^\varepsilon(t) \overset{*}{\rightharpoonup} x^0(t)$ with $x^0(0) = x_0 \in B(x_*, r)$. For any $\delta > 0$, there is a T and ε_0 such that $x^\varepsilon(t) \in B(x_0, \delta)$ for $t \in [-T, T]$, $\varepsilon \in [0, \varepsilon_0]$. Thus, $x^\varepsilon(t)$ solves the modified equation (34) within $B(x_0, \delta)$. Let $\eta = 1/4$. We use Lemma 5 to see that for suitable $\delta, \varepsilon_0 > 0$ the barriers persist, and the size of the largest invariant cell is not greater than $D(x_0) + 1$. Thus we obtain the uniform estimate (28). $\square$

6. Persistence of periodic orbits

In this section we derive macroscopic equations when at least one component of $f(x_0) - a(\theta)$ does not change sign. In this case we can study the problem using circle diffeomorphisms. Our results of this section may be summarized as follows. We derive the homogenized equation

$$\dot{x}(\theta) = \frac{1}{T(x, \theta)} \begin{pmatrix} 1 \\ \rho(x) \end{pmatrix}, \tag{35}$$

where ρ is the rotation number of the microscopic flow (defined below), and $T(x, \theta)$ depends delicately on whether ρ is rational or irrational. If $\rho \in \mathbb{Q}$, $\omega(\theta)$ is a periodic orbit and $T(x, \theta)$ depends on the time period of $\omega(\theta)$ (see Theorem 4). If $\rho \in \mathbb{R}\backslash\mathbb{Q}$, $T(x, \theta)$ is an ergodic average that is independent of θ.

Under the assumption that x_0 lies in a resonance band $S_{p/q} = \rho^{-1}\{p/q\}$ we can prove this rigorously, with a minimal smoothness assumption. Let us state the theorems at the outset.

Theorem 3. *Suppose that $A \in C^2(\mathbb{T}^2)$, $\rho(x_0)$ is rational, and there is at least one hyperbolic periodic orbit for the flow of* (40) *in $\mathbb{T}^2$. Then there exists $\delta > 0$ and $T(\delta) > 0$ such that*

$$\frac{v^0(t) - v_0}{u^0(t) - u_0} = \rho$$

for all $t \in [-T, T]$, and any weak limit $x^0(t) = (u^0(t), v^0(t))$ with $x^0(0) = x_0$.

The second theorem needs a slightly stronger hyperbolicity assumption, which is

Theorem 4. *Suppose that $A \in C^2(\mathbb{T}^2)$, $\rho(x_0)$ is rational, and all periodic orbits for the flow of* (40) *in $\mathbb{T}^2$are hyperbolic.*

(a) *If $\omega((\theta^0, \psi^0)) = \{\gamma_k^s\}$, then*

$$\lim_{t \to 0_+} \frac{u^0(t) - u_0}{t} = \frac{q}{T_k^s}. \tag{36}$$

(b) *If $\alpha((\theta^0, \psi^0)) = \{\gamma_k^u\}$, then*

$$\lim_{t \to 0_-} \frac{u^0(t) - u_0}{t} = \frac{q}{T_k^u}. \tag{37}$$

Conditions (a) and (b) above are statements about the basin of attraction of the microscopic flow: γ_k^{s}, (γ_k^{u}) are stable (unstable) periodic orbits and T_k^{s}, T_k^{u} their time periods. This theorem is the two-dimensional analogue of (25). We prove later that the hypotheses of the theorem are satisfied for $x_0 \in S_{p/q}$ a.e. under the further assumption that $A \in C^\infty$ (Remark 4).

Note that we have *not* proved (35) when ρ is irrational. This question remains open.

6.1. Preliminaries

We will need to distinguish between the coordinates in this section, and the notation is unfortunately cumbersome. At a first reading one may suppose for simplicity that $x^\varepsilon(0) = x^0(0)$, for this simplifies most computations significantly.

The variable x denotes a point in $\mathbb{R}^2$ and (u, v) its components. The blow-up transformation about $x_0 = (u_0, v_0)$ is written as

$$(y, z) = \frac{(u, v) - (u_0, v_0)}{\varepsilon} + (\alpha^\varepsilon, \beta^\varepsilon), \tag{38}$$

where $(\alpha^\varepsilon, \beta^\varepsilon) = (u_0, v_0)/\varepsilon \mod 1$. The equations of motion are written as

$$\begin{aligned} y' &= f(x_0 + \varepsilon((y,z) - (\alpha^\varepsilon, \beta^\varepsilon))) + a(y,z), \\ z' &= g(x_0 + \varepsilon((y,z) - (\alpha^\varepsilon, \beta^\varepsilon))) + b(y,z). \end{aligned} \tag{39}$$

In the limit $\varepsilon = 0$ we have the microscopic vector field

$$\begin{aligned} y' &= f(x_0) + a(y,z), \\ z' &= g(x_0) + b(y,z). \end{aligned} \tag{40}$$

In the rest of this section we assume for concreteness that $f(x_0) > \|a\|_\infty$. Thus, $y' > 0$, and there are no critical points. Since y is increasing, the trajectories of (39) are solutions to

$$\frac{dz}{dy} = \frac{g(x_0) + b(y,z)}{f(x_0) + a(y,z)}. \tag{41}$$

We shall denote a solution to (41) with initial condition $z(y_0) = z_0$, by $Z(y; (y_0, z_0))$. The graph $\{(y, Z(y; (y_0, z_0)))\}_{y \in \mathbb{R}}$ is a trajectory of (40). Since $y' > 0$ each trajectory intersects the z-axis transversely at a single point, and it is sufficient to study trajectories with $y_0 = 0$. In this case, we simplify notation and write

$$Z(y; (0, z_0)) = Z_{z_0}(y).$$

The solution Z is constrained by the periodicity. Suppose $n = (n_y, n_z) \in \mathbb{Z}^2$ and

$$(y_1, z_1) = (y_0, z_0) + (n_y, n_z).$$

Then

$$Z(y; (y_1, z_1)) = Z(y - n_y; (y_0, z_0)) + n_z. \tag{42}$$

6.2. *The rotation number*

The properties of the homogenized vector field are strongly dependent on the rotation number for the microscopic flow. Our results rely heavily on work by DENJOY [16], ARNOL'D [6], BRUNOVSKY [13] and HERMAN [27]. In this section we use only basic properties of the rotation number, but in the later sections the role of the stronger theorems will be clear.

There is a huge literature on circle maps because of their mathematical appeal and ubiquity in applications. HERMAN's beautiful thesis is an authoritative source on the subject of circle diffeomorphisms [27]. The first chapter of DE MELO & VAN STRIEN [15] provides a different perspective and notes on recent developments. YOCCOZ's article is an extremely concise and elegant survey of the state of the art (ca. 1989) [45].

Our approach and definitions are specific to the problem at hand. Following ARNOL'D [6] we define

Definition 1. The *rotation number* $\rho = \lim_{y \to \infty} \frac{Z_{z_0}(y)}{y}$.

The number ρ is well defined and is independent of z_0 [7]. We may also consider the Poincaré map, P, from $\{y = 0\}$ to $\{y = 1\}$ and an equivalent definition of the rotation number is

$$\rho = \lim_{n\to\infty} \frac{P^n(z) - z}{n}.$$

It is more common to consider $\rho_0 = \rho \bmod 1$ as the rotation number when studying circle maps. We use Definition 1 since it has the natural interpretation of the slope of the averaged vector field as in (35).

The rotation number ρ is a rational number if and only if (5) has a periodic orbit. Suppose $\rho = p/q$, with $\gcd(p, q) = 1$ (henceforth, we always assume this). Then there is a periodic orbit Z_{z_0} such that

$$Z_{z_0}(y + q) = Z_{z_0}(y) + p.$$

In this case we say that the flow is resonant. The following theorem characterizes the direction of the weak limits in a resonant region, under an assumption of hyperbolicity. The proof of this theorem relies on a construction of barriers as in Theorem 2. In this case, the barriers are constructed using the hyperbolic periodic orbit.

6.3. Modified equations

We choose $\delta > 0$ as follows. We have assumed that $f(x_0) > \|a\|_\infty$. Hence, $f(x) > \|a\|_\infty$ for $x \in B(x_0, \delta)$ if $\delta > 0$ is small. Suppose $x^\varepsilon(t) \overset{*}{\rightharpoonup} x^0(t)$ and $x^0(0) = x_0$. The speeds $|\dot{x}^\varepsilon(t)|$ are uniformly bounded. Hence, for any $\delta > 0$ there is a time $T(\delta)$ and $\varepsilon_0(\delta) > 0$ such that $x^\varepsilon(t) \in B(x_0, \delta)$ for each $t \in [-T, T]$, and $\varepsilon \in [0, \varepsilon_0]$. As in Section 5 we use the modified vector field (34). In all that follows we suppose that $|t| < T(\delta)$ so that all conclusions hold for the unmodified system. We first introduce simpler notation. Let

$$f^\varepsilon(y, z) = f(x_0 + \varepsilon((y, z) - (\alpha^\varepsilon, \beta^\varepsilon))\chi_\delta(\varepsilon(y, z))). \tag{43}$$

The function $g^\varepsilon(y, z)$ is defined similarly. If δ is small enough,

$$\sup_{y,z} |f^\varepsilon(y, z) - f(x_0)| \leqq \tfrac{1}{2}\left(f(x_0) - \|a\|_\infty\right). \tag{44}$$

Hence,

$$\begin{aligned} f^\varepsilon(y, z) - \|a\|_\infty &\geqq f(x_0) - \|a\|_\infty - |f^\varepsilon(y, z) - f(x_0)| \\ &\geqq \tfrac{1}{2}(f(x_0) - \|a\|_\infty) > 0. \end{aligned}$$

It follows that

$$L|t_2 - t_1| \leqq |u^\varepsilon(t_2) - u^\varepsilon(t_1)| \leqq M|t_2 - t_1| \tag{45}$$

with

$$L = \tfrac{1}{2}(f(x_0) - \|a\|_\infty) \qquad (46)$$
$$M = \tfrac{1}{2}(3f(x_0) + \|a\|_\infty).$$

Thus, u^ε, and hence y^ε, increase monotonically with time, and the trajectories of (34) also solve

$$\frac{dz}{dy} = \frac{g^\varepsilon(y,z) + b(y,z)}{f^\varepsilon(y,z) + a(y,z)}. \qquad (47)$$

The solution to (47) through $(y^\varepsilon(0), z^\varepsilon(0))$ is denoted by $Z^\varepsilon(y; (y^\varepsilon(0), z^\varepsilon(0)))$. In the limit $\varepsilon = 0$ we drop the superscript for Z.

6.4. Proofs

Let $x^\varepsilon(t) = (u^\varepsilon(t), v^\varepsilon(t))$ converge weakly to $x^0(t)$. We use the same time parametrization for microscopic coordinates, i.e., we write

$$(y^\varepsilon(t), z^\varepsilon(t)) = \frac{(u^\varepsilon(t), v^\varepsilon(t)) - (u_0, v_0)}{\varepsilon} + (\alpha^\varepsilon, \beta^\varepsilon).$$

The change of scale is singular. Thus, though $x^\varepsilon(0) \to x_0$, the rescaled version $(y^\varepsilon(0), z^\varepsilon(0))$ may diverge. The following lemma is used often to quell this minor annoyance.

Lemma 6. *Let $t \in [-T, T]$, $t \neq 0$. Then*

(a) $\lim_{\varepsilon\to 0} |y^\varepsilon(t)| = \infty$,
(b) $\lim_{\varepsilon\to 0} \frac{z^\varepsilon(0)}{y^\varepsilon(t)} = 0$,
(c) $\lim_{\varepsilon\to 0} \frac{y^\varepsilon(0)}{y^\varepsilon(t)} = 0$.

Proof. We have $|u^\varepsilon(t) - u^\varepsilon(0)| \geqq L|t|$ by (45). Therefore,

$$\begin{aligned}|y^\varepsilon(t)| &= \left|\frac{(u^\varepsilon(t) - u_0)}{\varepsilon} + \alpha^\varepsilon\right| \geqq \frac{|u^\varepsilon(t) - u_0|}{\varepsilon} - 1 \\ &\geqq \frac{|u^\varepsilon(t) - u^\varepsilon(0)| - |u^\varepsilon(0) - u_0|}{\varepsilon} - 1 \\ &\geqq \frac{1}{\varepsilon}\left(L|t| - \left|u^\varepsilon(0) - u_0\right|\right) - 1.\end{aligned}$$

But $|u^\varepsilon(0) - u_0| \to 0$, and we have (a). To obtain (b), write

$$\left|\frac{z^\varepsilon(0)}{y^\varepsilon(t)}\right| = \left|\frac{v^\varepsilon(0) - v_0 + \varepsilon\beta^\varepsilon}{u^\varepsilon(t) - u_0 + \varepsilon\alpha^\varepsilon}\right| \leqq \frac{|v^\varepsilon(0) - v_0| + \varepsilon}{L|t| - |u^\varepsilon(0) - u_0| - \varepsilon}.$$

Since $x^\varepsilon(0) \to x_0$, we obtain (b). The proof of (c) is similar, and is omitted. □

The next lemma is the crucial uniform estimate that we need to prove Theorem 3. It says that we can estimate the solutions of (47) in terms of solutions to the periodic limit (41).

Lemma 7. *Assume the hypotheses of Theorem 3. Then there exists an* $\eta_0 > 0$ *such that for each* $\eta \in (0, \eta_0)$ *we can find* $\varepsilon_0(\eta)$ *such that*

$$\sup_{\varepsilon \in [0, \varepsilon_0(\eta)]} \sup_{y \in \mathbb{R}} |Z^{\varepsilon}(y; (y_0, z_0)) - Z(y; (y_0, z_0))| \leqq 1 + 4\eta$$

uniformly in the initial conditions (y_0, z_0).

Proof. Let γ denote a hyperbolic periodic orbit in $\mathbb{T}^2$. Without loss of generality, suppose that γ is attracting. The images $\Pi^{-1}\{\gamma\}$ are a family of curves in $\mathbb{R}^2$ that are integer translates of one another. Let Γ be a curve in $\Pi^{-1}\{\gamma\}$ which passes through $[0, 1)^2$. We can write Γ as the graph of a C^1 map $\varphi : \mathbb{R} \to \mathbb{R}$. Then by (42), $\varphi(y + q) = \varphi(y) + p$. For any $\eta > 0$ define the strip

$$S(\varphi, \eta) = \left\{ (y, z) \in \mathbb{R}^2 \middle| |z - \varphi(y)| \leqq \eta. \right\}.$$

Since γ is hyperbolic, there is an $\eta_0 > 0$, such that for each $\eta \in (0, \eta_0)$ the strip of width η about γ is strictly absorbing and the vector field (5) is transverse to the boundary $\partial S(\varphi, \eta)$ (see Fig. 10). Precisely, let $\mathbf{v}(y, z)$ denote the vector field (5) at $(y, z) \in \mathbb{R}^2$, and $\mathbf{n}_{\pm}(y)$ denote the outward normal vectors to $\partial S(\varphi, \eta)$ at the points $(y, \varphi(y) \pm \eta)$. Then, it follows from the periodicity of (5) that

$$\inf_{y \in \mathbb{R}, m \in \mathbb{Z}} \mathbf{n}_{\pm}(y) \cdot \mathbf{v}(y, \varphi(y) \pm \eta + m) \leqq c(\eta) < 0,$$

where $c(\eta)$ is some negative constant.

By Lemma 4, the vector field (47), denoted by $\mathbf{v}^{\varepsilon}$, is $O(\varepsilon + \delta)$ close to (41) in the C^1 topology. Thus, there exist $\delta > 0$ and $\varepsilon_0(\delta, \eta) > 0$ such that for $\varepsilon \in [0, \varepsilon_0]$, we have

$$\inf_{y \in \mathbb{R}, m \in \mathbb{Z}} \mathbf{n}_{\pm}(y) \cdot \mathbf{v}^{\varepsilon}(y, \varphi(y) \pm \eta + m) \leqq \frac{c(\eta)}{2} < 0.$$

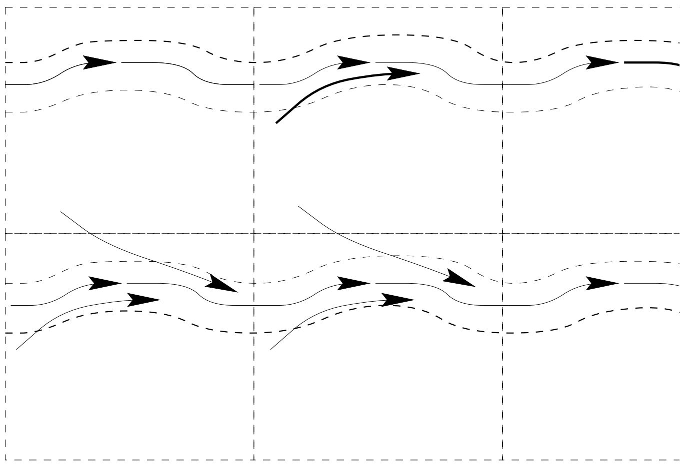

Fig. 10. The absorbing strip of width η about Γ.

This shows that the strips $S(\varphi, \eta) + m, m \in \mathbb{Z}$ remain positively invariant. Thus, for $0 \leqq \varepsilon \leqq \varepsilon_0$, $\mathbb{R}^2$ is separated by positively invariant strips of width 2η that are a unit distance apart in the z-direction. This implies the lemma. $\square$

Proof of Theorem 3. Choose $\delta > 0$ so that (44) and Lemma 7 are true, and define $T(\delta)$ as in Section 6.3. Since $x^\varepsilon(t) \to x^0(t)$ uniformly on $[-T, T]$,

$$\lim_{\varepsilon\to 0} \frac{v^\varepsilon(t) - v_0}{u^\varepsilon(t) - u_0} = \frac{v^0(t) - v_0}{u^0(t) - u_0}.$$

But by equation (38) and Lemma 6, the left-hand side is

$$\lim_{\varepsilon\to 0} \frac{z^\varepsilon(t) - \beta^\varepsilon}{y^\varepsilon(t) - \alpha^\varepsilon} = \lim_{\varepsilon\to 0} \frac{z^\varepsilon(t)}{y^\varepsilon(t)}.$$

We rewrite $z^\varepsilon(t)$ using (41) and apply Lemma 7 to obtain

$$z^\varepsilon(t) = Z^\varepsilon(y^\varepsilon(t); (y^\varepsilon(0), z^\varepsilon(0))) = Z(y^\varepsilon(t); (y^\varepsilon(t), z^\varepsilon(0))) + O(1).$$

By Lemma 6(a), we see that it suffices to evaluate

$$\lim_{\varepsilon\to 0} \frac{Z(y^\varepsilon(t); (y^\varepsilon(0), z^\varepsilon(0)))}{y^\varepsilon(t)}.$$

We exploit the periodicity of Z as follows. Let $[(y, z)] \in \mathbb{Z}^2$ denote the integer part of (y, z). Let $(\theta^\varepsilon, \psi^\varepsilon) = (y^\varepsilon(0), z^\varepsilon(0)) - [(y^\varepsilon(0), z^\varepsilon(0))]$. Then, by (42)

$$Z(y; y^\varepsilon(0), z^\varepsilon(0)) = Z(y - [y^\varepsilon(0)]; (\theta^\varepsilon, \psi^\varepsilon)) + [z^\varepsilon(0)].$$

Applying Lemma 6 again, we see that

$$\lim_{\varepsilon\to 0} \frac{Z(y^\varepsilon(t); (y^\varepsilon(0), z^\varepsilon(0)))}{y^\varepsilon(t)} = \lim_{\varepsilon\to 0} \frac{Z(y^\varepsilon(t) - [y^\varepsilon(0)]; (\theta^\varepsilon, \psi^\varepsilon))}{y^\varepsilon(t) - [y^\varepsilon(0)]}.$$

Furthermore, $(\theta^\varepsilon, \psi^\varepsilon) \in [0, 1)^2$ so that

$$\sup_y |Z(y, (\theta^\varepsilon, \psi^\varepsilon)) - Z(y; (0, 0))| \leqq 2$$

uniformly in ε. Thus, by Lemma 6 the limit is

$$\lim_{\varepsilon\to 0} \frac{Z(y^\varepsilon(t) - [y^\varepsilon(0)]; (0, 0))}{y^\varepsilon(t) - [y^\varepsilon(0)]} = \lim_{y\to\infty} \frac{Z(y; (0, 0))}{y} = \rho. \qquad \square$$

6.5. Calculation of the limiting speed

Theorem 3 tells us $\dot{x}^0$ is in the direction $(1, \rho)^t$. Therefore, in order to compute its magnitude, $|\dot{x}^0|$, it is sufficient to evaluate $\dot{u}^0$. We make the further assumption that all periodic orbits of the microscopic flow are hyperbolic. Then there are an even number of periodic orbits on $\mathbb{T}^2$ which we denote by $\{\gamma_1^{\mathrm{s}}, \ldots, \gamma_m^{\mathrm{s}}, \gamma_1^{\mathrm{u}}, \ldots, \gamma_m^{\mathrm{u}}\}$. The superscripts "s" and "u" mean "stable" and "unstable" respectively.

We assume that δ and $T(\delta)$ are chosen as in Section 6.3. For $0 < \varepsilon \leqq \varepsilon_0(\delta)$, we may solve (47) for z as a function of y, and invert the blow-up transformation (38) to obtain

$$\dot{u} = f(u, v^\varepsilon(u)) + a\left(\frac{u}{\varepsilon}, \frac{v^\varepsilon(u)}{\varepsilon}\right), \tag{48}$$

where

$$v^\varepsilon(u) = v_0 - \varepsilon\beta^\varepsilon + \varepsilon Z^\varepsilon(y^\varepsilon(u); (y^\varepsilon(0), z^\varepsilon(0))), \tag{49}$$

with

$$y^\varepsilon(u) = \frac{u - u_0}{\varepsilon} + \alpha^\varepsilon, \quad \text{and} \tag{50}$$

$$(y^\varepsilon(0), z^\varepsilon(0)) = \frac{(u^\varepsilon(0), v^\varepsilon(0)) - (u_0, v_0)}{\varepsilon} + (\alpha^\varepsilon, \beta^\varepsilon). \tag{51}$$

Equation (48) can be inverted to obtain

$$\int_{u^\varepsilon(0)}^{u^\varepsilon(t)} \frac{du}{f(u, v^\varepsilon(u)) + a(u/\varepsilon, v^\varepsilon(u)/\varepsilon)} = t.$$

We take the limit $\varepsilon \downarrow 0$ on the left-hand side, and use the uniform convergence of $u^\varepsilon(t)$ to $u^0(t)$ to obtain

$$\lim_{\varepsilon \to 0} \int_{u_0}^{u^0(t)} \frac{du}{f(u, v^\varepsilon(u)) + a(u/\varepsilon, v^\varepsilon(u)/\varepsilon)} = t. \tag{52}$$

We shall compute the limit on the left-hand side using the microscopic flow. The following sequence of lemmas progressively simplifies the calculation.

Lemma 8. *The following inequatlity holds:*

$$\limsup_{\varepsilon \to 0} \left| \int_{u_0}^{u^0(t)} \frac{du}{f(u, v^\varepsilon(u)) + a(u/\varepsilon, v^\varepsilon(u))} - \int_{u_0}^{u^0(t)} \frac{du}{f(x_0) + a(u/\varepsilon, v^\varepsilon(u))} \right| \leqq Ct^2.$$

Proof. Since f and u^ε are Lipschitz

$$\begin{aligned}|f(x^\varepsilon(s)) - f(x_0)| &\leqq |f(x^\varepsilon(s)) - f(x^\varepsilon(0))| + |f(x^\varepsilon(0)) - f(x_0)|\\ &\leqq \mathrm{Lip}(f)(M|s| + |x^\varepsilon(0) - x_0|),\end{aligned}$$

by (45). The denominators in the integrands are bounded away from zero because $f(x) - \|a\|_\infty > 0$ in $B(x_0, \delta)$. Thus it is enough to estimate the difference

$$\begin{aligned}\int_{u_0}^{u^0(t)} |f(u, v^\varepsilon(u)) - f(x_0)|\, du &\leqq M \int_0^{|t|} |f(u(s), v^\varepsilon(u(s))) - f(x_0)|\, ds\\ &\leqq \mathrm{Lip}(f) M \int_0^{|t|} (Ms + |x^\varepsilon(0) - x_0|)\, ds\\ &\leqq M\mathrm{Lip}(f)\left(\frac{Mt^2}{2} + |t||x^\varepsilon(0) - x_0|\right).\end{aligned}$$

The linear term in $|t|$ vanishes in the limit $\varepsilon \downarrow 0$. □

The phase of the initial conditions plays an important role in evaluating the limit. As earlier, write

$$(y^\varepsilon(0), z^\varepsilon(0)) - [(y^\varepsilon(0), z^\varepsilon(0))] = (\theta^\varepsilon, \psi^\varepsilon) \in [0, 1]^2.$$

Thus, there exists a convergent subsequence $(\theta^{\varepsilon_k}, \psi^{\varepsilon_k}) \to (\theta^0, \psi^0) \in [0, 1]^2$. The limit in (52) exists for all ε, so it may be computed by evaluating a subsequential limit. There are three distinct cases to consider:

(·1) (θ^0, ψ^0) lies on an attracting periodic orbit, γ_k^{s}, $1 \leqq k \leqq m$;
(·2) (θ^0, ψ^0) lies on a repelling periodic orbit, γ_k^{u}, $1 \leqq k \leqq m$;
(·3) (θ^0, ψ^0) lies in $\mathbb{T}^2/\{\gamma_k^{\mathrm{s}}, \gamma_k^{\mathrm{u}}\}_{k=1,\dots,m}$.

We begin with an analysis of the case when (θ^0, ψ^0) lies in the domain of attraction of an attracting periodic orbit γ_k^{s}. The following lemmas have obvious analogues for γ_k^{u} that we do not state explicitly (but see Theorem 4).

Lemma 9. *Suppose* $\omega((\theta^0, \psi^0)) = \{\gamma_k^{\mathrm{s}}\}$ *and* $u \geqq u_0$. *Let* $\eta \in [0, \eta_0]$ *be as in Lemma 7. Then*

$$\limsup_{\varepsilon \to 0} \left| a\left(\frac{u}{\varepsilon}, \frac{v^\varepsilon(u)}{\varepsilon}\right) - a(y^\varepsilon(u), \varphi_k^{\mathrm{s}}(y^\varepsilon(u)))\right| \leqq \mathrm{Lip}(a)\eta.$$

Proof. Consider the flow in $\mathbb{R}^2$. The point $(\theta^0, \psi^0) \in [0, 1]^2$ is absorbed into the η-strip about $\Gamma^s \in \Pi^{-1}\{\gamma_k^s\}$. Thus, there is a constant $C(\theta^0, \psi^0)$ such that

$$\mathrm{dist}(Z(y; (\theta^0, \psi^0)), \Pi^{-1}\{\gamma_k^{\mathrm{s}}\}) \leqq \eta \quad \text{for } y \geqq C.$$

Since $(\theta^\varepsilon, \psi^\varepsilon) \to (\theta^0, \psi^0)$ we can choose $\varepsilon_0 > 0$ so that for $\varepsilon \in [0, \varepsilon_0]$

$$\mathrm{dist}(Z(y; (\theta^\varepsilon, \psi^\varepsilon)), \Pi^{-1}\{\gamma_k^{\mathrm{s}}\}) \leqq \eta \quad \text{for } y \geqq 2C.$$

This estimate is equivalent to

$$\mathrm{dist}(Z(y-[y^\varepsilon(0)];(y^\varepsilon(0),z^\varepsilon(0))),\Pi^{-1}\{\gamma_k^s\})\leqq\eta\quad\text{for } y-[y^\varepsilon(0)]\geqq 2C$$

by the periodicity of Z (see (42)). We can say that the "time" $y-[y^\varepsilon(0)]$ taken to enter an absorbing strip is uniform over the initial conditions $(y^\varepsilon(0), z^\varepsilon(0))$. By Gronwall's inequality, solutions to (41) and (47) are $O(\varepsilon+\delta)$ close. Thus, for small $\delta(\eta)>0$ and $\varepsilon_0(\eta,\delta)>0$ the "time" taken for solutions of (47) to enter the absorbing strip is not greater than $4C$. Thus, we have the uniform estimate

$$\mathrm{dist}(Z^\varepsilon(y-[y^\varepsilon(0)];(y^\varepsilon(0),z^\varepsilon(0))),\Pi^{-1}\{\gamma_k^s\})\leqq\eta\quad\text{for } y-[y^\varepsilon(0)]\geqq 4C. \tag{53}$$

By (50), $(u,v)/\varepsilon=(y^\varepsilon(u),Z^\varepsilon(y^\varepsilon(u);(y^\varepsilon(0),z^\varepsilon(0))))$ mod $\mathbb{Z}$. Thus,

$$\begin{aligned}a\left(\frac{u}{\varepsilon},\frac{v^\varepsilon(u)}{\varepsilon}\right)&=a(y^\varepsilon(u),Z^\varepsilon(y^\varepsilon(u);(y^\varepsilon(0),z^\varepsilon(0))))\\&=a(y^\varepsilon-[y^\varepsilon(0)],Z^\varepsilon(y^\varepsilon(u)-[y^\varepsilon(0)];(y^\varepsilon(0),z^\varepsilon(0)))).\end{aligned}$$

Since $u>u_0$, Lemma 6 asserts that $y^\varepsilon(u)-[y^\varepsilon(0)]\to\infty$. The Lemma now follows from (53). □

Lemma 10. *Assume the hypotheses of Lemma 9. Then*

$$\begin{aligned}\limsup_{\varepsilon\to 0}&\left|\int_{u_0}^{u^0(t)}\frac{du}{f(x_0)+}-\int_{u_0}^{u^0(t)}\frac{du}{f(x_0)+a(y^\varepsilon(u),\varphi_k^s(y^\varepsilon(u)))}\right|\\&\leqq C\eta t.\end{aligned}$$

Proof. By Lemma 9,

$$\begin{aligned}\limsup_{\varepsilon\to 0}&\left|\frac{1}{f(x_0)+a(u/\varepsilon,v^\varepsilon(u)/\varepsilon)}-\frac{1}{f(x_0)+a(y^\varepsilon(u),\varphi_k^s(y^\varepsilon(u)))}\right|\\&\leqq\frac{\mathrm{Lip}(a)\eta}{(f(x_0)-\|a\|_\infty)^2}.\end{aligned}\tag{54}$$

It remains to interchange the integral and the limit, and this is justified as follows. Let $0\leqq g^\varepsilon\leqq 1$ be a sequence of measurable functions on $[0,1]$. Applying Fatou's lemma to the functions $1-g^\varepsilon\geqq 0$, we see that

$$\begin{aligned}1-\int\limsup_{\varepsilon\to 0}g^\varepsilon\,dx&=\int\liminf_{\varepsilon\to 0}(1-g^\varepsilon)\,dx\\&\leqq\liminf_{\varepsilon\to 0}\int(1-g^\varepsilon)\,dx=1-\limsup_{\varepsilon\to 0}\int g^\varepsilon\,dx.\end{aligned}$$

Therefore,

$$\limsup_{\varepsilon\to 0}\int g^\varepsilon dx\leqq\int\limsup_{\varepsilon\to 0}g^\varepsilon dx. \tag{55}$$

Combining (54) and (55) we see that the difference in the integrals is not greater than

$$\frac{\mathrm{Lip}(a)\eta}{(f(x_0)-\|a\|_\infty)^2}|u^0(t)-u_0| \leqq \frac{M\mathrm{Lip}(a)\eta t}{(f(x_0)-\|a\|_\infty)^2}. \qquad \square$$

We have thus reduced the problem to a computation with the microscopic flow. Recall that the rotation number is p/q, graph(φ_k^{s}) is a component of $\Pi^{-1}\{\gamma_k^{\mathrm{s}}\}$, and $\varphi_k^{\mathrm{s}}(y+q)=\varphi_k^{\mathrm{s}}(y)+p$.

Lemma 11. *Let T_k^{s} be the time period of γ_k^{s}. Then*

$$\lim_{\varepsilon\to 0}\int_{u_0}^{u^0(t)} \frac{du}{f(x_0)+a(y^\varepsilon(u),\varphi_k^{\mathrm{s}}(y^\varepsilon(u)))} = \frac{T_k^{\mathrm{s}}}{q}(u^0(t)-u_0). \tag{56}$$

Proof. The time T_k^{s} is the time taken by y to increase from 0 to q. Therefore,

$$T_k^s = \int_0^q \frac{dy}{f(x_0)+a(y,\varphi_k^{\mathrm{s}}(y))}. \tag{57}$$

Change coordinates using (38). Then

$$\begin{aligned}\int_{u_0}^{u^0(t)} \frac{du}{f(x_0)+a(y^\varepsilon(u),\varphi_k^{\mathrm{s}}(y^\varepsilon(u)))} &= \int_0^{\frac{u^0(t)-u_0}{\varepsilon}+\alpha^\varepsilon} \frac{\varepsilon\, dy}{f(x_0)+a(y,\varphi_k^{\mathrm{s}}(y))}\\ &= \frac{T_k^{\mathrm{s}}}{q}(u^0(t)-u_0)+O(\varepsilon).\end{aligned}$$

$\square$

Lemmas 8, 9 and 10 enable us to compute the speed $\dot{u}^0$.

Proof of Theorem 4. It is sufficient to prove (a). Let $t>0$. By Lemma 8, and Lemma 10, we have the estimate

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Bigg|\int_{u_0}^{u^0(t)} &\frac{du}{f(x_0)+a(y^\varepsilon(u),\varphi_k^{\mathrm{s}}(y^\varepsilon(u)))}\\ &-\int_{u_0}^{u^0(t)} \frac{du}{f(u,v^\varepsilon(u))+a(u/\varepsilon,v^\varepsilon(u))}\Bigg|\\ \leqq C(\eta t+t^2).\end{aligned} \tag{58}$$

But the limits of both integrals have been computed explicitly in equations (52) and (36). Thus, (58) can be rewritten as

$$\left|\frac{T_k^{\mathrm{s}}}{q}\frac{u^0(t)-u_0}{t}-1\right| \leqq \eta + Ct.$$

Letting $t\to 0$, we find

$$\left|\limsup_{t\to 0_+}\frac{u^0(t)-u_0}{t} - \liminf_{t\to 0_+}\frac{u^0(t)-u_0}{t}\right| \leqq \frac{q}{T_k^{\mathrm{s}}}\eta.$$

The left-hand side is independent of η. Hence, $u^0(t)$ is differentiable from above with limit q/T_k^{s}. $\square$

The point x_0 has been chosen so that all periodic oribits for the microscopic vector field (5) are hyperbolic. If δ is sufficiently small, $\{\gamma_k^{\mathrm{s}}, \gamma_k^{\mathrm{u}}\}$, $1 \leqq k \leqq m$ and $\{T_k^{\mathrm{s}}, T_k^{\mathrm{u}}\}$ continue for $x \in B(x_0, \delta)$ in a C^1 manner to the periodic orbits $\{\gamma_k^{\mathrm{s}}(x), \gamma_k^{\mathrm{u}}(x)\}$ with periods $\{T_k^{\mathrm{s}}(x), T_k^{\mathrm{u}}(x)\}$. The function $x^0(t)$ is differentiable a.e. by Rademacher's theorem. Thus, for a.e. $t \in [-T(\delta), T(\delta)]$ the upper and lower derivatives $\dot{x}^0_{\pm}$ are equal and the derivative

$$\dot{x}^0(t) \in \begin{pmatrix} q \\ p \end{pmatrix} \left\{ \frac{1}{T_k^s(x^0(t))}, \frac{1}{T_k^u(x^0(t))} \right\}_{1 \leqq k \leqq m}. \tag{59}$$

6.6. Irrational rotation number

If $\rho \in \mathbb{R}\backslash\mathbb{Q}$, and a is C^2 the microscopic flow is ergodic. In this case, there are no underlying hyperbolic periodic orbits, and we cannot use these techniques to obtain a homogenized vector field. However, it is natural to proceed by analogy and compute the limiting equation (35). We may rewrite $T(\theta)$ as

$$T(\theta) = \lim_{u \to \infty} \frac{1}{u} \int_0^u \frac{du}{f(x_0) + a(u, Z(u; \theta))}, \tag{60}$$

a form that generalizes naturally to irrational ρ. When the flow is ergodic the limit above exists for all θ and is constant.

7. Weak continuity of invariant measures

In this section we show that the "vector field" (35) is continuous at points x_0 where $\rho(x_0) \in \mathbb{R}\backslash\mathbb{Q}$. This follows from the following theorem on weak continuity of the invariant measures for C^2 circle maps with irrational rotation number.

Theorem 5. *Let $P_t : S^1 \to S^1, t \in [0, 1]$, be a one-parameter family of orientation-preserving circle homeomorphisms such that*

(a) $P = P_0 \in C^2$ *and* $\rho(P) \in \mathbb{R}\backslash\mathbb{Q}$,
(b) $\lim_{t \to 0} \|P_t - P\|_{C^0} = 0$.

Let μ_t be probability measures invariant under P_t. Then the measures μ_t converge weakly to μ (written $\mu_t \rightharpoonup \mu$), where μ is the unique probability measure invariant under P.

Remark 2. The maps P_t need not be C^2, and the measures μ_t need not be unique.

By continuity of the limiting vector field we mean the following.

Corollary 1. *Suppose $\rho(x_0) \in \mathbb{R}\backslash\mathbb{Q}$ and $A, F \in C^3$. Then*

$$\lim_{x \to x_0} \frac{1}{T(x,\theta)} \begin{pmatrix} 1 \\ \rho(x) \end{pmatrix} = \frac{1}{T(x_0)} \begin{pmatrix} 1 \\ \rho(x_0) \end{pmatrix}$$

uniformly in $\theta \in \mathbb{T}^2$.

Weak convergence of measures is characterised by any of the following equivalent definitions (see [10, Section 2]):

(·1) $\lim_{t\to 0} \int_{S^1} f d\mu_t = \int_{S^1} f d\mu$, for any $f \in C(S^1)$;
(·2) $\mu(F) \geqq \limsup_{t\to 0} \mu_t(F)$ for all closed F;
(·3) $\mu(G) \leqq \liminf_{t\to 0} \mu_t(G)$ for all open G;
(·4) $\mu(B) = \lim_{t\to 0} \mu_t(B)$ for Borel sets B with $\mu(\partial B) = 0$.

The problem may be reduced to the study of irrational rotations as follows. A Borel measure, ν, and a homeomorphism, h, together induce a measure $h_*\nu$ defined on any Borel set B by

$$h_*\nu(B) = \nu(h(B)).$$

It is easy to check that $\mu_t \rightharpoonup \mu$ is equivalent to $h_*\mu_t \rightharpoonup h_*\mu$. Since P is C^2, it follows from Denjoy's theorem that P is topologically conjugate to the ergodic translation $R_\rho : x \mapsto x + \rho$. That is, there is a homeomorphism h of S^1 so that $h \circ P \circ h^{-1} = R_\rho$. Thus, to establish Theorem 5 it is sufficient to suppose that $P = R_\rho$.

Since $\mu_t(S^1) = 1$, the family of measures μ_t is weakly compact and tight, so that there exists a subsequence (also denoted μ_t) that converges weakly to a probability measure μ_*. We must prove that μ_* is μ, and because the invariant measure is unique, it suffices to show that μ_* is invariant under P. The proof follows from the following estimate.

Proposition 2. *For any $l > 0$ let*

$$\eta_l = \sup_I \limsup_{t\to 0} \mu_t(I), \tag{61}$$

where the suprememum is taken over all closed intervals $I \subset S^1$ with $m(I) \leqq l$. Then

$$\lim_{l\to 0} \eta_l = 0. \tag{62}$$

The proof of Proposition 2 requires some estimates, and we return to it in a moment. Theorem 5 is obtained from it as follows.

Proof of Theorem 5. Firstly, it is an immediate consequence of Proposition 2 that μ_* is non-atomic. Therefore, property (·4) of weak convergence implies that $\mu_*(G) = \lim_{t\to 0} \mu_t(G)$ for any open interval G. Hence,

$$\begin{aligned} \mu_*(G) &= \lim_{t\to 0} \mu_t(G) = \lim_{t\to 0} \mu_t(P_t(G)) \\ &\leqq \limsup_{t\to 0} \left[\mu_t(P(G)) + \mu_t(P_t(G)\backslash P(G))\right]. \end{aligned}$$

Since $\mu_*(P(G)) = \lim_{t\to 0} \mu_t(P(G))$, we see that

$$\mu_*(G) \leqq \mu_*(P(G)) + \limsup_{t\to 0} \mu_t(P_t(G)\backslash P(G)).$$

Now, it follows from Proposition 2 that the second term is zero, so that we have $\mu_*(G) \leqq \mu_*(P(G))$. Repeating the argument with P_t replaced by P_t^{-1}, we find that $\mu_*(G) \leqq \mu_*(P^{-1}(G))$ or $\mu_*(P(G)) \leqq \mu_*(G)$. Thus, $\mu_*(P(G)) = \mu_*(G)$ for all open intervals G, and hence for all open sets G. Taking complements of open sets, we find $\mu_*(P(K)) = \mu_*(K)$ for all closed sets K.

Let $\eta > 0$. Any Borel set B may be approximated by open and closed sets so that $K \subset B \subset G$, and

$$\mu_*(G) - \eta \leqq \mu_*(B) \leqq \mu_*(K) + \eta.$$

Clearly, $\mu_*(P(K)) \leqq \mu_*(P(B)) \leqq \mu_*(P(G))$. Combining these estimates, we have

$$-\eta \leqq \mu_*(B) - \mu_*(P(B)) \leqq \eta.$$

This shows that μ_* is an invariant measure for P. □

7.1. Proof of Proposition 2

In the following lemmas we fix a rational number c/d and a closed interval $I = [a, a + c/d]$. The strategy of the proof is to estimate $\limsup_{t\to 0} \mu_t(I)$ using the invariance of μ_t and the known limit $P_0 = R_\rho$. Since we are interested in the limit $m(I) \to 0$, we shall suppose that $c/d < \rho$.

Definition 2. For $(x, t) \in I \times [0, 1]$ we define the *first return time* under the map P_t by

$$N_t(x) = \inf_{k\geqq 1}\{P_t^k(x) \in I\}.$$

Lemma 12. *The return time $N_t(x)$ is jointly lower semicontinuous in x and t.*

Proof. Evidently, $N_t(x) \neq -\infty$. If $N_t(x)$ is finite, then by definition $P_t^k(x) \in S^1\backslash I$ for $1 \leqq k \leqq N_t(x) - 1$, and therefore for y near x, and s near t, we have $P_s^k(y) \in S^1\backslash I$ for $1 \leqq k \leqq N_t(x) - 1$. Thus, $N_t(x) \leqq N_s(y)$, and hence $N_t(x) \leqq \liminf_{y\to x, s\to t} N_s(y)$. Finally, suppose $N_t(x) = +\infty$, but $\liminf_{y\to x,s\to t} N_s(y) = \bar{N} < \infty$. Then there exists a sequence $y_n \to x$ and $s_n \to t$ such that $N_{s_n}(y_n) = \bar{N}$. By the definition of N_t this means that $P_{s_n}^{\bar{N}}(y_n) \in I$. But then, $\lim_{n\to\infty} P_{s_n}^{\bar{N}}(y_n) = P_t^{\bar{N}}(x) \in I$, so that $N_t(x) \leqq \bar{N} < \infty$. This contradiction shows that $\liminf_{y\to x,s\to r} N_s(y) = \infty$. □

Corollary 2. *There exists $x_t \in I$ so that $M_t \stackrel{\text{def}}{=} \inf_{x\in I} N_t(x) = N_t(x_t)$.*

The following lemma is a consequence of the monotonicity of circle maps.

Lemma 13. *For $t = 0$, the minimum M_0 is attained on a closed interval $J \subset I$ with nonempty interior. Furthermore, M_0 is independent of a, and it depends only on the length of I.*

Proof. Observe that M_0 must be finite, since $P = R_\rho$ is an ergodic translation. Therefore, $P^{M_0}(I)$ is a closed interval with $P^{M_0}(I) \bigcap I \neq \phi$. Furthermore, $P^k(a) \neq a$ or $a + c/d$ for any k. Thus, $P^{M_0}(I) \bigcap I$ is a closed interval with nonempty interior. Then J is the interval $I \bigcap P^{-M_0}(I)$. The second assertion is clear: R_ρ is spatially uniform. □

Lemma 14. *There exists $t_0 \in [0, 1]$ such that $M_t = M_0$ for $0 \leqq t \leqq t_0$.*

Proof. By the definition of M_0, we have $P_0^k(I) \bigcap I = \phi$ for $0 \leqq k \leqq M_0 - 1$. Therefore, for small t_0 we must have $P_t^k(I) \bigcap I = \phi, t \in [0, t_0], 0 \leqq k \leqq M_0 - 1$. It follows that for any $y \in I$, we have $N_t(y) \geqq M_0$, and hence $M_t = \inf_{y \in I} N_t(y) \geqq M_0$.

Let $J = [x_1, x_2]$ be as in Lemma 13. Let $J' = [y_1, y_2]$ with $x_1 < y_1 < y_2 < x_2$. Then for small t we have $P_t^{M_0}(J') \subset I$, therefore $N_t(y) \leqq M_0$ for $y \in J'$. It follows that $M_t = \inf_y N_t(y) \leqq M_0$. □

Lemma 15. *The following inequality holds:*

$$\limsup_{t \to 0} \mu_t(I) \leqq \frac{1}{M_0}.$$

Proof. Let $t \in (0, t_0]$ so that $M_t = M_0$. Since μ_t is an invariant measure, there is x_t such that

$$\mu_t(I) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_I(P_t^k(x_t)),$$

where χ_I is the characteristic function for the interval I. Let $\{b_1 b_2 \dots b_n \dots\}$ denote the binary string $\{\chi_I(P_t^k(x_t))\}$. If $b_i = 1$ then $b_{i+j} = 0$ for $1 \leqq j \leqq M_t$, by the definitions of the first return time, and M_t. Therefore, if we evaluate the limit along a subsequence of the form $n = pM_t$ we have

$$\mu_t(I) = \lim_{p \to \infty} \frac{1}{pM_t} \sum_{k=0}^{pM_t - 1} \chi_I(P_t^k(x_t)) \leqq \frac{1}{M_t} = \frac{1}{M_0}$$

for $0 < t \leqq t_0$. □

Lemma 16. *Let I_n be a sequence of closed intervals with length $m(I_n) = c_n/d_n \to 0$. Then $M_0(I_n) \to \infty$.*

Proof. Suppose there are intervals I_n such that $\sup_n M_0(I_n) < \infty$. Passing to a subsequence we may suppose that $M_0(I_n) = M < \infty$ for all n. Furthermore, since $M_0(I)$ depends only on the length of I we may translate all the intervals I_n so that they are nested, i.e., $I_{n+1} \subset I_n$ for all n. By Lemma 13 we know that $J_n = P^M(I_n) \bigcap I_n$ is a nonempty compact interval and $J_{n+1} \subset J_n$ for all n, so that

$\bigcap_{n=1}^{\infty} J_n \neq \phi$. Since, the intervals I_n are shrinking, this intersection is a singleton, say $\{p\}$ and $\{p\} = \bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} P^M(I_n) = \{P^M(p)\}$, so that we have a periodic orbit. This contradicts $\rho \in \mathbb{R}\backslash\mathbb{Q}$. $\square$

We have proved Proposition 2 when $m(I) = c/d \in \mathbb{Q}$. Let $m(I)$ be irrational. Then we choose a sequence of intervals $[a_k, a_k + c_k/d_k] \supset I$, with $\bigcap_{k=1}^{\infty}[a_k, a_k + c_k, d_k] = I$, and apply Lemma 15. This completes the proof of Proposition 2.

Remark 3. We have not considered the question of higher regularity of the convergence. For example, suppose P_t is a Lipschitz path in $C^0(S^1)$ at 0, that is $\|P_0 - P_t\|_\infty \leqq C|t|$. An equivalent definition of the rotation number is (see [45])

$$\rho(P_0) = \int_{S^1} (P_0(\theta) - \theta)\, d\mu_0 \equiv \langle P_0 - Id, \mu_0 \rangle.$$

Brunovsky has shown that $|\rho(P_t) - \rho(P_0)| \leqq \|P_t - P_0\|_\infty$ when P_0 is an irrational rotation [13]. Therefore, $|\langle P_0, \mu_0\rangle - \langle P_0, \mu_t\rangle| \leqq 2\|P_0 - P_t\|_\infty \leqq C|t|$, so certain functionals of the invariant measure certainly have higher regularity. It is also amusing to use Theorem 5 and this definition to show that the rotation number is continuous at x_0. Since $P_t \to P_0$ in $C^0(S^1)$ and $\mu_t \rightharpoonup \mu_0$, by the well-known combination of weak and strong convergence $\langle P_t, \mu_t\rangle \to \langle P_0, \mu_0\rangle$. Thus, $\rho(P_t) \to \rho(P_0)$.

7.2. Proof of Corollary 1

Proof. We extend Theorem 5 to flows as follows. Fix $x_0 \in \mathbb{R}^2$ and consider the C^2 vector field, $(y', z') = (f, g)(x_0) + (a, b)$, on the torus with $y' > 0$ as in Section 6. For any $\alpha \in [0, 1)$ we have a Poincaré map $P_\alpha : \{y = \alpha\} \to \{y = \alpha + 1\}$. Notice that P_α are diffeomorphic and have the same rotation number $\rho \in \mathbb{R}\backslash\mathbb{Q}$.

Let $g : \mathbb{T}^2 \to \mathbb{R}$ be a continuous function. For any $\theta \in [0, 1)^2$ the limit $\lim_{u\to\infty} \frac{1}{u}\int_0^u g(s, Z(s;\theta))ds$ exists. For brevity of notation we drop the dependence on θ. Then, the limit can be rewritten using the periodicity of g as

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \sum_{k=0}^{n-1} g(\alpha, P_\alpha^k(Z(\alpha))\, d\alpha = \int_0^1 \int_0^1 g(\alpha, \beta)\, d\mu_\alpha(\beta)\, d\alpha, \qquad (63)$$

where μ_α is the unique invariant probability measure for P_α. Thus the invariant measure on the torus factors into Lebesgue measure $\times \mu_\alpha$.

Let $x_n \to x_0$. It is clear that the corresponding Poincaré maps $P_\alpha(x_n) \to P_\alpha$. Therefore, by Theorem 5, any corresponding invariant measures $\mu_{\alpha,n} \rightharpoonup \mu_\alpha$. A continuous function on the torus can be written as the limit of a sum of products of the form $g(\alpha, \beta) = g_1(\alpha)g_2(\beta)$. For any product of this type we have, by the dominated convergence theorem,

$$\begin{aligned}\int_0^1 \int_0^1 g_1(\alpha) g_2(\beta)\, d\mu(\beta)\, d\alpha &= \int_0^1 g_1(\alpha) \int_0^1 g_2(\beta)\, d\mu(\beta)\, d\alpha \\ &= \int_0^1 g_1(\alpha) \left(\lim_{n\to\infty} \int_0^1 g_2(\beta)\, d\mu_{\alpha,n}(\beta) \right) d\alpha \\ &= \lim_{n\to\infty} \int_0^1 g_1(\alpha) g_2(\beta)\, d\mu_{\alpha,n}(\beta)\, d\alpha.\end{aligned}$$

Therefore, the invariant measures for the flow converge weakly. In particular, choosing $g(\alpha, \beta) = 1/(f(x_0) + a(\alpha, \beta))$ we see that the time periods $T(x_n, \theta)$ converge to $T(x_0)$, and since $\rho(x_n) \to \rho(x_0)$, the homogenized vector field (35) is continuous at irrationals. $\square$

8. The bifurcation problem

For generic $A \in C^r(\mathbb{T}^2)$, $r \geqq 3$, what can we say about the bifurcations of the vector field

$$\theta' = \Omega - \nabla A(\theta) = \Omega + a(\theta) \tag{64}$$

as $\Omega \in \mathbb{R}^2$ varies? This may be considered the simplest problem, since $F(x) = |x|^2/2$ is, in some sense, the simplest macroscopic energy. The condition $r \geqq 3$ is imposed to discount DENJOY's counterexample [16]. It is clear from the previous sections that the qualitative nature of the microscopic flow strongly influences the macroscopic dynamics. Thus, knowledge of the bifurcation diagram is but a preliminary step in the determination of the weak limits and the study of a two-parameter bifurcation problem arises naturally. Our results here are incomplete since the problem of generic bifurcations in two-parameter circle maps is not fully understood. In fact, even the basic problem of generic properties of one-parameter circle maps is not completely understood. BRUNOVSKY studied this problem in 1974 [13], but there was a subtle gap in his proof pointed out twenty years later by DE MELO & PUGH [14]. Nevertheless, by combining known results and some heuristic arguments, we can obtain a good idea of the complexity of the averaged equations.

8.1. Bifurcation of circle maps

In the notation of Section 6 we restrict attention to the case where $y' = \Omega_1 + a(y, z) > 0$. We only study $\rho(\Omega)$ in the open region $U = \{\Omega \in \mathbb{R}^2 | \Omega_1 > \|a\|_\infty\}$. This analysis extends immediately to the other case. It is more convenient to work with the Poincaré maps $P : \mathbb{R} \times U \to \mathbb{R}$, where $P = P(\cdot, \Omega)$ is the Poincaré map from $\{y = 0\}$ to $\{y = 1\}$ of the flow for (64).

Let us recall some well-known definitions and results [6, 13, 27]. A continuous one-parameter family of circle maps, $P_t, t \in [0, 1]$ is said to be *increasing* if for each $x \in \mathbb{R}$, $P_{t_1}(x) > P_{t_2}(x)$ when $t_1 > t_2$. *Decreasing paths* are defined in a similar manner. Notice that for fixed Ω_1, the family $P(\cdot, \Omega)$ is an increasing family in Ω_2.

Lemma 17. *Fix Ω_1 and consider the increasing family $P_{(\Omega_1, \cdot)}$. Then,*

(i) *ρ is a continuous and increasing function of Ω_2 with range $\mathbb{R}$;*
(ii) *$\rho^{-1}\{\alpha\}$ is a singleton for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.*
(iii) *Let $J_{p/q}(\Omega_1) = \{\Omega_2 | \rho(\Omega) = p/q\}$. Then $J_{p/q}(\Omega_1)$ has nonempty interior if and only if $P^q_\Omega \neq R_p$, where $R_p : x \mapsto x + p$.*
(iv) *If $\Omega_2 \in \mathrm{Int}(J_{p/q}(\Omega_1))$, then there is at least one hyperbolic cycle of periodic points.*

Proofs can be found in [6,13,27]. All of the properties are direct consequences of the monotonicity of circle maps. These results hold for any vector field a. We are specifically interested in $a = -\nabla A$, and results that hold for generic $A : \mathbb{T}^2 \to \mathbb{R}$. Since, it is easier to study one-parameter families, we begin with the following proposition.

Proposition 3. *Fix $\Omega_1 > \|a\|_\infty$. Then for A in a residual subset of $C^r(\mathbb{T}^2)$, $r \geqq 3$, all the intervals $J_{p/q}(\Omega_1)$, $p/q \in \mathbb{Q}$ have nonempty interior.*

Proof. Let $p/q \in \mathbb{Q}$. We show that $O_{p/q} = \{A \in C^r(\mathbb{T}^2) | \mathrm{Int}(J_{p/q}(\Omega_1)) \neq \phi\}$ is open and dense. The set $O_{p/q}$ is clearly open. Let A be chosen so that $J_{p/q}(\Omega_1)$ is a singleton. Let P_0 denote the corresponding Poincaré map. Then $P^q_0(x) = x + p$ for all x by Lemma 17 above. Thus every point lies on a periodic orbit. To prove the Proposition it suffices to construct $A_\varepsilon \in C^r(\mathbb{T}^2)$ that is $O(\varepsilon)$ close to A, has rotation p/q, and whose Poincaré map does *not* satisfy $P^q_\varepsilon(x) = x + p$ for all x. We shall accomplish this by a perturbation that destroys at least one periodic orbit, but not all.

Let γ_0 be a periodic orbit for A. In general, γ_0 is only a C^{r-1} curve. However, we may choose a C^∞ curve γ that is arbitrarily close to γ_0 in the C^{r-1} topology. Further, we may choose a C^∞ coordinate system, (t, n), in a small neighborhood of γ where (t, n) denote the tangential and normal components respectively. Let $\chi : \mathbb{T}^2 \to \mathbb{R}$ be a C^∞ cut-off function that is 1 in the strip of width δ about γ, and 0 outside the strip of width 2δ. Let $\psi : \mathbb{T}^2 \to \mathbb{R}$ be defined in a 2δ neighborhood of γ by $\psi(t, n) = n$, and extended continuously to the rest of the torus. Then $A_\varepsilon = A + \varepsilon\psi\chi$ is C^r for sufficiently small $\delta > 0$, and $\|A_\varepsilon - A\|_{C^r} \leqq \varepsilon\|\psi\chi\|_{C^r}$. Furthermore, $\nabla A_\varepsilon \cdot n(\theta) = \varepsilon$ for all $\theta \in \gamma$. It follows that $\nabla A_\varepsilon \cdot n_0(\theta) > \varepsilon/2$ for all $\theta \in \gamma_0$ (here n_0 denotes the normal to γ_0), if γ is chosen sufficiently close to γ_0. In particular this means that $P^q_\varepsilon(x) > x + p$, where x denotes the starting point $\gamma_0 \bigcap \{y = 0\}$, and we have destroyed the periodic orbit γ_0. On the other hand, for small δ the perturbation is localized, and all periodic orbits of A outside the support of ψ persist. Thus, the rotation number is unchanged by this perturbation. Finally, by Lemma 17 (iii), the intervals $J^\varepsilon_{p/q}$ have nonempty interior, and $A_\varepsilon \in O_{p/q}$. □

Corollary 3. *The conclusions of Proposition 3 remain valid for Ω_1 in a countable, dense subset of $(|a|_\infty, \infty)$.*

Thus, ρ is typically a singular function on vertical slices. However, we can control the regularity of ρ with an extra degree of smoothness. Suppose that $A \in C^4(\mathbb{T}^2)$, ie. $P_\Omega \in C^3$. Then

(a) $\rho(\Omega_1, \cdot)$ is absolutely continuous;
(b) $\rho^{-1}(\mathbb{R}\backslash\mathbb{Q})$ has positive measure.
(c) For Ω_2 in a set of full measure we have one of the following. Either $\rho \in \mathbb{Q}$, or $\rho \in \mathbb{R}\backslash\mathbb{Q}$ and satisfies the diophantine condition: for every $\beta > 0$ there is a C_β such that

$$\left|\rho - \frac{p}{q}\right| \geqq \frac{C_\beta}{q^{2+\beta}}.$$

Consequently, by the theorems of Herman and Yoccoz, the map P_Ω is $C^{2-\beta}$ conjugate to a rotation.

Properties (a) and (b) are results of HERMAN [26]; (c) is due to TSUJII [42]. Applying (c) to every vertical slice, and using Fubini's theorem, we see that for almost all $\Omega \in U$ the microscopic flow has a periodic orbit, or is smoothly linearizable. Let us now obtain finer information on the structure of resonance zone boundaries.

8.2. *Transversality conditions*

In this section we only work with C^∞ functions. Fix $0 < p/q \in \mathbb{Q}$. We define $S_{p/q}$, the *resonance zone* of *order* p/q, to be the set $\rho^{-1}\{p/q\} \subset U$. The zone $S_{p/q}$ is closed, and $\Omega \in S_{p/q}$ if and only if there exists $x \in \mathbb{R}$ such that $P^q(x) = x + p$; e $S_{p/q}$ is the natural projection onto U of the set

$$V_{p/q} = \{(x, \Omega) \in \mathbb{R}^1 \times U | P^q(x) - (x + p) = 0\},$$

and we write $S_{p/q} = \pi_2 \circ V_{p/q}$. The boundaries of the resonance zones are degenerate critical points, so consider the following subset of $V_{p/q}$:

$$W_{p/q} = \{(x, \Omega) \in V_{p/q} | DP^q_\Omega(z) - 1 = 0\}.$$

Theorem 6. (a) *For every $A \in C^\infty(\mathbb{T}^2)$, $V_{p/q}$ is a two-dimensional C^∞ submanifold of $\mathbb{R}^1 \times U$.*
(b) *For A in a residual subset of $C^\infty(\mathbb{T}^2)$, $W_{p/q}$ is a one-dimensional submanifold of $\mathbb{R}^1 \times U$.*

Remark 4. Of particular interest are the boundaries of the resonance zones. The projection of the manifold $W_{p/q}$ into U are smooth curves that may meet at isolated cusps. Away from the cusps we have saddle-node bifurcations. Moreover, this also shows that for $x \in S_{p/q}$ a.e. the hypotheses of Theorem 3 and Theorem 4 are satisfied for $A \in C^\infty$.

Proof. The proof uses basic transversality theory. Part (a) is a simple consequence of the regular value theorem [28, Theorem 3.2, p. 22], whereas Part (b) follows from an infinite dimensional version of the parametrized transversality theorem [28, Theorem 2.7, p. 79]. In both cases, we write $V_{p/q}$ and $W_{p/q}$ as zero-level sets of suitable C^∞ maps, and it suffices to show that zero is a regular value.

The maps are as follows. Let $g : \mathbb{R} \times U :\to \mathbb{R}^1$ by $g(x, \Omega) = P^q(x, \Omega, A) - (x + p)$. Then $V_{p/q} = g^{-1}\{0\}$. To prove (b) we consider the map $G : \mathbb{R} \times U \times C^\infty(\mathbb{T}^2) \to \mathbb{R}^2$ given by $G(x, \Omega, A) = (P^q(x, \Omega, A) - (x+p), D_1 P^q(x, \Omega, A) - 1)$. We show that G is transverse to $(0, 0)$. In this case, the parametrized transversality theorem allows us to conclude that for a residual set of $A \in C^\infty(\mathbb{T}^2)$, the restricted map $G_A : \mathbb{R} \times U \to \mathbb{R}^2$ is transverse to zero. This will prove (b).

Let $Z(y; x, \Omega, A)$, (or $Z_x(y)$ for brevity) denote the solution to

$$\frac{dz}{dy} = \frac{\Omega_2 + b(y, z)}{\Omega_1 + a(y, z)}, \quad z(0) = x. \tag{65}$$

Then $P^q(x, \Omega, A) = Z_x(q)$. We use the following notation. $D_1 G$ is the derivative with respect to x, $D_2 G$ and $D_3 G$ are the derivatives with respect to Ω_1 and Ω_2 respectively, and $D_4 G$ is the derivative with respect to A.

We calculate the derivatives using the equation of variations. First, the derivative with respect to x is given by

$$\frac{dD_x Z_x}{dy} = \partial_z \left(\frac{\Omega_2 + b(y, z)}{\Omega_1 + a(y, z)} \right) \Bigg|_{z = Z_x(y)} D_x Z_x \stackrel{\text{def}}{=} h(y) D_x Z_x. \tag{66}$$

Let $H(y) = \int_0^y h(s)ds$. Then $D_x Z_x(y) = e^{H(y)} D_x Z_x(0) = e^{H(y)}$. In particular, $D_1 g = e^{H(q)} - 1$.

We next calculate the derivative $D_2 g$. Differentiating (65) with respect to Ω_i, $i = 1, 2$ we obtain the equation of variations,

$$\frac{dD_{\Omega_i} Z_x}{dy} = h(y) D_{\Omega_i} Z_x + h_i(y), \quad i = 1, 2, \tag{67}$$

where

$$h_1(y) = -\frac{(\Omega_2 + b(y, Z_x(y)))}{(\Omega_1 + a(y, Z_x(y)))^2}, \quad h_2(y) = \frac{1}{\Omega_1 + a(y, Z_x(y))}. \tag{68}$$

Solving (67) we find that

$$D_2 G = \int_0^q e^{H(q) - H(y)} h_1(y) dy, \quad D_3 G = \int_0^q e^{H(q) - H(y)} h_2(y) dy. \tag{69}$$

In particular, we have $D_2 g < 0$ and $D_3 g > 0$. This proves (a).

Notice that we did not need the freedom in A for (a). To prove (b) we will need to vary A. We show that the derivative DG, written explicitly as

$$DG = \begin{pmatrix} D_1 G & D_2 G & D_3 G & D_4 G \\ D_1^2 G & D_2 D_1 G & D_3 D_1 G & D_4 D_1 G \end{pmatrix},$$

has $(0, 0)$ as a regular value. Notice that for $(x, \Omega, A) \in G^{-1}\{(0, 0)\}$ we have $D_1 G = 0$, $D_2 G$, $D_3 G \neq 0$. Thus in order to prove the result it suffices to show that there is a direction $\eta \in C^\infty(\mathbb{T}^2)$ such that $\langle D_4 G, \eta\rangle = 0$ and $\langle D_1 D_4 G, \eta\rangle \neq 0$. The derivative $D_4 G(x, \Omega, A)$ is a bounded linear operator whose action on $\eta \in C^\infty(\mathbb{T}^2)$ is denoted by the duality pairing $\langle\cdot\,,\cdot\rangle$. It satisfies the equation of variations

$$\frac{d\langle D_A Z_x, \eta\rangle}{dy} = h\langle D_A Z_x, \eta\rangle + h_1 \partial_y \eta + h_2 \partial_z \eta, \tag{70}$$

with the solution

$$\langle D_A Z_x(y), \eta\rangle = \int_0^y e^{H(y)-H(y')} \big(h_1 \partial_y \eta + h_2 \partial_z \eta\big)\, dy'. \tag{71}$$

We will choose a function η so that $\nabla\eta = 0$ on $(y, Z_x(y))$. In this case, the solution to (71) is $\langle D_A Z_x(y), \eta\rangle \equiv 0$ which implies $\langle D_4 G, \eta\rangle = 0$. A similar calculation for the second derivative $\langle D_x D_A Z_x, \eta\rangle$ gives the equation of variations

$$\begin{aligned}\frac{d\langle D_x D_A Z_x, \eta\rangle}{dy} &= h\langle D_x D_A Z_x, \eta\rangle + \langle D_A Z_x, \eta\rangle D_x h D_x Z_x \\ &\quad + (D_x h_1 \partial_y \eta + D_x h_2 \partial_z \eta) + (h_1 D_x \partial_y \eta + h_2 D_x \partial_z \eta).\end{aligned}$$

If we choose η as earlier, this simplifies to

$$\frac{d\langle D_x D_A Z_x, \eta\rangle}{dy} = h\langle D_x D_A Z_x, \eta\rangle + (h_1 D_x \partial_y \eta + h_2 D_x \partial_z \eta), \tag{72}$$

with the solution

$$\langle D_x D_A Z_x, \eta\rangle(y) = \int_0^y e^{H(y)-H(y')} (h_1 D_x \partial_y \eta + h_2 D_x \partial_z \eta) dy'. \tag{73}$$

It remains to construct η. The preimage of Z_x in $\mathbb{T}^2$ is a periodic orbit, say γ. Choose a C^∞ coordinate system (t, n), and a cut-off function ψ in the vicinity of γ as in Proposition 3, and define

$$\eta = \tfrac{1}{2}\psi n.^2.$$

Then $\nabla\eta = 0$ on γ, and the only nonzero second derivative of η is $\partial_n^2 \eta \equiv 1$ on γ. The bracketed term in (73) is as follows. First

$$D_x \partial_y \eta(y, Z_x(y; x, \Omega, A) = \partial_{yz}^2 \eta(y, Z_x(y)) D_x Z_x(y, Z_x(y)) = e^{H(y)} \partial_{yz}^2 \eta.$$

Similarly, $D_x \partial_z \eta = e^{H(y)} \partial_z^2 \eta$. The derivatives $\partial_{yz}^2 \eta$ and $\partial_z^2 \eta$ are obtained by changing basis from (t, n) to (y, z) and using $\partial_n^2 \eta = 1$. Using the explicit expressions for h_1 and h_2 and the fact that t is parallel to the vector field on γ we find, after some calculation, that

$$\langle D_x D_A Z_x, \eta\rangle(y) = e^{H(y)} \int_0^y (\Omega_1 + a(y, Z_x(y')))dy'.$$

In particular $D_1 D_4 G = \langle D_x D_A Z_x, \eta\rangle(q) > 0$. □

8.3. *Behavior at* ∞

In the limit when $\Omega_i \gg 1$, the bifurcation problem can be understood more completely, as it reduces to the bifurcation of two parameter circle maps of the kind studied by ARNOL'D [6].

We take the limit $\Omega_1 \to \infty$ along the line $\Omega_2/\Omega_1 = \alpha$. Equation (65) can be rewritten as

$$\frac{dz}{dy} = \frac{\alpha + \dfrac{b(y,z)}{\Omega_1}}{1 + \dfrac{a(y,z)}{\Omega_1}} = \alpha + \frac{1}{\Omega_1}\frac{b - \alpha a}{1 + \dfrac{a}{\Omega_1}}.$$

Let $\eta = \frac{1}{\Omega_1}$. The Poincaré map associated with this differential equation is

$$P_{\alpha,\eta}(x) = R_\alpha(x) + \psi(x,\eta) = x + \alpha + \eta\psi(x,\eta), \tag{74}$$

where ψ is 1-periodic in x. In the limit $\eta = 0$, $P_{\alpha,\eta}$ is the rigid rotation R_α. Arnol'd studied (74) for analytic ψ. For typical ψ we have the following properties. For small $\eta > 0$, the set $\{\alpha | \rho(\alpha,\eta) \in \mathbb{Q}\}$ consists of a countable number of nontrivial closed intervals. However, for α in a nowhere dense set of almost full measure, the rotation number is irrational. Theorem 2 of [6] states that for irrational α satisfying the diophantine condition $|\alpha - p/q| \geqq K|q|^{-3}$, there exists an analytic curve $\alpha(\eta)$ so that $\rho(P_{\alpha(\eta),\eta}) = \alpha$ for small η. These curves separate the resonant regions. The width of the resonant zones shrinks rapidly to zero as $\eta \to 0$, and they have picturesque names such as horns [25], tongues [6] or wedges [23]. The assumptions of analyticity are not necessary, and for $\psi \in C^1$, the boundaries of the resonance zones are Lipschitz [27]. HALL showed that the irrational curves $\alpha(\eta)$ and the resonance zone boundaries are differentiable at $\eta = 0$ [25].

The above limit is important for the following reason. Consider the homogenization of

$$\dot{x} = -x + ra\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^2, \quad \nabla A = a : \mathbb{T}^2 \to \mathbb{R}^2. \tag{75}$$

We may assume that $\|a\|_\infty = 1$. The parameter r is a measure of the amplitude of the wiggles, and we term it the *roughness*. We are interested in the limit $r \to 0$. All the homogenization problems for $r > 0$ are equivalent to the case $r = 1$. Indeed let $x = ry$, so that

$$\dot{y} = -y + a\left(\frac{y}{r\varepsilon}\right),$$

and for fixed r, the homogenized limit $\varepsilon \to 0$ is the same as (75) with $r = 1$. Therefore, for small $r > 0$, most of the macroscopic phase space $\mathbb{R}^2$ is filled with Arnol'd tongues and their complements.

8.4. Variation of ρ on Lipschitz paths

Let $h : [0, 1] \to D^r(S^1), r \geqq 3$ be a C^1 path in the space of orientation-preserving circle diffeomorphishms. Associated with h we can define the map $\rho : t \mapsto \rho(h(t))$. Herman showed that if $\rho(0) \neq \rho(1)$, and ρ is of bounded variation, then ρ is absolutely continuous. The proof requires the full strength of Herman's linearization theorems.

A careful examination of Herman's proof shows that it is sufficient for the paths to be uniformly Lipschitz (see [26, Section 4, Section 6]), as the C^1 hypothesis is only used to derive a uniform Lipschitz estimate. The theorem may then be applied to any weak limit $x^0(t)$, for these define Lipschitz paths in $D^r(S^1)$ in a natural way. Therefore, if $\rho(x^0(t))$ is of bounded variation, it follows that it changes in an absolutely continuous manner on a weak limit $x^0(t)$. But $\rho(t)$ is itself the direction of the tangent $\dot{x}^0(t)$, and this gives us a highly nontrivial improvement of the regularity of $x^0(t)$. This "proof" is incomplete, since we have not proved that (35) holds when $\rho(t) \in \mathbb{R}\backslash\mathbb{Q}$ or that $\rho(x^0(t))$ is BV.

8.5. Bifurcations of equilibria

There are some simplifications in the bifurcation analysis for equilibria. Let y_* be an equilibrium. The linearization of (64) at y_* is

$$v' = Da(y_*)v. \tag{76}$$

A necessary condition for a local bifurcation of y_* is that $Da(y_*)$ be non-degenerate. But $Da(y_*) = -D^2A(y_*)$, the Hessian matrix of A at y_*. The Hessian matrix is symmetric and its eigenvalues λ_i, $i = 1, 2$ must be real. Let $\gamma_i = \{y \in \mathbb{T}^2 | \lambda_i(y) = 0\}$. For generic a we can choose γ_i to be continuous curves that are smooth except when they intersect each other. For a local bifurcation of equilibria to occur, they must lie on the curves γ_i. Let us call the γ_i, *curves of degeneracy*. If a rest point y_* lies on a curve of degeneracy, then a bifurcation must occur as we vary Ω. This property allows us to determine curves in the Ω-plane that correspond to bifurcations. Let

$$\Gamma_i = \{\Omega \in \mathbb{R}^2 | \Omega = -a(y) \ \text{ for some } \ y \in \gamma_i\}.$$

Thus Γ_i is the negative of the range of a on the curve of degeneracy. Typically, we expect that Γ_i are themselves continuous closed curves. The curves Γ_i cannot intersect $B(0, r)$, for if $\Omega \in B(0, r)$ then all rest points are non-degenerate. We can use a homotopy argument to prove that Γ_i must surround the origin. Suppose this were not true. Then we could draw a line from the origin to infinity, say $t\Omega, 0 \leqq t < \infty$, that does not intersect any of the Γ_i. Let n_0 be the number of zeros of $a(y)$. Then the set $Z = \{t \in [0, \infty) | a(y) + t\Omega \ \text{ has } n_0 \text{ zeros}\}$ is closed. But Z is also open. This is because $t\Omega$ never hits Γ_i, ensuring that all zeros of $a + t\Omega$ have non-degenerate linearization. Thus they can be uniquely continued for a neigbhorhood $(t - \eta, t + \eta)$ and are the only zeros in this neighborhood. Thus, $Z = [0, \infty)$ which contradicts our earlier conclusion that all zeros vanish as $|\Omega| \to \infty$.

9. Transport equations

The wiggly energy problem admits a formulation in terms of a linear PDE (partial differential equation) with oscillating coefficients. We might reasonably expect that homogenization methods should play some role in this problem. In this section we show that these methods do not provide sufficiently fine information.

9.1. Homogenized transport equation

Consider the linear transport equation

$$u_t^\varepsilon + \Big(f(x) + a\Big(\frac{x}{\varepsilon}\Big)\Big) \cdot \nabla_x u^\varepsilon = 0. \tag{77}$$

The characteristics of (77) are the solutions to (2), thus the study of the two equations is equivalent.

Homogenization for linear transport equations with *incompressible* vector fields has been studied rigorously by several authors [4, 12, 17, 29] (similar results were anticipated in [34]). E's results are the strongest [17], and in this section we shall repeat his argument, with the obvious modifications, to derive a homogenized equation. His argument is an application of Tartar's oscillatory test function method [40] coupled with a compactness theorem of Nguetseng [36]. Nguetseng's theorem formalizes the heuristic notion that u^ε depends on two scales, x and $\frac{x}{\varepsilon}$. A later paper by Allaire contains an excellent exposition of this theorem with simplified proofs and generalizations [3]. We need the following version.

Theorem 7 (Nguetseng [36], Allaire [3]). *Let* $\sup_\varepsilon \|u^\varepsilon\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)} \leqq C < \infty$. *Then there exists a subsequence* (*also denoted* ε) *and a function* $U : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$ *with*

$$\|U\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{T}^d)} \leqq C,$$

such that

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u^\varepsilon(t,x)\psi\Big(t,x,\frac{x}{\varepsilon}\Big)\,dx\,dt = \int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} U(t,x,\theta)\psi(t,x,\theta)d\theta\,dx\,dt$$

for all $\psi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{T}^d)$. *In this case* u^ε *is said to two-scale converge to its two-scale limit,* U.

Note that $u^\varepsilon \overset{*}{\rightharpoonup} \int_{\mathbb{T}^d} U(t,x,\theta)\,d\theta$ in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$. Thus the two-scale limit has at least as much information as the weak limit.

Unlike previous studies, in our problem the oscillatory vector field $a(\theta)$ is *not* divergence free. In fact, since $a = -\nabla A$, if $\nabla \cdot a = 0$, then $\Delta A = 0$. But by Liouville's theorem all harmonic, periodic functions are constant and a is trivial if it is divergence free. This is a serious problem, and we show that the evolution equations for U are ill posed.

Fix a set of initial conditions $u_0^\varepsilon(x)$ that is uniformly bounded in $L^\infty(\mathbb{R}^d)$. The solutions to (77) are given by $u^\varepsilon(t,x) = u_0(X_{-t}^\varepsilon(x))$, where X_t^ε is the flow generated by (2). Thus,

$$\sup_\varepsilon \|u^\varepsilon(t,x)\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)} < \infty.$$

The boundedness criterion of Theorem 7 is satisfied, hence there exists a subsequence $\varepsilon_n \downarrow 0$ and an associated two-scale limit $U(t,x,\theta)$.

Let $\phi^\varepsilon \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$. A weak solution of (77) solves

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u^\varepsilon(t,x)\left[\phi_t^\varepsilon + \nabla_x \cdot \left(\phi^\varepsilon\left(f(x) + a\left(\frac{x}{\varepsilon}\right)\right)\right)\right] dtdx + \int_{\mathbb{R}^d} u_0^\varepsilon(x)\phi^\varepsilon(0,x)\,dx = 0. \quad (78)$$

Let $\psi \in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{T}^d)$ and put $\phi^\varepsilon = \varepsilon\psi(t,x,x/\varepsilon)$, i.e., ϕ^ε is a low-amplitude, rapidly oscillating function. Substituting in (78), we find

$$\begin{aligned}
&\varepsilon\int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u^\varepsilon(t,x)\left[\psi_t + \nabla_x\psi \cdot \left(f(x) + a\left(\frac{x}{\varepsilon}\right)\right) + \psi\nabla_x \cdot f(x)\right] dx\,dt \qquad (79)\\
&\quad + \varepsilon\int_{\mathbb{R}^d} u_0^\varepsilon(x)\psi\left(0, x, \frac{x}{\varepsilon}\right) dx\\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}^d} u^\varepsilon(t,x)\left[\left(f(x) + a\left(\frac{x}{\varepsilon}\right)\right)\cdot\nabla_\theta\psi + \nabla_\theta\cdot a(x/\varepsilon)\psi\right] dx\,dt = 0.
\end{aligned}$$

The terms on the first two lines are $O(\varepsilon)$. Letting $\varepsilon \downarrow 0$ and using Theorem 7 we have

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} U(t,x,\theta)\nabla_\theta\cdot[\psi(t,x,\theta)(f(x)+a(\theta))]d\theta\,dx\,dt = 0. \quad (80)$$

And since $\psi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{T}^d)$ was arbitrary, this means that $U(t,x,\theta)$ is a weak solution to

$$(f(x) + a(\theta)) \cdot \nabla_\theta U = 0, \quad \theta \in \mathbb{T}^d. \quad (81)$$

Equation (81) identifies the behavior of U on the microscale. And U is a solution if and only if it is constant on the characteristics of (81). The characteristics are solutions to the ordinary differential equations

$$\frac{d\theta}{d\tau} = f(x) + a(\theta), \quad \theta \in \mathbb{T}^d. \quad (82)$$

and we recover the vertical flow on the torus given by (14).

We also need equations to describe the evolution of U on the macroscale. To this end, let ψ solve the adjoint equation of (81)

$$\nabla_\theta \cdot [(f(x) + a(\theta))\psi] = 0, \quad \theta \in \mathbb{T}^d, \quad (83)$$

and let $\phi^\varepsilon(t, x) = \psi(t, x, x/\varepsilon)$ in (79). Then the $O(1)$ term in (79) is zero, so that we may divide by ε and take the limit to obtain

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} U(t, x, \theta)[\psi_t + \nabla_x \cdot ((f(x) + a(\theta))\psi)]d\theta\, dx\, dt \\ + \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} U(0, x, y)\psi(0, x, y)d\theta\, dx = 0. \quad (84)$$

9.2. Ill-posed limit

E proved that equations (81), (83), and (84), along with an ergodic hypothesis are sufficient to determine the evolution of the two-scale limit for incompressible vector fields [17]. We will show that this does not hold if the wiggly energy A satisfies the generic hypothesis, Property 1 of Section 5.

Hou and Xin [29], assume ergodicity of the flow generated by a to solve the linear PDE (81) when $d = 2$. The following theorem is in this vein. But it is stronger since it relies on generic hypotheses. For incompressible vector fields there is no difference between the solvability of equations (81) and (83). The situation is different for gradient dynamics.

Theorem 8. *Let x be fixed so that the flow of*

$$\frac{d\theta}{d\tau} = f(x) + a(\theta), \quad \theta \in \mathbb{T}^d$$

is gradient-like, and the ω-limit set of any point θ_0 is a non-degenerate critical point. Then

(a) *Any continuous solution of* (81) *is independent of θ.*
(b) *Any continuous solution of* (83) *satisfies $\psi(x, \theta) = 0$ for all θ.*

Remark 5. Theorem 8(b) is obvious when $d = 1$. In this case (83) reduces to

$$\frac{d}{d\theta}((f(x) + a(\theta))\psi) = 0,$$

so that $(f(x) + a(\theta))\psi$ is only a function of x. This is incompatible with the boundedness of ψ if $f(x) + a(\theta) = 0$ for some θ. The content of Theorem 8 is that under some global hypothesis on the microscopic flow, solutions to (83) must blow up in higher dimensions too.

Proof. The number of critical points must be finite, and we denote the stable critical points by $\{\theta_1, \dots, \theta_n\}$. The union of the basins of attraction of $\{\theta_1, \dots, \theta_n\}$ is open and dense in $\mathbb{T}^d$. Let θ_0 lie in the basin of attraction of θ_j, and let $\theta(\tau)$ denote the solution to the ODE (82) with initial condition θ_0. We use the method of characteristics to solve (83) for ψ (see [18]):

$$\psi(x, \theta(\tau)) = \exp\left(-\int_0^\tau \nabla \cdot a(\theta(s))\, ds\right)\psi(x, \theta_0).$$

But θ_j is a non-degenerate sink, so that $\nabla_\theta \cdot a(\theta_j) \leqq -2\beta < 0$ for some positive number β. So we may choose a trapping ball $B(\theta_j, r_j)$ about θ_j in which $\nabla_\theta \cdot a(\theta) \leqq -\beta$. The time, T, that $\theta(\tau)$ takes to enter this ball is finite. For all $\tau \geqq T$ we have

$$|\psi(x, \theta(\tau))| \geqq \exp\left(- \int_0^T \nabla \cdot a(\theta(s))\, ds \right) |\psi(x, \theta_0)| e^{\beta(\tau - T)}.$$

The term $\psi(x, \theta(\tau)) \to \psi(x, \theta_j)$ as $\tau \to \infty$ because ψ is continuous. Hence, $\psi(x, \theta_j)$ is finite if and only if $\psi(x, \theta_0) = 0$. Therefore, $\psi(x, \theta)$ vanishes for θ in the basin of attraction of θ_j. Since θ_j was arbitrary, we have proved that $\psi(x, \cdot)$ vanishes on a dense set. Thus $\psi(x, \theta) = 0$ for all θ. This proves (b).

The proof of (a) is similar. Let U solve (81). The method of characteristics implies that $U(x, \theta_j) = U(x, \theta_0)$ on the basin of attraction of θ_j. Thus $U(x, \theta)$ takes only a finite number of values on a dense set and its range is discrete. Connectedness requires that $U(x, \theta)$ is independent of θ. □

Corollary 4. *Suppose $f(x_*) = 0$. There is an $r > 0$ such that if ψ is a smooth solution to* (83)*, then $\psi(t, x, \theta) = 0$ for $|x - x_*| < r$.*

Proof. Since $a(\theta)$ is Morse-Smale, the flows of the vector fields $a(\theta)$ and $f(x) + a(\theta)$ are topologically conjugate for sufficiently small $|f(x)|$. Thus, the hypotheses of Theorem 8 are satisfied for sufficiently small $r > 0$. □

We now construct a nontrivial solution to the evolution equations with zero initial data. Let $\eta : [0, \infty) \to \mathbb{R}$ be C^∞ with $\eta(0) = 0$. Choose $r > 0$ as in Corollary 4 and $\zeta \in C_0^\infty(\mathbb{R}^d)$ that is supported within $B(x_0, r/2)$. Finally, let $U(t, x, \theta) = \eta(t)\zeta(x)$. By construction, supp$(U)$ has empty intersection with the support of any ψ that solves (83). We may then verify that (81), (83), and (84) are true. This shows that the homogenized transport equation does not have a unique solution.

10. Dynamics and microstructure

Our work is but one example of a mathematical model for dynamics and microstructure. It is a coarse model, since we are only interested in the evolution of bulk properties. However, as we have seen, the presence of microstructure even in such a simple form leads to several interesting questions. We now summarize some future directions, and some problems of modeling.

10.1. A probabilistic interpretation

The homogenized equations are not well posed and we must consider some other interpretation. We have derived differential equations of the form

$$\dot{x}(\theta) = g(x, \theta), \; x(0) = x_0, \quad x \in \mathbb{R}^2 \quad \theta \in \mathbb{T}^2. \tag{85}$$

The space $\mathbb{T}^2$ is a natural probability space with a probability measure given by Lebesgue measure. We aim to solve (85) holding θ fixed, thus obtaining a realization of a trajectory starting at x_0. If $g(x, \theta)$ is continuous for fixed θ, then this can be done by Peano's theorem. However, $g(x, \theta)$ depends on the ω-limit set of θ for the microscopic flow at x, and it is not continuous in general. This is not as bad as it may seem, at least in regions of slip. In $\rho^{-1}\{\mathbb{R}\backslash\mathbb{Q}\}$ $g(x, \theta)$ is continuous (see Section 7). Furthermore, on open and dense subsets of $S_{p/q}$ the basins of attraction of the stable periodic orbits are open and dense in $\mathbb{T}^2$, therefore we expect $g(x, \theta)$ to have only countably many discontinuities. Uniqueness cannot be obtained within this framework since g is definitely not Lipschitz.

10.2. Comparison with differential inclusions

At points $x_0 \in \mathbb{R}^2$ where the microscopic flow has distinct invariant measures, $g(x, \theta)$ takes distinct values. For example, if x_0 lies in a resonance zone $S_{p/q}$ where all periodic orbits are hyperbolic, we have the differential inclusion

$$\dot{x} \in K(x) = \begin{pmatrix} 1 \\ \rho \end{pmatrix} \left\{ \frac{1}{T_1^s}, \dots, \frac{1}{T_m^s}, \frac{1}{T_1^u}, \dots \frac{1}{T_m^u} \right\}. \tag{86}$$

Near x_0, each term in the set above is a Lipschitz function of x. It then follows from a theorem of Filippov [20, Theorem 3] that the weak$*$ closure of solutions to the inclusion (86) is the set of solutions to

$$\dot{x} \in \operatorname{conv}(K(x)), \tag{87}$$

the convex closure of K. Therefore, the weak$*$ closure of solutions to the wiggly energy problem is strictly contained within the set of solutions obtained by differential inclusions.

10.3. Separation of scales

While it is possible to extract interesting mathematical features for the limit, it has limited efficacy for quantitative predictions. The heuristic idea that averaging simplifies a problem fails here. There is no resolution of scales in the model, i.e., the fast and slow variables do not decouple, and in the limit fine number theoretic properties such as the degree of rationality and irrationality determine the homogenized limit. These properties cannot be resolved on a computer. There is also good reason to expect that these conditions matter for $d \geqq 3$. For large Ω, or equivalently small r, we can reduce the problem to a study of diffeomorpisms of $\mathbb{T}^{d-1}$. Herman has shown that higher dimensional analogues of Arnol'd tongues exist in such problems, i.e., a Cantor set of non-resonant points of large measure separating resonant zones, and these depend on diophantine conditions [26]. We should also note that apparently technical smoothness requirements have a qualitative influence on the model. The most tractable approach seems to be to avoid periodic homogenization altogether and work instead with statistical models of roughness.

This is almost a form of modeling chaos. Fine details that cannot be measured experimentally exert a significant influence on the dynamics. Can microstructure really have such an influence on dynamics? And if so, what is the best way to model it? We take a hint from physics, and mention some examples where we believe these questions are more than mathematical curiosities.

10.4. Fine structure in martensitic phase transitions

The limiting behavior of our simple model is rough; however the roughness is due to the change in rotation number, and not to jumps between metastable states. In the experiments of Chu and James, the transition is driven by the sequential splitting of martensite needles, and this gives rise to the stick-slip character of the transition. In another set of experiments, VIVES *et al.* [43] studied the acoustic emission generated during a thermal martensitic tranformation. In both sets of experiments the phase transformation progresses as an avalanche of jumps between metastable states. Each avalanche corresponds to the nucleation and motion of one or more phase (or twin) boundaries and has an acoustic signature. Vives *et al.* observed scaling behavior in the statistics of the amplitude and lifetime of these avalanches. The number of avalanches $N(A)$ with a specific amplitude A scales like $N \sim A^{-\alpha}$, $\alpha > 0$. Similarly $N \sim \tau^{-\beta}$ where τ is the lifetime of an avalanche. The scaling behavior shows that the evolution of the phase fraction is a very rough function of time. It would be extremely interesting to obtain a clear continuum description of such phenomenon, and a rigorous explanation of the roughness of the dynamics.

10.5. Random landscapes

Another class of phenomenological models called "landscape paradigms" has been used by condensed-matter physicists to study disordered systems [21]. In mathematical language, these are stochastic perturbations of gradient dynamical systems with rough Lyapunov functions. For example, LEDOUSSAL & VINOKUR [32] consider the following equation as a model for creep of flux lines in superconductors

$$\dot{x} = -\nabla V(x) + c + \sqrt{2T}\,\dot{B}.$$

Here $\dot{B}$ is white noise, i.e., the derivative of Brownian motion, T is the temperature, c is a constant forcing term, and $V(x)$ is a spatially random field ("quenched disorder"). They derive equations for the limiting velocity v as functions of T and c, after averaging over the spatial disorder. Again, it would be very interesting to obtain rigorous results for these systems.

Acknowledgements. I thank R. D. JAMES for introducing me to these problems, and for his enthusiastic encouragement. I am also indebted to J. MALLET-PARET for his help. This article formed part of my dissertation at Brown University. I thank my thesis advisor G. HALLER, and also R. ABEYARATNE and G. C. YUAN for useful discussions. Finally, I thank P. BRUNOVSKY for bringing the article [14] to my attention, and his help with Theorem 6.

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(*Accepted November 1, 2001*)
Published online April 11, 2002 – © Springer-Verlag (2002)