Random Matrix Theory and Numerical Algorithms

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Contents

Notation and Abbreviations

$\mathsf{Quart}(n)$ The space of all $n \times n$ Hermitian quarternionic matrices 10

$\mathrm{d}M$ The differential of M 27

$\mathsf{S})$ The symmetric group 21

Σ_n The simplex $\{c \in \mathbb{R}^n : \sum_j c_j = 1, \quad c_j > 0\}$ 50

Sine_β Sine-β point process 14

$\mathsf{Symm}(n)$ The space of all $n \times n$ real-symmetric matrices 10

Hermite(β) Tridiagonal Hermite(β) ensemble 46

$\mathsf{Tridiag}(n)$ $n \times n$ tridiagonal matrices 45

$\mathsf{U}(n)$ The group of all $n \times n$ complex unitary matrices 10

$\mathsf{USp}(n)$ The group of all $n \times n$ unitary quarternionic matrices 10

$\mathrm{D}M$ Induced volume form 27

$\mathcal{W}^n$ Weyl chamber 33

$\{\pi_k\}_{k=0}^{\infty}$ Monic orthogonal polynomials 53

$\{p_k\}_{k=0}^{\infty}$ Orthonormal polynomials 53

$A_{j,\ell:p}$ $A_{j:j,\ell:p}$

$A_{j:k,\ell:p}$ The submatrix of A consisting of all entries in rows j through k and columns ℓ through p.

$A_{j:k,\ell}$ $A_{j:k,\ell:\ell}$

$B_\epsilon(x)$ Open ball of radius ϵ centered at x 29

e_j^p jth standard basis vector for $\mathbb{F}^p$ 47

M A generic point in a manifold 27

P_v $I - 2vv^*$ 47

$T_M(\mathbb{M})$ The tangent space at M 27

GOE Gaussian Orthogonal Ensemble 10

GSE Gaussian Symplectic Ensemble 10

GUE Gaussian Unitary Ensemble 9

LOE Laguerre Orthogonal Ensemble 75

LUE Laguerre Unitary Ensemble 75

Chapter 1

Fundamentals of random matrix theory and numerical linear algebra

ch:background

1.1 What is a random matrix?

There are two distinct points of view that one may adopt. On one hand, our intuitive ideas of randomness are intimately tied to the notion of sampling a realization of a random variable. Thus, given a random number generator, one may build a random Hermitian matrix, $M \in \mathsf{Her}(n)$, by choosing its real diagonal and complex upper-triangular entries independently at random. It is conventional to assume further that all the diagonal entries have the same law, that all the upper-triangular entries have the same law, and that the real and imaginary parts of each off-diagonal entry are independent. For example, we may assume that the entries on the diagonal are ± 1 with probability 1/2, and that the upper-triangular entries are $\pm 1 \pm \mathrm{i}$ with probability 1/4. It is also conventional to have the variance of the diagonal entries to be twice that of the real part of the off-diagonal entries. Random matrices of this kind, are said to be drawn from Wigner ensembles.

On the other hand, one may adopt a more analytic view. The Hilbert–Schmidt inner product of two Hermitian matrices, $\mathrm{Tr}(M^*N) = \sum_{j,k=1}^{n} \bar{M}_{jk} N_{jk}$, defines a natural metric $\mathrm{Tr}(\mathrm{d}M^2)$ and volume form $\mathrm{D}M$ on $\mathsf{Her}(n)$ (see Chapter 2). ch:integration In this text, unless otherwise stated, $\|M\| = \sqrt{\mathrm{Tr}\, M^* M}$. Thus, each positive function $p : \mathsf{Her}(n) \to [0, \infty)$ that decays sufficiently fast as $\|M\| \to \infty$, may be normalized to define a probability measure. A fundamental example is the law of the Gaussian Unitary Ensemble (GUE)

$$p_{\mathrm{GUE}}(M)\mathrm{D}M = \frac{1}{Z_n} \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}(M^2)} \mathrm{D}M. \qquad (1.1.1)$$ eq:gue

Here Z_n is a normalization constant that ensures p_{GUE} is a probability density (we use the same notation for different ensembles; thus the numerical value of Z_n must be inferred from the context). The term GUE was introduced by Freeman Dyson Dyson1 [Dys62], and refers to an important invariance property of p_{GUE}. Each $U \in \mathsf{U}(n)$ defines a transformation $\mathsf{Her}(n) \to \mathsf{Her}(n)$, $M \mapsto UMU^*$. It is easily checked that the volume form $\mathrm{D}M$ is invariant under the map $M \mapsto UMU^*$, as is the measure $p_{\mathrm{GUE}}(M)\mathrm{D}M$. More generally, a probability measure on $\mathsf{Her}(n)$ is said to be invariant if $p(M)\,\mathrm{D}M$ remains invariant under the map $M \mapsto UMU^*$, for each $U \in \mathsf{U}(n)$. Important examples of invariant ensembles are defined by polynomials in one-variable of the form

$$g(x) = a_{2N}x^{2N} + a_{2N-1}x^{2N-1} + \ldots + a_0, \quad a_j \in \mathbb{R}, \quad j = 0, 1, \ldots, 2N, \; a_{2N} > 0. \tag{1.1.2}$$

eq:inv1

Then the following probability measure is invariant

$$p(M)\mathrm{D}M = \frac{1}{Z_n}\mathrm{e}^{-\operatorname{Tr} g(M)}\mathrm{D}M. \tag{1.1.3}$$

eq:inv-ensemble

We have assumed that all matrices are Hermitian for simplicity. The above notions extend to ensembles of matrices from $\mathsf{Symm}(n)$ and $\mathsf{Quart}(n)$. The notion of invariance in each case is distinct: for $\mathsf{Symm}(n)$, the natural transformation is $M \mapsto QMQ^T$, $Q \in \mathsf{O}(n)$; for $\mathsf{Quart}(n)$ it is $M \mapsto SMS^D$, $S \in \mathsf{USp}(n)$. The standard Gaussian ensembles in these cases are termed GOE (the Gaussian Orthogonal Ensemble) and GSE (the Gaussian Symplectic Ensemble), and they are normalized as follows:

Is this $\mathsf{USp}(n)$? Or $\mathsf{Sp}(2n,\mathbb{R})$?

$$p_{\mathrm{GOE}}(M)\mathrm{d}M = \frac{1}{Z_n}\mathrm{e}^{-\frac{1}{4}\operatorname{Tr}(M^2)}\mathrm{d}M, \quad p_{\mathrm{GSE}}(M)\mathrm{d}M = \frac{1}{Z_n}\mathrm{e}^{-\operatorname{Tr}(M^2)}\mathrm{D}M. \tag{1.1.4}$$

eq:goe

The differing normalizations arise from the different volume forms on $\mathsf{Symm}(n)$, $\mathsf{Her}(n)$ and $\mathsf{Quart}(n)$ as will be explained in Chapter ch:integration 2. For now, let us note that the densities for all the Gaussian ensembles may be written in the unified form

$$Z_n(\beta)^{-1}\mathrm{e}^{-\frac{\beta}{4}\operatorname{Tr}(M^2)} \tag{1.1.5}$$

eq:gue2

where $\beta = 1$,2 and 4 for GOE, GUE and GSE respectively. While it is true that there are no other ensembles that respect fundamental physical invariance (in the sense of Dyson), many fundamental results of random matrix theory can be established for all $\beta > 0$. These results follow from the existence of ensembles of tridiagonal matrices, whose eigenvalues have a joint distribution that interpolates those of the $\beta = 1$,2 and 4 ensembles to all $\beta > 0$ Dumitriu-Edelman [DE02].

1.2 The Ginbre ensemble

subsec:ginibre

The so-called Ginibre Ensemble can be seen as a fundamental building block to define GOE and GUE. The real Ginbre ensemble, denoted $\mathrm{Gin}_{\mathbb{R}}$ is an $n \times n$ matrix of iid standard normal random variables. The Complex Ginibre Ensemble is then defined as

$$X \sim \mathrm{Gin}_{\mathbb{C}}(n) \quad \Leftrightarrow \quad X \sim \frac{1}{\sqrt{2}}(X_1 + \mathrm{i}X_2), \quad X_1, X_2 \sim \mathrm{Gin}_{\mathbb{R}}(n),$$

where X_1 and X_2 are independent. Then it follows that

$$\mathrm{GOE}(n) \sim \frac{1}{\sqrt{2}}(X + X^T), \quad X\ \mathrm{Gin}_{\mathbb{R}}(n),$$
$$\mathrm{GUE}(n) \sim \frac{1}{\sqrt{2}}(X + X^*), \quad X\ \mathrm{Gin}_{\mathbb{C}}(n).$$

There is a related construction for GSE that avoids direct use of quarternions using a $2n \times 2n$ complex matrix. See [DE02] for the details.

1.3 The main limit theorems

subec:limits

The basic question in random matrix theory is the following: what can one say about the statistics of the eigenvalues of a random matrix? For example, what is the probability that the largest eigenvalue lies below a threshold? Or, what is the probability that there are no eigenvalues in a given interval? The difficulty here is that even if the entries of a random matrix are independent, the eigenvalues are strongly coupled.

Gaussian ensembles play a very special, and important, role in random matrix theory. These are the only ensembles that are both Wigner and invariant (see Theorem 19 below). Pioneering, ingenious calculations by Dyson [Dys62], Gaudin and Mehta [MG60, Meh04], on the Gaussian ensembles served to elucidate the fundamental limit theorems of random matrix theory. In this section we outline these theorems, assuming always that the ensemble is GUE. Our purpose is to explain the form of the main questions (and their answers) in the simplest setting. All the results hold in far greater generality as is briefly outlined at the end of this section.

By the normalization (1.1.1), a GUE matrix has independent standard normal entries on its diagonal (mean zero, variance 1). The complex off-diagonal entries have independent real and imaginary parts with mean zero and variance 1/2. We denote the ordered eigenvalues of the GUE matrix by $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$. A fundamental heuristic for GUE matrices (that will be proven later, and may be easily simulated) is that the largest and smallest eigenvalues have size $O(\sqrt{n})$. In fact, $\lambda_1 \approx -2\sqrt{n}$ and $\lambda_n \approx 2\sqrt{n}$ as $n \to \infty$. Since there are n eigenvalues, the gap between these eigenvalues is typically $O(1/\sqrt{n})$. There are thus two natural scaling limits to consider as $n \to \infty$:

1. Rescale $M \mapsto n^{-1/2}M$ so that the spectral radius is $O(1)$. In this scaling limit, n eigenvalues are contained within a bounded interval, and we obtain a deterministic limit called the semicircle law.

2. Rescale $M \mapsto n^{1/2}M$ so that the gaps between eigenvalues are $O(1)$. In this scaling limit, we expect a random limiting point process. The limiting point process is a determinantal point process called the Sine_2 process.

In fact, the situation is more subtle. While the expected value of the gap between eigenvalues for a GUE matrix is indeed $O(n^{-1/2})$, the gaps are $O(n^{-2/3})$ about the edge of the spectrum. There is an an entirely different scaling limit called the Airy_2 process obtained by rescaling the spectrum of $M \pm 2\sqrt{n}I$.

In all that follows, we consider a sequence of random matrices of size n sampled from GUE(n). To make this explicit, the matrix is denoted M_n, and its ordered eigenvalues are denoted $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \cdots \leq \lambda_n^{(n)}$.

1.3.1 The semicircle law

Definition 1. The probability density and distribution function

$$p_{\mathrm{sc}}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbb{1}_{|x|\leq 2}, \quad F_{\mathrm{sc}}(x) = \int_{-\infty}^{x} p_{\mathrm{sc}}(x')\,\mathrm{d}x', \tag{1.3.1}$$

eq:psc

are called the semicircle density and the semicircle distribution respectively.

thm:wigner

Theorem 2. *Let M_n be a sequence of GUE matrices of size n. The rescaled empirical spectral measures*

$$\mu_n(\mathrm{d}x) = \frac{1}{n}\sum_{j=1}^{n} \delta_{n^{-1/2}\lambda_j^{(n)}}(\mathrm{d}x) \tag{1.3.2}$$

converge weakly to the semicircle density almost surely.

Theorem 2 may also be interpreted as the statement that the empirical spectral distribution of the matrices $M_n/\sqrt{n}$ converges to the semicircle distribution. The shortest proof of Theorem 2 uses the following integral transform.

Definition 3. Assume μ is a measure on $\mathbb{R}$ that satisfies the finiteness condition

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^2}}\mu(\mathrm{d}x) < \infty. \tag{1.3.3}$$

eq:stieltjes1

The Stieltjes transform of μ is the function

$$\hat{\mu}(z) = \int_{-\infty}^{\infty} \frac{1}{x-z}\mu(\mathrm{d}x), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{1.3.4}$$

eq:stieltjes2

The Stieltjes transform is of fundamental importance in the theory of orthogonal polynomials and spectral theory. This is because there are natural Stieltjes transforms associated to the resolvent $(M-z)^{-1}$, such as

$$\mathrm{Tr}(M-z)^{-1} \quad \text{and} \quad v^*(M-z)^{-1}v, \quad v \in \mathbb{C}^n, \quad |v| = 1. \tag{1.3.5}$$

The general proof of Theorem thm:wigner 2 uses a recursive expression for the law of $\mathrm{Tr}(z-M_n)^{-1}$. As $n \to \infty$, the fixed point of this recursion, R_{sc} solves the quadratic equation

$$R^2 - zR + 1 = 0. \tag{1.3.6}$$

eq:stieltjes3

It is then easy to verify that

$$R_{\mathrm{sc}}(z) = \frac{1}{2}\left(-z + \sqrt{z^2-4}\right), \quad z \in \mathbb{C}\backslash[-2,2]. \tag{1.3.7}$$

We recover the semicircle law from $R_{\mathrm{sc}}(z)$ by evaluating the jump in $\mathsf{Im}(R_{\mathrm{sc}}(z))$ across the branch cut $[-2,2]$.

rem:gaps **Remark 4.** The heuristic to determine the typical spacings is the following. Define $\gamma_j^{(n)} \in [-2,2]$ by the relation

$$\frac{j}{n} = \int_{-\infty}^{\gamma_j^{(n)}} p_{\mathrm{sc}}(x)\mathrm{d}x, \quad j = 1,2,\ldots,n.$$

Then the approximation $\lambda_j^{(n)} \approx \sqrt{n}\gamma_j^{(n)}$ should hold[1]. We have

$$\frac{1}{n} = \int_{\gamma_j^{(n)}}^{\gamma_{j+1}^{(n)}} p_{\mathrm{sc}}(x)\mathrm{d}x \approx (\gamma_{j+1}^{(n)} - \gamma_j^{(n)})p_{\mathrm{sc}}(\gamma_j^{(n)}). \tag{1.3.8}$$

If $j = j(n)$ is chosen so that $\gamma_j^{(n)} \to r$, $r \in (-2,2)$ (i.e. in the "bulk") we have

$$\lambda_{j+1}^{(n)} - \lambda_j^{(n)} \approx \frac{1}{\sqrt{n}p_{\mathrm{sc}}(r)}.$$

At the edge, consider (noting that $\gamma_1^{(n)} > -2$)

$$\begin{gathered}\frac{1}{n} = \int_{-2}^{\gamma_1^{(n)}} p_{\mathrm{sc}}(x)\mathrm{d}x \approx \int_{-2}^{\gamma_1^{(n)}} \frac{1}{\pi}\sqrt{2+x}\,\mathrm{d}x = \frac{2}{3\pi}\left(\gamma_1^{(n)}+2\right)^{3/2},\\ \gamma_1^{(n)} + 2 \approx \frac{c}{n^{2/3}},\\ 2\sqrt{n} + \lambda_1^{(n)} = O(n^{-1/6}), \quad \lambda_n^{(n)} - 2\sqrt{n} = O(n^{-1/6}),\end{gathered} \tag{1.3.9}$$

eq:largestscaling

where the last equation follows from $\lambda_j^{(n)} \approx \sqrt{n}\gamma_j^{(n)}$ and the natural symmetry between $\lambda_1^{(n)}$ and $\lambda_n^{(n)}$.

[1] This is made rigorous and quantitative by Erdős, Yau and Yin Erdos2012a [EYY12].

1.3.2 Fluctuations in the bulk: the sine process

We now rescale so that the gaps between eigenvalues is $O(1)$ and the scaling limit is a random process. This random process will be denoted Sine_2 (and Sine_β for the general β-ensembles). Each realization of the Sine_2 process is a countable set of points $\{x_k\}_{k=-\infty}^{\infty}$. One of the fundamental statistics associated to a point process is the probability of having k points in an interval. In order to state a typical fluctuation theorem that describes these probabilities, we must define the sine-kernel and its Fredholm determinants.

def:sine-kernel **Definition 5.** The sine-kernel is the integral kernel on $\mathbb{R} \times \mathbb{R}$ given by

$$K_{\mathsf{sine}}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}, \quad x \neq y, \tag{1.3.10}$$ eq:sinek

and $K_{\mathsf{sine}}(x,x,) = 1$.

In the following theorem we will assume that x and y are restricted to a finite interval (a,b). The sine-kernel defines an integral operator on $L^2(a,b)$ that we denote by $K_{\mathsf{sine}}\mathbb{1}_{(a,b)}$. The kernel $K_{\mathsf{sine}}(x,y)$ is clearly continuous, thus bounded, for $x,y \in (a,b)$. Thus, $K_{\mathsf{sine}}\mathbb{1}_{(a,b)}$ defines an integral operator on $L^2(a,b)$ that is trace-class, and it has a well-defined Fredholm determinant

$$\begin{aligned} &\det\left(1 - K_{\mathsf{sine}}\mathbb{1}_{(a,b)}\right) \qquad (1.3.11) \\ &\quad = 1 + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{(a,b)^m} \det\left(K_{\mathsf{sine}}(x_j,x_k)_{1\le j,k\le m}\right) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_m. \end{aligned}$$ eq:fred-series

Though perhaps mysterious at first sight, the origin of this formula is rather simple. Integral operators with some smoothness and boundedness (in particular, continuous integral operators K whose trace $\int_a^b |K(x,x)|\mathrm{d}x$ is finite) may be approximated on a discrete-grid of size h by a finite-dimensional discretization K_h. The determinant $(I - K_h)$ is then the usual determinant of a matrix and we may use the definition of the determinant to expand $\det(I - K_h)$ in a finite series, which is nothing but the infinite series above in the instance when all terms beyond $m = \mathrm{rank}(K_h)$ vanish. This approach was pioneered by Fredholm in 1900 before the development of functional analysis and is turned into an efficient numerical method in Section sec:det-numerics 10.1. From a probabilistic point of view, this formula arises from the Inclusion-Exclusion Principle, taken to the limit. The operator theory introduced by Fredholm allows for that limit to be understood.

thm:dyson **Theorem 6** (Gaudin-Mehta GM [MG60])**.** *For each finite interval $(a,b) \subset \mathbb{R}$, and $r \in (-2,2)$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\sqrt{n}p_{\mathrm{sc}}(r)\left(\lambda_k^{(n)} - r\sqrt{n}\right) \notin (a,b),\, 1 \le k \le n\right) = \det\left(1 - K_{\mathsf{sine}}\mathbb{1}_{(a,b)}\right). \tag{1.3.12}$$ eq:sine

The probabilities of the Sine_2 process can be expressed without reference to the matrices M_n. For each interval (a,b) let $N_{(a,b)} = \sum_{k=-\infty}^{\infty} \mathbb{1}_{\{x_k \in (a,b)\}}$. Then,

$$\mathbb{P}\left(N_{(a,b)} = 0\right) = \det\left(1 - K_{\mathsf{sine}}\mathbb{1}_{(a,b)}\right). \tag{1.3.13}$$ eq:sine2

For comparison, if we had a Poisson process $\{\tilde{x}_k\}_{k=-\infty}^{\infty}$ with rate $\lambda(\mathrm{d}x)$, the associated count $\tilde{N}_{(a,b)}$ would satisfy

$$\mathbb{P}\left(\tilde{N}_{(a,b)} = 0\right) = \exp\left(-\int_a^b \lambda(\mathrm{d}x)\right).$$

1.3.3 Fluctuations at the edge: the Airy point process

Remark 4 and Theorem 6 reveal that the gaps between consecutive eigenvalues $\lambda_j^{(n)}$ and $\lambda_{j+1}^{(n)}$ is of $O(n^{-1/2})$. However, the fluctuations at the edge are much larger, of $O(n^{-1/6})$. The point process of shifted and scaled eigenvalues converges in distribution to a limiting point process, $\{y_k\}_{k=1}^{\infty}$ called the Airy_2 process. In order to describe the law of this process, we must define the Airy function and the Airy kernel.

def:airy-function **Definition 7.** The Airy function is defined by the oscillatory integral

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{\mathrm{i}k^3/3}\,\mathrm{d}k. \tag{1.3.14}$$ eq:def-airy

The Airy function is one of the classical special functions [AS72]. It admits several alternative definitions. For instance, the oscillatory integral in (1.3.14) may be deformed into an absolutely convergent integral in the complex plane. This argument allows us to establish that the Airy function is entire and to determine its asymptotic expansions as $x \to \pm\infty$.

These properties may also be established using the theory of ordinary differential equations in the complex plane [Hil97]. It is easy to verify from (1.3.14), after deformation, that $\mathrm{Ai}(x)$ satisfies the differential equation

$$\varphi''(x) = x\varphi, \quad -\infty < x < \infty. \tag{1.3.15}$$ eq:ode-airy

Equation (1.3.15) admits two linearly independent solutions, only one of which decays as $x \to \infty$. Up to a (fixed by convention, but otherwise arbitrary) normalization constant, the decaying solution is $\mathrm{Ai}(x)$.

def:airy-kernel **Definition 8.** The Airy kernel is the continous integral kernel on $\mathbb{R}\times\mathbb{R}$ given by

$$K_{\mathsf{Airy}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, \quad x \neq y,$$

and by continuity at $x = y$.

Observe that both the sine and Airy kernel have the form

$$K(x,y) = \frac{f(x)f'(y) - f'(x)f(y)}{x-y}, \quad x \neq y \tag{1.3.16}$$ eq:int-structure

where f solves a second-order linear differential equation. Similar kernels arise in various limiting models in random matrix theory. For instance, the Bessel kernel – corresponding to $f(x) = J_\alpha(x)$, the Bessel function with parameter α – describes fluctuations about the singular values of random matrices.

thm:TW **Theorem 9.** *For each interval $(a,b) \subset \mathbb{R}$, $-\infty < a < b \leq \infty$,*

$$\lim_{n\to\infty} \mathbb{P}\left(n^{1/6}\left(\lambda_k^{(n)} - 2\sqrt{n}\right) \notin (a,b),\, 1 \leq k \leq n\right) = \det\left(1 - K_{\mathsf{Airy}} \mathbb{1}_{(a,b)}\right). \tag{1.3.17}$$

eq:airy-det

As in the remarks following Theorem 6, the expression $\det\left(1 - K_{\mathsf{Airy}} \mathbb{1}_{(a,b)}\right)$ gives the probability that no points of a realization of the Airy_2 point process lie in (a,b).

1.3.4 Fredholm determinants, Painlevé equations, and integrable systems

It is immediate from Theorem 6 and Theorem 9 that the Fredholm determinants $\det\left(1 - K_{\mathsf{sine}} \mathbb{1}_{(a,b)}\right)$ and $\det\left(1 - K_{\mathsf{Airy}} \mathbb{1}_{(a,b)}\right)$ are positive for all (a,b). This is astonishing if one treats (1.3.11) as a starting point, since it is by no means clear that the signed infinite series sums to a positive number! It is in fact rather challenging to extract meaningful information, such as the asymptotics of tails, from the expression of probabilities as Fredholm determinants. A crucial piece of the puzzle lies in the connection between Fredholm determinants and the theory of integrable systems. More precisely, the Fredholm determinants satisfy differential equations in a and b (or more generally in endpoints of intervals, when one considers the obvious extensions of Theorem 6 and Theorem 9 to a collection of intervals $\prod_{j=1}^m (a_m, b_m)$). These ordinary differential equations have a special, integrable structure, that allows their analysis. The following theorems illustrate this aspect of random matrix theory.

eq:JMMS **Theorem 10** (Jimbo-Miwa-Mori-Sato [JMMS80])**.** *For all $t > 0$,*

$$\det\left(1 - K_{\mathsf{sine}} \mathbb{1}_{(-\frac{t}{2}, \frac{t}{2})}\right) = \exp\left(\int_0^t \frac{\sigma(s)}{s}\, \mathrm{d}s\right), \tag{1.3.18}$$

where $\sigma(t)$ is the solution to the Painlevé-5 equation

$$\left(t\sigma''\right)^2 + 4\left(t\sigma' - \sigma\right)\left(t\sigma' - \sigma + \sigma^2\right) = 0, \tag{1.3.19}$$

eq:p5-jmms

which satisfies the asymptotic condition

$$\sigma(t) = -\frac{t}{\pi} - \frac{t^2}{\pi^2} - \frac{t^3}{\pi^3}, \quad t \downarrow 0. \tag{1.3.20}$$

eq:p5-ic

eq:TW **Theorem 11** (Tracy–Widom distribution [TW94])**.** *For all real t,*

$$F_2(t) := \det\left(1 - K_{\mathsf{Airy}} \mathbb{1}_{(t,\infty)}\right) = \exp\left(-\int_t^\infty (s-t) q^2(s)\, \mathrm{d}s\right), \tag{1.3.21}$$

eq:tw-formula

where q is the solution to the Painlevé-2 equation

$$q'' = tq + 2q^3, -\infty < t < \infty \tag{1.3.22}$$

eq:TW-p2

which satisfies the asymptotic condition

$$q(t) \sim \mathrm{Ai}(t), \quad t \to \infty. \tag{1.3.23}$$

eq:TW-p2-ic

We will discuss the basic properties of Painlevé equations and integrable systems in Lecture ??. Here is a brief preview.

The Painlevé differential equations are a special family of nonlinear ordinary differential equations that generalize the classical theory of linear differential equations in the complex plane and the associated theory of special functions [Hil97]. For example, the Painlevé-2 equation (1.3.22) may be viewed as a nonlinear analogue of the Airy differential equation (1.3.15). Broadly, the Painlevé differential equations represent a complete classification of second-order differential equations with the Painlevé property — their only movable singularities (movable by changing initial conditions) are poles — that are not solvable with elementary functions. The theory of Painlevé equations was developed in the early years 1900's, by Boutroux and Painlevé, but fell into obscurity[2]. It was reborn in the 1970s with the discovery of their importance in integrable systems and exactly solvable models in statistical mechanics, such as the Ising model in 2D [MW73]. We illustrate these links with a fundamental integrable system: the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0, \quad -\infty < x < \infty,\ t \geq 0. \tag{1.3.24}$$

Despite the fact that KdV is nonlinear, it may be solved explicitly through the inverse scattering transform. We will not discuss this method in detail. But in order to make the connection with random matrix theory, let us note that if one seeks self-similar solutions to KdV of the form

$$u(x,t) = \frac{1}{(3t)^{2/3}} q\left(\frac{x}{(3t)^{2/3}}\right) \tag{1.3.25}$$

then $q = v^2 + v'$ and v satisfies the Painlevé-2 equation (1.3.22). It is in this context that Hastings and McLeod established the existence of a solution to (1.3.22) that satisfies the asymptotic condition (1.3.23) [HM80]. It is remarkable that it is exactly this solution that describes the Tracy-Widom distribution $F_2(t)$!

1.3.5 Universality

We have restricted attention to matrices from GUE to present some of the central theorems in the subject in an efficient manner. One of the main achievements of the past decade has been the establishment of universality – informally, this is the notion that the limiting fluctuations in the bulk and edge described by the Sine_2 and Airy_2 processes, hold for both Wigner and invariant ensembles which satisfy natural moment assumptions. The idea of universality is of clear practical importance (we need understand only a few universal limits). It also appears to hold the key to some of the connections between random matrix theory and other areas of mathematics. The explanation of these connections may

[2]Paul Painlevé was rather restless: he began in mathematics, became an early aviation enthusiast and then turned to politics. He rose to become the Prime Minister of France for part of World War I and was later the designer of the disastrous Maginot line.

lie in the fact that determinantal point processes, such as the Sine_2 and Airy_2 process, have the simplest structure of strongly interacting point processes. By contrast, Poisson processes, while universal, describe non-interacting points.

1.4 Connections to other areas of mathematics

Random matrix theory has deep connections with many areas of mathematics, many of which are still poorly understood. A brief overview of some of these connections is presented below. While some of these notions, such as the connections with stochastic PDE require more background than we assume, some other connections (e.g. with quantum gravity) are in fact more elementary (and fundamental) than one may naively expect. Our purpose here is to present a small sample of the rich set of ideas that make the subject so attractive.

1.4.1 Number theory

The Riemann zeta function is defined by the infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathsf{Re}(s) > 1. \tag{1.4.1}$$

eq:zeta1

The function $\zeta(s)$ is central to number theory, since it provides a generating function for the distribution of the prime numbers via Euler's product formula

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \quad \mathsf{Re}(s) > 1. \tag{1.4.2}$$

For instance, the divergence of the harmonic series at $s = 1$ provides a proof that there are infinitely many primes. The study of $\zeta(s)$ by complex analysis is the cornerstone of analytic number theory. The basic facts are as follows. The function $\zeta(z)$ extends to a meromorphic function on $\mathbb{C}$ by analytic continuation, which has a simple pole at $s = 1$ where the residue is 1. A closely related function is the Riemann ξ-function

$$\xi(s) = \frac{1}{2\pi^{s/2}} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{1.4.3}$$

eq:zeta2

Recall that the Γ function is a 'continuous interpolation' of the factorial, defined by the integral

$$\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-x} x^{s-1}\, \mathrm{d}x, \quad \mathsf{Re}(s) > 0. \tag{1.4.4}$$

eq:zeta4

The Γ-function extends to a meromorphic function $\mathbb{C}$, which has simple poles at $\ldots, -2, -1, 0$ where the residue is 1. These poles cancel the 'trivial' zeros of the ζ function, and the essential difficulties related to the study of the ζ function are more transparent for the ξ function. It satisfies the functional equation

$$\xi(s) = \xi(1-s), \quad s \in \mathbb{C}. \tag{1.4.5}$$

eq:zeta3

The celebrated Riemann Hypothesis is the conjecture that all zeros of the ξ function lie on the critical line $\mathsf{Re}(s) = 1/2$ (this line is the symmetry axis for the functional equation above). In his fundamental paper on the distribution of prime numbers (translated in Edwards [Edw74] and Riemann [Rie53]) Riemann presented a series of asymptotic expansions that would imply rigorous bounds on the distribution of primes if the Riemann Hypothesis is true.

The connection between random matrix theory and the Riemann Hypothesis is two-fold. First, if one could construct a Hermitian operator with point spectrum whose eigenvalues coincide with the zeros of $\xi(i(s-1/2)$ then the Riemann Hypothesis would follow immediately (since all eigenvalues of a Hermitian operator are real). The catch, of course, is to determine such an operator. Nevertheless, as we discuss below, random matrix theory has shed new light on the spectral theory of several operators, deterministic and random. Thus, the theory provides a catalog of 'guesses'. Second, if one assumes the Riemann hypothesis, the fluctuations in the zeros of $\zeta(s)$ are described by the sine-kernel! Under the Riemann hypothesis, the non-trivial zeros of $\zeta(s)$ may be written $\gamma_n = \frac{1}{2} \pm i t_n$ with $0 < t_1 < t_2 < \ldots$. Let

$$w_n = \frac{t_n}{2\pi} \log\left(\frac{t_n}{2\pi}\right), \quad \text{and} \quad N(x) = \sum_{k=1}^{\infty} \mathbb{1}_{w_n \le x}. \tag{1.4.6}$$

eq:zeta5

This rescaling is chosen so that $\lim_{x\to\infty} N(x)/x = 1$ in accordance with the Prime Number Theorem.

Despite the fact that the zeros w_n are deterministic, we may introduce probabilistic notions by counting the (rescaled) zeros upto a level $x > 0$. For example, we may define the empirical probability measure

$$\mu_1(\mathrm{d}w; x) = \frac{1}{N(x)} \sum_{k=1}^{N(x)} \delta_{w_k}(\mathrm{d}w). \tag{1.4.7}$$

eq:zeta6

In order to study the gaps between eigenvalues, we must consider instead the empirical measures

$$\mu_2(\mathrm{d}l; x) = \frac{1}{x} \sum_{1\le j,k \le N(x); j\neq k} \delta_{w_j - w_k}(\mathrm{d}l). \tag{1.4.8}$$

eq:zeta7

The expectation of a continuous function with respect to $\mu_2(\mathrm{d}l; x)$ is denoted

$$R_2(f; x) = \int_{-\infty}^{\infty} f(l)\mu_2(\mathrm{d}l; x) = \frac{1}{x} \sum_{1\le j<k\le N(x)} f\left(w_j - w_k\right). \tag{1.4.9}$$

eq:zeta8

Under the assumption that f is band-limited, i.e. that its Fourier transform has compact support, Montgomery established the following

thm:montgomery

Theorem 12 (Montgomery)**.** *Assume the Riemann Hypothesis. Assume f is a Schwartz function whose Fourier transform $\hat{f}$ is supported in $[-1, 1]$. Then*

$$\lim_{x\to\infty} R_2(f; x) = \int_{-\infty}^{\infty} f(l)\mu_2(\mathrm{d}l), \quad \mu_2(\mathrm{d}l) = \left(1 - \left(\frac{\sin \pi l}{\pi l}\right)^2\right) \mathrm{d}l. \tag{1.4.10}$$

eq:montgomery

The point here is that the right hand side of (1.4.10) is precisely the 2-point function for the sine process. More generally, Montgomery's theorem is now known to hold for the distribution of n-consecutive gaps. That is, the rescaled fluctuations converge to the Sine_2 process in distribution. Bourgade's thesis provides an excellent introduction to these topics [Bou09].

1.4.2 Combinatorics and enumerative geometry

We will present two problems of enumerative combinatorics that connect with random matrix theory. As a first example, we note that the $2m$-th moment of the semicircle law

$$\int_{-2}^{2} x^{2m} p_{\mathrm{sc}}(x)\,\mathrm{d}x = \frac{1}{m+1}\binom{2m}{m} = C_m, \tag{1.4.11}$$

eq:catalan1

the m-th Catalan number. An analytic proof of this identity follows from a comparison between the Stieltjes transform $R_{\mathrm{sc}}(z)$, and the generating function

$$\hat{C}(x) = \sum_{m\geq 0} C_m x^m = \frac{1-\sqrt{1-4x}}{x}. \tag{1.4.12}$$

eq:catalan2

The Catalan numbers describe the solution to many combinatorial problems [3]. For example, C_m enumerates the number of Bernoulli excursions or Dyck paths of length $2m$: these are walks S_k, $1 \leq k \leq 2m$ such that $S_0 = S_{2m} = 0$, $S_k \geq 0$, $0 \leq k \leq 2m$, and $|S_{k+1} - S_k| = 1$.

A deeper set of connections between integrals on $\mathsf{Her}(n)$ and geometry was first noticed by the physicist 't Hooft [tH74]. Ignoring for now the physicists' motivation, let us illustrate a particular computational technique that underlies their work. Consider a matrix integral of the form

$$Z_n(z) = \int_{\mathsf{Her}(n)} \mathrm{e}^{\mathrm{Tr}(-zM^4)} p_{\mathrm{GUE}}(M)\,\mathrm{D}M, \quad \mathsf{Re}(z) > 0. \tag{1.4.13}$$

eq:IZ1

The quartic nonlinearity prevents us from expressing this integral in closed form. Nevertheless, this integral may be expanded in a Taylor series

$$Z_n(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \int \left(\mathrm{Tr}(M^4)\right)^k p_{\mathrm{GUE}}(M)\,\mathrm{D}M, \quad \mathsf{Re}(z) > 0. \tag{1.4.14}$$

eq:IZ2

A fundamental lemma on Gaussian integrals (on $\mathbb{R}^N$) (Wick's lemma) allows us to reduce each integral above to a sum over pairings of indices. It is convenient to keep track of these pairings with a graphical description, called a Feynman diagram. 't Hooft observed that when $\mathbb{R}^N \equiv \mathsf{Her}(n)$ the Feynman diagram associated to each term in (1.4.14) enumerates embedded graphs on a Riemann surface. This characterization was independently discovered by mathematicians.

[3] Stanley lists 66 examples in [Sta11, Exercise 6.19].

Lemma 1 (Harer-Zagier [HZ86])**.** *Let $\varepsilon_g(m)$ denote the number of ways to pair the edges of a symmetric $2m$-gon to form an orientable surface with genus g. Then*

$$f(m,n) = \sum_{g=0}^{\infty} \varepsilon_g(m) n^{m+1-2g} = \int_{\mathsf{Her}(n)} \mathrm{Tr}(M^{2m}) p_{\mathrm{GUE}}(M)\, \mathrm{D}M. \tag{1.4.15}$$

Note that only finitely many terms in the sum are non-zero. The series above is an instance of a genus-expansion. It illustrates the beautiful fact that matrix integrals serve as the generating functions for Riemann surfaces with a given combinatorial decomposition.

1.4.3 Random permutations

Consider the symmetric group $\mathsf{S}(n)$ of permutations of size n. Every element of $\mathsf{S}(n)$ can be represented as a reordering of the integers $1, 2, \ldots, n$. For example, three elements of $\mathsf{S}(5)$ are

$$\pi_1 = 54312, \quad \pi_2 = 12435, \quad \pi_3 = 45123.$$

We define a function $\ell : \mathsf{S}(5) \to \mathbb{N}$ by $\ell(\pi) =$ length of the longest increasing subsequence of π. For example,

$$\ell(\pi_1) = 2, \quad \ell(\pi_2) = 4, \quad \ell(\pi_3) = 3.$$

There is a natural probability distribution $\mathrm{Uni}(n)$ on $\mathsf{S}(n)$, the uniform distribution, or Haar measure. If $\Pi_n \sim \mathrm{Uni}(n)$ then $\mathbb{P}(\Pi_n = \pi) = \frac{1}{n!}$ for any $\pi \in S_n$ since $|\mathsf{S}(n)| = n!$.

The law of $l(\Pi_n)$ when $\Pi_n \sim \mathrm{Uni}(n)$ was one of the first problems to be investigated by Monte Carlo simulation on a computer. Ulam performed simulations in the early 60's [Ula61] and conjectured that

$$\frac{1}{\sqrt{n}} \mathbb{E}\left[\ell(\Pi_n)\right] \to c.$$

It was later independently established by Vershik and Kerov [VK77], and Logan and Shepp [LS77] that $c = 2$. The detailed numerical computations of Odlyzko and Rains [OR00] indicated

$$\mathbb{E}\left[\ell(\Pi_n)\right] - 2\sqrt{n} = O(n^{-1/6}). \tag{1.4.16}$$

The comparison between (1.3.9) and (1.4.16) should be striking. Indeed, the following is often called the Baik–Deift–Johansson Theorem and it makes this scaling rigorous.

Theorem 13 ([BDJ99])**.** *Let $\mathsf{S}(n)$, ℓ and Π_n be as above. Then for all $t \in \mathbb{R}$*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell(\Pi_n) - 2\sqrt{n}}{n^{1/6}} \le t \right) = \det(1 - K_{\mathsf{Airy}} \mathbb{1}_{(t,\infty)}).$$

That is, the limit is the same as the largest eigenvalue of a random Hermitian matrix.

This theorem is discussed in great detail in [BDS17]. This surprising connection was explored further by Johansson [Joh00] leading to many connections to random growth processes and the KPZ equation.

1.4.4 Spectral and inverse spectral theory of operators

While Theorem 2–Theorem 9 associate limits to the spectrum of the operators M_n, it is natural to ask if there are limiting operators that may be naturally associated to the limiting spectra. Thus, for Theorem 2 we ask for a 'natural' operator that has spectral density given by the semicircle law, p_{sc}, and for Theorem 6 and Theorem 9 we seek 'natural' random operators that have pure point spectra with the law of the Sine_2 and Airy_2 point processes. What is a 'natural' operator is, of course, a subjective idea, but convincing candidates operators are suggested by inverse spectral theory.

We say that a matrix $T \in \mathsf{Symm}(n)$ is a Jacobi matrix if all its off-diagonal entries are strictly positive. The spectral measure of a Jacobi matrix is the measure whose Stieltjes transform is $e_1^T(T-z)^{-1}e_1$. There is a $1-1$ correspondence between the space of $n \times n$ Jacobi matrices and probability measures on the line with n atoms. This correspondence extends naturally, but with some caveats, to semi-infinite Jacobi matrices. The essence of this theory (due to Stieltjes) is that the entries of T may be determined from the continued fraction expansion of $e_1^T(T-z)^{-1}e_1$. This correspondence will be considered in detail in Chapter 3, but here is a concrete example. By applying Stieltjes' procedure[4] to the semicircle law, we discover that $p_{\mathrm{sc}}(x)$ is the spectral density for the seminfinite tridiagonal matrix that is 1 on the off-diagonal, and 0 in all other entries. This follows from the continued fraction expansion

$$R_{\mathrm{sc}}(-z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{1}{z - \dots}}} \tag{1.4.17}$$

eq:cont-frac-sc

Ensembles of tridiagonal matrices are of practical important in numerical linear algebra. For instance, a key pre-processing step while solving symmetric linear systems is to transform the matrix to tridiagonal form by Householder's procedure (see Chapter 3). Dumitriu and Edelman pushed forward the Gaussian measures under this procedure to obtain a family of tridiagonal ensembles, known as the general-β ensembles [DE02]. Further, Edelman and Sutton made a formal expansion of these operators, and observed that as $n \to \infty$, the tridiagonal operators appeared to converge to the stochastic Airy operator [ES07]:

$$H_\beta = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x + \frac{2}{\sqrt{\beta}}\dot{b}, \quad 0 < x < \infty \tag{1.4.18}$$

eq:stoch-airy

[4]The Stieltjes' procedure is the procedure by which an orthonormal basis of polynomials is constructed by the Gram–Schmidt process, exploiting a three-term recurrence relation. It is intimately connected with the Lanczos iteration.

with Dirichlet boundary conditions at $x = 0$. Here $\dot{b}$ denotes (formally) white noise (it is not hard to define H_β rigorously).

Theorem 14 (Ramirez-Rider-Virag RRV [RRV11])**.** *The spectrum $\sigma(H_\beta)$ of the operator H_β is almost surely a countably infinite number of eigenvalues $\mu_1 < \mu_2 < \mu_3 < \ldots$. Moreover, $\sigma(H_\beta)$ has the same law as the Airy_β point process.*

In particular, for $\beta = 2$, the spectrum of the stochastic Airy operator describes the limiting fluctuations at the edge of the spectrum of GUE matrices. Despite the simplicity of this characterization, it is not completely understood how to recover the explicit determinantal formulas of Tracy and Widom from this formulation (see Rumanov [Rum15]).

1.4.5 Universality in numerical computation

It is quite natural to ask about other aspects of random matrices beyond properties of their eigenvalues. For example, what distributions "arise" in the computation of the eigenvalues of a random matrix. And are these distributions universal?

Before one can truly ask this question (and make it more precise) an algorithm needs to be set. For example, one can use the power method to compute the top eigenvalue, the QR eigenvalue algorithm to compute the entire spectrum, or if one is truly trying to be competative with the state of the art, the implicitly shifted QR algorithm (i.e. Francis' algorithm) Francis [Fra61] (see also Watkins2011 [Wat11]) should be used.

And then the question of how a distribution can "arise" in a deterministic algorithm needs to be addressed. The most natural way is to examine the runtime distribution also called the halting time. Since eigenvalue computation amounts to polynomial rootfinding, abstract theory (Galois theory, specifically) tells us that if the degree is five or larger then any general algorithm to compute the eigenvalues must be iterative.

It turns out that the so-called Toda algorithm is a natural algorithm to use on symmetric indefinite matrices[5]. The Toda algorithm to compute the eigenvalues of a symmetric (or Hermitian) matrix H is derived by discretizing and numerically solving the matrix flow

$$X'(t) = X(t)B(X(t)) - B(X(t))X(t), \quad B(X) = X_- - X_-^*, \quad X(0) = H,$$

where X_- is the strictly lower-triangular part of X. It follows from the fact that $B(X(t))$ is skew-symmetric that the eigenvalues of $X(t)$ do not change in time — the flow is isospectral.

A measure of the error in computing the top eigenvalue of the matrix is

$$E(t) = \sum_{j=2}^{n} |X_{1j}(t)|^2$$

[5]This is not because of its efficiency but rather because of its mathematical properties.

because if $E(t) = 0$ then we are guaranteed that $X_{11}(t)$ is an eigenvalue of $X(0) = H$. The associated halting time is given by

$$T_\epsilon(H) = \min\{t \geq 0 : E(t) \leq \epsilon^2\}.$$

The following is a consenquence of Deift2016 [DT18].

Theorem 15. *Suppose* $H \sim \mathrm{GOE}(n)$ $(\beta = 1)$ *or* $H \sim \mathrm{GUE}(n)$ $(\beta = 1)$ *and* $\epsilon \leq n^{-5/3-\sigma}$ *for* $\sigma > 0$. *Then*

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{T_\epsilon(n^{-1/2}H)}{n^{2/3}(\log \epsilon^{-1} - 2/3 \log n)} \leq t \right) = F_\beta^{\mathrm{gap}}(t), \tag{1.4.19}$$

eq:toda-thm

where

$$F_\beta^{\mathrm{gap}}(t) := \lim_{n\to\infty} \mathbb{P}\left(\frac{1}{n^{1/6}(\lambda_n^{(n)} - \lambda_{n-1}^{(n)})} \leq t \right).$$

The existence of the limit $F_\beta^{\mathrm{gap}}(t)$ in the $\beta = 2$ case is implied by Theorem t:joint 59 below. An important additional fact that is also established in Deift2016 [DT18] is that this theorem also holds, up to the modification of some ensemble-dependent constants, if H is from a so-called generalized Wigner matrix or invariant ensemble.

Part I

Exact theory for random matrices and numerical linear algebra

Chapter 2

Integration on spaces of matrices

ch:integration

In this section, we review the geometry of the classical Lie groups, as well as the spaces $\mathsf{Symm}(n)$, $\mathsf{Her}(n)$ and $\mathsf{Quart}(n)$ and explain how to integrate over these groups and spaces. Given an point on a manifold $M \in \mathbb{M}$, we use $\mathrm{d}M$ to denote the differential of M, i.e., an infintesimal element on the tangent space $T_M(\mathbb{M})$ at M. We reserve $\mathrm{D}M$ to refer to a (naturally induced) volume form defined using an inner product on the tangent space. Note that for $x \in \mathbb{R}$, $\mathrm{d}x = \mathrm{D}x$. Our main goal is the following

thm-weyl-formula

Theorem 16 (Weyl's formula)**.**

$$\mathrm{D}M \propto |\triangle(\Lambda)|^{\beta}\, \mathrm{D}\Lambda\, \mathrm{D}U \tag{2.0.1}$$

eq:weyl-formula

where $\triangle(\Lambda)$ is the *Vandermonde determinant*

$$\triangle(\Lambda) = (-1)^{\frac{n(n-1)}{2}} \prod_{1\le j<k\le n} (\lambda_j - \lambda_k), \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \tag{2.0.2}$$

eq:vandermonde

$\mathrm{D}\Lambda$ *is Lebesgue measure on* $\mathbb{R}^n$*, and* $\mathrm{D}U$ *denotes (unnormalized) Haar measure*[1] *on* $\mathsf{O}(n)$*, and an appropriately defined measure on* $\mathsf{U}(n)/\mathbb{T}^n \cong \{U \in \mathsf{U}(n) : u_{1j} > 0, \quad j = 1, 2, \ldots, n\}$*, and* $\mathsf{USp}(n)/\mathbb{T}^n \cong \{V \in \mathsf{USp}(n) : v_{1j} > 0, \quad j = 1, 2, \ldots, n\}$ *in the cases* $\beta = 1, 2$ *and* 4 *respectively.*

Since the set of all diagonal unitary matrices is not a normal subgroup of $\mathsf{U}(n)$, the quotient is not a group. So we cannot define Haar measure on it.

The main strategy to prove Theorem 16 (thm-weyl-formula) is to treat the mapping from matrices with distinct eigenvalues to their eigenvalues and eigenvectors. Then we identify the tangent spaces, and give a formula that relates the tangent space for the eigenvalues and the tangent space for the eigenvectors to the tangent space for the matrix. This formula allows one to change variables in the metric tensor and therefore in the volume form.

[1]See Section B.4 (s:Haar) for a discussion of Haar measure.

Remark 17. It is common to normalize the Haar measure such that it is a probability measure. We have ignored this constant here, though is is explored in the exercises. The essential aspect of (2.0.1) is that the Jacobian for diagonalization is given by $|\triangle(\Lambda)|^\beta$. This has far-reaching consequences for random matrix theory and has the interesting physical interpretation of eigenvalue repulsion.

In what follows, we first present a detailed description of integration on $\mathsf{O}(n)$ and $\mathsf{Symm}(n)$. The ideas are then extended to $\mathsf{Her}(n)$ and $\mathsf{Quart}(n)$.

2.1 Integration on $\mathsf{O}(n)$ and $\mathsf{Symm}(n)$

sec:snon

A (linear) isometry of $\mathbb{R}^n$ is a linear transformation that leaves the inner-product invariant The Lie group $\mathsf{O}(n)$ is the group, under composition, of linear transformations of $\mathbb{R}^n$ that preserve the standard metric $g = I$. For each $O \in \mathsf{O}(n)$ and each $x \in \mathbb{R}^n$ we must have $(Ox)^T(Ox) = x^T x$. Thus, $\mathsf{O}(n)$ is equivalent to the group of matrices O such that $O^T O = I$. The group operation is matrix multiplication. It is easy to check that the group axioms are satisfied, but a little more work is required to check that $\mathsf{O}(n)$ is a differentiable manifold, and that the group operation is smooth.

We now introduce the natural volume forms on $\mathsf{Symm}(n)$ and $\mathsf{O}(n)$. We first note that the space $\mathsf{Symm}(n)$ is isomorphic to $\mathbb{R}^p$, $p = n(n+1)/2$ via the map

$$M \mapsto (M_{11}, \dots, M_{nn}, M_{12}, \dots, M_{n-1,n}). \tag{2.1.1}$$ ee:inp2b

Thus, all that is needed to define integrals over $\mathsf{Symm}(n)$ is a choice of inner-product. We will always use the Hilbert–Schmidt inner product

$$\mathsf{Symm}(n) \times \mathsf{Symm}(n) \to \mathbb{R}, \quad (M, N) \mapsto \mathrm{Tr}(M^T N) = \mathrm{Tr}(MN). \tag{2.1.2}$$ eq:hsnorm

The associated infinitesimal length element is

$$\mathrm{d}s^2 = \mathrm{Tr}(\mathrm{d}M^T \mathrm{d}M) = \sum_{j=1}^n \mathrm{d}M_{jj}^2 + 2\sum_{j<k} \mathrm{d}M_{jk}^2. \tag{2.1.3}$$ eq:int4

In ξ coordinates on $\mathbb{R}^p$, the associated metric tensor g is diagonal and takes the value 1 for the first n coordinates (diagonal terms), and the value 2 for all the other coordinates (off-diagonal terms). Thus, the metric tensor $g \in \mathsf{Symm}_+(p)$ has determinant $2^{n(n-1)/2}$. We apply formula (B.2.2) to find the following volume form on $\mathsf{Symm}(n)$,

$$\mathrm{D}M = 2^{n(n-1)/4} \prod_{j=1}^n \mathrm{d}M_{jj} \prod_{1 \le j < k \le n} \mathrm{d}M_{jk}. \tag{2.1.4}$$ eq:int5

Each $O \in \mathsf{O}(n)$ defines a map $\mathsf{Symm}(n) \to \mathsf{Symm}(n)$, $M \mapsto OMO^T$. This map is an isometry on $\mathsf{Symm}(n)$ with the metric above. It is in this sense that (2.1.6) is the natural inner-product. Since this map is an isometry, the volume element $\mathrm{D}M$ is also invariant.

$\mathsf{O}(n)$ is a differentiable manifold that is not flat. Thus, in order to define a volume form on $\mathsf{O}(n)$, we must identify its tangent space $T_Q\mathsf{O}(n)$, $Q \in \mathsf{O}(n)$, and then introduce an inner-product on $T_Q\mathsf{O}(n)$. Further, the 'natural' inner-product should be invariant under the group operations. The tangent space at the identity to $\mathsf{O}(n)$, $T_I\mathsf{O}(n)$, is isomorphic to the Lie algebra, $\mathfrak{o}(n)$, of $\mathsf{O}(n)$. In order to compute $\mathfrak{o}(n)$ we consider smooth curves $(-a, a) \to \mathsf{O}(n)$, $a > 0$, $t \mapsto Q(t)$ with $Q(0) = I$, differentiate the equation $Q(t)^T Q(t) = I$ with respect to t, and evaluate at $t = 0$ to find

$$\dot{Q}(0)^T = -\dot{Q}(0). \tag{2.1.5}$$
eq:int2a

Thus, each matrix in $\mathfrak{o}(n)$ is antisymmetric. Conversely, given an antisymmetric matrix A, the curve $t \mapsto e^{tA}$ gives a smooth curve in $\mathsf{O}(n)$ that is tangent to I at $t = 0$. Thus,

$$T_I\mathsf{O}(n) = \mathfrak{o}(n) = \{A \,|A = -A^T\}. \tag{2.1.6}$$
eq:int3

The tangent space at arbitrary $O \in \mathsf{O}(n)$ is obtained by replacing (2.2.2) eq:int2 with the condition that $O^T\dot{O}$ is antisymmetric. Thus,

$$T_O\mathsf{O}(n) = \{OA \,|A \in \mathfrak{o}(n)\}. \tag{2.1.7}$$
eq:int3b

Finally, given $A, \tilde{A} \in \mathfrak{o}(n)$, we define their inner product $\langle A, \tilde{A}\rangle = \operatorname{Tr}(A^T\tilde{A}) = -\operatorname{Tr}(A\tilde{A})$. This inner-product is natural, because it is invariant under left-translation. That is, for two vector $OA, O\tilde{A} \in T_O\mathsf{O}(n)$ we find $\operatorname{Tr}\left((OA)^T(O\tilde{A}\right) = \operatorname{Tr}(A^T\tilde{A})$. The associated volume form on $\mathsf{O}(n)$ is called Haar measure. It is unique, up to a normalizing factor, and we write

$$\mathrm{D}O = 2^{n(n-1)/4} \prod_{1 \le j < k \le n} \mathrm{d}A_{jk}. \tag{2.1.8}$$
eq:haar

Now let $f : \mathsf{O}(n) \to \mathbb{R}$ be a bounded, measurable function. Define a neighborhood of $O \in \mathsf{O}(n)$ by $B_\varepsilon(O) = \{\tilde{O} \in \mathsf{O}(n) : \|O - \tilde{O}\| < \varepsilon\}$. Then for $\varepsilon > 0$, sufficiently small, we can find a diffeomorphism (i.e., a chart) $\varphi_O : U_O \to B_\varepsilon(O) \subset \mathsf{O}(n)$, U_O open satisfying

$$0 \in U_O \subset T_O\mathsf{O}(n), \quad \varphi_O(0) = O \tag{2.1.9}$$

Then for such $\varepsilon > 0$ define

$$\int_{B_\varepsilon(O)} f \mathrm{D}O = 2^{n(n-1)/4} \int_{\varphi_O^{-1}(B_\varepsilon(O))} f(\varphi_O(A)) \prod_{1 \le j < k \le n} \mathrm{d}A_{jk}. \tag{2.1.10}$$

It can be verified that this is independent of the choice of φ_O. So, now consider such mapping at the identity, φ_I. And choose

$$\varphi_O(A) = O\varphi_I(O^T A). \tag{2.1.11}$$

We find

$$\int_{B_\varepsilon(O)} f \mathrm{D}O = 2^{n(n-1)/4} \int_{O\varphi_I^{-1}(B_\varepsilon(I))} f(O\varphi_I(O^T A)) \prod_{1 \le j < k \le n} \mathrm{d}A_{jk}. \tag{2.1.12}$$

We use the fact that O furnishes an isometry from $T_I\mathsf{O}(n)$ to $T_O\mathsf{O}(n)$ so that

$$\int_{B_\varepsilon(O)} f \mathrm{D}O = 2^{n(n-1)/4} \int_{\varphi_I^{-1}(B_\varepsilon(I))} f(O\varphi_I(A)) \prod_{1\le j<k\le n} \mathrm{d}A_{jk}. \tag{2.1.13}$$

In particular, if we choose $f \equiv 1$, then $\int_{B_\varepsilon(O)} \mathrm{D}O$ does not depend on $O \in \mathsf{O}(n)$, showing that this is indeed uniform measure on $\mathsf{O}(n)$.

2.2 Weyl's formula on $\mathsf{Symm}(n)$

Let us now recall some basic facts about $\mathsf{Symm}(n)$. Each matrix $M \in \mathsf{Symm}(n)$ has n real eigenvalues and an orthonormal basis of real eigenvectors. We write Λ for the matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ of eigenvalues, and Q for a matrix whose k-th column is a normalized eigenvector of M associated to the eigenvalue λ_k, $1 \le k \le n$. Since the columns of Q are orthogonal and normalized to length 1, it is immediate that $Q \in \mathsf{O}(n)$. Thus,

$$MQ = Q\Lambda \quad \text{and} \quad M = Q\Lambda Q^T. \tag{2.2.1}$$

eq:int1

In what follows, we will view the transformation $M \mapsto (\Lambda, Q)$ as a change of variables, from $\mathsf{Symm}(n) \to \mathbb{R}^n \times \mathsf{O}(n)$. Strictly speaking, this change of variables is not well-defined since (eq:int1 2.2.1) is unaffected if we replace the k-th column Q_k of Q by $-Q_k$. This issue is considered more carefully in Lemma le:coords 3 and Lemma le:coords-pert 5 below. In a loose sense, diagonalization is analogous to polar coordinates in $\mathbb{R}^n$,

$$\mathbb{R}^n \to [0,\infty) \times S^{n-1}, \quad x \mapsto (r,u)\,, \quad r = |x|, u = \frac{x}{r}. \tag{2.2.2}$$

eq:int2

Polar coordinates are natural for rotation invariant probability density on $\mathbb{R}^n$. For example, the standard Gaussian measure on $\mathbb{R}^n$ may be written

$$e^{-\frac{|x|^2}{2}} \mathrm{D}x = C_n e^{-\frac{r^2}{2}} r^{n-1}\,\mathrm{d}r\,\mathrm{D}u, \tag{2.2.3}$$

eq"int3

where $\mathrm{D}u$ denotes the normalized $n-1$-dimensional measure on S^{n-1} and C_n is a universal constant. The factor r^{n-1} is the Jacobian of this transformation. Weyl's formula shows that the Jacobian for (eq:int1 2.2.1) is $|\triangle(\Lambda)|$. The proof of Weyl's formula relies on an orthogonal decomposition of $T_M\mathsf{Symm}(n)$.

le:decomp **Lemma 2.** *Let M have distinct eigenvalues. Then*

$$T_M\mathsf{Symm}(n) \cong \mathbb{R}^n \oplus \mathfrak{o}(n). \tag{2.2.4}$$

eq:tang-isomorph

with respect to the Hilbert–Schmidt inner product.

In what sense are $\dot\Lambda$ and $\dot Q$ orthogonal? I guess $\dot Q$ has zero diagonals, so this is true in the sense of Hilbert-Schmidt.

I'm a bit confused by the following: If $\Lambda(0) = 0$, then $\dot M = \dot\Lambda$. But then my mapping from $\mathbb{R}^n \oplus \mathfrak{o}(n) \to T_M\mathsf{Symm}(n)$ is $(\dot\Lambda, \dot Q) \to \dot\Lambda$ which is not invertible, which tells me they are not isomorphic. So I added the additional assumption.

Proof. We first assume that $M = \Lambda$ is diagonal. Consider a smooth curve $(-a,a) \to \mathsf{Symm}(n)$, $a > 0$, $t \mapsto M(t) = Q(t)\Lambda(t)Q(t)^T$ such that $M(0) =$

$\Lambda(0) = \Lambda$, and $Q(0) = I$. We differentiate[2] this expression with respect to t and evaluate it at $t = 0$ to find the following expression for a tangent vector in $T_\Lambda \mathsf{Symm}(n)$:

$$\dot{M} = \dot{\Lambda} + [\dot{Q}, \Lambda]. \tag{2.2.5}$$

eq:tang1

Here $\dot{\Lambda}$ can be an arbitrary diagonal matrix, and $\dot{Q}$ an arbitrary antisymmetric matrix. By the assumption of distinct eigenvalues, given $\dot{M}$, $\dot{\Lambda} = \text{diagonal}(\dot{M})$ and Λ, $\dot{Q}$ is uniquely determined. Since the diagonal terms in the commutator $[\dot{Q}, \Lambda]$ vanish, of $\dot{Q}$ vanish, $\dot{Q}$ and $\dot{\Lambda}$ are orthogonal with respect to the Hilbert–Schmidt inner product. Thus,

$$T_\Lambda \mathsf{Symm}(n) \cong \mathbb{R}^n \oplus \mathfrak{o}(n). \tag{2.2.6}$$

eq:tang2a

When $M = QMQ^T$ is not diagonal, we consider a curve $M(t)$ as above, with $M(0) = M$, $\Lambda(0) = \Lambda$ and $Q(0) = Q$. Now equation (2.2.5) is replaced by

$$\dot{M} = Q\left(\dot{\Lambda} + [Q^T\dot{Q}, \Lambda]\right)Q^T. \tag{2.2.7}$$

eq:tang2

The matrices $Q^T\dot{Q}$ are antisymmetric and span $\mathfrak{o}(n)$. Again, $\dot{Q}$ is uniquely determined by $\dot{M}, \dot{\Lambda}$ and Λ. Moreover, the matrices $[Q^T\dot{Q}, \Lambda]$ and $\dot{\Lambda}$ are orthogonal as before. For arbitrary $\dot{\Lambda}$ and A we find $M(t) := Q\mathrm{e}^{tA}(\Lambda + t\dot{\Lambda})\mathrm{e}^{-tA}Q^T$ is a smooth curve in $\mathsf{Symm}(n)$, satisfying $M(0) = M$.

Lecture Note 1. In the above calculation we have implicitly assumed that $t \to \Lambda(t)$ and $t \to Q(t)$ are also smooth. Certainly, such smooth curves exist. If one only assumes that $M(t)$ is smooth but ignores the distinct eigenvalue assumption, because $M(t)$ is always symmetric perturbation theory gives that the eigenvalues (which are ordered) and associated projections can be chosen to be differentiable functions of t [Kat95, Theorem 5.4]. But note that symmetry is actually unnecessary as we have assumed distinct eigenvalues. The main point is that one can build a matrix $Q(t)$ by applying the projections to the standard basis, allowing the computation of $\dot{\Lambda}$ and $\dot{Q}$, in a well-defined way.

Lecture Note 2. The fact that the commutator $[Q^T\dot{Q}, \Lambda]$ must be symmetric implies that $(Q^T\dot{Q})_{ij} + (Q^T\dot{Q})_{ji} = 0$ for $i \neq j$ and more specifically for $\lambda_i \neq \lambda_j$. The diagonal entries of this product must vanish. So, for given distinct eigenvalues we can define the mapping

$$\dot{M} \mapsto \left(\text{diagonal}(Q\dot{M}Q^T), Q^T\dot{Q}\right). \tag{2.2.8}$$

This maps $T_M \mathsf{Symm}(n)$ onto $\mathbb{R}^n \oplus \mathfrak{o}(n)$. The inverse map is, of course, given by

$$Q\left(\dot{\Lambda} + [\dot{A}, \Lambda]\right)Q^T, \quad (\dot{\Lambda}, \dot{A}) \in \mathbb{R}^n \oplus \mathfrak{o}(n).$$

□

rem:isospectral

Remark 18. The proof of Lemma 2 reveals that all matrices of the form

I changed this statement, it used to read $Q\left([Q^T\dot{Q}, \Lambda]\right)Q^T$ but this is a traceless matrix.

[2] Differentiability is guaranteed by classical perturbation theory [Kat95, Theorem 5.4].

$$M+\int_0^t Q(s)\left([Q(s)^T\dot{Q}(s),\Lambda]\right)Q(s)^T\,\mathrm{d}s \tag{2.2.9}$$

lie on an isospectral manifold – i.e. a manifold of matrices in $\mathsf{Symm}(n)$ with the same spectrum as Λ. And if one makes the ansatz $Q(t)=e^{tA}$ for an antisymmetric matrix A, one has

$$\dot{M}=[A,M]. \tag{2.2.10}$$

Lecture Note 3. Conversely, if $\dot{M}=[A,M]$ for a skew-symmetric matrix A then $M(t)$ is symmetric if $M(0)$ is. And the trace of all powers are conserved, using the cyclic properties of the trace,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}M^k(t)&=\operatorname{Tr}\sum_{j=1}^{k}M^{j-1}(t)\dot{M}(t)M^{k-j}(t)=k\operatorname{Tr}\dot{M}(t)M^{k-1}(t)\\&=k\operatorname{Tr}[A,M(t)]M^{k-1}(t)=k\left[A,M^k(t)\right]=0.\end{aligned} \tag{2.2.11}$$

This implies for $\lambda_j=\lambda_j(t)$ we have

$$\begin{pmatrix}1&1&\cdots&1\\2\lambda_1&2\lambda_2&\cdots&2\lambda_n\\\vdots&\vdots&\ddots&\vdots\\n\lambda_1^{n-1}&n\lambda_2^{n-1}&\cdots&n\lambda_n^{n-1}\end{pmatrix}\begin{pmatrix}\dot{\lambda}_1\\\dot{\lambda}_2\\\vdots\\\dot{\lambda}_n\end{pmatrix}=0. \tag{2.2.12}$$

If the eigenvalues are distinct, this matrix is non-singular (see (2.6.6)) and spectrum of M is constant.

Proof of Weyl's formula for $\beta=1$. We now have two coordinate systems on $T_M\mathsf{Symm}(n)$ provided that the eigenvalues of M are distinct. We will show that the set of all symmetric matrices with distinct eigenvalues is open, dense and of full Lebesgue measure (see Lemma 5 and Corollary 1). The coordinates ξ_α, $1\le\alpha\le p$ give the metric (2.1.3). A coordinate system, which is always locally defined, is $\left(\dot{\Lambda},\dot{A}\right)$, where $\dot{\Lambda}$ is a diagonal matrix and $\dot{A}$ is an antisymmetric matrix. We use (2.2.7) to find the infinitesimal length element in this coordinate system. On the subset of $\mathsf{Symm}(n)$ consisting of matrices with distinct eigenvalues, using that M is symmetric, and $Q^T\mathrm{d}Q=\mathrm{d}A$, $\quad A\in\mathfrak{o}(n)$,

$$\begin{aligned}\operatorname{Tr}\mathrm{d}M^2&=\operatorname{Tr}(\mathrm{d}M)^T\mathrm{d}M=\operatorname{Tr}Q(\mathrm{d}\Lambda+[\mathrm{d}A,\Lambda])^T(\mathrm{d}\Lambda+[\mathrm{d}A,\Lambda])Q^T\\&=\operatorname{Tr}\mathrm{d}\Lambda^2+2\operatorname{Tr}\mathrm{d}\Lambda[\mathrm{d}A,\Lambda]+\operatorname{Tr}[\mathrm{d}A,\Lambda]^2\\&=\operatorname{Tr}\mathrm{d}\Lambda^2+\operatorname{Tr}[\mathrm{d}A,\Lambda]^2.\end{aligned} \tag{2.2.13}$$

Expanding out this last trace, we find

$$\begin{aligned}
\operatorname{Tr}[\mathrm{d}A,\Lambda]^2 &= \operatorname{Tr}(\mathrm{d}A\Lambda - \Lambda\mathrm{d}A)^2 \\
&= \operatorname{Tr}\mathrm{d}A\Lambda\mathrm{d}A\Lambda + \operatorname{Tr}\Lambda\mathrm{d}A\Lambda\mathrm{d}A - \operatorname{Tr}\Lambda\mathrm{d}A^2\Lambda - \operatorname{Tr}\mathrm{d}A\Lambda^2\mathrm{d}A \\
&= 2\sum_{j=1}^{n}\sum_{k=1}^{n}\mathrm{d}A_{jk}\mathrm{d}A_{kj}\lambda_k\lambda_j - \sum_{j=1}^{n}\sum_{k=1}^{n}\mathrm{d}A_{jk}\mathrm{d}A_{kj}\lambda_j^2 - \sum_{j=1}^{n}\sum_{k=1}^{n}\mathrm{d}A_{jk}\mathrm{d}A_{kj}\lambda_k^2 \\
&= 2\sum_{j<k}(\lambda_j-\lambda_k)^2\mathrm{d}A_{jk}^2.
\end{aligned} \tag{2.2.14}$$

Therefore

$$\mathrm{d}s^2 = \operatorname{Tr}(\mathrm{d}M^2) = \sum_{j=1}^{n}\mathrm{d}\lambda_j^2 + 2\sum_{1\le j<k\le n}(\lambda_j-\lambda_k)^2\mathrm{d}A_{jk}^2. \tag{2.2.15}$$

eq:weyl-metric

Thus, the metric tensor in these coordinates is a diagonal matrix in $\mathsf{Symm}_+(p)$ that takes the value 1 on the first n coordinates, and the value $2(\lambda_j-\lambda_k)^2$ for each term A_{jk}. By (B.2.2), the volume form is

eq:inp2

$$\mathrm{D}M = 2^{n(n-1)/4}\prod_{j=1}^{n}\mathrm{d}\lambda_j\prod_{1\le j<k\le n}|\lambda_j-\lambda_k|\,\mathrm{d}A_{jk} = |\triangle(\Lambda)|\,\mathrm{D}\Lambda\,\mathrm{D}O. \tag{2.2.16}$$

eq:weyl-volume

□

To interpret Weyl's formula, in a neighborhood U_M of a matrix with distinct eigenvalues, one needs to construct an invertible map $\phi(M) = (\Lambda, Q)$ from symmetric matrices in this neighborhood to these "spectral" variables. Then for f with compact support in U_M

$$\int f(M)\mathrm{D}M = \int_{\phi(U_M)} f(Q\Lambda Q^T)|\triangle(\Lambda)|\mathrm{D}\Lambda\mathrm{D}O. \tag{2.2.17}$$

We now work to understand how to define such a map, and why matrices with repeated eigenvalues do not cause further issues.

2.3 Diagonalization as a change of coordinates

Some care is needed when treating the map $M \to (\Lambda, Q)$ as a change of variables. First, the map is not even well-defined in general, since the sign of each normalized eigenvector is arbitrary. Second, even if we fix the signs, the choice of eigenvectors is degenerate when M has repeated eigenvalues. Third, Λ is not uniquely defined if we do not specify an ordering of the eigenvalues. The following lemmas address this issue. Define the Weyl chamber

$$\mathcal{W}^n = \{\Lambda \in \mathbb{R}^n \,|\lambda_1 < \lambda_2 < \ldots < \lambda_n\}. \tag{2.3.1}$$

eq:weyl-chamber

le:coords **Lemma 3.** *Assume $M_0 \in \mathsf{Symm}(n)$ has distinct eigenvalues. Then there exists $\varepsilon > 0$ such that for each $s \in \{\pm 1\}^n$, there is a C^∞ map*

$$h^{(s)} : B_\varepsilon(M_0) \to \mathcal{W}^n \times \mathsf{O}(n), \quad M \mapsto \left(\Lambda,\, Q^{(s)}\right)$$

that is a C^∞ diffeomorphism onto its image.

Proof of Lemma 3. An outline of the proof is presented. The remaining details are left to the exercises. Standard perturbation theory (see [Kat95], for example) demonstrates that the map is C^∞. The choice of s corresponds to fixing the signs of the eigenvectors as follows. Let a basis of normalized eigenvectors of M_0 be fixed. Call the associated matrix of eigenvectors Q_0. For each s, let $Q_0^{(s)} = \mathrm{diag}(s_1, \ldots, s_n) Q_0$. Each $Q_0^{(s)}$ is also an eigenvector matrix for M_0. Since the eigenvalues of M are distinct, we may use the implicit function theorem to solve the algebraic equations that determine the eigenvalues and eigenvectors, in a way that is consistent with the choice of s. □

l:hoffman **Lemma 4** (Weilandt–Hoffman inequality)**.** *Let $M_1, M_2 \in \mathsf{Symm}(n)$ and use $\lambda_j(M_i)$ to denote the jth eigenvalue (in increasing order) of M_i. Then*

$$\sum_{j=1}^{n} |\lambda_j(M_1) - \lambda_j(M_2)|^2 \le \|M_1 - M_2\|^2.$$

Proof. See [Tao11, Section 1.3] for a particularly nice proof. □

le:coords-pert **Lemma 5.** *Assume that $M \in \mathsf{Symm}(n)$ has a repeated eigenvalue. Then for every $\varepsilon > 0$ there exists $M_\varepsilon \in \mathsf{Symm}(n)$, such that $\|M - M_\varepsilon\| < \varepsilon$ and M_ε has distinct eigenvalues. Furthermore, the set of all matrices in $\mathsf{Symm}(n)$ with distinct eigenvalues is open.*

Proof. Exercise. □

Lemma 3 shows that the map $M \mapsto (\Lambda, Q)$ provides a local coordinate system near each matrix with distinct eigenvalues. Lemma 5 shows that set of such matrices is dense. As has been noted, more is true. The set of all matrices with both distinct eigenvalues and non-vanishing first entries in its eigenvectors is of full measure. This follows from (3.3.3) and Lemma 8 below. One has to note that the procedure of reducing a full matrix to a tridiagonal matrix that is used to establish (3.3.3) does not affect the first row of the eigenvector matrix.

In fact, Weyl's formula shows that the set of $M \in \mathsf{Symm}(n)$ with repeated eigenvalues and at least one eigenvector with a zero first component has measure zero with respect to $\mathrm{D}M$. Let $O_> = \{O \in \mathsf{O}(n) \mid O_{1j} > 0, \;\; j = 1, \ldots, n\}$ and $W_R = \{M = O\Lambda O^T \in \mathsf{Symm}(n) \mid -R < \lambda_1 < \lambda_2 < \cdots < \lambda_n < R, O \in O_>\}$. Then define $\varphi(M) = (\Lambda, Q)$ uniquely by the convention that the first non-zero entry in each column of Q is positive. Then we find

$$\int_{W_R} \mathrm{D}M = \int_{\mathcal{W}_R} |\triangle(\Lambda)| \mathrm{D}\Lambda \left(\int_{O_>} \mathrm{D}O \right), \tag{2.3.2}$$

$$\mathcal{W}_R = \{\Lambda \in \mathbb{R}^n \mid -R < \lambda_1 < \lambda_2 < \cdots < \lambda_n < R\}. \tag{2.3.3}$$

But

$$\int_{\mathcal{W}_R} |\triangle(\Lambda)| \mathrm{D}\Lambda \left(\int_{O_>} \mathrm{D}O \right) = \int_{\overline{\mathcal{W}}_R} |\triangle(\Lambda)| \mathrm{D}\Lambda \left(\int_{\overline{O}_>} \mathrm{D}O \right). \tag{2.3.4}$$

This then implies, after changing variables back to M,

$$\int_{W_R} \mathrm{D}M = \int_{\overline{W}_R} \mathrm{D}M. \tag{2.3.5}$$

This then implies that for $S = \{M \in \mathsf{Symm}(n) | \lambda_j = \lambda_i \text{ for some } i \neq j\}$ has measure zero. To see that

$$\int_{O_>} \mathrm{D}O = \int_{\overline{O}_>} \mathrm{D}O,$$

make a local change of variables to the tangent space in the neighborhood of a matrix $O \in \overline{O}_> \setminus O_>$ via $\varphi_O(A) = O(I - A)(A + I)^{-1}$. For O fixed, the condition $e_1^T \varphi_O(A) e_1 = 0$ is the zero set of a function that is real-analytic in the components of A. If it vanished on a set of positive measure (Lebesgue measure on the entries of A), it would have to vanish identically. This shows the measure of $\overline{O}_> \setminus O_>$ that lies in this neighborhood is zero. Compactness of $\overline{O}_> \setminus O_>$ can then be used to prove it is measure zero.

2.4 Independence and Invariance implies Gaussian

Fix $M \in \mathsf{Symm}(n)$ with spectrum $\sigma(M)$. Fix an interval $(a, b) \subset \mathbb{R}$ and let $\mathsf{Symm}(n)_{(a,b)}$ denote the set of $M \in \mathsf{Symm}(n)$ with spectrum $\sigma(M) \subset (a, b)$. Each function $f : (a, b) \to \mathbb{R}$ extends naturally to a map $\mathsf{Symm}(n)_{(a,b)} \to \mathsf{Symm}(n)$ as follows:

$$f(M) = Qf(\Lambda)Q^T, \quad M = Q\Lambda Q^T, \quad f(\Lambda) = \mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n)). \tag{2.4.1}$$

eq:class-function

Clearly, $\mathrm{Tr}(f(M)) = \mathrm{Tr}(f(\Lambda)) = \sum_{j=1}^n f(\lambda_j)$. Each $f : \mathbb{R} \to \mathbb{R}$ that grows sufficiently fast as $x \to \pm\infty$ defines an invariant distribution on $\mathsf{Symm}(n)$

$$\mu(\mathrm{D}M) = \frac{1}{Z} \exp\left(-\mathrm{Tr}(f(M))\right) \mathrm{D}M. \tag{2.4.2}$$

eq:class-function-invariant

This is the most general form of an invariant probability distribution.

By contrast, a Wigner distribution relies on independence of the entries of M. This means that if a Wigner distribution has a density, then it must be of the form

$$\mu(\mathrm{D}M) = \frac{1}{Z} \left(\prod_{j=1}^n f_j(M_{jj}) \prod_{1 \le j < k \le n} f_{jk}(M_{jk}) \right) \mathrm{D}M. \tag{2.4.3}$$

eq:wigner-general

thm:gauss **Theorem 19.** *Assume a probability measure μ on $\mathsf{Symm}(n)$ is both a Wigner distribution and an invariant distribution. Assume further that $\mu(\mathrm{D}M)$ has a strictly positive, smooth density of the form (2.4.2) and (2.4.3). Then $\mu(\mathrm{D}M)$ is a Gaussian ensemble,*

$$\mu(\mathrm{D}M) = \frac{1}{Z} \mathrm{e}^{-\frac{1}{2\sigma^2}\operatorname{Tr}(M-\gamma I)^2} \mathrm{D}M, \tag{2.4.4}$$

eq:prob-gauss

with variance σ^2 and mean γI, for some $\gamma \in \mathbb{R}$.

Proof. We first illustrate the essential calculation for 2×2 matrices. Suppose

$$\mu(\mathrm{D}M) = p(M)\,\mathrm{D}M = \frac{1}{Z} f(M_{11}) g(M_{22}) h(M_{12}) \mathrm{d}M_{11} \mathrm{d}M_{12} \mathrm{d}M_{22}. \tag{2.4.5}$$

eq:gauss2

We compute the variation in μ along an isospectral curve (see Remark 18). Consider the curve $M(t) = Q(t) M Q(t)^T$ with

$$Q(t) = \mathrm{e}^{tR}, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.4.6}$$

The matrix R spans $\mathfrak{so}(2)$. We differentiate $M(t)$ with respect to t to obtain

$$\dot{M}(0) = [R, M] = \begin{pmatrix} -2M_{12} & M_{11} - M_{22} \\ M_{11} - M_{22} & 2M_{12} \end{pmatrix} \tag{2.4.7}$$

eq

Thus, the infinitesimal change in the density $p(M(t))$ is

$$\begin{aligned} \frac{1}{p} \frac{\mathrm{d}p}{\mathrm{d}t}\bigg|_{t=0} &= \frac{f'(M_{11})}{f(M_{11})} \dot{M}_{11} + \frac{g'(M_{22})}{g(M_{22})} \dot{M}_{22} + \frac{h'(M_{12})}{h(M_{12})} \dot{M}_{12} \\ &= -2M_{12} \left(\frac{f'(M_{11})}{f(M_{11})} - \frac{g'(M_{22})}{g(M_{22})} \right) + (M_{11} - M_{22}) \frac{h'(M_{12})}{h(M_{12})}. \end{aligned} \tag{2.4.8}$$

On the other hand, since $\mu(\mathrm{D}M)$ is invariant, $p(M(t)) = p(M)$ and

$$\frac{\mathrm{d}p}{\mathrm{d}t}\bigg|_{t=0} = 0. \tag{2.4.9}$$

eq:gauss3a

We equate (2.4.8) and (2.4.9), and separate variables to obtain

$$\frac{1}{M_{11} - M_{22}} \left(\frac{f'(M_{11})}{f(M_{11})} - \frac{g'(M_{22})}{g(M_{22})} \right) = c = \frac{1}{2M_{12}} \frac{h'(M_{12})}{h(M_{12})}, \tag{2.4.10}$$

eq:gauss4

for some constant $c \in \mathbb{R}$. Equation (2.4.10) immediately implies that

$$h(M_{12}) = h(0) \mathrm{e}^{cM_{12}^2}. \tag{2.4.11}$$

eq:gauss5

Separating variables again in (2.4.10), we find with a second constant $b \in \mathbb{R}$,

$$\frac{f'}{f} = cM_{11} + b, \quad \frac{g'}{g} = cM_{22} + b, \tag{2.4.12}$$

eq:gauss6

which integrates to

$$f(M_{11}) = f(0)\mathrm{e}^{\frac{cM_{11}^2}{2}}\mathrm{e}^{bM_{11}}, \quad g(M_{22}) = g(0)e^{\frac{cM_{22}^2}{2}}\mathrm{e}^{bM_{22}}. \tag{2.4.13}$$

eq:gauss7

We combine all the terms to obtain

$$p(M) = f(0)g(0)h(0)\mathrm{e}^{c\frac{\mathrm{Tr}(M^2)}{2}}\mathrm{e}^{b\,\mathrm{Tr}(M)}. \tag{2.4.14}$$

eq:gauss8

Since $p(M)$ integrates to 1, we must have $c < 0$, say $c = -1/\sigma^2$. The scalar b is arbitrary and contributes a shift in the mean that is a scalar multiple of I. The combination of constants $f(0)g(0)h(0)$ may be absorbed into the normalization constant Z^{-1}. We have thus proved Theorem 19 for $n = 2$.

In order to prove Theorem 19 for arbitrary n we generalize the above argument as follows. Fix a pair of off-diagonal indices $1 \leq l < m \leq n$. We consider a rotation in $\mathbb{R}^n$ that rotates the $x_l x_m$ plane as above, and leaves the other coordinates invariant. This entails replacing the matrix R in the argument above with the matrix $R^{lm} \in \mathfrak{so}(n)$ with coordinates $R^{lm}_{jk} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$. The argument above now shows that the density of p in the M_{ll}, M_{lm} and M_{mm} coordinates is a Gaussian distribution of the form (2.4.14):

$$p(M^{lm}) = \mathrm{e}^{c\frac{\mathrm{Tr}((M^{lm})^2)}{2}}\mathrm{e}^{b\,\mathrm{Tr}(M^{lm})}, \tag{2.4.15}$$

eq:gd

where M^{lm} denotes the 2×2 matrix

$$M^{lm} = \begin{pmatrix} M_{ll} & M_{lm} \\ M_{lm} & M_{mm} \end{pmatrix}.$$

At this stage, the constants c and b depend on l and m. But now note that since the same argument applies to <u>every</u> pair of indices $1 \leq l < m \leq n$, the constants c and b must be independent of l and m. □

Lecture Note 4. Then

$$(R^{lm})_{jk} = \begin{cases} 0 & (j,k) \neq (l,m) \text{ or } (k,j) \neq (l,m), \\ 1 & j = l,\ k = m, \\ -1 & j = m,\ k = l. \end{cases} \tag{2.4.16}$$

Then for $M(t) = Q(t)MQ(t)^T$, $Q(t) = e^{tR}$ we have

$$\begin{aligned} &\dot{M}(0) = [M, R^{lm}], \\ &(MR^{lm})_{jk} = \sum_{\ell=1}^n M_{j\ell}(R^{lm})_{\ell k} = \begin{cases} 0 & k \neq l, m, \\ M_{jl} & k = m, \\ -M_{jm} & k = l, \end{cases} \\ &(R^{lm}M)_{jk} = \sum_{\ell=1}^n (R^{lm})_{j\ell}M_{\ell k} = \begin{cases} 0 & j \neq l, m, \\ -M_{lk} & j = m, \\ M_{mk} & j = l. \end{cases} \end{aligned} \tag{2.4.17}$$

Then we compute

$$C_{jk} := ([M, R^{lm}])_{jk} = \begin{cases} 0 & j \neq l, m \text{ and } k \neq l, m, \\ 2M_{ml} & k = m,\ j = m, \\ -2M_{ml} & k = l,\ j = l, \\ M_{ll} - M_{mm} & k = m, j = l, \\ M_{mm} - M_{ll} & k = l, l = m, \\ M_{jl} & k = m,\ j \neq l, m, \\ -M_{jm} & k = l,\ j \neq l, m, \\ -M_{lk} & j = m,\ k \neq l, m, \\ M_{mk} & j = l,\ k \neq l, m. \end{cases} \tag{2.4.18}$$

Differentiating the density, using invariance, we find

$$0 = \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}\bigg|_{t=0} = \sum_{1 \leq j \leq k \leq n} C_{jk} \frac{f'_{jk}(M_{jk})}{f_{jk}(M_{jk})}, \qquad f_{jj} = f_j. \tag{2.4.19}$$

Separating out terms, using $l < m$, we find

$$\begin{aligned} &C_{ll}\frac{f'_l(M_{ll})}{f_l(M_{ll})} + C_{mm}\frac{f'_m(M_{mm})}{f_m(M_{mm})} + C_{lm}\frac{f'_{lm}(M_{lm})}{f_{lm}(M_{lm})} \\ &\qquad + \sum_{k=l+1, k\neq m}^{n} C_{lk}\frac{f'_{lk}(M_{lk})}{f_{lk}(M_{lk})} + \sum_{j=1, j\neq l}^{m-1} C_{jm}\frac{f'_{jm}(M_{jm})}{f_{jm}(M_{jm})} = 0. \end{aligned} \tag{2.4.20}$$

Using the expressions for C_{jk} we have

$$\begin{aligned} &2M_{ml}\left(\frac{f'_m(M_{mm})}{f_m(M_{mm})} - \frac{f'_l(M_{ll})}{f_l(M_{ll})}\right) + (M_{ll} - M_{mm})\frac{f'_{lm}(M_{lm})}{f_{lm}(M_{lm})} \\ &\qquad + \sum_{k=l+1, k\neq m}^{n} M_{mk}\frac{f'_{lk}(M_{lk})}{f_{lk}(M_{lk})} + \sum_{j=1, j\neq l}^{m-1} M_{jl}\frac{f'_{jm}(M_{jm})}{f_{jm}(M_{jm})} = 0 \end{aligned} \tag{2.4.21}$$

Then one separates variables. The last line of this equation contains only variables M_{jk} that are not included on the first line of this equation. So, we may separate variables as in the 2×2 case to find

$$f_l(M_{ll}) f_m(M_{mm}) f_{lm}(M_{lm})$$

is given by (2.4.15) eq:gd. It is straightforward to argue that if this is true for every pair (l, m) then c and b cannot depend on l or m.

2.5 Integration on $\mathsf{Her}(n)$ and $\mathsf{U}(n)$

sec:weyl-hermite

The space of Hermitian matrices $\mathsf{Her}(n)$ is a vector-space of real dimension n^2, as may be seen by the isomorphism $\mathsf{Her}(n) \to \mathbb{R}^{n^2}$,

$$M \mapsto (M_{11}, \ldots, M_{nn}, \mathsf{Re}M_{12}, \ldots, \mathsf{Re}M_{n-1,n}, \mathsf{Im}M_{12}, \ldots, \mathsf{Im}M_{n-1,n}). \tag{2.5.1}$$

eq:herm1

The Hilbert-Schmidt inner product on $\mathsf{Her}(n)$ is

$$\mathsf{Her}(n) \times \mathsf{Her}(n) \to \mathbb{C}, \quad (M,N) \mapsto \mathrm{Tr}(M^*N). \tag{2.5.2}$$

eq:herm2

The associated infinitesimal length element is

$$\mathrm{d}s^2 = \mathrm{Tr}(\mathrm{d}M^2) = \sum_{j=1}^n \mathrm{d}M_{jj}^2 + 2\sum_{1\le j<k\le n}\left(\mathrm{d}\,\mathsf{Re}M_{jk}^2 + \mathrm{d}\,\mathsf{Im}M_{jk}^2\right). \tag{2.5.3}$$

eq:herm3

Thus, in the coordinates ξ, the metric is an $n^2 \times n^2$ diagonal matrix whose first n entries are 1 and all other entries are 2. We apply (2.2.1) to obtain the volume form on $\mathsf{Her}(n)$

$$\mathrm{D}M = 2^{n(n-1)/2}\prod_{j=1}^n \mathrm{d}M_{jj}\prod_{1\le j<k\le n}\mathrm{d}\,\mathsf{Re}M_{jk}\,\mathrm{d}\,\mathsf{Im}M_{jk}. \tag{2.5.4}$$

eq:herm4

The unitary group, $\mathsf{U}(n)$ is the group of linear isometries of $\mathbb{C}^n$ equipped with the standard inner-product $\langle x,y\rangle = x^*y$. Thus, $\mathsf{U}(n)$ is equivalent to the group of matrices $U \in \mathbb{C}^{n\times n}$ such that $U^*U = I$. The inner-product (2.5.3) and volume form (2.5.4) are invariant under the transformation $M \mapsto UMU^*$.

The Lie algebra $\mathfrak{u}(n)$ is computed as in Section 2.1. We find

$$\mathfrak{u}(n) = T_I\mathsf{U}(n) = \left\{A \in \mathbb{C}^{n\times n}\,|A = -A^*\right\}, \quad T_U\mathsf{U}(n) = \{UA \,|\, A \in \mathfrak{u}(n)\}. \tag{2.5.5}$$

eq:herm5

The transformation $M \mapsto \mathrm{i}M$ is an isomorphism between Hermitian and anti-Hermitian matrices. In fact, the map $\mathsf{Her}(n) \to \mathsf{U}(n)$, $M \mapsto \mathrm{e}^{\mathrm{i}M}$ is onto and locally one-to-one. The inner-product $A, \tilde{A} \mapsto \mathrm{Tr}(A^*\tilde{A})$ is invariant under left application of $\mathsf{U}(n)$. Thus, we obtain the volume form for Haar measure on $\mathsf{U}(n)$

$$\mathrm{D}\tilde{U} = 2^{n(n-1)/2}\prod_{j=1}^n \mathrm{d}A_{jj}\prod_{1\le j<k\le n}\mathrm{d}\,\mathsf{Re}A_{jk}\,\mathrm{d}\,\mathsf{Im}A_{jk}. \tag{2.5.6}$$

eq:herm6

However, when viewing diagonalization $M \mapsto U\Lambda U^*$ as a change of variables on $\mathsf{Her}(n)$, it is necessary to quotient out the following degeneracy: For each $\theta = (\theta_1,\ldots,\theta_n) \in \mathbb{T}^n$, the diagonal matrix $D = \mathrm{diag}\left(e^{i\theta_1},\ldots,e^{i\theta_n}\right)$ is unitary and $M = U\Lambda U^*$ if and only if $M = UD\Lambda D^*U^*$. Thus, for $\mathsf{Her}(n)$, the measure $D\tilde{U}$ must be replaced by Haar measure a measure on $\mathsf{U}(n)/\mathbb{T}^n$. The form of this measure on Haar measure on $\mathsf{U}(n)/\mathbb{T}^n$ follows from the following assertion, which is proved as in Section 2.1.

le:herm-decomp

Lemma 6. *Each matrix $\dot{M} \in T_M\mathsf{Her}(n)$ is of the form*

$$\dot{M} = U\left(\dot{\Lambda} + [U^*\dot{U},\Lambda]\right)U^*, \quad \dot{\Lambda} \in T_\Lambda\mathbb{R}^n, \quad \dot{U} \in T_U\mathsf{U}(n), \quad \mathrm{diagonal}(U^*\dot{U}) = 0. \tag{2.5.7}$$

eq:herm7

The matrices $\dot{\Lambda}$ and $U^*\dot{U}$ are orthogonal under the inner-product (2.5.2).

Thus, the volume form on the quotient $\mathsf{U}(n)/\mathbb{T}^n$ is locally equivalent to a volume form on the subspace of anti-Hermitian matrices consisting of matrices with zero diagonal:

$$\mathrm{D}U = 2^{n(n-1)/2} \prod_{1\le j<k\le n} \mathrm{d}\,\mathsf{Re}A_{jk}\,\mathrm{d}\,\mathsf{Im}A_{jk}. \tag{2.5.8}$$

eq:herm6

Furthermore, $B \mapsto \varphi(B) = U\mathrm{e}^{U^*B}$ provides a locally one-to-one mapping from $PT_U\mathsf{U}(n) = UT_I\mathsf{U}(n)$ to $\mathsf{U}(n)/\mathbb{T}^n$.

le:herm-decomp

Lemma 6 shows that the mapping $\mathbb{R}^n \oplus PT_I\mathsf{U}(n) \to T_M\mathsf{Her}(n)$, $PT_I\mathsf{U}(n) = \{A \in T_I\mathsf{U}(n) \mid \mathrm{diag}(A) = 0\}$, defined by $(\dot\Lambda, \dot A) \mapsto U(\dot\Lambda + [\dot A, \Lambda])U^*$ maps onto $T_M\mathsf{Her}(n)$. Again, the two spaces are isomorphic if M has distinct eigenvalues.

Proof of Weyl's formula for $\beta = 2$. We write, on the subset of $\mathsf{Symm}(n)$ consisting of matrices with distinct eigenvalues, using that M is Hermitian, and $U^*\mathrm{d}U = \mathrm{d}A, \quad A \in T_I\mathsf{U}(n), \mathrm{diag}(A) = 0$,

$$\begin{aligned}\mathrm{Tr}\,\mathrm{d}M^2 &= \mathrm{Tr}\,\mathrm{d}\Lambda^2 + 2\,\mathrm{Tr}\,\mathrm{d}\Lambda[\mathrm{d}A,\Lambda] + \mathrm{Tr}[\mathrm{d}A,\Lambda]^*[\mathrm{d}A,\Lambda]\\ &= \mathrm{Tr}\,\mathrm{d}\Lambda^2 + \mathrm{Tr}[\mathrm{d}A,\Lambda]^*[\mathrm{d}A,\Lambda].\end{aligned} \tag{2.5.9}$$

check

Expanding out this last trace, using that $\mathrm{d}A = \mathrm{d}\mathsf{Re}A + \mathrm{i}\,\mathrm{d}\mathsf{Im}A$, we need only collect the real part

$$\begin{aligned}\mathrm{Tr}[\mathrm{d}A,\Lambda]^*[\mathrm{d}A,\Lambda] &= \mathrm{Tr}(\mathrm{d}\,\mathsf{Re}A)\Lambda(\mathrm{d}\,\mathsf{Re}A)\Lambda + \mathrm{Tr}\,\Lambda(\mathrm{d}\,\mathsf{Re}A)\Lambda(\mathrm{d}\,\mathsf{Re}A)\\ &\quad - \mathrm{Tr}\,\Lambda(\mathrm{d}\,\mathsf{Re}A)^2\Lambda - \mathrm{Tr}(\mathrm{d}\,\mathsf{Re}A)\Lambda^2(\mathrm{d}\,\mathsf{Re}A)\\ &\quad + \mathrm{Tr}(\mathrm{d}\,\mathsf{Im}A)\Lambda(\mathrm{d}\,\mathsf{Im}A)\Lambda + \mathrm{Tr}\,\Lambda(\mathrm{d}\,\mathsf{Im}A)\Lambda(\mathrm{d}\,\mathsf{Im}A)\\ &\quad - \mathrm{Tr}\,\Lambda(\mathrm{d}\,\mathsf{Im}A)^2\Lambda - \mathrm{Tr}(\mathrm{d}\,\mathsf{Im}A)\Lambda^2(\mathrm{d}\,\mathsf{Im}A)\\ &= 2\sum_{j<k}(\lambda_j-\lambda_k)^2\mathrm{d}\mathsf{Re}A_{jk}^2 + 2\sum_{j<k}(\lambda_j-\lambda_k)^2\mathrm{d}\mathsf{Im}A_{jk}^2.\end{aligned} \tag{2.5.10}$$

eq:power2

Then it follows that the associated volume form satisfies

$$\mathrm{D}M = |\triangle(\Lambda)|^2\mathrm{D}\Lambda\mathrm{D}U. \tag{2.5.11}$$

□

2.6 Integration on $\mathsf{Quart}(n)$ and $\mathsf{USp}(n)$

sec:weyl-quart

The field of quaternions, $\mathbb{H}$, is the linear space

$$x = c_0 + c_1\mathfrak{e}_1 + c_2\mathfrak{e}_2 + c_3\mathfrak{e}_3, \quad c_i \in \mathbb{R},\ i = 0,1,2,3, \tag{2.6.1}$$

eq:quart1

equipped with the non-commutative rules of multiplication

$$\mathfrak{e}_1^2 = \mathfrak{e}_2^2 = \mathfrak{e}_3^2 = \mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_3 = -1. \tag{2.6.2}$$

eq:quart2

These rules ensure that the product of any two quaternions is again a quaternion. Each $x \in \mathbb{H}$ has a complex conjugate $\bar{x} = c_0 - c_1\mathfrak{e}_1 - c_2\mathfrak{e}_2 - c_3\mathfrak{e}_3$, and its absolute value $|x|$ is determined by

$$|x|^2 = \bar{x}x = c_0^2 + c_1^2 + c_2^2 + c_3^2. \tag{2.6.3}$$

eq:quart3

Each non-zero $x \in \mathbb{H}$ has a multiplicative inverse $1/x = \bar{x}/|x|^2$. Thus, $\mathbb{H}$ is indeed a field.

The normed linear vector space $\mathbb{H}^n$ consists of vectors $x = (x_1, \ldots, x_n)^T$ with inner product $\langle x, y\rangle = \sum_{j=1}^n \bar{x}_j y_j$. The adjoint, $M^\dagger$ of a linear transformation $M : \mathbb{H}^n \to \mathbb{H}^n$ is defined by the inner-product

$$\langle M^\dagger x, y\rangle := \langle x, My\rangle. \tag{2.6.4}$$

eq:quart4

It follows that the entries of $M^\dagger$ are $M_{jk} = \bar{M}_{kj}$. We say that an operator is self-adjoint if $M = M^\dagger$. It is anti self-adjoint if $M = -M^\dagger$. The space of self-adjoint operators is denoted $\mathsf{Quart}(n)$. We equip this space with the Hilbert-Schmidt norm as before.

The group $\mathsf{USp}(n)$ is the set of linear transformations of $\mathbb{H}^n$ that preserve this inner product. We thus require that for each $x, y \in \mathbb{H}^n$

$$\langle x, y\rangle = \langle Ux, Uy\rangle = \langle U^\dagger Ux, y\rangle. \tag{2.6.5}$$

eq:quart4

Thus, $\mathsf{USp}(n)$ is equivalent to $U \in \mathbb{H}^{n\times n}$ such that $U^\dagger U = I$. As for $\mathsf{U}(n)$ we find that its Lie algebra $\mathfrak{usp}(n)$ is the space of anti self-adjoint matrices. The inner-product on $\mathfrak{usp}(n)$ and Haar measure are defined exactly as in Section 2.5, as is the analogue of Lemma 6 and the Weyl formula. It is also clear from how the proof of Weyl's formula extends to $\beta = 2$, that because the field of quarternions is a four-dimensional space, $|\triangle(\Lambda)|^4$ will arise, see (2.5.10).

Exercises

2.1. Show that

$$\triangle(\Lambda) = \det \begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1}. \end{pmatrix}. \tag{2.6.6}$$

eq:vdm

2.2. The Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.6.7}$$

eq:Pauli-matrix

allow a representation of the quarternions in terms of Hermitian matrices.

(a) Show that the Pauli matrices together with the identity matrix span $\mathsf{Her}(2)$.

(b) Show that the matrices $\{\mathrm{i}\sigma_1, \mathrm{i}\sigma_2, \mathrm{i}\sigma_3\}$ form a basis of $\mathfrak{su}(2)$. (This is the subalgebra of $\mathfrak{u}(2)$ consisting of trace-free matrices).

(c) Verify that if $\mathfrak{e}_j = i\sigma_j$, the rules (2.6.2) hold (replace 1 by I_2).

2.3. The canonical symplectic matrix of size $2n \times 2n$ denoted J_n, or simply J, is the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.6.8}$$

eq:J-def

where 0 and I denote the $n \times n$ zero and identity matrices. The symplectic group $\mathsf{Sp}(2n, \mathbb{R})$ (not to be confused with the unitary symplectic group $\mathsf{USp}(n)$!) is

$$\mathsf{Sp}(2n, \mathbb{R}) = \left\{ S \in \mathbb{R}^{n\times n} \,\middle|\, S^T J S = J \right\}. \tag{2.6.9}$$

eq:Spn

Verify that $\mathsf{Sp}(2n, \mathbb{R})$ is a group and compute its Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$.

2.4. Use the Gaussian integral

$$\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|x|^2}{2}} \, \mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

to compute the $n-1$-dimensional volume ω_{n-1} of the unit sphere S^{n-1}. Determine the asymptotic behavior of ω_{n-1} as $n \to \infty$.
Hint: Do the integral two ways– once in Cartesian coordinates, and once in polar coordinates.

2.5. Assume given a C^1 function $f : (a, b) \to \mathbb{R}$, and extend it to a function $f : \mathsf{Symm}(n) \to \mathsf{Symm}(n)$ as in (2.4.1). Compute the Jacobian of this transformation. Apply this formula to the function $f(x) = e^{ix}$ to compute the analogue of Weyl's formula on $\mathsf{U}(n)$ (note that each $U \in \mathsf{U}(n)$ is of the form e^{iM} for some $M \in \mathsf{Her}(n)$).

2.6. Prove Lemma 4.

2.7. Let $A \in \mathbb{R}^{m\times n}$ for $m < n$. Show that $\{x \mid Ax = 0\} \subset \mathbb{R}^n$ has zero Lebesgue measure.

2.8. Assume $f : \mathbb{R} \to (0, \infty)$ satisfies the functional equation

$$f(x + y) = f(x)f(y), \quad x, y \in \mathbb{R}. \tag{2.6.10}$$

eq:fe

It is easy to check that for each $a \in \mathbb{R}$ functions of the form $f(x) = e^{ax}$ solve (2.6.10). Show that these are the only solutions to (2.6.10) assuming only that f is continuous. (Do not assume that f is differentiable).

Remark 20. The use of row operations in Problem (1) underlies the introduction of orthogonal polynomials. Problems (2) and (3) may be combined to show that $\mathsf{Sp}(2n, \mathbb{C}) \cap \mathsf{U}(n) \cong \mathsf{USp}(n)$. The approach in Problem (4) yields the volume of $\mathsf{O}(n)$, $\mathsf{U}(n)$ and $\mathsf{USp}(n)$ when applied to GOE, GUE and GSE. The assumptions of Problem (7) may be weakened further – measurability is enough! You could try to develop a similar approach for the functional equation implicit in the proof of Theorem 19. That is, can you establish a stronger form of Theorem 19 that does not assume differentiability ?

2.9. Every $V \in \mathsf{U}(n)$ is of the form $V = \exp(\mathrm{i}M)$ for $M \in \mathsf{Her}(n)$ (you can assume this fact, but try and prove it). Thus, show that $V = U\mathrm{diag}(e^{\mathrm{i}\alpha_1}, \dots, e^{\mathrm{i}\alpha_n})U^*$ for real numbers $\alpha_1, \dots, \alpha_n$. Combine this with problem (4) to derive ...

know how to finish this...

2.10. Show that the mapping $A \mapsto (I - A)(A + I)^{-1}$ from $\mathfrak{o}(n)$ to $\mathsf{O}(n)$ is bijective in a neighborhood of 0 to a neighborhood of the identity. Construct an atlas of $\mathsf{O}(n)$ using this mapping.

2.11. Using the Submersion Theorem [BL05, Proposition 3.42] (also called the Regular Value theorem) show that $\mathsf{O}(n)$ is a smooth manifold.
Hint: Consider $\phi : \mathbb{R}^{n\times n} \to \mathsf{Symm}(n)$ defined by $\phi(X) = X^T X$. Then show that I is a regular value and therefore $\phi^{-1}(I) = \mathsf{O}(n)$ is a smooth manifold.

Chapter 3

Jacobi matrices and tridiagonal ensembles

ch:tridiag

3.1 Jacobi ensembles

sec:tridiagonalize

The space of real $n \times n$ tridiagonal matrices is denoted $\mathsf{Tridiag}(n)$. A typical matrix in $\mathsf{Tridiag}(n)$ is written

$$T = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & & 0 \\ 0 & b_2 & a_3 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \dots & b_{n-1} & a_n \end{pmatrix}. \tag{3.1.1}$$ eq:defjac

Jacobi matrices, and their closure within the space $\mathsf{Tridiag}(n)$ are the manifolds

$$\begin{aligned} \mathsf{Jac}(n) &= \{T \in \mathsf{Tridiag}(n) \,|\, b_j > 0,\, 1 \le j \le n\}, \qquad (3.1.2) \\ \overline{\mathsf{Jac}(n)} &= \{T \in \mathsf{Tridiag}(n) \,|\, b_j \ge 0,\, 1 \le j \le n\}. \end{aligned}$$

Jacobi matrices, or more generally Jacobi operators, are of fundamental importance in spectral theory. A self-adjoint operator K on a Hilbert space can be decomposed using its cyclic subspaces. On each of these cyclic subspaces an orthonormal basis for $\operatorname{span}\{K^j x \mid j = 0, 1, 2, \dots\}$ can be found and the operator K becomes tridiagonal in this basis. This is an idea used by conjugate gradient algorithm Hestenes1952 [HS52]. They also play an important role in approximation theory, the theory of orthogonal polynomials, and more widely in numerical linear algebra. An essential step in the symmetric eigenvalue problem is the reduction of a full symmetric matrix to an isospectral tridiagonal matrix (tridiagonalization) by a sequence of orthogonal reflections. Under this procedure, the Gaussian ensembles push forward to ensembles of tridiagonal matrices whose laws have the following simple description.

defn:te1 **Definition 21** (Dumitriu–Edelman Dumitriu-Edelman [DE02]). For each $\beta > 0$, the Hermite(β) ensemble consists of $T \in \mathsf{Tridiag}(n)$ such that a_k, $1 \le k \le n$, are iid normal random variables with mean zero and variance $2/\beta$, and b_k, $1 \le k \le n-1$ where are independent $\chi_{(n-k)\beta}(1/\beta)$ random variables.

I think I fixed this to say $\chi_{(n-k)\beta}$ instead of $\chi_{k\beta}$

Use this notation or just normalize them appropriately?

The density for $\chi_k(\sigma^2)$ is supported on $[0,\infty)$ and is proportional to

$$t^{k-1}\mathrm{e}^{-\frac{t^2}{2\sigma^2}}.$$

In matrix form, we have

$$T \sim \begin{bmatrix} \mathcal{N}(0,2/\beta) & \chi_{\beta(n-1)}(1/\beta) & & & \\ \chi_{\beta(n-1)}(1/\beta) & \mathcal{N}(0,2/\beta) & \chi_{\beta(n-2)}(1/\beta) & & \\ & \chi_{\beta(n-2)}(1/\beta) & \ddots & \ddots & \\ & & \ddots & & \chi_{\beta}(1/\beta) \\ & & & \chi_{\beta}(1/\beta) & \mathcal{N}(0,2/\beta), \end{bmatrix},$$

where all entries are independent.

The point here is that the Hermite(β) ensembles are the push-forwards of the Gaussian ensembles when $\beta = 1$, 2 or 4. Then they interpolate Dyson's classification of ensembles to every $\beta > 0$. When combined with classical spectral theory, they provide a distinct, and important, perspective on the limit theorems of random matrix theory. Our immediate goal in this chapter is the following

thm:beta-ensembles **Theorem 22.** *Fix $\beta > 0$ and assume $T \sim$ Hermite(β). Then the marginal distribution of its eigenvalues is*

$$p_{\mathrm{Hermite}(\beta)}(\Lambda)\mathrm{D}\Lambda = \frac{1}{Z_{n,\beta}}\mathrm{e}^{-\frac{\beta}{4}\operatorname{Tr}(\Lambda^2)}|\triangle(\Lambda)|^{\beta}\,\mathrm{D}\Lambda. \qquad (3.1.3)$$ eq:general-beta

The chapter concludes with a more refined version of Theorem thm:beta-ensembles 22 that includes the distribution of the spectral measure of matrices $T \sim$ Hermite(β).

3.2 Householder tridiagonalization on $\mathsf{Symm}(n)$

sec:house

Each $M \in \mathsf{Symm}(n)$ may be diagonalized $M = Q\Lambda Q^T$. However, the computation of Λ depends on the solvability of the characteristic polynomial $\det(zI - M) = 0$. For $n \ge 5$, this equation is not solvable by radicals[1]. Nevertheless, every matrix always admits the following reduction that requires only a finite number of algebraic operations.

thm:tridiagonal **Theorem 23.** *For every $M \in \mathsf{Symm}(n)$ there exists a tridiagonal matrix T and $Q \in \mathsf{O}(n)$ such that*

$$M = QTQ^T. \qquad (3.2.1)$$ eq:tridiag

[1]Practical numerical schemes for eigenvalue decomposition are unaffected by this algebraic obstruction, since they rely on iteration.

A decomposition (3.2.1) is given by a change of variables

$$\mathsf{Symm}(n) \to \overline{\mathsf{Jac}(n)} \times \left(S^{n-2} \times S^{n-3} \times \dots S^1\right). \tag{3.2.2}$$ eq:tridiag1

under which the volume form $\mathrm{D}M$ *on* $\mathsf{Symm}(n)$ *transforms as follows:*

$$\mathrm{D}M = C_n \prod_{j=1}^{n} \mathrm{d}a_j \prod_{k=1}^{n-1} b_k^{(n-k)-1} \mathrm{d}b_k \prod_{l=1}^{n-2} \mathrm{D}\omega_l \tag{3.2.3}$$ eq:tridiag2

where $\mathrm{D}\omega_l$ *denotes uniform measure on the sphere* S^{l-1}*, and* C_n *is the normalization constant.*

To interpret this theorem one needs a mapping h

$$h : S^{n-2} \times S^{n-3} \times \cdots \times S^1 \mapsto \mathsf{O}(n), \tag{3.2.4}$$

so that $h(\omega_{n-2}, \omega_{n-3}, \dots, \omega_1) = Q$. This mapping is given explicitly below in terms of Householder reflections. As the dimension of the domain for this mapping is less than $\frac{1}{2}n(n-1)$, the dimension of $\mathsf{O}(n)$, not all matrices in $\mathsf{O}(n)$ are attainable.

Remark 24. The space $\mathsf{Tridiag}(n)$ clearly inherits the inner-product $\mathrm{Tr}(T^2) = \sum_{j=1}^{n} a_j^2 + 2\sum_{j=1}^{n-1} b_j^2$ from $\mathsf{Symm}(n)$. However, the volume form obtained from this metric is not the same as the volume form in (3.2.3) above.

Remark 25. (For algebraists!) The proof will also show that T and Q may be computed with a finite number of the following algebraic operations: addition, multiplication and square-roots.

eq:householder **Definition 26.** Suppose $v \in \mathbb{R}^n$ is a unit vector. The Householder reflection in v is the matrix

$$P_v = I - 2vv^T. \tag{3.2.5}$$ eq:householder

le:householder **Lemma 7.** *The matrix* P_v *has the following properties:*

(a) $P_v^2 = I$.

(b) $P_v \in \mathsf{O}(n)$.

Proof. Decompose $\mathbb{R}^n$ into the orthogonal subspaces $\mathrm{span}\{v\}$ and $v^\perp$. Then $P_v v = -v$ and $P_v|_{v^\perp} = I$. Thus, $P_v^2 = I$. This proves (a). By construction $P_v^T = P_v$. Thus, by (a), we also have $P_v^T P_v = I$. □

Proof of Theorem 23. 1. The proof relies on a sequence of Householder reflections that progressively introduce zeros in a sequence of matrices similar to M. The first such matrix is the following. Let $w_1 = (M_{21}, \dots, M_{n1})^T \in \mathbb{R}^{n-1}$ denote the last $n-1$ entries of the first column of M. If the first coordinate of w_1 is non-negative, and all other coordinates vanish there is nothing to do. If not, we may choose a Householder reflection (in $\mathbb{R}^{n-1}$) that maps w_1 to $\|w_1\|_2 e_1^{(n-1)}$ (here the superscript $n-1$ denotes that we consider the basis vector $e_1 \in \mathbb{R}^{n-1}$). Geometrically, such a reflection is obtained by choosing v_1 to be the unit vector that lies in between w_1 and $\|w_1\|_2 e_1^{(n-1)}$. Explicitly, we set[2]

$$\tilde{v}_1 = \|w_1\|_2 e_1^{(n-1)} - w_1, \quad v_1 = \frac{\tilde{v}_1}{\|\tilde{v}_1\|_2}, \quad P^{(1)} = P_{v_1}. \tag{3.2.6}$$

eq:house-reflect

By Lemma 7, $P^{(1)} \in \mathsf{O}(n-1)$ is a Householder reflection that maps w_1 to $|w_1| e_1^{(n-1)}$. It may be extended to a Householder reflection in $\mathsf{O}(n)$, by defining

$$Q^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & P^{(1)} \end{pmatrix}. \tag{3.2.7}$$

eq:house1

Then the matrix

$$M^{(1)} := Q^{(1)} M \left(Q^{(1)}\right)^T = Q^{(1)} M Q^{(1)}, \tag{3.2.8}$$

eq:house2

is similar to M. By construction, the first row of $M^{(1)}$ is $(M_{11}, |w_1|, 0, \ldots, 0)$, and the first column is $(M_{11}, |w_1|, 0, \ldots, 0)^T$. Thus, we may write

$$M^{(1)} = \begin{pmatrix} T^{(1)} & |w_1|(e_1^{(n-1)})^T \\ |w_1| e_1^{(n-1)} & N^{(1)} \end{pmatrix}, \tag{3.2.9}$$

eq:house3

where $T^{(1)}$ is a (trivial) 1×1 tridiagonal matrix and $N^{(1)} \in \mathrm{Symm}(n-1)$. See Exercise 3.2 for the appropriate generalization of this step for the complex case $\mathbb{C}^{n \times n}$.

2. The proof is completed by induction. Assume that $M^{(k)} \in \mathsf{Symm}(n)$ has the form

$$M^{(k)} = \begin{pmatrix} T^{(k)} & |w_k|(e_1^{(n-k)})^T \\ |w_k| e_1^{(n-k)} & N^{(k)}) \end{pmatrix}, \tag{3.2.10}$$

eq:house4

where $T^{(k)} \in \mathsf{Tridiag}(k)$ and $N^{(k)} \in \mathrm{Symm}(n-k)$, $1 \le k \le n-1$. We apply the procedure of step 1 to $N^{(k)}$ to obtain a vector v_k, a Householder reflection $P^{(k)} = P_{v_k}$, and an orthogonal transformation of $M^{(k)}$,

$$Q^{(k)} = \begin{pmatrix} I_k & 0 \\ 0 & P^{(k)} \end{pmatrix} \in \mathsf{O}(n), \quad M^{(k+1)} = Q^{(k)} M^{(k)} Q^{(k)}. \tag{3.2.11}$$

eq:house5

Note that $Q^{(k)}$ leaves the first k rows and columns of $M^{(k)}$ unchanged, thus it does not destroy the tridiagonal structure of the first k rows and columns. Thus, $M^{(k+1)}$ has the form (3.2.10) with the index k replaced by $k+1$.

The procedure terminates when $k = n-2$, and yields

$$M = QTQ^T, \quad Q = Q^{(n-2)} Q^{(n-3)} \ldots Q^{(1)}. \tag{3.2.12}$$

eq:house6

[2] If one is using this method numerically and $|\tilde{v}_1|$ is small, instabilities can be introduced. In this case one should use $-|w_1| e_1^{(n-1)} - w_1$.

3. It is simplest to prove (3.2.3) probabilistically. Informally, the k-th step of the procedure above is a change to polar coordinates in $\mathbb{R}^{n-k}$, with $b_k \geq 0$ playing the role of the radius, and the factor $b_k^{n-k-1} \mathrm{d}b_k \mathrm{D}\omega_{n-1-k}$ being the pushforward of Lebesgue measure in $\mathbb{R}^{n-k}$ to polar coordinates. More precisely, assume that $M \sim \mathrm{GOE}(n)$. We note that the first step of the above procedure leaves M_{11} alone. Thus, $a_1 = M_{11} \sim \mathcal{N}(0,1)$. Moreover, the term b_1 is the length of the first column of M, not including the diagonal term M_{11}. Since a χ_m^2 random variable has the same law as the length of a vector in $\mathbb{R}^m$ whose entries are iid $\mathcal{N}(0,1)$ random variables, we see that $b_1 \sim \chi_{n-1}$. Further, the vector $\omega_1 = w_1/|w_1|$ is uniformly distributed on S^{n-2} and is independent of both a_1 and b_1 (see Exercise 3.1). We next observe that by the independence and invariance of the Gaussian ensembles, the matrix $N^{(1)}$ in (3.2.9) $\sim \mathrm{GOE}(n-1)$. Indeed, $\tilde{M}_1$, the lower-right $(n-1)\times(n-1)$ block of M, is a $\mathrm{GOE}(n-1)$ matrix, and the reflection $P^{(1)}$ is independent of $\tilde{M}_1$. Thus, $N^{(1)} = P^{(1)}\tilde{M}_1 P^{(1)}$ has law $\mathrm{GOE}(n-1)$ (note that b_1, a_1 and ω_1 are independent; see Exercise 3.3). Thus, $a_2 \sim \mathcal{N}(0,1)$ and $b_2 \sim \chi_{n-2}$. An obvious induction now shows that if $M \sim \mathrm{GOE}$ then $T \sim \mathrm{Hermite}(1)$, and the vectors $\omega_k = w_k/|w_k|$, are uniformly distributed on S^{n-1-k}, $1 \leq k \leq n-2$. Comparing the two laws, we find (with $\beta = 1$)

$$\mathrm{e}^{-\frac{\beta \operatorname{Tr}(M^2)}{2}} \mathrm{D}M = C_n \mathrm{e}^{-\frac{\beta \operatorname{Tr}(T^2)}{2}} \mathrm{d}a_j \prod_{k=1}^{n-1} b_k^{n-k-1} \mathrm{d}b_k \prod_{l=1}^{n-2} \mathrm{D}\omega_l \tag{3.2.13}$$ eq:tridiag3

The exponential weights cancel, and yield the Jacobian formula (3.2.3). □

3.3 Tridiagonalization on Her(n) and Quart(n)

Theorem 23 admits a natural extension to Her(n) and Quart(n).

thm:tridiag-her **Theorem 27.** *For every $M \in \mathsf{Her}(n)$ (resp. $\mathsf{Quart}(n)$) there exists a tridiagonal matrix $T \in \mathsf{Jac}(n)$ and $Q \in \mathsf{U}(n)$ (resp. $\mathsf{USp}(n)$) such that*

$$M = QTQ^*. \tag{3.3.1}$$ eq:tridiag-her

The transformation (3.3.1) is given by a change of variables

$$\mathsf{Her}(n) \to \overline{\mathsf{Jac}(n)} \times \left(S_{\mathbb{F}}^{n-2} \times S_{\mathbb{F}}^{n-3} \times \ldots S_{\mathbb{F}}^{1}\right), \tag{3.3.2}$$ eq:tridiag1

where $S_{\mathbb{F}}^l$ denotes the unit sphere in $\mathbb{F}^l$, with $\mathbb{F} = \mathbb{C}$ (resp. $\mathbb{H}$). The volume form DM on $\mathsf{Her}(n)$ (resp. $\mathsf{Quart}(n)$) transforms as follows:

I added β to the exponents for the b_k's. I think this is correct...

$$\mathrm{D}M = C_n \prod_{j=1}^{n} \mathrm{d}a_j \prod_{k=1}^{n-1} b_k^{\beta(n-k)-1} \mathrm{d}b_k \prod_{l=1}^{n-2} \mathrm{D}\omega_l \tag{3.3.3}$$ eq:tridiag2u

where $\mathrm{D}\omega_l$ denotes uniform measure on the sphere $S_{\mathbb{F}}^l$, and C_n is a normalization constant.

For a vector $w \in \mathbb{C}^n$ with independent standard normal complex entries, $w_j \sim \frac{1}{\sqrt{2}}(N_1 + \mathrm{i}N_2)$, where $N_1, N_2 \sim \mathcal{N}(0,1)$ are independent, $|w| \sim \frac{1}{\sqrt{2}}\chi_{2n}$. For a quarternion vector w, one finds $|w| \sim \frac{1}{2}\chi_{4n}$. So, β is introduced in this way.

Remark 28. Note that the matrix T is always real, whereas the entries of M and Q are in $\mathbb{C}$ or $\mathbb{H}$.

The proof of Theorem thm:tridiag-her 27 is in the same vein as that of Theorem thm:tridiagonal 23. It is only necessary to replace the Householder projections in $\mathsf{O}(n)$ with projections in $\mathsf{U}(n)$ and $\mathsf{USp}(n)$. For example, given $v \in \mathbb{C}^n$ with $|v| = 1$, the associated Householder projection in $\mathsf{U}(n)$ is $P_v = I - 2vv^*$. Step 3 in the proof of Theorem thm:tridiag-her 27 also explains the role of the parameter β in the definition of the Hermite-β ensembles. The k-th step of the Householder transformation maps a standard Gaussian vector in $\mathbb{C}^{n-k}$ to its magnitude and direction. The law of the magnitude is now $\chi_{2(n-k)}$ (or $\chi_{\beta(n-k)}$ with $\beta = 2$). Similarly, the direction of the Gaussian vector is uniformly distributed on the unit sphere in $\mathbb{C}^{n-k-1}$.

3.4 Inverse spectral theory for Jacobi matrices

sec:jacobi-inv

Bounded Jacobi operators admit a complete and beautiful spectral theory that is intimately tied to orthogonal polynomials and continued fractions. We first introduce this theory for finite Jacobi matrices, since it underlies Theorem thm:beta-ensembles 22. As usual, write

$$T = Q\Lambda Q^T, \quad Q \in \mathsf{O}(n), \tag{3.4.1}$$ eq:spec-map-1

for the diagonalization of T. We also recall the

$$\mathcal{W}^n = \{\Lambda \in \mathbb{R}^n \,|\, \lambda_1 < \lambda_2 < \ldots < \lambda_n\}. \tag{3.4.2}$$

For each $\Lambda \in \mathcal{W}^n$, its <u>isospectral manifold</u> is the set

$$\mathcal{M}_\Lambda = \{T \in \overline{\mathsf{Jac}(n)} \,\big|\, T = Q\Lambda Q^T, \text{ for some } Q \in \mathsf{O}(n)\}. \tag{3.4.3}$$ eq:isospectral-mfld

The following theorem shows that the interior of the isospectral manifold is diffeomorphic to the positive orthant $S_+^{n-1} = \{u \in \mathbb{R}^n \,|\, \|u\|_2 = 1,\ u_j > 0,\ j = 1, 2, \ldots, n\}$ of the unit sphere. Given T, we uniquely define Q by forcing the first non-zero entry in each column to be positive.

thm:Jacobi-spec **Theorem 29.** *The spectral mapping*

$$\mathcal{S} : \mathsf{Jac}(n) \to \mathcal{W}^n \times S_+^{n-1}, \quad T \mapsto (\Lambda, Q^T e_1), \tag{3.4.4}$$ eq:spec-map

is an analytic diffeomorphism.

We prove this in stages below. See Figure fig:specmap 3.4.1.

The isospectral manifold admits several distinct parametrizations. First, it is clear that we could use the simplex Σ_n instead of the orthant S_+^{n-1}. Indeed, let

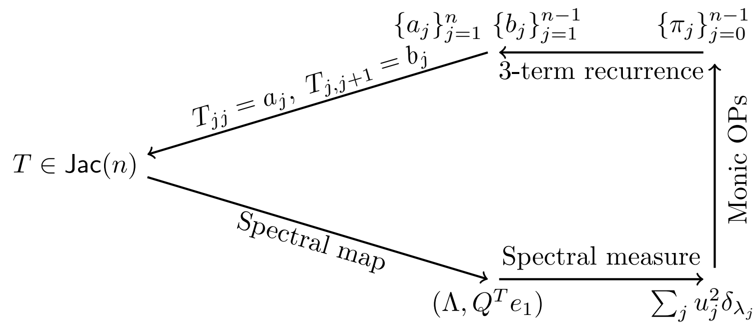

fig:specmap Figure 3.4.1: The construction of the spectral map and its inverse. The transformation to spectral variables is computed by computing eigenvalues and taking the first component of the (normalized) eigenvectors. Then a spectral measure (3.4.5) is created from this data and is used to define monic orthogonal polynomials (3.4.16). These polynomials satisfy a three-term recurrence relation (see Lemma 11) and the coefficients in the relation allow for the (unique) reconstruction of T, see (3.4.21). This shows the spectral map from $\mathsf{Jac}(n)$ to $\mathcal{W}^n \times S_+^{n-1}$ is invertible.

$u = Q^T e_1$ denote the first row of the matrix of eigenvectors and define $c_j = u_k^2$, $1 \le k \le n$. Since $Q \in \mathsf{O}(n)$, $\sum_{k=1}^n u_k^2 = 1$. Thus, $u \in S^{n-1}$ and $c \in \Sigma_n$. But, we shall use S_+^{n-1}. Lemma 8 below shows that u_k can be chosen to be strictly positive, which allows us to restrict attention to the positive orthant S_+^{n-1}

Theorem 29 may also be viewed as a mapping to the spectral measure

$$T \mapsto \mu(T) = \sum_{j=1}^n u_j^2 \delta_{\lambda_j} = \sum_{j=1}^n c_j \delta_{\lambda_j}. \qquad (3.4.5)$$ eq:spec-map2

It is often more convenient to work with the Cauchy transform of the spectral measure, μ. Define the τ-function,

$$\mu \mapsto \tau(z) = \int_{\mathbb{R}} \frac{1}{x-z}\mu(\mathrm{d}x) = \sum_{j=1}^n \frac{u_j^2}{\lambda_j - z}, \quad z \in \mathbb{C}\backslash\{\lambda_1, \ldots, \lambda_n\}. \qquad (3.4.6)$$ eq:tau

The inverse $\tau \mapsto \mu$ is obtained by computing the poles and residues of τ.

The τ-function may also be written as a ratio of polynomials of degree $n-1$ and n respectively. Let $T_k \in \mathsf{Jac}(k)$ denote the lower-right $k \times k$ submatrix of T, $1 \le k \le n$. It follows from Cramer's rule that

$$\tau(z) = e_1^T (T-z)^{-1} e_1 = \frac{\det(T_{n-1} - zI)}{\det(T - zI)} = \frac{\prod_{j=1}^{n-1}(\lambda_j^{(n-1)} - z)}{\prod_{j=1}^{n}(\lambda_j^{(n)} - z)}, \qquad (3.4.7)$$ eq:tau-interlace

where $\Lambda^{(k)}$ denotes the diagonal matrix of eigenvalues of T_k and $\Lambda^{(n)} = \Lambda$. We will show that the ordered eigenvalues of T_{n-1} and T_n interlace, i.e.

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \lambda_2^{(n)} < \ldots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)}. \qquad (3.4.8)$$ eq:interlace

Thus, interlacing sequences provide another parametrization of $\mathsf{Jac}(n)$. A convenient visal description of interlacing sequences, called diagrams, was introduced by Kerov and Vershik [Ker03]. The importance of these alternate parametrizations (spectral measures, τ-function, diagrams) is that they provide a transparent framework for the analysis of the limit $n \to \infty$.

The surprising aspect of Theorem 29 is that the spectral data (Λ, u) provides enough information to reconstruct the matrix T. There are three reconstruction procedures. The first involves orthogonal polynomials, the second uses the theory of continued fractions and a third involves the explicit solution of the equation $TQ = \Lambda Q$ for T. We explain the use of orthogonal polynomials below, and outline the theory of continued fractions in the exercises. In order to develop these procedures, it is first necessary to establish basic properties of the eigenvalues of Jacobi matrices.

le:jacobi-signs **Lemma 8.** *Assume $T \in \mathsf{Jac}(n)$. Then*

1. *The first entry of each non-zero eigenvector is non-zero. In particular, we may normalize the eigenvectors to ensure $u_k > 0$ for $1 \le k \le n$.*

2. *The eigenvalues of T are distinct.*

Proof. We write the eigenvalue equation $Tv = zv$ in coordinates.

$$b_{k-1}v_{k-1} + (a_k - z)\, v_k + b_k v_{k+1} = 0, \quad 1 \le k \le n, \tag{3.4.9}$$ eq:jacobi-eig

with the convention $b_0 = b_n = 0$. Since the off-diagonal terms b_k are strictly positive, we may solve this linear system recursively. Given v_1, we find

$$v_2 = \frac{v_1(z-a_1)}{b_1}, \quad v_3 = \frac{v_1}{b_1 b_2}\left((a_2 - z)(a_1 - z) - b_1^2\right), \text{ etc.} \tag{3.4.10}$$

Thus, $v \equiv 0 \in \mathbb{R}^n$ if $v_1 = 0$. Further, the solution space to the eigenvalue equation $Tv = \lambda v$ has dimension at most 1. □

The next statement follows from Lemma 8(2) and (3.3.3), (3.2.3) because the probability that any b_j vanishes is zero.

cor:distinct **Corollary 1.** *Suppose $M \sim \mathrm{GOE}(n), \mathrm{GUE}(n)$ or $\mathrm{GSE}(n)$. Then the eigenvalues of M are distinct with probability one.*

le:3termd **Lemma 9.** *The characteristic polynomials $d_k(z) = \det(zI - T_k)$ satisfy the recurrence relations*

$$d_{k+1}(z) = (z - a_{n-k})d_k(z) - b_{n-k}^2\, d_{k-1}(z), \quad 1 \le k \le n-1, \tag{3.4.11}$$ eq:3term-det

with the initial condition $d_0(z) \equiv 1$ and the convention $b_n = 0$.

Proof. Expand the determinant $\det(zI - T_k)$ about the k-th row, and compute the minors associated to $z - a_{n-k}$ and b_{n-k}. □

le:interlace **Lemma 10.** *The eigenvalues of T_k and T_{k+1} interlace, $1 \le k \le n-1$.*

Proof. We consider the τ-functions for the minors T_k,

$$\tau_k(z) = \frac{\det(T_k - zI)}{\det(T_{k+1} - zI)} = -\frac{d_k(z)}{d_{k+1}(z)}. \tag{3.4.12}$$

eq:rec-tau1

By the recurrence relation (3.4.11), we have

$$-\frac{1}{\tau_k(z)} = z - a_{n-k} + b_{n-k}^2 \tau_{k-1}(z). \tag{3.4.13}$$

eq:rec-tau2

We claim that on the real line, $\tau_k(x)$ is strictly increasing between the zeros of d_k. Indeed, it is clear that $\tau_1(x) = (a_n - x)^{-1}$ has this property, and upon differentiating (3.4.13) we find that

$$\frac{1}{\tau_k^2}\tau_k' = 1 + b_{n-k}^2 \tau_{k-1}' > 0,$$

except at poles. The claim follows by induction.

Since τ_k is strictly increasing between poles, by the intermediate value theorem, it has exactly one zero between any two poles. By (3.4.12), the zeros of τ_k are the eigenvalues of T_k, and the poles of τ_k are the eigenvalues of T_{k+1}. Thus, they interlace. $\square$

A remarkable feature of the spectral theory of Jacobi matrices is that the orthogonal polynomials associated to the spectral measure $\mu(T)$ may be used to reconstruct T. In order to state this assertion precisely, let us recall some basic facts about orthogonal polynomials. Assume given a probability measure μ on $\mathbb{R}$ that has finite-moments of all orders, i.e.,

$$\int_{\mathbb{R}} |x|^\alpha \, \mu(\mathrm{d}x) < \infty, \quad \alpha > 0. \tag{3.4.14}$$

eq:moments-OP

We may apply the Gram-Schmidt procedure to the monomials $\{x^k\}_{k=0}^\infty$ to construct a sequence of polynomials that are orthogonal in $L^2(\mathbb{R}, \mu)$. There are two standard normalizations that one may adopt.

Orthonormal polynomials, denoted $\{p_k\}_{k=0}^\infty$, have the property that p_k is of degree k, $k = 0, 1, 2, \ldots$, with positive leading coefficient, and

$$\int_{\mathbb{R}} p_k(x) p_l(x) \, \mu(\mathrm{d}x) = \delta_{kl}. \tag{3.4.15}$$

eq:orthonormal

Monic polynomials, denoted $\{\pi_k\}_{k=0}^\infty$ have the property that $\pi_k(x)$ is of degree k and the coefficient of x^k is 1. Further,

$$\int_{\mathbb{R}} \pi_k(x)\pi_l(x) \, \mu(\mathrm{d}x) = 0, \quad k \neq l. \tag{3.4.16}$$

eq:orthonormal-mon

le:3term **Lemma 11** (Three-term recurrence for orthogonal polynomials)**.** *Given* $(\Lambda, u) \in \mathcal{W}^n \times S_+^{n-1}$*, let* $\mu(\Lambda, u) = \sum_{k=1}^n u_k^2 \delta_{\Lambda_k}$*. Then the associated monic orthogonal polynomials* $\{\pi_k\}_{k=0}^n$*, satisfy the three-term recurrence (3.4.17)*

$$\pi_k(z) = (z - a_k)\pi_{k-1}(z) - b_{k-1}^2\, \pi_{k-2}(z), \quad 1 \leq k \leq n, \tag{3.4.17}$$ eq:3term

where the coefficients a_k *and* b_k *are given by*

$$a_k = \frac{\int_{\mathbb{R}} x\pi_{k-1}^2\, \mu(\mathrm{d}x)}{\int_{\mathbb{R}} \pi_{k-1}^2(x)\, \mu(\mathrm{d}x)}, \quad b_k^2 = \frac{\int_{\mathbb{R}} x\pi_k(x)\pi_{k-1}(x)\, \mu(\mathrm{d}x)}{\int_{\mathbb{R}} \pi_{k-1}^2(x)\, \mu(\mathrm{d}x)}, \quad k = 1, \ldots, n, \tag{3.4.18}$$ eq:three-term2

with $\pi_{-1} = 0$ *and hence* $b_0 = 0$*. Recall that* $\pi_1 = 1$*. The recurrence (3.4.18) defines a Jacobi matrix* $T(\mu)$*.*

rem:three-term **Remark 30.** If μ is not a discrete measure of the form (3.4.5), but has bounded support, the recurrence (3.4.17) defines a bounded Jacobi operator on $l^2(\mathbb{C})$.

Proof. Given any μ as in (3.4.14), we obtain the sequence $\{\pi_k\}$ using the Gram-Schmidt procedure. When μ is of the form (3.4.5) with (3.4.5), the vector space $L^2(\mathbb{R}, \mu)$ has dimension n and the Gram-Schmidt procedure yields an orthogonal basis $\{\pi_0, \pi_1, \ldots, \pi_{n-1}\}$ for $L^2(\mathbb{R}, \mu)$.

The three-term recurrence for the orthogonal polynomials is obtained as follows. Since $x\pi_k(x)$ is a polynomial of degree $k+1$ it can be expressed as a linear combination $x\pi_k(x) = \sum_{j=0}^{k+1} c_{j,k}\pi_j(x)$. Since the π_j are monic, we must have $c_{k+1,k} = 1$. Moreover, for $j = 0, \ldots, k-2$

$$\int_{\mathbb{R}} x\pi_k(x)\pi_j(x)\, \mu(\mathrm{d}x) = \int_{\mathbb{R}} \pi_k(x) x\pi_j(x)\, \mu(\mathrm{d}x) = 0,$$

since $x\pi_j$ lies in the span of $\{\pi_0, \ldots, \pi_{k-1}\}$. Thus, $c_{j,k} = 0$ for $j = 0, \ldots, k-2$ and we find

$$x\pi_k(x) = \pi_{k+1}(x) + c_{k,k}\pi_k(x) + c_{k-1,k}\pi_{k-1}(x). \tag{3.4.19}$$ eq:three-term-OP

It remains to show that $c_{k-1,k} > 0$. By orthogonality, $\int_{\mathbb{R}} x\pi_k(x)\pi_{k+1}(x)\, \mu(\mathrm{d}x) = \int_{\mathbb{R}} \pi_{k+1}^2(x)\, \mu(\mathrm{d}x)$. Thus, $c_{k,k-1} > 0$ for all k such that $\pi_{k-1}(x)$ does not vanish in $L^2(\mathbb{R}, \mu)$: Assume π_l does not vanish in $L^2(\mathbb{R}, \mu)$ for $l = 0, 1, 2, \ldots, k-1 < n-1$. Then this recurrence defines π_k which is not the zero polynomial since it is monic. For $\Lambda \in \mathcal{W}^n$, it has distinct diagonal entries, so $p(x) \neq 0$ implies $\int p^2(x)\mu(\mathrm{d}x) > 0$ if p is a polynomial of degree less than n. This is (3.4.17) aside from a change in notation. □

Proof of Theorem 29. We have defined a forward map $T \mapsto \mu(T)$ as follows. The matrix T defines a τ-function $\tau(z) = e_1^T(T - zI)^{-1}e_1$, which is expressed as a ratio of characteristic polynomials in (3.4.7). The poles of $\tau(z)$ are the eigenvalues of T. The norming constants are the residues at the poles, and are given by

$$u_k^2 = -\frac{d_{n-1}(\lambda_k)}{d_n'(\lambda_k)}, \quad 1 \leq k \leq n. \tag{3.4.20}$$ eq:norming-constant

The inverse map $\mu \to T(\mu)$ is given by Lemma 11. The orthogonal polynomials defined by μ satisfy a three-term recurrence whose coefficients determine T.

We only need to show that the map $\mu \mapsto T(\mu) \mapsto \mu\,(T(\mu))$ is the identity. Let $\mu \cong (\Lambda, u)$ be given and define $T(\mu)$ by the recurrence relations. We will show that

$$e_1^T (T - zI)^{-1} e_1 = \int_{\mathbb{R}} \frac{1}{x-z}\mu(\mathrm{d}x) = \sum_{k=1}^n \frac{u_k^2}{\lambda_k - z}. \tag{3.4.21}$$

eq:moments1

We first show that the eigenvalues of T coincide with $\{\lambda_k\}$. Define $p_j(x) = \pi_j(x)\prod_{k=1}^j b_k^{-1}$, $\pi_0(x) = p_0(x)$, then

$$\begin{aligned} xp_0(x) &= a_1 p_0(x) + b_1 p_1(x), \\ xp_k(x) &= b_k p_{k-1}(x) + a_{k+1} p_k(x) + b_{k+2} p_{k+1}(x), \quad k > 0. \end{aligned}$$

Because $p_n(\lambda_j) = 0$ for all j, we conclude that

$$(p_0(\lambda_j), p_2(\lambda_j), \ldots, p_{n-1}(\lambda_j))^T$$

is a non-trivial eigenvector for eigenvalue λ_j. This shows that both the left and right-hand sides of (3.4.21) are rational functions of z with simple poles in the same locations. We expand both sides of (3.4.21) for large z, and if we establish the relation

$$e_1^T T^k e_1 = \int_{\mathbb{R}} x^k \mu(\mathrm{d}x), \quad 0 \le k \le n-1, \tag{3.4.22}$$

eq:moments2

To establish what we want, don't we need something like k up to $2n-1$? I showed that the eigenvalues are equal, then we show these moments are equal

then it follows that

$$e_1^T (T - zI)^{-1} e_1 - \sum_{k=1}^n \frac{u_k^2}{\lambda_k - z} = O(z^{-n-1})$$

as $z \to \infty$. And therefore

$$\left(\prod_{k=1}^n (z - \lambda_k)\right)\left(e_1^T (T - zI)^{-1} e_1 - \sum_{k=1}^n \frac{u_k^2}{\lambda_k - z}\right)$$

is an entire function that decays at infinity — it must be identically zero.

To see why (3.4.22) holds, consider

$$\begin{aligned} Te_1 &= a_1 e_1 + b_1 e_2, \\ Te_k &= b_{k-1} e_{k-1} + a_k e_k + b_k e_{k+1}, \quad k > 1. \end{aligned}$$

Define new basis vectors $f_j = e_j \prod_{k=1}^{j-1} b_k$, $f_1 = e_1$ because $b_j > 0$ for all $j = 1, 2, \ldots, n-1$. We then have

$$\begin{aligned} Tf_1 &= a_1 f_1 + f_2, \\ Tf_k &= b_{k-1}^2 f_{k-1} + a_k f_k + f_{k+1}, \quad k > 1. \end{aligned}$$

We then diagonalize this, setting $T = Q\Lambda Q^T$, $\hat{f}_j = Q^T f_j$ to find

$$\begin{aligned}\Lambda \hat{f}_1 &= a_1 \hat{f}_1 + b_1^2 \hat{f}_1,\\ \Lambda \hat{f}_k &= b_{k-1}^2 \hat{f}_{k-1} + a_k \hat{f}_k + \hat{f}_{k+1}, \quad k > 1.\end{aligned}$$

Component-wise, this is the same three-term recurrence as the monic polynomials. So, taking into account $f_1 = e_1$, we find

$$\hat{f}_j = \pi_{j-1}(\Lambda) Q^T e_1, \quad f_j = \pi_{j-1}(T) e_1.$$

Then because $x^k = \sum_{j=0}^k c_{jk}\pi_j(x)$ we have $T^k e_1 = \sum_{j=0}^k c_{jk}\pi_j(T)e_1 = \sum_{j=0}^k c_{jk} e_{j+1}$ and

$$e_1^T T^k e_1 = c_{0k}.$$

Similarly,

$$\int_{\mathbb{R}} x^k \mu(\mathrm{d}x) = \sum_{j=0}^k c_{jk} \int_{\mathbb{R}} \pi_j(x)\mu(\mathrm{d}x) = c_{0k}.$$

This proves the theorem and this approach extends to the semi-infinite Jacobi operators [Dei99]. $\square$

Lecture Note 5. *Alternate proof of Theorem 29.* Lemma 8 establishes that $Q^T e_1 \in S_+^{n-1}$ and $\Lambda \in \mathcal{W}^n$. Now, we explicitly construct the inverse map $(\Lambda, Q^T e_1) \to T$. We follow [DNT83]. The algorithm to construct T uniquely is as follows. We use conjugates so that it is clear how this generalizes.

1. Compute

$$\begin{aligned} T_{11} &= \sum_{j=1}^n \lambda_j |Q_{1j}|^2,\\ T_{12}^2 &= \sum_{j=1}^n |Q_{1j}|^2 |\lambda_j - T_{11}|^2,\\ Q_{2j} &= \frac{1}{T_{21}}(\lambda_j Q_{1j} - T_{11}Q_{1j}), \quad j = 1, 2, \ldots, n.\end{aligned}$$

2. For $k = 2, 3, \ldots, n-1$, compute

$$\begin{aligned} T_{kk} &= \sum_{j=1}^n \lambda_j |Q_{kj}|^2,\\ T_{k,k+1}^2 &= \sum_{j=1}^n |(\lambda_j - T_{kk})Q_{kj} - T_{k,k-1}Q_{k-1,j}|^2,\\ Q_{k+1,j} &= \frac{1}{T_{k,k+1}}(\lambda_j Q_{kj} - T_{k,k-1}Q_{k-1,j} - T_{kk}Q_{kj}), \quad j = 1, 2, \ldots, n.\end{aligned}$$

3. Compute

$$T_{nn} = \sum_{j=1}^{n} \lambda_j |Q_{nj}|^2.$$

This procedure will succeed if $T_{k,k+1} \neq 0$ for all k. And if it succeeds it is easy to check that the tridiagonal matrix T will satisfy $TQ = Q\Lambda$ and by assumption no column of Q vanishes identically. It remains to prove that $T_{k,k+1} \neq 0$ and that Q has no columns that vanish identically. It easily follows that $T_{12} \neq 0$, since the eigenvalues are distinct. We perform induction on k. Assume $T_{l,l+1} \neq$ for $l = 1, 2, \ldots, k-1$. Also assume the vectors $\{q_l\}_{l=1}^{k}$, $q_l = (Q_{l1}, Q_{l2}, \ldots, Q_{ln})^T$ are orthonormal. If $T_{k,k+1} = 0$, we would have

$$(\lambda_j - T_{kk})Q_{kj} - T_{k,k-1}Q_{k-1,j} = 0, \quad j = 1, 2, \ldots, n. \tag{3.4.23}$$

This then implies that the $k \times k$ tridiagonal matrix $(T_{lj})_{1 \leq l,j \leq k}$, $T_{lj} = 0$ if $|l-j| > 1$, has n distinct eigenvalues, which is impossible. The fact that the diffeomorphism is analytic follows[3] from the fact that these expressions for T are rational functions of Λ and $Q^T e_1$. $\square$

Remark 31. Observe that the recurrence relation (eq:3term 3.4.17) may be rewritten as the matrix equation,

$$\begin{pmatrix} a_1 - z & 1 & 0 & \ldots & 0 \\ b_1^2 & a_2 - z & 1 & \ldots & 0 \\ 0 & b_2^2 & a_3 - z & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \ldots & b_{n-1}^2 & a_n - z \end{pmatrix} \begin{pmatrix} \pi_0(z) \\ \pi_1(z) \\ \vdots \\ \pi_{n-1}(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -\pi_n(z) \end{pmatrix}. \tag{3.4.24}$$

eq:matrix-recurrence

Since $\pi_0(z) = 1$, each zero of $\pi_k(z)$ is an eigenvalue of the matrix above. Thus, $\pi_k(z) = \det(zI - \tilde{T}_k)$ where $\tilde{T}_k$ denotes the upper-left $k \times k$ submatrix of T (compare with T_k and $d_k(z) = \det(zI - T_k)$).

Thus, given μ, the entries of T are obtained from "top to bottom". However, given T, the τ-function is the limit of τ-functions $-d_k(z)/d_{k+1}(z)$ computed 'bottom-to-top'.

r:ONP **Remark 32.** Consider the sequence of orthogonal polynomials

$$p_k(x) = \left(\prod_{j=1}^{k} b_k \right)^{-1} \pi_k(x), \quad k = 1, 2, \ldots, n-1. \tag{3.4.25}$$

[3]The simplest way to do this is to realize this as a mapping from $\mathcal{W}^n \times U$ where U is an subset of $\mathbb{R}^{n-1}$ found by mapping $S_+^{n-1} \to \Sigma_n$, and then mapping Σ_n to $\mathbb{R}^{n-1}$. Then the analytic implicit function theorem can be applied.

This is actually an orthonormal sequence which satisfies the three-term recurrence

$$b_k p_k(x) = (z - a_k)p_{k-1}(z) - b_{k-1}p_{k-2}(x). \tag{3.4.26}$$

3.5 Jacobians for tridiagonal ensembles

We can now combine Theorem 29 with the definition of Hermite-β ensembles to state a refined version of Theorem 22.

thm:dumitriu-edelman **Theorem 33.** *For each $\beta > 0$, the law of the* Hermite(β) *ensembles in spectral variables $(\Lambda, u) \in \mathcal{W}^n \times S_+^{n-1}$ is given by*

$$p_{\text{Hermite}}(\Lambda, u)\mathrm{D}\Lambda\mathrm{D}u = \frac{1}{Z_{n,\beta}}\left(e^{-\frac{\beta}{4}\operatorname{Tr}(\Lambda^2)}|\triangle(\Lambda)|^\beta \mathrm{D}\Lambda\right)\left(\prod_{j=1}^n u_j^{\beta-1}\right)\mathrm{D}u, \tag{3.5.1}$$

where Du *refers to uniform measure on S_+^{n-1}, see Example 110. In particular, Λ and u are independent.*

Theorem 33 follows from a computation of the Jacobian of the spectral map $\mathcal{S} : \mathsf{Jac}(n) \to \mathcal{W}^n \times S_+^{n-1}$.

thm:jacobian-spectral **Theorem 34.** *The volume forms on $\mathsf{Jac}(n)$ and $\mathcal{W}^n \times S_+^{n-1}$ are related by*

$$\mathrm{D}T = \prod_{j=1}^n \mathrm{d}a_j \prod_{k=1}^{n-1} b_k^{n-k-1}\,\mathrm{d}b_k = C_n \triangle(\Lambda)\mathrm{D}\Lambda\left(\prod_{k=1}^n u_j\right)\mathrm{D}u. \tag{3.5.2}$$ eq:jacobian-spectral

where C_n is a normalization constant.

Remark 35. We have suppressed the explicit form of the normalization constants in the statement of the lemma to focus on the marginals on $\mathcal{W}^n$ and S_+^{n-1} respectively. The computation of the constants is an interesting exercise (see [DE02]).

While Theorem 34 is an analytic/geometric assertion, the simplest proof uses probabilistic reasoning, as in step 3 of the proof of Theorem 23. Since we have computed the Jacobian for the diagonalizing map $\mathsf{Symm}(n) \to \mathbb{R}^n \times \mathsf{O}(n)$ (Weyl's formula) and the tridiagonalizing map $\mathsf{Symm}(n) \to \mathsf{Jac}(n)$ (Theorem 23), the ratio of these Jacobians may be used to compute the Jacobian of the spectral map $\mathsf{Jac}(n) \to \mathcal{W}^n \times S_+^{n-1}$. The main point is that by the $\mathsf{O}(n)$ invariance of GOE, the top row of the eigenvector matrix must be uniformly distributed on S^{n-1} and is independent of Λ. This gives the term $\prod_{k=1}^n u_j\,\mathrm{d}u_j$ in equation (3.5.2). As Dumitriu and Edelman remark, this is a 'true random matrix theory' calculation. Another approach to (3.5.2) uses symplectic geometry.

le:van-jac **Lemma 12** (Vandermonde determinant in (a,b) coordinates)**.**

$$\triangle(\Lambda) = \prod_{j<k}(\lambda_j - \lambda_k) = \frac{\prod_{k=1}^{n-1} b_k^{n-k}}{\prod_{j=1}^{n} u_j}. \tag{3.5.3}$$ eq:van-jac

Proof. 1. Recall that $\Lambda^{(l)}$ denotes the diagonal matrix of eigenvalues of T_l and that $d_l(x) = \prod_{j=1}^{l}(x-\lambda_j^{(l)})$. Therefore, we have the identity

$$\prod_{j=1}^{l}\prod_{k=1}^{l-1}\left|\lambda_j^{(l)} - \lambda_k^{(l-1)}\right| = \left|\prod_{j=1}^{l} d_{l-1}\left(\lambda_j^{(l)}\right)\right| = \left|\prod_{k=1}^{l-1} d_l\left(\lambda_k^{(l-1)}\right)\right|. \tag{3.5.4}$$ eq:dum-edel1

Since d_{l-1} and d_l are related through the three-term recurrence

$$d_l(x) = (x-a_l)d_{l-1}(x) - b_{n-l+1}^2 d_{l-2}(x),$$

we have

$$\left|\prod_{k=1}^{l-1} d_l(\lambda_k^{(l-1)})\right| = b_{n-l+1}^{2(l-1)}\left|\prod_{k=1}^{l-1} d_{l-2}(\lambda_k^{(l-1)})\right| = b_{n-l+1}^{2(l-1)}\left|\prod_{j=1}^{l-2} d_{l-1}(\lambda_j^{(l-2)})\right|.$$

We apply this identity repeatedly, starting with $l=n$ to obtain

$$\begin{aligned}\left|\prod_{k=1}^{n-1} d_n(\lambda_k^{(n-1)})\right| &= b_1^{2(n-1)}\left|\prod_{j=1}^{n-2} d_{n-1}(\lambda_j^{(n-2)})\right| \\ &= b_1^{2(n-1)} b_2^{2(n-2)}\left|\prod_{k=1}^{n-3} d_{n-2}(\lambda_k^{(n-3)})\right| = \cdots = \prod_{k=1}^{n-1} b_k^{2(n-k)}.\end{aligned}$$

2. The coefficients u_j^2 are the residue of $\tau_n(z)$ at the poles λ_j, i.e.

$$u_k^2 = \left|\frac{d_{n-1}(\lambda_k)}{d_n'(\lambda_k)}\right|, \quad 1 \le k \le n. \tag{3.5.5}$$ eq:dum-edel3

Observe also that

$$d_n'(\lambda_k) = \prod_{j\neq k}(\lambda_j - \lambda_k), \text{ and } \prod_{k=1}^{n} d_n'(\lambda_k) = \triangle(\Lambda)^2. \tag{3.5.6}$$

Therefore,

$$\prod_{j=1}^{n} u_j^2 = \frac{1}{\triangle(\lambda)^2}\prod_{k=1}^{n}|d_{n-1}(\lambda_k)| = \frac{\prod_{k=1}^{n-1} b_k^{2(n-k)}}{\triangle(\lambda)^2}. \tag{3.5.7}$$ eq:dum-edel4

□

$$\begin{tikzcd}
 & \text{Hermite-1 } \mathsf{Jac}(n) \\
\text{GOE } \mathsf{Symm}(n) \arrow[ur, "\text{Householder}"] \arrow[dr, "\text{Spectral variables}"'] & \\
 & (\Lambda, Q^T e_1) \arrow[uu, "\text{Inv. spectral map}"']
\end{tikzcd}$$

fig:push Figure 3.5.1: We have already computed the push-forward of GOE under Householder reflections (eq:tridiag3 3.2.13) and the push-forward of GOE onto spectral variables via Weyl's formula (eq:weyl-volume 2.2.16). The composition of the map to spectral variables and the inverse spectral map must give us the reduction to tridiagonal form via Householder reflections. This allows the computation of the Jacobian of the inverse spectral map.

Proof of Theorem thm:jacobian-spectral *34.* 1. Our goal is to compute the Jacobian of the spectral mapping $\mathcal{S}$,

$$\mathrm{D}T = \frac{\partial(T(a,b))}{\partial(\Lambda,u)} \mathrm{D}\Lambda \mathrm{D}u, \tag{3.5.8}$$ eq:3.5.8

where $\mathrm{D}u$ is uniform measure on $\{u \in \mathbb{R}^n \mid |u| = 1, u_j > 0 \text{ for all } j\}$. Rather than compute the change of variables directly, we will compute the push-forward of GOE onto $\mathsf{Jac}(n)$ and $\mathcal{W}^n \times S_+^{n-1}$ separately, and obtain the Jacobian above, see Figure fig:push 3.5.1.

2. Consider the push-forward of GOE under the map $M \mapsto (\Lambda, u)$, where $M = Q\Lambda Q^T$ is the diagonalization of M, with the normalization that the first non-zero entry in each column is positive. Since Λ and the matrix of eigenvalues Q are independent, Λ and $u = Q^T e_1$ are independent. Since Q is distributed according to Haar measure on $\mathsf{O}(n)$, the vector u is uniformly distributed on S_+^{n-1} and the push-forward of GOE is the measure

$$p(\Lambda, u)\mathrm{D}\Lambda\mathrm{D}u = c_n \mathrm{e}^{-\frac{1}{4}\operatorname{Tr}(\Lambda)^2} \triangle(\Lambda)\mathrm{D}\Lambda\mathrm{D}u. \tag{3.5.9}$$ eq:3.5.9

3. Next consider the push-forward of GOE under the map $M \mapsto T$, where $M = QTQ^T$ denotes the tridiagonalization of M. As we have seen in the proof of Theorem 20, T and U are independent, and the marginal distribution of T is given by

$$\tilde{p}(T)\,\mathrm{D}T = C_n \mathrm{e}^{-\frac{1}{4}\operatorname{Tr}(T^2)} \prod_{j=1}^n \mathrm{d}a_j \prod_{k=1}^{n-1} b_k^{n-k-1}\,\mathrm{d}b_k. \tag{3.5.10}$$ eq:3.5.10

4. Since $T \in \mathsf{Jac}(n)$ and $(\Lambda, u) \in \mathcal{W}^n \times S_+^{n-1}$ are in bijection, we have

$$p(\Lambda, u) = \tilde{p}(T(\Lambda, u)) \frac{\partial(T(a,b))}{\partial(\Lambda, u)}. \tag{3.5.11}$$ eq:3.5.11

We compare the expressions in (3.5.9) and (3.5.10) and use Lemma 12 to obtain

$$\frac{\partial(T(a,b))}{\partial(\Lambda,u)} = \frac{C_n}{c_n}\frac{\prod_{k=1}^{n-1} b_k}{\prod_{j=1}^{n} u_j}. \tag{3.5.12}$$

The constants are computed in [DE02]. □

Proof of Theorem 33. The law of We change variables using the spectral mapping and Theorem 34 to obtain the following identity for the law of the Hermite−β ensembles

$$C_{n,\beta} \mathrm{e}^{-\frac{\beta}{4}\operatorname{Tr}(T^2)} \prod_{k=1}^{n-1} b_k^{(\beta-1)(n-k)}\, \mathrm{D}T \tag{3.5.13}$$

$$= C_{n,\beta}\left(\mathrm{e}^{-\frac{\beta}{4}\operatorname{Tr}(\Lambda^2)} \triangle(\Lambda)^{\beta}\, \mathrm{D}\Lambda\right)\left(\prod_{j=1}^{n} u_j^{\beta-1}\right)\mathrm{D}u. \tag{3.5.14}$$

Since the distribution factors, Λ and u are independent with the laws stated in Theorem 33. □

Exercises

3.1. Let $w \in \mathbb{R}^n$ have iid $\mathcal{N}(0,1)$ components. Show that $|w|$ and $w/|w|$ are independent.

3.2. Suppose that $x \in \mathbb{C}^n$. And suppose that $x_1 \notin \mathbb{R}$. Define

- $\tilde{x} = \left(\frac{|x_1|}{x_1}\right) x$,
- $w = \|\tilde{x}\|_2 e_1^n + \tilde{x}$, and
- $v = w/\|w\|_2$.

Show that $P = -\frac{|x_1|}{x_1}(I - 2vv^*)$ satisfies

- $P \in \mathsf{U}(n)$,
- $Px = \|x\|_2 e_1^n$, and
- P is a function of $x/\|x\|_2$ alone.

3.3. Let $U \in \mathsf{O}(n)$ be a random orthogonal matrix. For example U could be a Householder reflection associated to a random vector w. Then assume $A \sim \mathrm{GOE}$. Show that $B := UAU^T \sim \mathrm{GOE}$ and B is independent of U. Hint: Use Lemma 34.

3.4. Write a numerical code to sample matrices from both GOE and the Hermite−1 ensemble. Verify numerically that a suitably normalized density of eigenvalues for the GOE matrix approaches the semicircle law as n increases ($n = 100$ should be ample). Is the same true for the Hermite − 1 ensemble? Why or why not?

3.5. Consider the tridiagonal matrix $T \in \mathsf{Jac}(n)$ that has entries $a_j = 0$, $1 \leq j \leq n$, $b_k = 1$, $1 \leq k \leq n-1$.

(a) Compute explicitly the spectral measure using Chebyshev polynomials (compare T with the recurrence relations for the Chebyshev polynomials).

(b) Plot histograms of two distributions related to T for $n = 100$: (i) the empirical distribution of eigenvalues $\left(\frac{1}{n}\sum_{k=1}^{n}\delta_{\lambda_k}\right)$; (ii) the spectral density $\sum_{k=1}^{n} u_k^2 \delta_{\lambda_k}$. Can you identify the limit in (i)?

(This exercise will be relevant for an enumeration problem relating Brownian excursion to the Riemann-ζ function).

3.6. Establish uniqueness and smoothness in the proof of Theorem thm:Jacobi-spec 29.

3.7. Use equation (eq:rec-tau1 3.4.12) to recursively expand τ_n as a continued fraction. Combine this with the uniqueness step in Q.2 to deduce an alternative approach to Theorem thm:Jacobi-spec 29 that avoids the theory of orthogonal polynomials.

3.8. The following property of the function $-z^{-1}$ is relevant in the continued fraction scheme. Symmetric matrices have a partial order: Given $A, B \in \mathsf{Symm}(n)$ we say that $A \geq B$ if $u^T A u \geq u^T B u$ for every $u \in \mathbb{R}^n$. Suppose $A \geq B \geq 0$. Show that $-A^{-1} \geq -B^{-1}$.

3.9. This problem is a follow-up to exercise 5 in HW 1. Given a map f as in that exercise, compute an (explicit) expression for its derivative Df.

3.10. Compute the following normalization constants:

(a) The normalization constants $Z_{n,\beta}$ in the standard definitions of GOE, GUE and GSE with exponential weight $e^{-\frac{\beta}{4}\operatorname{Tr}(M^2)}$.

(b) The constant $C_{n,\beta}$ in (eq:3.5.13 3.5.13).

(c) The constant C_n in the Jacobian for ensembles (eq:tridiag2 3.2.3) (compare with your calculation of the volume of the unit sphere in HW1).

3.11. The proofs of Dumitriu and Edelman finesse the following issue: given $T \in \mathsf{Jac}(n)$ it requires some care to find a decomposition for the tangent space $T_T\mathsf{Jac}(n)$, especially the isospectral manifold, $\mathcal{M}_T$, that is analogous to Lemma 2. As in that lemma, we may split $T_T\mathsf{Jac}(n)$ into orthogonal subspaces that correspond to diagonal matrices $\dot{\Lambda}$ and $Q^T\dot{Q} \in \mathfrak{o}(n)$. However, while each $Q^T\dot{Q} \in \mathfrak{o}(n)$ generates a curve in $T_T\mathsf{Symm}(n)$, not all $Q^T\dot{Q}$ give rise to curves in $T_T\mathsf{Jac}(n)$. Verify this. Explore this issue further by trying to find a basis for the isospectral manifold $\mathcal{M}_T$ (see equation (eq:isospectral-mfld 3.4.3)).

Chapter 4

Beyond the symmetric eigenvalue problem

ch:nonnormal

In this chapter we discuss other random matrix ensembles that differ fundamentally from GUE, GOE and GSE. For this discussion we concentrate on real and complex matrices. The first ensembles we consider are the real and complex Ginibre ensembles[1], $\mathrm{Gin}_{\mathbb{R}}(m,n)$ on $\mathbb{R}^{m\times n}$ and $\mathrm{Gin}_{\mathbb{C}}(m,n)$ on $\mathbb{C}^{m\times n}$. These are ensembles of real and complex matrices of size $m \times n$. without symmetry conditions. Their densities are given by

$$p_{\mathrm{Gin},\mathbb{R}}(Y)\mathrm{D}Y = \frac{1}{Z_{\mathbb{R},m,n}}\mathrm{e}^{-\frac{1}{2}\operatorname{Tr} Y^T Y}\mathrm{D}Y, \quad p_{\mathrm{Gin},\mathbb{C}}(X)\mathrm{D}X = \frac{1}{Z_{\mathbb{C},m,n}}\mathrm{e}^{-\operatorname{Tr} X^* X}\mathrm{D}X.$$

Thus, the entries are distributed as independent (real or complex) normal random variables. The definition $\mathrm{D}Y$ and $\mathrm{D}X$ in each case follows directly from the volume forms associated to the length elements $\operatorname{Tr}(\mathrm{d}Y^T\mathrm{d}Y)$ and $\operatorname{Tr}(\mathrm{d}X^*\mathrm{d}X)$. When $m = n$ we use the notation $\mathrm{Gin}_{\mathbb{C}}(n)$ and $\mathrm{Gin}_{\mathbb{R}}(n)$ and $Z_{\mathbb{R},n}$ and $Z_{\mathbb{C},m}$.

Our first task is to generalize Weyl's formula to the Ginibre ensembles $\mathrm{Gin}_{\mathbb{R}}(n)$ and $\mathrm{Gin}_{\mathbb{C}}(n)$. To compute this, we use the Schur decomposition. The Schur decomposition is often seen as a numerical tool to perform a spectral decomposition of non-normal matrices. The eigenvalue decomposition is unstable to compute: matrices with distinct eigenvalues are dense and so, computing a Jordan block of a non-normal matrix is a precarious task when round-off errors are present. An arbitrarily small perturbation will lead to an $O(1)$ change in the eigenvalue matrix.

t:schur **Theorem 36.** *All matrices $Y \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ have decompositions*

$$Y = OSO^T, \quad X = UTU^*,$$

[1] Often, the term Ginibre ensemble is reserved for square matrices, but we find it convenient to keep it for all rectangular matrices.

where $O \in \mathsf{O}(n)$, $U \in \mathsf{U}(n)$. Here $T \in \mathbb{C}^{n\times n}$ is upper-triangular and $S \in \mathbb{R}^{n\times n}$ is block-upper triangular with blocks of size 1 or 2. These 2×2 blocks have the form

$$\begin{pmatrix} \alpha & -\gamma \\ \delta & \alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}, \ \ \delta, \gamma > 0. \tag{4.0.1}$$ eq:2x2-form

Furthermore, if the eigenvalues are distinct with a given ordering, and the eigenvectors are normalized (say, first non-zero component is positive), the decomposition is unique.

Unify the following notation

This can be proved by first performing an eigenvalue decomposition and second, performing a QR factorization of the eigenvector matrix. We now describe the QR decomposition algorithm, using Householder reflections, for real matrices. Another numerically viable, but less stable, algorithm is the modified Gram–Schmidt procedure. Both algorithms extend to complex matrices in a straightforward way (see Exercise ex:complexhouse 3.2). Given a matrix $Y \in \mathbb{R}^{m\times n}$, $Y = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}$, define $v(y)$ by

$$v(y) = \tilde{v}/\|\tilde{v}\|_2, \quad \tilde{v} = \|y\|_2 e_1^m - y$$

if $y \neq 0$ and $v(0) = 0$. Then,

$$P_{v(y_1)} Y = \begin{pmatrix} \|y_1\|_2 e_1^n & P_{v(y_1)} y_2 & \cdots & P_{v(y_1)} y_n \end{pmatrix}. \tag{4.0.2}$$

Let I_j be the $j \times j$ identity matrix, then given $y \in \mathbb{R}^p$, define $Q_y \in \mathsf{O}(m)$ by

$$Q_y = \begin{pmatrix} I_{m-p} & 0 \\ 0 & P_{v(y)} \end{pmatrix}, \tag{4.0.3}$$ eq:Qy

The QR factorization of a matrix Y is then given via

$$\begin{aligned} Y^{(0)} &:= Y, \\ Y^{(1)} &:= Q_{v_1} Y^{(0)}, \quad v_1 = Y^{(0)}_{1:m,1}, \\ Y^{(2)} &:= Q_{v_2} Y^{(1)}, \quad v_2 = Y^{(1)}_{2:m,2}, \\ &\ \ \vdots \\ Y^{(j)} &:= Q_{v_j} Y^{(j-1)}, \quad v_j = Y^{(j-1)}_{j:m,j}. \end{aligned} \tag{4.0.4}$$ eq:QR-alg

It follows that $R = Y^{(\min\{m,n\})}$ is upper-triangular and $Y = QR$ where $Q = (Q_{v_{\min\{m,n\}}} \cdots Q_{v_2} Q_{v_1})^T$. We arrive at the following.

t:QRfac **Theorem 37.** *Every matrix $Y \in \mathbb{R}^{m\times n}$, $X \in \mathbb{C}^{m\times n}$ has a factorization $Y = QR$, $X = UT$ such that $Q \in \mathsf{O}(m)$, $U \in \mathsf{U}(m)$ where R, T are upper-triangular with non-negative diagonal entries. The factorization is unique if X (resp. Y) is invertible. This is called the QR factorization, or decomposition, of the matrix.*

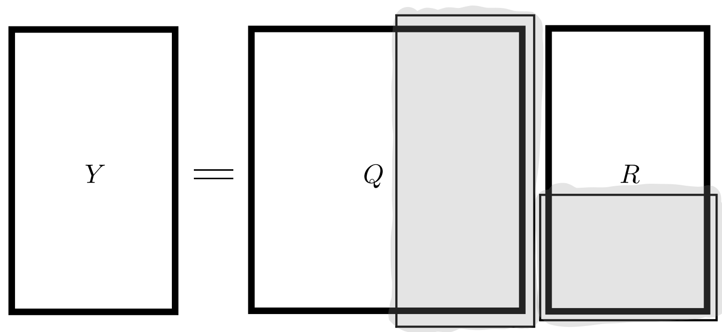

fig:QR Figure 4.0.1: The full QR decomposition in the case $m > n$. The shaded area columns and rows are removed to create the reduced QR decomposition.

This theorem gives the full QR decomposition. If $m > n$, then a $m - n$ columns of Q, U are redundant, and $m - n$ rows of R, T are as well, see Figure fig:QR 4.0.1. After dropping these columns and rows, one obtains the reduced QR decomposition.

If $m > n$, one can count the number of degrees of freedom to see that neither Q nor U could ever be distributed according to Haar measure on $\mathsf{U}(m)$ or $\mathsf{O}(n)$ for $X \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$ or $Y \sim \mathrm{Gin}_{\mathbb{R}}(m,n)$, respectively. So, we instead consider the QR factorization of the augmented matrices

$$\begin{pmatrix} X & X' \end{pmatrix} \text{ and } \begin{pmatrix} Y & Y' \end{pmatrix}, \quad X' \sim \mathrm{Gin}_{\mathbb{C}}(m, m-n), \quad Y' \sim \mathrm{Gin}_{\mathbb{R}}(m, m-n), \tag{4.0.5}$$

eq:extend

for X' and Y' independent of X and Y, respectively. This can be performed even if X and Y are deterministic matrices. So, in the real case, and similarly in the complex case,

$$Y \mapsto \begin{pmatrix} Y & Y' \end{pmatrix} = QR' \mapsto QR := QR' \begin{pmatrix} I_n \\ 0 \end{pmatrix} = Y.$$

Since it is a non-classical theorem for the Schur decomposition, we state the following which establishes the same claim for the eigendecomposition along the way.

t:schur-diff **Theorem 38.** *Let $X(t)$, $X : (-a, a) \to \mathbb{F}^{n \times n}$, $a > 0$, be a C^k matrix function. Assume $X(0)$ has distinct eigenvalues. Then the induced factors $X(t) \mapsto (T(t), U(t))$ or $X(t) \mapsto (S(t), O(t))$ obtained by the Schur decomposition for $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ can be chosen so that they are C^k in a neighborhood of $t = 0$.*

Furthermore, if the first row of $U(0)$ or $O(0)$ has all its entries being positive, then by enforcing the same of $U(t)$ or $O(t)$, for t sufficiently small, one obtains the C^k decomposition.

Proof. Because these eigenvalues are distinct, we can suppose δ is taken sufficiently small so that the sets $\Omega_j = \{z : |\lambda_j - z| < \delta\}$ are disjoint. Then consider

$$P_j(t) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial\Omega_j} (zI - X(t))^{-1} \mathrm{d}z.$$

Since $\|X(t) - X(0)\|_2 \to 0$ as $t \to 0$ and $\max_{z\in\partial\Omega_j} \|(zI - X(0))^{-1}\|_2 < \infty$, it follows that $zI - X(t)$ is invertible for sufficiently small t. This then implies that $P_j(t)$ is as smooth as $X(t)$ is.

Next, form a matrix $V(t) = [v_1(t), v_2(t), \ldots, v_n(t)]$ where

$$v_j(t) = P_j(t)v_j(0), \quad \operatorname{span}\{v_j(0)\} = \operatorname{ran} P_j(0).$$

Provided none of these vectors vanish, we are guaranteed linear independence. Continuity of $P_j(t)$ guarantees that none of these vectors vanish. Then let $V(t) = Q(t)R(t)$ be the (unique) QR decomposition of $V(t)$. It follows that, by the Gram–Schmidt process, the entries of $Q(t)$ and $R(t)$ are smooth functions of the entries of $V(t)$, for t sufficiently small. Then it follows that

$$Q(t)^* X(t) Q(t),$$

is upper-triangular and is as smooth as $X(t)$ is.

For the last claim, we note that the condition that the first row of $U(t)$ or $O(t)$ being positive is sufficient for uniqueness. This construction produces the correct $O(t)$. And this construction can be modified slightly for $U(t)$. This establishes the theorem. □

Finally, before we proceed to pushing forward measure via these decompositions, we prove an elementary result for Ginibre ensembles using the QR factorization.

t:fullrank

Theorem 39. *If $X \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$, $Y \sim \mathrm{Gin}_{\mathbb{R}}(m,n)$, $m \geq n$ then*

$$\mathbb{P}(\operatorname{rank} X < n) = 0 \quad \textit{and} \quad \mathbb{P}(\operatorname{rank} Y < n) = 0.$$

Proof. We use induction on n for the real case. The complex case is similar. If $n = 1$, then a Gaussian vector in $\mathbb{R}^n$ is non-zero with probability one. If $n > 1$, $n \leq m - 1$, assume

$$\mathbb{P}(\operatorname{rank} Y < n) = 0, \quad Y \sim \mathrm{Gin}_{\mathbb{R}}(m,n).$$

Let $b \in \mathbb{R}^m$ be an independent Gaussian vector ($b \sim \mathrm{Gin}_{\mathbb{R}}(m,1)$). Then

$$\mathbb{P}\left(\operatorname{rank}\begin{pmatrix} Y & b \end{pmatrix} < n+1\right) = \mathbb{E}\left[\mathbb{P}\left(\operatorname{rank}\begin{pmatrix} Y & b \end{pmatrix} < n+1 \mid Y\right)\right].$$

On a set of full probability $\operatorname{rank} Y = n$. For such a matrix consider

$$\mathbb{P}\left(\operatorname{rank}\begin{pmatrix} Y & b \end{pmatrix} < n+1 \mid Y\right).$$

Solve

$$Yx = b = QRx = b, \quad Rx = Q^T b =: \tilde{b},$$

and therefore $\tilde{b} \sim \mathrm{Gin}_{\mathbb{R}}(m, 1)$. For this equation to have a solution x, $Rx = \tilde{b}$, since $R \in \mathbb{R}^{m\times n}$, triangular, and $n < m$, the last entry of $\tilde{b}$ must vanish. Thus

$$\mathbb{P}\left(\mathrm{rank}\begin{pmatrix} Y & b \end{pmatrix} < m+1 \,|\, Y\right) = 0$$

almost surely. To truly make this rigorous, one should use Lemma 34. This proves the claim. $\square$

Finally, we want to know that the probability of finding a Ginibre matrix with an eigenvector that has a zero first component is zero.

Theorem 40. *Assume* $X \sim \mathrm{Gin}_{\mathbb{C}}(n)$, $Y \sim \mathrm{Gin}_{\mathbb{R}}(n)$. *Then*

$$\mathbb{P}\left(\exists \lambda \in \mathbb{C}, v \in \mathbb{C}^n, v \neq 0, Xv = \lambda v \text{ and } v_1 = 0\right) = 0,$$
$$\mathbb{P}\left(\exists \lambda \in \mathbb{C}, v \in \mathbb{R}^n, v \neq 0, Yv = \lambda v \text{ and } v_1 = 0\right) = 0.$$

Proof. We prove this for Y. The proof for X is similar. First, we write

$$Y = \begin{pmatrix} y_0 & y_1^T \\ y_2 & Y' \end{pmatrix},$$
$$y_0 \sim \mathrm{Gin}_{\mathbb{R}}(1),\ y_1, y_2 \sim \mathrm{Gin}_{\mathbb{R}}(n-1, 1),\ Y' \sim \mathrm{Gin}_{\mathbb{R}}(n-1, n-1),$$

mutually independent. Let

$$E = \left\{\exists \lambda \in \mathbb{C}, v \in \mathbb{R}^{n-1}, Y'v = \lambda v \text{ and } Y\begin{pmatrix} 0 \\ v \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ v \end{pmatrix}\right\}.$$

It then follows that

$$\mathbb{P}\left(\exists \lambda \in \mathbb{C}, v \in \mathbb{R}^n, Yv = \lambda v \text{ and } v_1 = 0\right) = \mathbb{P}(E) = \mathbb{E}\left[\mathbb{P}(E|Y')\right].$$

Then

$$\mathbb{P}(E|Y') = \mathbb{P}\left(\exists v \in \mathbb{R}^n,\ y_1^T v = 0,\ v \text{ is an eigenvector of } Y'|Y'\right).$$

For the eigenvalue λ_j of Y', let $V_j = \left(v^{(1)}, \ldots, v^{(\ell)}\right)$, $\ell \leq n-1$ be a basis of eigenvectors for this eigenvalue. Then

$$\mathbb{P}\left(\exists \{c_j\} \text{ so that } y_1^T\left(\sum_{j=1}^{\ell} c_j v^{(j)}\right) = 0 \middle| X'\right) = 0,\ a.s.$$

Because, given X', perform a QR factorization of $V_j = QR$, and consider $y_1^T QRc = 0$, $c = (c_1, \ldots, c_j)^T$. But as R has rank ℓ, this amounts to the condition that (at least) one component of the Gaussian vector $x^T = y_1^T Q$ has to vanish, a probability zero event. A union bound over all the distinct eigenvalues proves the result. $\square$

This theorem has an interesting implication. If a matrix Y has a repeated eigenvalue and two linearly independent eigenvectors, then an eigenvector can be constructed that has a zero first component. By the theorem, this event occurs with probability zero for $\mathrm{Gin}_{\mathbb{R}}(n)$, $\mathrm{Gin}_{\mathbb{C}}(n)$. And so, if one shows that Y is diagonalizable with probability one, then Y has distinct eigenvalues with probability one. Nevertheless, it is actually easier to directly show this.

Theorem 41. *Assume $X \sim \mathrm{Gin}_{\mathbb{C}}(n)$, $Y \sim \mathrm{Gin}_{\mathbb{R}}(n)$. Then*

$$\mathbb{P}\left(X \text{ has distinct eigenvalues }\right) = 1,$$
$$\mathbb{P}\left(Y \text{ has distinct eigenvalues }\right) = 1.$$

Proof. We show that the Vandermonde squared $\triangle(\Lambda)^2$ is a polynomial in the entries of the matrix. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Y and consider

$$V = (V_{ij}), \quad V_{jk} = \lambda_j^{k-1}.$$

Then

$$\triangle(\Lambda)^2 = \det(V)^2 = \det(V^T V), \quad (V^T V)_{jk} = \sum_{\ell=1}^{n} \lambda_\ell^{j+k-2} = \operatorname{Tr} Y^{j+k-2}.$$

Now consider a rectangle $R = [a,b]^{n^2} \subset \mathbb{R}^{n^2}$, and assume that

$$\int_R \mathbb{1}_{\{Y \in \mathbb{R}^{n\times n} \ | \ |\triangle(\Lambda)|=0\}} \mathrm{D}Y > 0.$$

Since the set of matrices with distinct eigenvalues is dense, $\triangle(\Lambda) \neq 0$ for some Y. But the only way for the zero locus of a polynomial in n variables to have positive n-dimensional Lebesgue measure is for the polynomial to vanish identically. The theorem follows. □

4.1 Schur decomposition of $\mathrm{Gin}_{\mathbb{C}}(n)$

Theorems 36 and 38 allow us to compute the distribution induced on U and T in the Schur decomposition. We first identify the tangent space.

Theorem 42. *Assume $X \in \mathbb{C}^{n\times n}$ has distinct eigenvalues. Then*

$$T_X(\mathbb{C}^{n\times n}) \cong \mathbb{R}^{n(n-1)} \oplus PT_I \mathsf{U}(n).$$

Proof. A straightforward computation, using the differentiability of the Schur decomposition gives

$$\dot{X} = U(\dot{T} + [U^*\dot{U}, T])U^*, \tag{4.1.1}$$

after using $X(t)$, $t \in (-a,a)$, $a > 0$, differentiating and evaluating at $t = 0$. It follows that $S := U^*\dot{U}$ is skew-Hermitian. We then decompose $T = \Lambda + T_+$ and

$S = S_0 + S_- + S_+$, where the $\pm$ refers to strict upper- and lower- triangular parts. We can first solve for S_- of S in the following way. Define $S_- \mapsto \zeta \in \mathbb{C}^{n(n-1)/2}$ by ordering the entries of using the following relations:

$$\begin{aligned} (i,j) &< (i',j') \quad \text{if} \quad i-j < i'-j', \\ (i,j) &< (i',j') \quad \text{if} \quad i-j = i'-j' \text{ and } i < i'. \end{aligned} \tag{4.1.2}$$

eq:lower-order

The first inequality orders entries by which diagonal they lie on. The second orders within the diagonal. Then

$$\dot{X}_- = [S_-, \Lambda] + [S_-, T_+].$$

With the chosen ordering

$$\zeta \mapsto [S_-, T_+] =: M_- \zeta \tag{4.1.3}$$

eq:M-

is strictly lower triangular. Thus provided $\lambda_i \neq \lambda_j$ for $i \neq j$, we can solve this for S_-. If we then make the choice that $S_0 = 0$, we can clearly solve for $\dot{T}$ once S is known. Finally, by adjusting $\dot{T}$ accordingly, it is clear that any $\dot{X}$ can be achieved with $S_0 = 0$. $\square$

Now, we give the analogue of Weyl's formula for $\mathbb{C}^{n \times n}$.

Theorem 43. *For* $X \in \mathbb{C}^{n\times n}$,

$$\mathrm{D}X = |\triangle(\Lambda)|^2 \, \mathrm{D}T \, \mathrm{D}U, \tag{4.1.4}$$

eq:Weyl-Ginc

where $\mathrm{D}T = \prod_{j=1}^n \mathsf{d}\mathsf{Re}\lambda_j \mathsf{d}\mathsf{Im}\lambda_j \prod_{j<k} \mathsf{d}\mathsf{Re}T_{jk} \mathsf{d}\mathsf{Im}T_{jk}$ *and* $\mathrm{D}U$ *refers to the same distribution as that of the eigenvectors of* $\mathrm{GUE}(n)$.

Proof. We first map X to $\mathbb{C}^{n^2}$ in a consistent way. We order X_- using (eq:lower-order 5.5.2) giving ζ^{X_-}. We then order diagonal(X) in the usual way. Then, finally we order X_+ using

$$(i,j) \prec (i',j') \text{ if and only if } (j,i) < (j',i'),$$

giving ζ^{X_+}, and $X \mapsto [\zeta^{X_-}, \eta, \zeta^{X_+}]^T$. We use ζ^{S_-} and ζ^{T_+} in same way for S_- and T_+, respectively. It then follows that, after ordering $U^*\mathrm{d}XU$,

$$U^*\mathrm{d}XU = \begin{pmatrix} \tilde{\Lambda} + M_- & 0 & 0 \\ D & I & 0 \\ M_+ & 0 & I \end{pmatrix} \begin{pmatrix} \mathrm{d}\zeta^{S_-} \\ \mathrm{d}\Lambda \\ \mathrm{d}\zeta^{T_+} \end{pmatrix}.$$

where $\tilde{\Lambda}\zeta^{S_-}$ is defined through $\zeta^{S_-} \mapsto [S_-, \Lambda]$, which is diagonal, $S_{jk} \mapsto (\lambda_k - \lambda_j) S_{jk}$. M_+ and D are matrices whose exact form is irrelevant. Decomposing all differentials into real and imaginary parts and computing the metric tensor

$$\operatorname{Tr} \mathrm{d}X^* \mathrm{d}X,$$

we find (eq:Weyl-Ginc 4.1.4) by using $\det(\tilde{\Lambda} + M_-) = \prod_{j<k}(\lambda_k - \lambda_j)$ and computing the associate volume form. Here one has to use that if $A : \mathbb{C}^n \to \mathbb{C}^n$ induces $B : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ (by separating real and imaginary parts), then $\det B = |\det A|^2$. $\square$

Theorem 44. *The Schur decomposition of* $\mathrm{Gin}_{\mathbb{C}}(n)$ *is given by*

$$p_{\mathrm{Gin},\mathbb{C}}(X)\mathrm{D}X = \frac{1}{Z_{\mathbb{C},n}} \mathrm{e}^{-\operatorname{Tr} T^*T} |\triangle(\Lambda)|^2 \,\mathrm{D}T\,\mathrm{D}U. \tag{4.1.5}$$

Note that this implies that the strict upper-triangular entries of T are all iid complex normal random variables.

4.2 Eigenvalues and eigenvectors of $\mathrm{Gin}_{\mathbb{R}}(n)$

Computing the analogue of Weyl's formula for $\mathrm{Gin}_{\mathbb{R}}(n)$ is much more complicated. This comes from the fact that complex eigenvalues must arise as complex conjugate pairs. Furthermore, for finite n there is a non-zero probability that the matrix with have k real eigenvalues. Thus the distribution on the eigenvalues is not absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$. We first compute the tangent space, under the assumption of k real eigenvalues.

Theorem 45. *Assume that* Y *has exactly* k *real eigenvalues. Assume further that the real part of all the eigenvalues of* $Y(0) = Y$ *in the closed upper-half plane are distinct. Finally, assume that each* 2×2 *block in the real Schur factorization has* $\gamma \neq \delta$ *in* (4.0.1). *Then*

$$T_Y\mathbb{R}^{n\times n} \cong \mathbb{R}^{n(n-1)/2} \oplus \mathfrak{o}(n).$$

Proof. Assume $Y(t)$ is a smooth curve in $\mathbb{R}^{n\times n}$ such that $Y(t)$ has k real eigenvalues for all t. As before, we have the relation

$$\dot{Y} = O(\dot{S} + [O^T\dot{O}, S])O^T.$$

We need to show that the entries of $\dot{S}$ and $\dot{O}$ are uniquely determined by this relation. We assume

$$S = \left(\begin{array}{cccc|ccc} R_1 & \times & \cdots & & & \cdots & \times \\ 0 & R_2 & \times & \cdots & & \cdots & \times \\ \vdots & \ddots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & R_\ell & \times & \cdots & \times \\ \hline 0 & \cdots & \cdots & 0 & \lambda_1 & \cdots & \times \\ \vdots & & & & \ddots & \ddots & \\ 0 & \cdots & & & \cdots & 0 & \lambda_k \end{array}\right), \quad R_j = \begin{pmatrix} \alpha_j & -\gamma_j \\ \delta_j & \alpha_j \end{pmatrix},$$

where $\ell = (n-k)/2$ and $n-k$ is assumed to be even. The ordering is fixed by $\alpha_j < \alpha_{j+1}$ and $\lambda_j < \lambda_{j+1}$. We also refer to the location of all the imposed zeros in S as the generalized lower-triangular part of S, denoted $L_G(S)$. Similarly, $U_G(S) = (L_G(S^T))^T$ and $D_G(S) = S - U_G(S) - L_G(S)$. So, we have

$$L_G(O^T\dot{Y}O) = L_G\left([A, S]\right), \quad A^T = -A.$$

After careful consideration, we find

$$L_G([A,S]) = L_G([L_G(A), U_G(S)] + [L_G(A), D_G(S)])$$

by noting that

$$\begin{aligned}[A,S] = {} & [L_G(A), L_G(S)] + [D_G(A), L_G(S)] + [U_G(A), L_G(S)] \\ & + [L_G(A), D_G(S)] + [D_G(A), D_G(S)] + [U_G(A), D_G(S)] \\ & + [L_G(A), U_G(S)] + [D_G(A), U_G(S)] + [U_G(A), U_G(S)],\end{aligned}$$

$L_G(S) = 0$, and any term involving only D_G and U_G or only U_G does not contribute to $L_G([A,S])$. Then, it is a non-trivial but straightforward calculation to find that $L_G([D_G(A), D_G(S)]) = 0$. This gives a linear system of equations for $L_G(A)$. Since it will be of use in computing the metric tensor below, we compute the determinant of this matrix in the following lemma.

Lemma 13. *There exists a trivial mapping $L_G(A) \to \xi \in R^{n(n-1)/2-\ell}$ defined by ordering the elements of $L_G(A)$ so that when M is the matrix representation for $\xi \mapsto L_G([A,S])$ we have*

$$\det M = \triangle_k(\Lambda) := \left(\prod_{1\le i<j\le k} (\lambda_j - \lambda_i)\right)\left(\prod_{1\le j<k\le \ell} \triangle_{ij}^{(1)}\right)\left(\prod_{1\le i\le k, 1\le j\le \ell} \triangle_{ij}^{(2)}\right)$$

where $\lambda_1, \ldots, \lambda_k$ are the real eigenvalues, $\mu_j = \alpha_j + \mathrm{i}\beta_j$, $\beta_j > 0$ are the complex eigenvalues (in the upper half plane) and

$$\begin{aligned}\triangle_{ij}^{(1)} &= |\mu_j - \mu_i|^2 |\mu_j - \bar{\mu}_i|^2 = |\mu_j - \mu_i|^2 |\bar{\mu}_j - \mu_i|^2, \\ \triangle_{ij}^{(2)} &= |\mu_j - \lambda_i|^2.\end{aligned}$$

Proof of Lemma 13. The important aspect of this is to choose the ordering. First split

$$L_G(A) = \left(\begin{array}{c|c} A^{(1,1)} & 0 \\ A^{(2,1)} & A^{(2,2)} \end{array}\right).$$

We order the 2×2 blocks of $A^{(1,1)}$ according to (5.5.2). Within each block we use this same ordering. We then order the entries of $A^{(2,2)}$ according to (5.5.2). Finally, we order the 1×2 blocks of $A^{(2,1)}$ according to (5.5.2) and within each block we use this same ordering. This defines $L_G(A) \mapsto \xi \in \mathbb{R}^{n(n-1)/2-\ell}$. Define $L = L_G(L_G(A), U_G(S))$ and decompose L into $L^{(i,j)}$, $i = 1,2$, $j = 1,2$ in the same was as for $L_G(A)$. From the reasoning[2] that went into (4.1.3), we have that the (i,j) block of $L^{(1,1)}$ depends only on blocks (i',j') of $A^{(1,1)}$ for $(i',j') > (i,j)$ and entries in $A^{(2,1)}$. Similarly, the (i,j) entry of $L^{(2,2)}$ depends

[2] The commutator of lower-triangular and upper triangular matrices at entry (i,j) only depends on entries (i',j') of the lower-triangular matrix for $j' \le j$ with $i = i'$ and $i' \ge i$ with $j = j'$. With strict triangularity, fewer dependencies occur.

only on entries (i',j') of $A^{(2,2)}$ for $(i',j') > (i,j)$ and entries in $A^{(2,1)}$. Lastly, one checks that block (i,j) of $L^{(2,1)}$ depends only on blocks (i',j') of $A^{(2,1)}$ for $(i',j') > (i,j)$. This gives a strong form of strict lower-triangularity for $\xi \mapsto L$.

We now show that $\xi \mapsto K := L_G(L_G(A), D_G(S))$ is block-diagonal in a way that does not overlap with this strict lower-triangularity. First, decompose K into $K^{(i,j)}$, $i = 1,2,\ j = 1,2$ in the same was as for $L_G(A)$ and L. We obtain the following relations for blocks of size 2×2, 1×1 and 1×2, respectively:

$$\begin{aligned} K_{ij}^{(1,1)} &= A_{ij}^{(1,1)} R_j - R_i A_{ij}^{(1,1)}, \\ K_{ij}^{(2,2)} &= A_{ij}^{(2,2)} (\lambda_j - \lambda_i), \\ K_{ij}^{(2,1)} &= A_{ij}^{(2,1)} R_j - \lambda_i A_{ij}^{(2,1)}. \end{aligned}$$

The determinants of each of these linear transformations are

$$\begin{gathered} (\alpha_j - \alpha_i)^4 + (\delta_j\gamma_j - \delta_i\gamma_i)^2 + 2(\alpha_j - \alpha_i)^2(\delta_j\gamma_j + \delta_i\gamma_i), \\ (\lambda_j - \lambda_i), \\ (\alpha_j - \lambda_i)^2 + \delta_j\gamma_j, \end{gathered}$$

respectively. For the non-real eigenvalues in the upper-half plane, we have $\mu_j = \alpha_j + \mathrm{i}\sqrt{\gamma_j\delta_j}$. This proves the lemma. $\square$

From this lemma, with our assumptions, we can uniquely find $L_G(A)$. But as A is skew-symmetric, we have ℓ entries left undetermined. So, we consider

$$\begin{aligned} (O^T\dot{Y})_{2j,2j} &= (\dot{S} + [A,S])_{2j,2j} = (\dot{\alpha}_j + (\gamma_j - \delta_j)\dot{s}_{2j+1,2j}) + f_{2j}(L_G(A)), \\ (O^T\dot{Y})_{2j+1,2j+1} &= (\dot{S} + [A,S])_{2j+1,2j+1} = (\dot{\alpha}_j + (\delta_j - \gamma_j)\dot{s}_{2j+1,2j}) + f_{2j}(L_G(A)). \end{aligned} \tag{4.2.1}$$

eq:diagent

for some functions f_j. As $L_G(A)$ is known, this gives a solvable system for $\dot{\alpha}_j$ and $s_{2j+1,2j}$, with determinant $2^\ell \prod_{j=1}^{\ell}(\gamma_j - \delta_j)$. The remaining entries of $\dot{S}$ are given through the relation

$$\dot{S} = O^T\dot{Y}O - [A,S].$$

$\square$

We now can compute the volume form.

Theorem 46. *For $Y \in \mathbb{R}^{n\times n}$ with k real eigenvalues,*

$$\mathrm{D}Y = 2^\ell |\triangle_k(\Lambda)| \left(\prod_{j=1}^{\ell} |\gamma_j - \delta_j| \right) \mathrm{D}S\, \mathrm{D}O, \tag{4.2.2}$$

where

$$\mathrm{D}S = \prod_{j=1}^{\ell} \mathrm{d}\alpha_j \mathrm{d}\gamma_j \mathrm{d}\delta_j \prod_{j=1}^{k} \mathrm{d}\lambda_j \prod_{s\in U_G(S)} \mathrm{d}s, \tag{4.2.3}$$

and DO *refers to the same distribution as that of the eigenvectors of* GOE(n), *i.e., Haar measure on* $\mathsf{O}(n)$.

When we restrict to k real eigenvalues we use the notation

$$p_{\mathrm{Gin},\mathbb{R},k}(Y)\mathrm{D}Y = \frac{1}{Z_{\mathbb{R},n}^{(k)}} \mathrm{e}^{-\frac{1}{2}Y^TY} \mathbb{1}_{\{Y \text{ has } k \text{ real eigenvalues}\}} \mathrm{D}Y. \tag{4.2.4}$$

Theorem 47. *The real Schur decomposition of* $\mathrm{Gin}_{\mathbb{R}}(n)$ *given* k *real eigenvalues is*

$$p_{\mathrm{Gin},\mathbb{R},k}(Y)\mathrm{D}Y = \frac{2^\ell}{Z_{\mathbb{R},n}^{(k)}} \mathrm{e}^{-\frac{1}{2}\operatorname{Tr} S^TS} |\triangle_k(\Lambda)| \left(\prod_{j=1}^{\ell} |\gamma_j - \delta_j| \right) \mathrm{D}S\, \mathrm{D}O. \tag{4.2.5}$$

Note that this implies that the generalized upper-triangular entries of S are all iid normal random variables.

lec:nonsym

Compute probability of k real eigenvalues?

Exercises

4.1. Complete the proof of Theorem 39 using Lemma 34.

4.2. Show that any given eigenvector of $X\mathrm{Gin}_{\mathbb{R}}(n)$ can be taken to be uniformly distributed on S^{n-1}

Chapter 5

Additional matrix factorizations

ch:fact

Expand on this.

The Ginibre ensembles allow us to define the Laguerre ensembles as transformations of $\mathrm{Gin}_{\mathbb{C}}(m,n)$ and $\mathrm{Gin}_{\mathbb{R}}(m,n)$.

Definition 48. The ensembles of positive (semi-) definite matrices defined by X^*X/m where $X \sim \mathrm{Gin}_{\mathbb{C}}(m,n), \mathrm{Gin}_{\mathbb{R}}(m,n)$ are called the Laguerre Unitary Ensemble (LUE(m,n)) and the Laguerre Orthogonal Ensemble (LOE(m,n)), respectively.

The Laguerre ensembles, LOE and LUE, are often referred to as Wishart and complex Wishart matrices, respectively Wishart [?]. They get their name from the close connection to Laguerre polynomials.

Sample covariance interpretation

Next, we turn to understanding the singular value decomposition of $\mathrm{Gin}_{\mathbb{C}}(m,n)$ and $\mathrm{Gin}_{\mathbb{R}}(m,n)$ which will give the eigenvalue distribution of the Laguerre ensembles. The following is the famous singular value decomposition.

Theorem 49. *Every matrix $Y \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{C}^{m\times n}$ has a decomposition*

$$Y = Q\Sigma O^T, \quad X = U\Sigma V^*,$$

where $Q \in \mathsf{O}(m)$, $O \in \mathsf{O}(n)$, $U \in \mathsf{U}(m)$, $V \in \mathsf{U}(m)$ and $\Sigma \in \mathbb{R}^{m\times n}$ is a diagonal matrix with non-negative diagonal entries.

The non-zero entries of Σ are called the singular values of the matrix in question. The singular values of matrix X are precisely the square roots of the non-zero eigenvalues of X^*X.

A main task of this chapter is to establish the following fact.

Theorem 50. *Let $x_1,\ldots,x_n$ be the unordered eigenvalues of* LUE(m,n) *($\beta = 2$) or* LOE(m,n) *($\beta = 1$). The following gives the joint marginal distribution* *on the eigenvalues*

$$\frac{1}{Z_n(\beta)}\prod_{j=1}^{n} x_j^{\frac{\beta}{2}(m-n+1)-1}\prod_{j<k}|x_j-x_k|^{\beta}\mathrm{e}^{-\frac{\beta m}{2}\sum_{j=1}^{n}x_j}\mathbb{1}_{\{x_j\geq 0,\ \textit{for all } j\}}\mathrm{d}x_1\cdots\mathrm{d}x_n. \tag{5.0.1}$$

eq:scm

Cite Wishart

5.1 QR decomposition of $\mathrm{Gin}_{\mathbb{C}}(m,n)$

sec:QRGinc

We now consider the distribution induced on U and T by $\mathrm{Gin}_{\mathbb{C}}(m,n)$. Following the discussion in (4.0.5), we assume $n \geq m$. We follow the push forward of the distributions under the algorithm in (4.0.4). If $X \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$ then it follows that if we replace Q_j with U_j and Y_j with X_j in (4.0.5) then X_j and U_j are independent for every j using the fact that the length of a Gaussian vector is independent of its angle and UX is independent of $U \in \mathsf{U}(m)$ if U is independent of X. And therefore, for $X = UT$, U is independent of T.

From the discussion in Section 3.2 it follows that the induced volume form on T is

$$\propto \mathrm{e}^{-\frac{\beta}{2}\operatorname{Tr}T^*T}\prod_{j=1}^{m}T_{jj}^{2(m-j+1)-1}\mathrm{D}T, \quad \beta = 2,$$

where $\mathrm{D}T$ refers to standard Lebesgue measure on $\mathbb{R}_+^m \times \mathbb{C}^{m(m-1)/2+m(m-n)}$. Note that all the strictly upper-triangular entries are standard complex normal random variables and the entries on the diagonal are all chi-distributed. To understand the distribution on U all we need to do use to use that for $O \in \mathsf{U}(m)$, $OX \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$ if $X \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$. Then factorize

$$X = UT \quad OX = U'T'.$$

From the uniqueness of the QR factorization (on set of full probability where X is invertible), $T = T'$ and $U = O^T U'$. But U and U' have the same distribution and this distribution must therefore be invariant under left multiplication by any element of $\mathsf{U}(m)$. We conclude U is distributed according to Haar measure on $\mathsf{U}(m)$ [Nac76] and to proportionality constants:

$$\mathrm{e}^{-\frac{\beta}{2}\operatorname{Tr}X^*X}\mathrm{D}X \xrightarrow{\mathrm{QR}} \mathrm{e}^{-\frac{\beta}{4}\operatorname{Tr}T^*T}\prod_{j=1}^{\tilde{n}}T_{jj}^{2m-2j+1}\mathrm{D}T\mathrm{D}\tilde{U}, \quad \tilde{n} = \min\{m,n\},$$

where $\mathrm{D}\tilde{U}$ is defined in (2.5.8). The normalization constant is easily computed in terms of Γ-functions. This can be seen as an equality when $m \leq n$. For $m \geq n$, we add additional degrees of freedom to find $\mathrm{D}\tilde{U}$, and so this is the push-forward under a random transformation.

5.2 QR decomposition of $\mathrm{Gin}_{\mathbb{R}}(m,n)$

sec:QRGinr

It follows from the discussion in Section 5.1 that up to proportionality constants

$$\mathrm{e}^{-\frac{\beta}{2}\operatorname{Tr}Y^TY}\mathrm{D}Y \xrightarrow{\mathrm{QR}} \mathrm{e}^{-\frac{\beta}{2}\operatorname{Tr}R^TR}\prod_{j=1}^{\tilde n} R_{jj}^{\beta(m-j+1)-1}\mathrm{D}R\,\mathrm{D}Q, \quad \beta=1, \quad \tilde n = \min\{m,n\},$$

where $\mathrm{D}R$ refers to standard Lebesgue measure on $\mathbb{R}_+^m \times \mathbb{R}^{m(m-1)/2+m(m-n)}$, and $\mathrm{D}Q$ is Haar measure on $\mathsf{O}(n)$.

5.3 Bidiagonalization of Ginibre

sec:tridiag-w

We first consider the reduction of $\mathrm{Gin}_{\mathbb{C}}(m,n)$ and $\mathrm{Gin}_{\mathbb{R}}(m,n)$ to bidiagonal matrices and, in the process, find a generalization of (5.0.1) to general β. This is sometimes called Golub–Kahan bidiagonalization. The aim here is not to preserve eigenvalues, but to preserve singular values as transformations are performed. So, we can perform independent Householder reflections from the left and the right. Recall the definition of Q_y from (4.0.3). Let $Y \sim \mathrm{Gin}_{\mathbb{R}}(m,n)$ for $m \geq n$. Consider the transformations

$$\begin{aligned}
Y^{(0)} &:= Y,\\
\tilde Y^{(1)} &:= Q_{v_1}Y^{(0)}, \quad v_1 = Y^{(0)}_{1:m,1},\\
(Y^{(1)})^T &:= Q_{\tilde v_1}(\tilde Y^{(1)})^T, \quad \tilde v_1 = (\tilde Y^{(1)}_{1,2:n})^T,\\
\tilde Y^{(2)} &:= Q_{v_2}Y^{(1)}, \quad v_2 = Y^{(1)}_{2:m,2},\\
(Y^{(2)})^T &:= Q_{\tilde v_2}(\tilde Y^{(2)})^T, \quad \tilde v_2 = (\tilde Y^{(2)}_{2,3:n})^T,\\
&\ \ \vdots\\
\tilde Y^{(j)} &:= Q_{v_j}Y^{(j-1)}, \quad v_j = Y^{(j-1)}_{j:m,j},\\
(Y^{(j)})^T &:= Q_{\tilde v_j}(\tilde Y^{(j)})^T, \quad \tilde v_j = (\tilde Y^{(j)}_{j,j+1:n})^T.
\end{aligned} \tag{5.3.1}$$

eq:bidiag

Each step in this process eliminates all entries below the diagonal entry in the jth column and all entries to the right of the superdiagonal entry in the jth row. The algorithm terminates when $j=n-1$, returning $Y^{(n-1)}$ which is a bidiagonal matrix with non-negative entries. Let $(Y^{(n-1)})_{jj} = c_j$ and $(Y^{(n-1)})_{j,j+1} = d_j$ for $j=1,2,\ldots$. We find that $(Q_j, \tilde Q_j, c_j, d_j)_{j\geq 1}$ is an independent set of random variables, with Q_j being defined by $v_j \in S_{\mathbb{R}}^{n-j}$ and $\tilde Q_j$ being defined by $\tilde v_j \in S_{\mathbb{R}}^{n-j-1}$ (Q_{n-1} gives a sign flip of one entry). Under this change of variables, following the arguments for (3.3.3), we have

$$\mathrm{D}Y \propto \prod_{j=1}^{n} c_j^{m-j}\mathrm{d}c_j \prod_{k=1}^{n-1} d_k^{n-k-1}\mathrm{d}d_k \prod_{l=1}^{n-2}\mathrm{D}\tilde\omega_k \prod_{p=1}^{n-1}\mathrm{D}\omega_p,$$

where $\mathrm{D}\tilde{\omega}_l$ and $\mathrm{D}\omega_p$ denote uniform measure on $S^l_{\mathbb{R}}$ and $S^p_{\mathbb{R}}$, respectively. Similarly, by applying this algorithm to $X \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$ we find

$$\mathrm{D}X \propto \prod_{j=1}^{n} c_j^{2(m-j)+1}\mathrm{d}c_j \prod_{k=1}^{n-1} d_k^{2(n-k)-1}\mathrm{d}d_k \prod_{l=1}^{n-2}\mathrm{D}\tilde{\omega}_k \prod_{p=1}^{n-1}\mathrm{D}\omega_p,$$

where $\mathrm{D}\tilde{\omega}_l$ and $\mathrm{D}\omega_p$ denote uniform measure on $S^l_{\mathbb{C}}$ and $S^p_{\mathbb{C}}$, respectively.

5.4 The Cholesky decomposition

To compute the singular value decomposition of $\mathrm{Gin}_{\mathbb{R}}(m,n)$ and $\mathrm{Gin}_{\mathbb{C}}(m,n)$ (the square roots of the eigenvalues of the Laguerre ensembles) we follow the approach of Edelman [Ede89] and first compute the Cholesky decomposition.

Theorem 51. *Every strictly positive definite matrix $A \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) has a unique decomposition*

$$A = LL^T \quad (A = LL^*),$$

where $L \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) is a lower-triangular matrix with positive diagonal entries.

Proof. We concentrate on the real case and we first show uniqueness. Assume $A = LL^T = L_1L_1^T$ for two different factorizations. Then

$$L_1^{-1}L = L_1^TL^{-T}, \quad \text{where} \quad L^{-T} = (L^{-1})^T.$$

Since the non-singular upper- and lower-triangular matrices for groups, the left-hand (right-hand) side is lower-triangular (upper-triangular). Therefore $L_1^{-1}L$ is a diagonal matrix that is equal to its own transpose-inverse: $e_jL_1^{-1}Le_j = \pm 1$. Positivity of the diagonal entries gives $L_1 = L$. Now, by Gaussian elimination, without pivoting[1] $A = \tilde{L}U$ where $\tilde{L}$ is lower-triangular and U is upper-triangular. Here $\tilde{L}$ has ones on the diagonal. We know that $e_j^TAe_j > 0$ and therefore $e_j^T\tilde{L}Ue_j = U_{jj} > 0$. Then Let $U_d = \mathrm{diagonal}(U)^{1/2}$ and $A = \tilde{L}U_dU_d^{-1}U$. It follows from the symmetry of A that $L = \tilde{L}U_d$ gives the Cholesky factorization. Similar considerations follow for $A \in \mathbb{C}^{m\times n}$. $\square$

5.5 Change of variables for $\mathrm{Gin}_{\mathbb{C}}(m,n)$

We now consider the change of variables that closely resembles the singular value decomposition, but differs in a fundamental way. For $X \in \mathbb{C}^{m\times n}$, full rank, define

$$X = UT \overset{QR}{\mapsto} (U,T) \overset{\text{Inv. Cholesky}}{\mapsto} (U, A = T^*T) = (U, V\Lambda V^*) \overset{\text{Spectral map}}{\mapsto} (U,\Lambda,V). \quad (5.5.1)$$ eq:mapp

[1]Pivoting is not required for strictly positive definite matrices because the upper left $\ell\times\ell$ blocks are non-singular for every ℓ.

This is a well-defined, invertible mapping, provided that the first row of V contains non-vanishing entries. It will follow from Section sec:tridiag-w 5.3 that the probability of this is one. But we emphasize that for this decomposition $X \neq U\Lambda V^*$, generally. We now show that if $X \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$ then U, Λ, V are independent and we then characterize the distribution of Λ and V.

Lemma 14 (Spectral variables for $\mathsf{Her}_+(n)$)**.** *If* $A \in \mathsf{Her}_+(n)$ *is non-singular with distinct eigenvalues then*

$$T_A\mathsf{Her}_+(n) \cong \mathbb{R}^n \oplus PT_I\mathsf{U}(n).$$

Proof. The proof is essentially the same as Lemma le:herm-decomp 6, just using that the set of strictly positive definite matrices is open. □

We define $\mathrm{D}A$ in the natural way as the volume form induced by the metric tensor $\mathrm{Tr}\, dA^2$. We then have the analogous formula to Theorem thm-weyl-formula 16:

$$\mathrm{D}A = |\triangle(\Lambda)|^2\, \mathrm{D}\Lambda\, \mathrm{D}U.$$

Next, we compute the volume form associated with the change Cholesky change of variables.

Lemma 15. *Let* $A = LL^*$ *be the Cholesky decomposition for a non-singular* $A \in \mathsf{Her}_+(n)$*. Let* $\mathrm{D}L$ *be the natural volume form induced by* $\mathrm{Tr}(\mathrm{d}L^*\mathrm{d}L)$*. Then*

$$\mathrm{D}A = 2^n \prod_{j=1}^{n} L_{jj}^{2(n-j)+1}\mathrm{D}L.$$

Proof. We prove this by identifying that the Jacobian of the transformation is triangular, and computing the diagonal entries. We first compute for $j \geq k$

$$\frac{\partial A}{\partial \mathsf{Re} L_{jk}} = e_j e_k^T L^* + L e_k e_j^T, \quad \frac{\partial A}{\partial \mathsf{Im} L_{jk}} = e_j e_k^T L^* - L e_k e_j^T.$$

Examine the structure of these matrices. Since $e_j e_k^T L^*$ is the matrix that contains the kth row of L^* in its jth row, with all other row being zero we find the following picture

$$\frac{\partial A}{\partial \mathsf{Re} L_{jk}} = \left(\begin{array}{ccccccc|c|c} & & & & & & & 0 & \\ & & & & & & & \vdots & \\ & & & & & & & 0 & \\ & & & & & & & L_{kk} & \\ & & & & & & & L_{k+1,k} & \\ & & & & & & & \vdots & \\ & & & & & & & L_{j-1,k} & \\ \hline 0 & \cdots & 0 & \overline{L}_{kk} & \overline{L}_{k+1,k} & \cdots & \overline{L}_{j-1,k} & 2\mathsf{Re} L_{jk} & \cdots \\ \hline & & & & & & & \vdots & \end{array}\right).$$

Here only the jth row and jth column have non-zero entries. Here $2\mathsf{Re}L_{jk}$ is in the (j,j) entry. A similar picture holds for $\frac{\partial A}{\partial \mathsf{Im} L_{jk}}$, with $2\mathsf{Im}L_{jk}$ in the (j,j) entry. We define a mapping $\mathsf{Re}L \mapsto \xi \in \mathbb{R}^{n(n-1)/2}$ and $\mathsf{Im}L \mapsto \eta \in \mathbb{R}^{n(n-3)/2}$ by the ordering of the non-zero elements of L:

$$\begin{aligned} (j,k) &< (j',k') \quad \text{if} \quad j < j', \\ (j,k) &< (j\ ,k') \quad \text{if} \quad k < k'. \end{aligned} \tag{5.5.2}$$

eq:lower-order

This orders first by row, and then by columns within each row. Assume $(i,\ell) < (j,k)$, $j \geq k$, $i \geq \ell$. Then

$$\frac{\partial A_{i\ell}}{\partial \mathsf{Re} L_{jk}} = 0, \quad \frac{\partial A_{i\ell}}{\partial \mathsf{Im} L_{jk}} = 0.$$

because either $i < j$ or $\ell < k'$ if $i = j$. And, it is clear that

$$\begin{aligned} &\frac{\partial A_{jk}}{\partial \mathsf{Re} L_{jk}} = L_{kk}, \quad j > k, \quad \frac{\partial A_{jk}}{\partial \mathsf{Re} L_{jk}} = 2L_{kk}, \quad j = k, \\ &\frac{\partial A_{jk}}{\partial \mathsf{Im} L_{jk}} = L_{kk}, \quad j > k. \end{aligned}$$

Then, if we define $L \mapsto \zeta$ where $\zeta = (\xi_1, \eta_1, \xi_2, \eta_2, \ldots)^T$ we find that the Jacobian is triangular and

$$\frac{\partial A}{\partial L} = 2^n \prod_{j=1}^n L_{jj}^{2(n-j)+1}.$$

$\square$

This theorem allows one to understand transformations of $\mathrm{Gin}_{\mathbb{C}}(m,n)$. Following the transformation (eq:mapp 5.5.1), with $X \in \mathbb{C}^{m\times n}$ with $m \geq n$ using $T = L^*$ noting that

$$T = \begin{pmatrix} \tilde{T} \\ 0 \end{pmatrix}.$$

CHECK THIS!!!!!

where $\tilde{T}$ is a upper-triangular matrix with positive diagonal entries. Then

$$\mathrm{D}X \overset{\mathrm{QR}}{\longrightarrow} \prod_{j=1}^n T_{jj}^{2(m-j+1)-1} \mathrm{D}T\,\mathrm{D}\tilde{U} = 2^{-n} \prod_{j=1}^n T_{jj}^{2(m-n)} \mathrm{D}A\,\mathrm{D}\tilde{U} \tag{5.5.3}$$

eq:mapp-com

$$= 2^{-n} \prod_{j=1}^n \sigma_j^{2(m-n)} |\triangle(\Sigma^2)|^2 \mathrm{D}(\Sigma^2)\,\mathrm{D}\tilde{U}\,\mathrm{D}V. \tag{5.5.4}$$

eq:mapp-com2

Here $\mathrm{D}\tilde{U}$ is Haar measure on $\mathsf{U}(n)$ and $\mathrm{D}V$ represents the same distribution as the eigenvectors of $\mathrm{GUE}(n)$. Also, $\mathrm{D}\Sigma$ is Lebesgue measure on $\mathbb{R}_+^n$. As noted below (eq:mapp 5.5.1), this is not the singular value decomposition for X, but we claim, it is in a distributional sense. For $X \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$, $m \geq n$ and consider

$$X = U_1 \Sigma V, \quad \tilde{X} := U \Sigma V$$

where (U, V, Σ) are independent with joint distribution (eq:mapp-com2 5.5.4), U_1 is the matrix of left singular vectors for X, and U is independent of U_1. Then $\tilde{X} = U U_1^* X$, but then by the invariance of U, for measureable sets $S_1 \subset \mathsf{U}(m)$, $S_2 \subset \mathbb{C}^{m\times m}$,

$$\begin{aligned}
&\mathbb{P}(UU_1^* \in S_1) = \mathbb{P}(U \in S_1 U_1) = \mathbb{P}(U \in S_1),\\
&\mathbb{P}(UU_1^* \in S_1, X \in S_2) = \mathbb{P}(U \in S_1 U_1, X \in S_2)\\
&\qquad\qquad = \int_{S_2} \left(\int_{S_1 U_1} \mathrm{D}U \right) p_{\mathrm{Gin},\mathbb{C}}(X) \mathrm{D}X = \mathbb{P}(U \in S_1)\mathbb{P}(X \in S_2).
\end{aligned}$$

So, UU_1^* is independent of X and therefore $\tilde{X}$ must have the same distribution as X. This implies the singular value decomposition of $\mathrm{Gin}_{\mathbb{C}}(m,n)$ is given by (eq:mapp-com2 5.5.4).

Remark 52. If one wants to match of dimensions, then $\mathrm{D}U$ should be replaced by the push-forward of uniform measure on $S_{\mathbb{C}}^{m-1} \times S_{\mathbb{C}}^{m-2} \times \cdots \times S_{\mathbb{C}}^{m-n-1}$ onto $\mathsf{U}(m)$ via Householder reflections.

Is this really correct? Dimensions match, but....

5.6 Change of variables for $\mathrm{Gin}_{\mathbb{R}}(m,n)$

Similar considerations show for $Y = Q\Sigma O^T \sim \mathrm{Gin}_{\mathbb{R}}(m,n)$ the singular value distributions are given by

$$\mathrm{D}Y \xrightarrow{\mathrm{QR}} 2^{-n} \prod_{j=1}^{n} \Sigma_j^{m-n-1} |\triangle(\Sigma^2)| \mathrm{D}\Sigma^2 \, \mathrm{D}Q \, \mathrm{D}O$$

where $\mathrm{D}O$ is Haar measure on $\mathsf{U}(n)$, $\mathrm{D}Q$ is Haar measure on $\mathsf{O}(m)$ and $\mathrm{D}\Sigma$ is as before.

In both cases, $\mathrm{Gin}_{\mathbb{R}}(m,n)$ or $\mathrm{Gin}_{\mathbb{C}}(m,n)$, if $m < n$, then same distributional description holds with the addition of $n-m$ point masses at zero for $\Sigma_1, \ldots, \Sigma_{n-m}$ (depending one's ordering convention) to indicate the deficiency of the matrix.

Exercises

5.1. Write a numerical code to compute the Cholesky decomposition of a symmetric positive-definite tridiagonal matrix.

Chapter 6

Determinantal formulas: From Vandermonde to Fredholm

ch:det

Our purpose in this section is to present the elegant determinantal formulas of Dyson, Gaudin and Mehta for invariant matrix ensembles on $\mathsf{Her}(n)$. These formulas combine three distinct elements: (i) the Weyl formula on $\mathsf{Her}(n)$; (ii) the theory of orthogonal polynomials; (iii) Fredholm determinants. We first introduce these formulas for GUE. We then use the asymptotic properties of Hermite polynomials to establish their scaling limits (Theorem 2, Theorem 6 and Theorem 9). While the eigenvalues of GOE and GSE do not have a determinantal structure, they have a related Pfaffian structure, which is described in a later chapter.

6.1 Probabilities as determinants

In what follows we will adopt the following notation. In order to avoid confusion, we let $x = (x_1(M), \dots, x_n(M)) \in \mathbb{R}^n$ denote the <u>unordered</u> eigenvalues[1] of M, and $\lambda = (\lambda_1(M), \dots, \lambda_n(M)) \in \mathcal{W}^n$ denote the <u>ordered</u> eigenvalues of M. We use $x_j = x_j(M)$ and $\lambda_j = \lambda_j(M)$ when M is clear from context. The probability density of x, denoted $P^{(n)}(x_1, \dots, x_n)$, is obtained from the Weyl's formula

$$P^{(n)}(x_1, \dots, x_n) = \frac{1}{Z_n} \triangle(x)^2 \mathrm{e}^{-\frac{1}{2}\sum_{k=1}^n x_k^2}. \tag{6.1.1}$$

eq:det-form1

Observe that $P^{(n)}$ is invariant under permutations $(x_1, \dots, x_n) \mapsto (x_{\sigma_1}, \dots, x_{\sigma_n})$, $\sigma \in \mathsf{S}(n)$. In practice, our interest lies not in the joint density of all n eigenvalues, but statistics such as the law of the largest eigenvalue. Thus, what is

[1] These is is clearly not well-defined. If M is random one can compute the eigenvalues and then randomly permute them.

required is an analytical technique to extract such information from (6.1.1) by integrating out degrees of freedom to obtain information on the joint distribution of m-eigenvalues, $1 \leq m \leq n$. More precisely, given m and a Borel function $f : \mathbb{R}^m \to \mathbb{R}$ that is symmetric under permutations,

I don't think we need this condition, the sum symmeterizes f, right?

$$f(x_1, \dots, x_m) = f(x_{\sigma_1}, \dots, x_{\sigma_m}), \quad \sigma \in \mathsf{S}(m), \tag{6.1.2}$$

eq:det-form2

we consider random variables of the type

$$N_f = \sum_{(j_1,\dots,j_m)\in[\![1,n]\!]^m,\ j_k \text{distinct}} f(x_{j_1}, \dots, x_{j_m}). \tag{6.1.3}$$

eq:det-form3

Expectations of random variables of the form (6.1.3) are given by

$$\mathbb{E}(N_f) = \int_{\mathbb{R}^m} f(x_1, \dots, x_m)\, R_m^{(n)}(x_1, \dots, x_m)\, \mathrm{d}x_1 \dots \mathrm{d}x_m, \tag{6.1.4}$$

eq:det-form6

where R_m is the <u>m-point correlation function</u>

$$\begin{aligned} &R_m^{(n)}(x_1, \dots, x_m) \\ &= \frac{n!}{(n-m)!} \int_{\mathbb{R}^{n-m}} P^{(n)}(x_1, \dots, x_m, x_{m+1}, \dots, x_n)\, \mathrm{d}x_{m+1} \dots \mathrm{d}x_n. \end{aligned} \tag{6.1.5}$$

eq:det-form4

The combinatorial factor in (6.1.3) arises as follows. There are $\binom{n}{m}$ ways of picking subsets of m distinct indices from $[\![1,n]\!]$. On the other hand,

$$R_m^{(n)}(x_1, \dots, x_m) = R_m^{(n)}(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_m}), \quad \sigma \in \mathsf{S}(m). \tag{6.1.6}$$

eq:det-form5

and the integral on the right hand side of (6.1.6) appears $m!$ times when integrating over the complementary $n-m$ variables for each choice of indices $\{j_1, \dots, j_m\} \in [\![1,n]\!]^m$. We state the following theorem which is proved in the following sections.

thm:det-formula

Theorem 53. *The joint density and m-point functions for* GUE(n) *are*

$$\begin{aligned} P^{(n)}(x_1, \dots, x_n) &= \frac{1}{n!} \det\left(K_n(x_j, x_k)_{1\leq j,k\leq n}\right), \qquad (6.1.7)\\ R_m^{(n)}(x_1, \dots, x_m) &= \det\left(K_n(x_j, x_k)_{1\leq j,k\leq m}\right), \qquad (6.1.8) \end{aligned}$$

where the integral kernel K_n is defined by the Hermite wave functions

$$K_n(x,y) = \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y). \tag{6.1.9}$$

eq:det-form9

Remark 54. The kernel K_n may be simplified using identities for the Hermite polynomials. The Christoffel-Darboux formula (E.1.16) allows us to write

$$K_n(x,y) = \sqrt{n}\frac{\psi_n(x)\psi_{n-1}(y) - \psi_n(y)\psi_{n-1}(x)}{x-y}. \tag{6.1.10}$$

eq:christoffel-darbou

Further, eliminating ψ_{n-1} with the identity (E.1.14) yields

$$K_n(x,y) = \frac{\psi_n(x)\psi_n'(y) - \psi_n(x)\psi_{n'}(y)}{x-y} - \frac{1}{2}\psi_n(x)\psi_n(y). \tag{6.1.11}$$

eq:christoffel-darbou

A particular consequence of Theorem 53 is the following fundamental formula. Assume S is a bounded Borel set, let $\mathbb{1}_S$ denote its indicator function, and let $A_m(S)$ denote the probability that the set S contains precisely m eigenvalues for $M \in \mathrm{GUE}(n)$.

thm:fred-det-prob **Theorem 55.** *The generating function of* $\{A_m(S)\}_{m=0}^{\infty}$ *is given by the formula*

$$\det\left(I - zK_n\mathbb{1}_S\right) = \sum_{m=0}^{\infty} A_m(S)(1-z)^m, \quad z \in \mathbb{C}, \tag{6.1.12}$$ eq:det-form-10b

where $\det\left(I - zK_n\mathbb{1}_S\right)$ *denotes the Fredholm determinant of the kernel*

$$K_n\mathbb{1}_S(x,y) = \sum_{k=0}^{n-1} \mathbb{1}_S(x)\psi_k(x)\psi_k(y)\mathbb{1}_S(y). \tag{6.1.13}$$ eq:det-form-10a

Theorem 53 and Theorem 55 illustrate the general spirit of determinantal formulas in random matrix theory. The density of a joint distribution is expressed as a determinant of an integral operator with finite rank. One may then use the theory of orthogonal polynomials, in particular, results on the asymptotics of orthogonal polynomials, to establish the basic limit theorems outlined in Chapter 1 (see Theorems 56 and Theorem 57 below).

Appendices E and D provide brief introductions to Hermite polynomials and Fredholm determinants respectively.

6.2 The m-point correlation function

Proof of Theorem 53. We form linear combinations of the rows of the Vandermonde matrix to obtain

Check ± 1 on Vandermonde throughout

$$\triangle(x) = \det\begin{pmatrix} \mathfrak{h}_0(x_1) & \mathfrak{h}_0(x_2) & \dots & \mathfrak{h}_0(x_n) \\ \mathfrak{h}_1(x_1) & \mathfrak{h}_1(x_2) & \dots & \mathfrak{h}_1(x_n) \\ \vdots & \vdots & & \vdots \\ \mathfrak{h}_{n-1}(x_1) & \mathfrak{h}_{n-1}(x_2) & \dots & \mathfrak{h}_{n-1}(x_n) \end{pmatrix}. \tag{6.2.1}$$ eq:det-form11

The calculations above would apply to any set of monic polynomials of degree $0, 1, 2, \dots, n-1$. The Hermite polynomials and wave functions are relevant because they satisfy the orthogonality relations

$$\int_{\mathbb{R}} \mathfrak{h}_j(x)\mathfrak{h}_k(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x = \delta_{jk}k!, \tag{6.2.2}$$ eq:det-form12

and allow the inclusion of an exponential weight. Precisely, the Hermite wave functions

$$\psi_k(x) = \frac{1}{\sqrt{k!}}\mathfrak{h}_k(x)\frac{\mathrm{e}^{-x^2/4}}{(2\pi)^{1/4}}, \tag{6.2.3}$$ eq:det-form13

satisfy the orthogonality relation

$$\int_{\mathbb{R}} \psi_j(x)\psi_k(x)\,\mathrm{d}x = \delta_{jk}, \tag{6.2.4}$$ eq:det-form14

and form a basis for $L^2(\mathbb{R})$. Let H denote the matrix with entries $H_{jk} = \psi_{j-1}(x_k)$. It follows from (eq:det-form11 6.2.1) and (eq:det-form13 6.2.3) that

$$\mathrm{e}^{-\frac{x^2}{2}}\triangle(x)^2 \propto \det H^2 = \det H^T H = \det\left[K_n(x_j,x_k)\right]_{1\le j,k\le n}, \tag{6.2.5}$$ eq:det-form14b

using the identity

$$\left(H^T H\right)_{jk} = \sum_{l=1}^{n} H_{lj}H_{lk} = \sum_{l=0}^{n-1}\psi_l(x_j)\psi_l(x_k) = K_n(x_j,x_k). \tag{6.2.6}$$ eq:det-form17

Therefore, the joint density $P^{(n)}(x)$ is proportional to $\det K_n$. To determine the constant of proportionality we recall that the determinant of a matrix $A = [a_{jk}]_{1\le j,k\le n}$ satisfies

$$\det A = \sum_{\sigma\in \mathsf{S}(n)} \operatorname{sgn}(\sigma)\prod_{j=1}^{n} a_{\sigma_j j} \tag{6.2.7}$$

where $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. We then evaluate the integral

$$\begin{aligned}\int_{\mathbb{R}^n}\det(H)^2\mathrm{d}x_1\ldots\mathrm{d}x_n &= \int_{\mathbb{R}^n}\left(\det\left[\psi_{j-1}(x_k)\right]_{1\le j,k\le n}\right)^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\ &= \sum_{\sigma,\tau\in\mathsf{S}(n)}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\int_{\mathbb{R}^n}\prod_{j=1}^{n}\psi_{\sigma_j-1}(x_j)\psi_{\tau_j-1}(x_j)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\ &= \sum_{\sigma,\tau\in\mathsf{S}(n)}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{j=1}^{n}\delta_{\sigma_j,\tau_j} = \sum_{\sigma,\tau\in\mathsf{S}(n)}\mathbb{1}_{\{\sigma=\tau\}} = n!\,.\end{aligned} \tag{6.2.8}$$

We combine (eq:det-form15 6.2.8) and (eq:det-form17 6.2.6) to obtain the first assertion in Theorem thm:det-formula 53:

$$P^{(n)}(x_1,\ldots,x_n) = \frac{1}{n!}\det\left[K_n(x_j,x_k)\right]_{1\le j,k\le n}.$$

The formulas for the correlation functions may be obtained by induction, beginning with

$$R^{(n)}(x_1,\ldots,x_n) = \det\left[K_n(x_j,x_k)\right]_{1\le j,k\le n}. \tag{6.2.9}$$ eq:det-form19

First, the orthonormality relations (eq:det-form14 6.2.4) imply

$$\int_{\mathbb{R}} K_n(x,x)\mathrm{d}x = n, \qquad \int_{\mathbb{R}} K_n(x,z)K_n(z,y)\,\mathrm{d}z = K_n(x,y). \tag{6.2.10}$$ eq:det-form20

Assume (eq:det-form8 6.1.8) holds for an index $m+1 \le n$. We then have

$$\begin{aligned} R_m^{(n)}(x_1,\dots,x_m) &= \frac{1}{n-m}\int_{\mathbb{R}} R_{m+1}^{(n)}(x_1,\dots,x_m,x_{m+1})\,\mathrm{d}x_{m+1} \\ &= \frac{1}{n-m}\int_{\mathbb{R}} \det\left[K_n(x_j,x_k)\right]_{1\le j,k\le m+1}\,\mathrm{d}x_{m+1} \qquad (6.2.11) \\ &= \frac{1}{n-m}\sum_{\sigma\in\mathsf{S}(m+1)}\mathrm{sgn}(\sigma)\int_{\mathbb{R}} K_n(x_1,x_{\sigma_1})\dots K_n(x_{m+1},x_{\sigma_{m+1}})\,\mathrm{d}x_{m+1}. \end{aligned}$$

If $\sigma_{m+1}=m+1$ in this sum, then the first equality in (eq:det-form20 6.2.10) implies

$$\begin{aligned} &\int_{\mathbb{R}} K_n(x_1,x_{\sigma_1})\cdots K_n(x_{m+1},x_{\sigma_{m+1}})\,\mathrm{d}x_{m+1} \qquad (6.2.12) \\ &\qquad = n\,K_n(x_1,x_{\sigma_1})\cdots K_n(x_m,x_{\sigma_m}). \end{aligned}$$

If $\sigma_{m+1}\neq m+1$, there exists $j\le m$ and $k\le m$ such that $\sigma_j=m+1$ and $\sigma_{m+1}=k$. We then use the second equality in (eq:det-form20 6.2.10) to find

$$\begin{aligned} &\int_{\mathbb{R}} K_n(x_1,x_{\sigma_1})\cdots K_n(x_{m+1},x_{\sigma_{m+1}})\,\mathrm{d}x_{m+1} \qquad (6.2.13) \\ &\quad = \int_{\mathbb{R}} K_n(x_1,x_{\sigma_1})\cdots K_n(x_j,x_{m+1})\cdots K_n(x_{m+1},x_k)\,\mathrm{d}x_{m+1} \\ &\quad = K_n(x_1,x_{\sigma_1'})\cdots K_n(x_m,x_{\sigma_m'}). \end{aligned}$$

eq:det-form22

where σ' is a permutation of $\{1,\dots,m\}$ such that $\sigma_j'=k$ and $\sigma_l'=\sigma_l$ if $l\neq j$. Each permutation $\sigma'\in S_m$ may come from m permutations $\sigma\in S_{m+1}$. Further, $\mathrm{sgn}(\sigma')=-\mathrm{sgn}(\sigma)$ since these permutations differ by a single swap. Therefore, using equations (eq:det-form21 6.2.12) and (eq:det-form22 6.2.13) we have

$$\int_{\mathbb{R}} \det\left[K_n(x_j,x_k)\right]_{1\le j,k\le m+1}\,\mathrm{d}x_{m+1} = (n-m)\det\left[K_n(x_j,x_k)\right]_{1\le j,k\le m}.$$

Lecture Note 6. Let $\sigma'\in\mathsf{S}(m)$ and for $1\le j\le m$, define $\sigma\in\mathsf{S}(m+1)$ by $\sigma_l=\sigma_l'$ for $1\le l\le m$ and $l\neq j$. Then $\sigma_l=m+1$ and $\sigma_{m+1}=\sigma_j'$. Then this procedure maps $\sigma\to\sigma'$ and j is is arbitrary. And, if we swap σ_j with σ_{m+1} we then see that $\mathrm{sgn}(\sigma)=-\mathrm{sgn}(\sigma')$.

□

6.3 Determinants as generating functions

Proof of Theorem thm:fred-det-prob *55.* The Fredholm determinant $\det(I-zK_n\mathbb{1}_S)$ is an entire function of z. Thus, equation (eq:det-form-10b 6.1.12) is equivalent to the statement

$$A_m(S)=\frac{1}{m!}\left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \det\left(I-zK_n\mathbb{1}_S\right)\Big|_{z=1}. \qquad (6.3.1)$$

eq:det-form-10

We first prove formula (6.3.1) in the case $m = 0$. Let $\mathbb{1}$ denote the characteristic function of the set S. The probability that all eigenvalues lie outside S is given by

$$\int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} (1 - \mathbb{1}_S(x_j)) \right) P^{(n)}(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n \tag{6.3.2}$$
$$= \sum_{j=0}^{n} (-1)^j \int_{\mathbb{R}^n} \rho_j^n(\mathbb{1}_S(x_1), \dots, \mathbb{1}_S(x_n)) P^{(n)}(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n,$$

where $\rho_j^n(x_1, \dots, x_n)$ is the j-th symmetric function in n variables. For example,

$$\rho_0^n(x) = 1, \quad \rho_1^n(x) = \sum_{j=1}^{n} x_j, \quad \rho_2^n(x) = \sum_{j<k}^{n} x_j x_k, \quad \rho_n^n(x) = \prod_{j=1}^{n} x_j.$$

Then, we can express

$$\rho_j^n(x) = \frac{1}{j!} \sum_{(j_1,\dots,j_m) \in [\![1,n]\!]^j,\ j_k \text{distinct}} \prod_k x_{j_k}.$$

Since each term in ρ_j^n consists of j products of the form $\mathbb{1}_S(x_{\sigma_j})$ for some permutation $\sigma \in S_n$, we integrate over the remaining $n - j$ variables, and use the permutation invariance of $R_j^{(n)}$ Using the m-point correlation function, we obtain using (6.1.4) with $f(x_1, \dots, x_m) = \prod_{j=1}^{m} \mathbb{1}_S(x_j)$,

$$\mathbb{E}\left(N_f\right) = \int_{\mathbb{R}^n} \rho_j^n(\mathbb{1}_S(x_1), \dots, \mathbb{1}_S(x_n)) P^{(n)}(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n \tag{6.3.3}$$
$$= \frac{1}{j!} \int_{\mathbb{R}^j} \det \left[K_n \mathbb{1}_S(x_k, x_l) \right]_{1 \le k,l \le j} \mathrm{d}x_1 \dots \mathrm{d}x_j.$$

In the last equality, we have used (6.1.8) and multiplied the kernel on the left and right by the diagonal matrix $d_S = \mathrm{diag}(\mathbb{1}_S(x_1), \dots, \mathbb{1}_S(x_j))$, so that

$$\mathbb{1}_S(x_1) \dots \mathbb{1}_S(x_j) R_j^{(n)}(x_1, \dots, x_j) = \mathbb{1}_S^2(x_1) \dots \mathbb{1}_S^2(x_j) R_j^{(n)}(x_1, \dots, x_j)$$
$$= \det \left(d_S \left[K_n(x_k, x_l) \right]_{1 \le k,l \le j} d_S \right) = \det \left[K_n \mathbb{1}_S(x_k, x_l) \right]_{1 \le k,l \le j},$$

where $K^{(n)} \mathbb{1}_S$ is defined in (6.1.13). We now combine (6.3.2) and (6.3.3) to obtain

$$\sum_{j=0}^{n} (-1)^j \int_{\mathbb{R}^n} \rho_j^n(\mathbb{1}_S(x_1), \dots, \mathbb{1}_S(x_n)) P^{(n)}(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n$$
$$= \det(I - K_n \mathbb{1}_S), \tag{6.3.4}$$

using the infinite series (D.1.8) for the Fredholm determinant (only n terms are non-zero, since $K^{(n)}$ has rank n, see Exercise 7.2).

We now turn to the case $m \geq 1$. Equation eq:fred1 (6.3.2) must now be modified to allow exactly m eigenvalues within S and $n-m$ eigenvalues outside S. Define

$$f(x_1,\ldots,x_n) = \prod_{j=1}^{m} \mathbb{1}_S(x_j) \prod_{j=m+1}^{n} (1-\mathbb{1}_S(x_j)).$$

Then from eq:det-form6 (6.1.4), when we take into account the $m!$ permutations of the first m elements, and the $(n-m)!$ permutations of the last $n-m$ elements

$$\begin{aligned} A_m(S) &= \frac{1}{m!(n-m)!}\mathbb{E}\left(N_f\right) \\ &= \frac{1}{m!(n-m)!}\int_{\mathbb{R}^n} f(x_1,\ldots,x_n) R_n^{(n)}(x_1,\ldots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n. \end{aligned}$$

We then write

$$f(x_1,\ldots,x_n) = \prod_{j=1}^{m} \mathbb{1}_S(x_j) \sum_{k=0}^{n-m} (-1)^k \rho_k^{n-m}(\mathbb{1}_S(x_{m+1}),\ldots,\mathbb{1}_S(x_{m+k}))$$

We use the fact that $\rho_k^{n-m}(\mathbb{1}_S(x_{m+1}),\ldots,\mathbb{1}_S(x_n))$ is given by a sum of $\binom{n-m}{k}$ terms, each of which is product of k terms, and all terms integrate to the same value. So,

$$\begin{aligned} &\int_{\mathbb{R}^n} \prod_{j=1}^{m} \mathbb{1}_S(x_j)\rho_k^{n-m}(\mathbb{1}_S(x_{m+1}),\ldots,\mathbb{1}_S(x_{m+k}))R_n^{(n)}(x_1,\ldots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n \\ &\quad= \binom{n-m}{k}\int_{\mathbb{R}^{m+k}} \prod_{j=1}^{m+k} \mathbb{1}_S(x_j)\left(\int_{\mathbb{R}^{n-m-k}} R_n^{(n)}(x_1,\ldots,x_{m+k})\mathrm{d}x_{m+k+1}\cdots\mathrm{d}x_n\right) \\ &\quad\times \mathrm{d}x_1\cdots\mathrm{d}x_{m+k} \\ &\quad= \frac{(n-m)!}{k!}\int_{\mathbb{R}^{m+k}} \prod_{j=1}^{m+k} \mathbb{1}_S(x_j)R_{m+k}^{(n)}(x_1,\ldots,x_{m+k})\mathrm{d}x_1\cdots\mathrm{d}x_{m+k} \\ &\quad= \frac{(n-m)!}{k!}\int_{\mathbb{R}^{m+k}} \det\left(K_n\mathbb{1}_S(x_j,x_l)_{1\leq j,l\leq m+k}\right)\mathrm{d}x_1\cdots\mathrm{d}x_{m+k}. \end{aligned}$$

Then, it follows that

$$\begin{aligned} A_m(S) &= \frac{1}{m!}\sum_{k=0}^{n-m}\frac{(-1)^k}{k!}\int_{\mathbb{R}^{m+k}} \det\left(K_n\mathbb{1}_S(x_j,x_l)_{1\leq j,l\leq m+k}\right)\mathrm{d}x_1\cdots\mathrm{d}x_{m+k} \\ &= \frac{1}{m!}\left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \det(I-zK_n\mathbb{1}_S)\Big|_{z=1}. \end{aligned}$$

Since there are $\binom{n}{m}$ ways of selecting m distinct eigenvalues, we find

$$
\begin{aligned}
&A_m(S) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.3.5)\\
&= \binom{n}{m} \int_{\mathbb{R}^n} \prod_{j=1}^{m} \mathbb{1}_S(x_j) \prod_{k=1}^{n-m} (1 - \mathbb{1}_S(x_{m+k}))\, P^{(n)}(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n \\
&= \binom{n}{m} \sum_{k=0}^{n-m} (-1)^k \int_{\mathbb{R}^n} \rho_k^{(n-m)} \left(\mathbb{1}_S(x_{m+1}), \dots, \mathbb{1}_S(x_n)\right) \\
&\qquad\qquad \times \left(\prod_{j=1}^{m} \mathbb{1}_S(x_j) \right) P^{(n)}(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n .
\end{aligned}
$$

eq:fred4

We then express

eq:fred1a eq:fred1a

As in (6.3.3) and (6.3.3) We use the fact that ρ_k^{n-m} is given by a sum of $\binom{n-m}{k}$ terms, each of which is product of k terms, and all terms integrate to the same value. Thus, the sum above is

$$
\begin{aligned}
&\binom{n}{m} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \int_{\mathbb{R}^n} \prod_{j=1}^{m+k} \mathbb{1}_S(x_j)\, P^{(n)}(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n \\
&\quad = \frac{1}{m!} \sum_{k=0}^{n-m} \frac{(-1)^k}{k!} \int_{\mathbb{R}^{m+k}} \prod_{j=1}^{m+k} \mathbb{1}_S(x_j) R_{m+k}^{(n)}(x_1, \dots, x_{m+k})\, \mathrm{d}x_1 \dots \mathrm{d}x_{m+k} \\
&\quad = \frac{1}{m!} \sum_{k=0}^{n-m} \frac{(-1)^k}{k!} \int_{\mathbb{R}^{m+k}} \det\left(K_n \mathbb{1}_S(x_p, x_q)_{1 \le p, q \le m+k}\right)\, \mathrm{d}x_1 \dots \mathrm{d}x_{m+k} \\
&\quad = \frac{1}{m!} \left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \det(I - zK_n \mathbb{1}_S)\big|_{z=1} . \qquad (6.3.6)
\end{aligned}
$$

In the second equality, we have simplified the combinatorial factors as follows:

$$
\frac{1}{m!} = \binom{n}{m} \binom{n-m}{k} \frac{(n-m-k)!}{n!} .
$$

eq:derivative-Fred

In the last line, we have used formula (D.1.11) for the derivative of a Fredholm determinant. □

Exercises

6.1. Plot the density $\frac{1}{n} K_n(x,x)$ for various choices of n. Compare the extrema of the density with the roots of the appropriately scaled Hermite wave functions.

Part II

Asymptotics and numerics in random matrix theory

Chapter 7

Scaling limits

ch:scaling

7.1 Scaling limits of independent points

Recall the semicircle density p_{sc} from (eq:psc 1.3.1). We show in the next section that the global eigenvalue density, or density of states, for GUE(n) is given by p_{sc} as $n \to \infty$. Before we describe this more precisely, we consider a situation of iid points to contrast with the distributions that arise in GUE(n).

Consider an iid vector $\Lambda = \sqrt{n}(\lambda_1, \lambda_2, \ldots, \lambda_n)^T \in \mathbb{R}^n$ where $\mathbb{P}(\lambda_j \in S) = \int_S p_{\mathrm{sc}}(x')\mathrm{d}x'$. We form the empirical measure

$$L_n\left(\mathrm{d}x\right) = \frac{1}{n}\sum_{k=1}^{n} \delta_{\Lambda_k}(\mathrm{d}x), \tag{7.1.1}$$

and consider the deterministic measure $\mathbb{E}L_n$ defined by

$$\int f(x)\mathbb{E}L_n(\mathrm{d}x) := \mathbb{E}\langle L_n, f\rangle = \mathbb{E}\frac{1}{n}\sum_{k=1}^{n} f(\lambda_k), \quad f \in C_0(\mathbb{R}). \tag{7.1.2}$$

But, it is clear, and effectively by definition, that $\mathbb{E}L_n(\mathrm{d}x') = p(x')\mathrm{d}x' = \frac{1}{\sqrt{n}} p_{\mathrm{sc}}\left(\frac{x'}{\sqrt{n}}\right)\mathrm{d}x'$ and hence $\sqrt{n}p(\sqrt{n}x')\mathrm{d}x' = p_{\mathrm{sc}}(x')\mathrm{d}x'$.

Next, we consider a gap probability in the "bulk". Let $s \in (-2,2)$, $I \subset \mathbb{R}$ be an interval and consider the rescaled interval $I_n = \sqrt{n}\left(s + \frac{I}{n p_{\mathrm{sc}}(s)}\right)$. Then by independence

$$\mathbb{P}\left(\text{ no } \lambda_j \in I_n\right) = \left(1 - \frac{1}{\sqrt{n}}\int_{I_n} p_{\mathrm{sc}}\left(\frac{x'}{\sqrt{n}}\right)\mathrm{d}x'\right)^n. \tag{7.1.3}$$

We directly find that

$$\frac{1}{\sqrt{n}}\int_{I_n} p_{\mathrm{sc}}\left(\frac{x'}{\sqrt{n}}\right)\mathrm{d}x' = \frac{|I|}{n}(1+o(1)) \text{ as } n \to \infty. \tag{7.1.4}$$

From this it follows that

$$\lim_{n\to\infty} \mathbb{P}\left(\text{ no } \lambda_j \in I_n \right) = \exp\left(-\int_I \mathrm{d}x'\right). \tag{7.1.5}$$

eq:condnum

This is, of course, the gap probability for a Poisson process.

We now consider the distribution of the maximum, i.e. at the "edge". Let $\hat{\lambda} = \max_j \Lambda_j$. Then, by independence,

$$\mathbb{P}(n^{1/6}(2\sqrt{n} - \hat{\lambda}) > t) = \left(1 - \int_{2-n^{-2/3}t}^{2} p_{\mathrm{sc}}\left(x'\right)\mathrm{d}x'\right)^n.$$

By direct calculation, replacing t with $\pi^{2/3}t^{2/3}(3/2)^{2/3}$ we find, for $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{P}(n^{1/6}(2\sqrt{n} - \hat{\lambda}) > t) = \mathrm{e}^{-\frac{2}{3\pi}t^{3/2}}. \tag{7.1.6}$$

From this we see a (trivial) scaling limit of the density of states after rescaling by $1/\sqrt{n}$, gaps on the order of $1/n$ after this rescaling and a largest "eigenvalue" that satisfies $\hat{\lambda} \sim 2\sqrt{n}+\xi n^{1/6}$ for an appropriate random variable ξ. All of these statements carry over to the random matrix setting, but the actual limits are very different for local statistics.

7.2 GUE scaling limits I: the semicircle law

The empirical measure of the eigenvalues of GUE(n) is

$$L_n\left(\mathrm{d}x\right) = \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k}(\mathrm{d}x) \tag{7.2.1}$$

eq:sclimit1

has the expected density

$$\mathbb{E}L_n\left(\mathrm{d}x\right) = \frac{1}{n}K_n(x,x)\,\mathrm{d}x. \tag{7.2.2}$$

This density is also referred to as the global eigenvalue density or the density of states. The above expression is somewhat more transparent in its weak form, using unordered $x_1, \ldots, x_n$. For every $f \in C_0(\mathbb{R})$, we have

$$\mathbb{E}\langle L_n, f\rangle = \frac{1}{n}\int_{\mathbb{R}} f(x)R_1^{(n)}(x)\,\mathrm{d}x = \frac{1}{n}\int_{\mathbb{R}} f(x)K_n(x,x)\,\mathrm{d}x, \tag{7.2.3}$$

eq:sclimit2

by Theorem 53 and equation (6.1.4). The value of the kernel K_n on the diagonal is determined by the Christoffel-Darboux relation (6.1.10) and L'Hospital's lemma:

$$K_n(x,x) = \sqrt{n}\left(\psi_n'(x)\psi_{n-1}(x) - \psi_n(x)\psi_{n-1}'(x)\right). \tag{7.2.4}$$

eq:sclimit3

The scaling limit of $\mathbb{E}L_n$ is the semicircle law defined in (1.3.1) le:scaling-semicircle **Lemma 16.**

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} K_n\left(x\sqrt{n}, x\sqrt{n}\right) = p_{\mathrm{sc}}(x), \quad x \in \mathbb{R}. \tag{7.2.5}$$

Further, for any $\varepsilon > 0$, the convergence is uniform on the set $\{x \,|\, |x-2| \geq \varepsilon\}$.

Proof. The lemma follows from the Plancherel-Rotach asymptotics for the Hermite wave functions (see Cases 1 and 2 and equations (E.1.39)–(??)) in Appendix E). Define the rescaled wave functions

$$\Psi_{n+p}(x) = n^{\frac{1}{4}} \psi_{n+p}(x\sqrt{n}), \quad p = -2, -1, 0. \tag{7.2.6}$$

eq:sclimit4

We use the identity (E.1.14) to eliminate ψ_n' and ψ_{n-1}' from (7.2.4) and find after a few computations that

$$\frac{1}{\sqrt{n}} K_n\left(x\sqrt{n}, x\sqrt{n}\right) = \Psi_{n-1}^2(x) - \sqrt{\frac{n-1}{n}}\,\Psi_{n-2}(x)\Psi_n(x). \tag{7.2.7}$$

eq:sclimit5

We now use the asymptotic relations (E.1.40) and (??) depending on whether $|x| < 2$ or $|x| > 2$. Since the region $|x| > 2$ corresponds to exponential decay with a rate proportional to n, we focus on the region $|x| < 2$. In order to simplify notation, let

$$\theta = n\left(\varphi - \frac{1}{2}\sin 2\varphi\right) - \frac{1}{2}\varphi - \frac{\pi}{4}. \tag{7.2.8}$$

eq:sclimit6

(This is the argument of the cosine in (E.1.54) when $p = -1$.) Then (7.2.7) and (E.1.40) yield

$$\frac{1}{\sqrt{n}} K_n\left(x\sqrt{n}, x\sqrt{n}\right) \sim \frac{1}{\pi \sin\varphi}\left(\cos^2\theta - \cos(\theta+\varphi)\cos(\theta-\varphi)\right) = \frac{1}{2\pi}\sqrt{4-x^2},$$

using $x = 2\cos\varphi$ and the trigonometric formulae $\cos 2\alpha = 2\cos^2\alpha - 1$ and $2\cos(\theta+\varphi)\cos(\theta-\varphi) = \cos 2\varphi + \cos 2\theta$. A similar calculation with (??) shows that the limit vanishes outside the set $|x| > 2$. The assertion of uniformity in the convergence follows from the assertion of uniform convergence in the Plancherel-Rotach asymptotics. □

Using Exercise 7.9, Lemma 16 implies that $\mathbb{E}L_n(\mathrm{d}x)$, after rescaling, converges weakly

Include the variance estimate to state this almost surely?

$$\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n} \delta_{x_k/\sqrt{n}}(\mathrm{d}x)\right) \to p_{\mathrm{sc}}(x)\mathrm{d}x, \quad \text{weakly}. \tag{7.2.9}$$

This is called the averaged semicircle law. It is also worth noting that if $f(x) = \mathbb{1}_S$ then

$$\mathbb{E}\left(\text{ fraction of eigenvalues that lie in } S\right) = \int f(x)\mathbb{E}L_n(\mathrm{d}x) = \frac{1}{n}\int_S K_n(x,x)\mathrm{d}x.$$

7.3 GUE scaling limits II: the sine kernel

Recall from Definition def:sine-kernel 5 that K_{sine} is the integral kernel on $\mathbb{R} \times \mathbb{R}$ given by

$$K_{\mathsf{sine}}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}, \quad x \neq y, \tag{7.3.1}$$ eq:ksine2

and $K_{\mathsf{sine}}(x,x,) = 1$. It defines an integral operator on $L^2(S)$ for every bounded, measurable set S. We can now prove a stronger version of Theorem thm:dyson 6.

thm:dyson2 **Theorem 56.** *For each integer $m = 0, 1, 2, \ldots$ and bounded, Borel set S and $r \in (-2, 2)$*

$$\lim_{n\to\infty} \mathbb{P}\left(M \sim \mathrm{GUE}(n) \text{ has } m \text{ eigenvalues in } \sqrt{n}\left(r + \frac{S}{np_{\mathrm{sc}}(r)}\right)\right) = \frac{1}{m!}\left(-\frac{d}{dz}\right)^m \det\left(I - zK_{\mathsf{sine}}\mathbb{1}_S\right)\Big|_{z=1}. \tag{7.3.2}$$ eq:rsc4

The proof of Theorem thm:dyson2 56 is a consequence of the following

le:hermite-sine **Lemma 17.** *Let S be a bounded measurable set. Then for $r \in (-2, 2)$*

$$\lim_{n\to\infty} \sup_{x,y\in S} \left| \frac{1}{p_{\mathrm{sc}}(r)\sqrt{n}} K_n\left(\sqrt{n}r + \frac{x}{p_{\mathrm{sc}}(r)\sqrt{n}}, \sqrt{n}r + \frac{y}{p_{\mathrm{sc}}(r)\sqrt{n}}\right) - K_{\mathsf{sine}}(x,y)\right| = 0. \tag{7.3.3}$$ eq:sine-conv1

Proof. For $r \in (-2, 2)$ define $\varphi(s)$ by $x = r + \frac{\pi s}{n \sin\varphi(0)} = 2\cos\varphi(s)$. We then note that $\sin\varphi(0)/\pi = p_{\mathrm{sc}}(r)$. We expand, for sufficiently large n,

$$\varphi(s) - \frac{1}{2}\sin 2\varphi(s) = \varphi(0) - \frac{1}{2}\sin 2\varphi(0) - \frac{\pi s}{n} + O(n^{-2}). \tag{7.3.4}$$

Define the new functions

$$\Psi_{n,p}(s) = n^{\frac{1}{4}}\psi_{n+p}\left(x\sqrt{n}\right), \tag{7.3.5}$$

From (eq:pr-proof8b E.1.40)

$$\Psi_{n,p}(s) \sim \frac{1}{\sqrt{\pi\sin\varphi(0)}}\cos\left[n\left(\varphi(0) - \frac{1}{2}\sin 2\varphi(0)\right) - \pi s + \left(p + \frac{1}{2}\right)\varphi(0) - \frac{\pi}{4}\right] \tag{7.3.6}$$

For fixed r, this is uniform for s in a compact set. We then use (eq:christoffel-darboux1 6.1.10) and

$y = r - \frac{\pi t}{n \sin\varphi(0)}$ to find, for $s \neq t$,

$$\begin{aligned}
&\frac{\pi}{\sin\varphi(0)\sqrt{n}} K_n(x\sqrt{n}, y\sqrt{n}) && (7.3.7)\\
&= \frac{\pi}{\sin\varphi(0)\sqrt{n}} \frac{\psi_n(x\sqrt{n})\psi_{n-1}(y\sqrt{n}) - \psi_n(y\sqrt{n})\psi_{n-1}(x\sqrt{n})}{x-y}\\
&= \frac{\Psi_{n,0}(s)\Psi_{n,-1}(t) - \Psi_{n,0}(t)\Psi_{n,-1}(s)}{s-t}\\
&\sim \frac{1}{\pi\sin\varphi(0)} \frac{\cos(\theta_n + s)\cos(\theta_n + t - \varphi(0)) - \cos(\theta_n + t)\cos(\theta_n + s - \varphi(0))}{t-s}\\
&= \frac{\sin\pi(s-t)}{\pi(s-t)}. && (7.3.8)
\end{aligned}$$

Here we set $\theta_n = n\left(\varphi(0) - \frac{1}{2}\sin 2\varphi(0)\right) + \frac{1}{2}\varphi(0) - \frac{\pi}{4}$ and used the identity

$$\cos\alpha\cos(\beta+\gamma) - \cos(\alpha+\gamma)\cos\beta = \sin\gamma\sin(\alpha-\beta). \tag{7.3.9}$$

This is uniform for $|t-s| \geq \delta$. For $|t-s| < \delta$, it is convenient to write

$$\frac{\psi_n(x)\psi_{n-1}(y) - \psi_n(y)\psi_{n-1}(x)}{x-y} = \begin{pmatrix}\psi_n(x) & \psi_{n-1}(x)\end{pmatrix} \int_0^1 \begin{pmatrix} -\psi_n'(\ell x + (1-\ell)y) \\ \psi_{n-1}'(\ell x + (1-\ell)y)\end{pmatrix} d\ell,$$

and establish uniform convergence of this, after rescaling as above, to

$$\frac{\sin\pi(s-t)}{\pi(s-t)} = \begin{pmatrix}\sin\pi s & \cos\pi s\end{pmatrix} \int_0^1 \begin{pmatrix}\sin(\pi\ell s + \pi(1-\ell)t) \\ \cos(\pi\ell s + \pi(1-\ell)t)\end{pmatrix} d\ell. \tag{7.3.10}$$

□

With this proof, I don't think we have any need for the small x Hermite asymptotics section in the appendix. But is is nice and clean...

Proof. Define the new rescaled wave function

$$\Psi_n(x) = n^{\frac{1}{4}}\psi_n\left(\frac{x}{\sqrt{n}}\right). \tag{7.3.11}$$

eq:rsc1

The identity (eq:hwave4 E.1.14) now takes the form

$$\Psi_{n-1}(x) = \Psi_n'(x) + \frac{x}{2n}\Psi_n(x), \tag{7.3.12}$$

eq:rsc2

and the rescaled kernel takes the form

$$K_n\left(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}}\right) = \frac{\Psi_n(x)\Psi'(y) - \Psi_n'(x)\Psi_n(y)}{x-y} - \frac{1}{2n}\Psi_n(x)\Psi_n(y). \tag{7.3.13}$$

eq:rsc3

We now use (eq:her-asympt3 E.1.21) (when n is even) and (eq:her-asympt3 E.1.21) (when n is odd) to obtain (eq:ksine2 7.3.1). □

Proof of Theorem 56. Let $\tilde{K}_n(x,y)$ denote the rescaled kernel $\frac{1}{p_{\mathrm{sc}}(r)\sqrt{n}}K_n(x\sqrt{n}, y\sqrt{n})$, $x = r - \frac{s}{np_{\mathrm{sc}}(r)}$, $y = r - \frac{t}{np_{\mathrm{sc}}(r)}$. It follows from Lemma 17, using Sections D.2.1 and D.2 that

$$\lim_{n\to\infty} \det\left(I - z\tilde{K}_n \mathbb{1}_S\right) = \det\left(I - zK_{\mathsf{sine}}\mathbb{1}_S\right), \quad z \in \mathbb{C}, \tag{7.3.14}$$

eq:rsc5

and that the convergence is uniform in z for z in a bounded set. In particular, the derivatives at $z = 1$ converge for all m, that is

$$\lim_{n\to\infty}\left(-\frac{d}{dz}\right)^m \det\left(I - z\tilde{K}_n\mathbb{1}_S\right)\Big|_{z=1} = \left(-\frac{d}{dz}\right)^m \det\left(I - zK_{\mathsf{sine}}\mathbb{1}_S\right)\big|_{z=1}. \tag{7.3.15}$$

eq:rsc6

By Theorem 55, this is equivalent to (7.13.3). $\square$

7.4 GUE scaling limits III: the Airy kernel

sec:AiryScale

Recall from Definition 8 that K_{Airy} is the continuous integral kernel on $\mathbb{R}\times\mathbb{R}$ given by

$$K_{\mathsf{Airy}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, \quad x \neq y. \tag{7.4.1}$$

eq:kairy2

The fluctuations at the edge of the spectrum are described as follows. Let $(x_1, \ldots, x_n)$ denote the unordered eigenvalues of a matrix $M \in \mathrm{GUE}(n)$ and let us consider the shifted and rescaled points

$$s_k = n^{\frac{1}{6}}\left(x - 2\sqrt{n}\right), \quad k = 1, \ldots, n. \tag{7.4.2}$$

eq:rsc7

For each nonnegative integer m and bounded, measurable set S, let $B_m^{(n)}(S)$ denote the probability that exactly m of the points $s_1, \ldots, s_n$ lie in S when $M \in \mathrm{GUE}(n)$. The following theorem is a consequence of Lemma 19 and the discussion in Section D.2.

thm:TW2

Theorem 57.

$$\lim_{n\to\infty} B_m^{(n)}(S) = \frac{1}{m!}\left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \det\left(I - zK_{\mathsf{Airy}}\mathbb{1}_S\right)\big|_{z=1}. \tag{7.4.3}$$

eq:rsc8

Remark 58. The assumption that S is bounded is necessary for K_{sine}. The sine-kernel has a (weak) rate of decay $|x|^{-1}$ as $|x| \to \infty$ and the Fredholm determinant $\det(I - zK_{\mathsf{sine}}\mathbb{1}_S)$ is not finite unless S is bounded. However, the Airy function, and the thus the Airy kernel, has strong decay as x and $y \to \infty$. The Fredholm determinant $\det(I - zK_{\mathsf{Airy}}\mathbb{1}_S)$ is well-defined in $L^2(S)$ for sets S that are bounded below, but not above, such as $S = (a, \infty)$ for any $a \in \mathbb{R}$. Such sets will be considered when we compute the Tracy-Widom distribution. See Exercise 5.

The proof of Theorem 57 follows from the Plancherel-Rotach asymptotics for the Hermite polynomials, in particular the Airy asymptotics in the transition zone (see Case 3 and (E.1.42)–(E.1.44) in Appendix E). The following lemma plays a role analogous to that of Lemma 17 in the proof of Theorem 56.

Lemma 18. *Let S be a bounded measurable set. Then*

$$\lim_{n\to\infty} \sup_{x,y\in S} \left| \frac{1}{n^{\frac{1}{6}}} K_n\left(2\sqrt{n} + \frac{x}{n^{\frac{1}{6}}}, 2\sqrt{n} + \frac{y}{n^{\frac{1}{6}}}\right) - K_{\mathsf{Airy}}(x,y)\right| = 0. \tag{7.4.4}$$

le:hermite-airy **Lemma 19.** *For $x \neq y$, uniformly on bounded sets,*

$$\lim_{n\to\infty} \left| \frac{1}{n^{\frac{1}{6}}} K_n\left(2\sqrt{n} + \frac{x}{n^{\frac{1}{6}}}, 2\sqrt{n} + \frac{y}{n^{\frac{1}{6}}}\right) - K_{\mathsf{Airy}}(x,y)\right| = 0 \tag{7.4.5}$$ eq:airy-conv1

and there exists a function $G(x,y) \in L^2([C,\infty)^2)$ for all $C \in \mathbb{R}$ such that

$$\left| \frac{1}{n^{\frac{1}{6}}} K_n\left(2\sqrt{n} + \frac{x}{n^{\frac{1}{6}}}, 2\sqrt{n} + \frac{y}{n^{\frac{1}{6}}}\right)\right| \leq G(x,y). \tag{7.4.6}$$

Proof. Convergence follows from (E.1.44). The function $G(x,y)$ can be constructed using (E.1.82) and (E.1.83), see Exercise 7.3. □

Proof. Let us define the rescaled wave-functions

$$\Psi_n(x) = n^{\frac{1}{12}} \psi_n\left(2\sqrt{n} + \frac{x}{n^{\frac{1}{6}}}\right). \tag{7.4.7}$$ eq:rsc9

We then use identity (E.1.14) to find

$$n^{\frac{1}{6}} K_n\left(2\sqrt{n} + \frac{x}{n^{\frac{1}{6}}}, 2\sqrt{n} + \frac{y}{n^{\frac{1}{6}}}\right) = \frac{\Psi_n(x)\Psi_n'(y) - \Psi_n'(x)\Psi_n(y)}{x-y} - \frac{1}{2n^{\frac{1}{3}}}\Psi_n(x)\Psi_n(y). \tag{7.4.8}$$

As noted in Appendix E, as $n \to \infty$, $\Psi_n(x)$ converges to $\mathrm{Ai}(x)$ and the convergence is uniform for x in compact subsets of $\mathbb{C}$. Thus, in addition, $\Psi_n'(x) \to \mathrm{Ai}'(x)$ uniformly in compact sets, and (7.4.5) follows. □

7.5 The eigenvalues and condition number of GUE

Let $M \sim \mathrm{GUE}(n)$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of M. A consequence of Theorem 57 is the following, for all $t \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}\left(n^{2/3}\left(\frac{\lambda_n}{\sqrt{n}} - 2\right) < t\right) = \det(1 - K_{\mathsf{Airy}} \mathbb{1}_{(t,\infty)}) =: F_2(t),$$
$$\lim_{n\to\infty} \mathbb{P}\left(-n^{2/3}\left(2 + \frac{\lambda_1}{\sqrt{n}}\right) < t\right) = F_2(t).$$

Then, Theorem thm:dyson2 56 gives for $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\sqrt{n}|\lambda_j|}{\pi} > t \text{ for all } j\right) = \det(1 - K_{\mathsf{sine}}\mathbb{1}_{(-t,t)}) := S(t). \tag{7.5.1}$$

eq:smallest

The singular values $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n$ of a matrix M are the square roots of the non-zero eigenvalues of M^*M. One can rewrite (eq:smallest 7.5.1) as

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\sqrt{n}\sigma_1}{\pi} > t\right) = S(t). \tag{7.5.2}$$

The condition number is defined as $\kappa(M) := \sigma_n/\sigma_1$.

Lemma 20. *If $M \sim \mathrm{GUE}(n)$, then for all $t > 0$*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\pi}{2n}\kappa(M) < t\right) = S(t^{-1}). \tag{7.5.3}$$

Proof. We first show that $\lambda_n/\sqrt{n} \to 2$, $\lambda_1/\sqrt{n} \to -2$ in probability. Fix $\epsilon > 0$, and let $L > 0$. Then

$$1 \leq \mathbb{P}\left(\left|\frac{\lambda_n}{\sqrt{n}} - 2\right| \leq \epsilon\right) = \mathbb{P}\left(n^{2/3}\left|\frac{\lambda_n}{\sqrt{n}} - 2\right| \leq n^{2/3}\epsilon\right) \geq \mathbb{P}\left(n^{2/3}\left|\frac{\lambda_n}{\sqrt{n}} - 2\right| \leq L\right),$$

provided $n^{2/3}\epsilon \geq L$. So we, find

$$1 \leq \liminf_{n\to\infty} \mathbb{P}\left(\left|\frac{\lambda_n}{\sqrt{n}} - 2\right| \leq \epsilon\right) \geq F_2(L) - F_2(-L).$$

Letting $L \to \infty$ gives convergence in probability for $\lambda_n/\sqrt{n}$. Similar arguments follow for $\lambda_1/\sqrt{n}$. Next, define

$$E_{\epsilon,n} = \left\{\left|\frac{\lambda_n}{\sqrt{n}} - 2\right| \leq \epsilon, \left|\frac{\lambda_1}{\sqrt{n}} + 2\right| \leq \epsilon\right\}.$$

We know that $\mathbb{P}(E_{\epsilon,n}) \to 1$ as $n \to \infty$. Then

$$\mathbb{P}\left(\frac{\pi}{2n}\kappa(M) < t\right) = \mathbb{P}\left(\frac{\pi}{2n}\kappa(M) < t, E_{\epsilon,n}\right) + \mathbb{P}\left(\frac{\pi}{2n}\kappa(M) < t, E_{\epsilon,n}^c\right).$$

Because the second term must vanish as $n \to \infty$, we focus on the first term. On $E_{\epsilon,n}$ it follows that $(2-\epsilon)\sqrt{n} \leq \sigma_n \leq (2+\epsilon)\sqrt{n}$ and

$$\mathbb{P}\left(\frac{\pi(2+\epsilon)}{2n\sigma_1} < t, E_{\epsilon,n}\right) \leq \mathbb{P}\left(\frac{\pi}{2n}\kappa(M) < t, E_{\epsilon,n}\right) \leq \mathbb{P}\left(\frac{\pi(2-\epsilon)}{2n\sigma_1} < t, E_{\epsilon,n}\right).$$

We find that for $\epsilon > 0$

$$\limsup_{n\to\infty} \mathbb{P}\left(\frac{\pi}{2n}\kappa(M) < t\right) = \limsup_{n\to\infty} \mathbb{P}\left(\frac{\pi}{2n}\kappa(M) < t, E_{\epsilon,n}\right) \leq S\left(\frac{2-\epsilon}{2}t^{-1}\right),$$

$$\liminf_{n\to\infty} \mathbb{P}\left(\frac{\pi}{2n}\kappa(M) < t\right) = \limsup_{n\to\infty} \mathbb{P}\left(\frac{\pi}{2n}\kappa(M) < t, E_{\epsilon,n}\right) \geq S\left(\frac{2+\epsilon}{2}t^{-1}\right).$$

If S is continuous at t, send $\epsilon \downarrow 0$ to obtain convergence in distribution. Since $S(t)$ is continuous, the result follows. $\square$

Discuss Folkmar's work on the singular values of GUE.

7.6 Tightness and joint distributions

In this section, we discuss some deeper topics concerning the convergence in distribution of the eigenvalues near the upper edge of the spectrum.

7.6.1 The joint distribution of the top two eigenvalues

We first discuss the joint distribution of the top two (ordered) eigenvalues of GUE noting that

$$\begin{aligned} F_{\text{Top }2}(t,s) &:= \mathbb{P}(\lambda_{n-1} \le t, \lambda_n \le s) = \mathbb{P}(\Omega_{t,s}),\\ \Omega_{t,s} &:= \{\text{ no eigenvalues in } (s,\infty),\ \text{no more than 1 eigenvalue in } (t,\infty))\}. \end{aligned}$$

For $s > t$ we have the disjoint union

$$\begin{aligned} \Omega_{t,s} &= \{\text{no eigenvalues in } (s,\infty), \text{no eigenvalues in } (t,\infty\}\\ &\cup \{\text{no eigenvalues in } (s,\infty), \text{exactly one in } (t,\infty)\}\\ &= \{\text{no eigenvalues in } (t,\infty)\}\\ &\cup \{\text{no eigenvalues in } (s,\infty), \text{exactly one in } (t,s]\}. \end{aligned}$$

For $s \le t$

$$\Omega_{t,s} = \{\text{no eigenvalues in } (s,\infty)\}.$$

The probability of the event $\{\text{no eigenvalues in } (s,\infty), \text{exactly one in } (t,s]\}$ needs to be computed for $s > t$. To this end, consider

$$\begin{aligned} f(x_1,\ldots,x_n) &= \left(\prod_{j=1}^n (1-\mathbb{1}_S(x_j))\right)\left(\prod_{j=2}^n (1-\mathbb{1}_T(x_j))\right)\mathbb{1}_T(x_1),\\ S &= (s,\infty), \quad T = (t,s]. \end{aligned}$$

Careful consideration reveals

$$\mathbb{E}(N_f) = (n-1)!\mathbb{P}(\text{no eigenvalues in } (s,\infty), \text{exactly one in } (t,s]).$$

One does this by simply computing N_f on $\Omega_{s,t}$ ($N_f = (n-1)!$)) and on $\Omega_{s,t}^c$ ($N_f = 0$). The function f must then be expanded using symmetric functions:

$$f(x_1,\ldots,x_n) \tag{7.6.1}$$

eq:sym-expand

$$= -\mathbb{1}_T(x_1)\sum_{k=0}^n\sum_{\ell=1}^n (-1)^{k+\ell}\rho_k^n(\mathbb{1}_S(x_1),\ldots,\mathbb{1}_S(x_n))\rho_{\ell-1}^{n-1}(\mathbb{1}_T(x_2),\ldots,\mathbb{1}_T(x_n)). \tag{7.6.2}$$

In order to further understand this expression, we consider matrix kernels as described in (eq:mat-fred D.1.12). Consider the determinant

$$d_{s,t}(z) := \det\left(\mathbf{1} - \begin{bmatrix} zK_n & K_n\\ zK_n & K_n\end{bmatrix}\Bigg|_{L^2(T)\oplus L^2(S)}\right)$$

By definition, we have

$$d_{s,t}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{i_1,\ldots,i_k=1}^{2} \int_{\mathbb{R}^k} \det[K_{i_j,i_\ell}(x_j,x_\ell)]_{1\le j,\ell\le k} \mathrm{d}x_1\cdots\mathrm{d}x_k,$$

where we extend $K_{j,\ell}=0$ on the complement of $I_j\times I_\ell$. Define

$$Q(i_1,\ldots,i_k) := \int_{\mathbb{R}^k} \det[K_{i_j,i_\ell}(x_j,x_\ell)]_{1\le j,\ell\le k} \mathrm{d}x_1\cdots\mathrm{d}x_k.$$

Then the set $\{(i_1,\ldots,i_k) : i_j \in 1,2\}$ can be expressed as the disjoint union of sets

$$\mathbb{I}_p = \{(i_1,\ldots,i_k) : i_j \in 1,2 \text{ and } \#\{j : i_j = 1\} = p\}.$$

Let $\sigma \in \mathsf{S}(k)$ and $(i_1,\ldots,i_k)\in\mathbb{I}_p$ and consider

$$\begin{aligned} Q(i_{\sigma(1)},\ldots,i_{\sigma(k)}) &= \int_{\mathbb{R}^k} \det[K_{i_{\sigma(j)},i_{\sigma(\ell)}}(x_{\sigma(j)},x_{\sigma(\ell)})]_{1\le j,\ell\le k}\mathrm{d}x_1\cdots\mathrm{d}x_k \\ &= \int_{\mathbb{R}^k} \det P[K_{i_j,i_\ell}(x_j,x_\ell)]_{1\le j,\ell\le k}P^T\mathrm{d}x_1\cdots\mathrm{d}x_k \end{aligned}$$

for a permutation matrix P. Thus $Q(i_{\sigma(1)},\ldots,i_{\sigma(k)}) = Q(i_1,\ldots,i_k)$ and

$$\#\mathbb{I}_p = \binom{k}{p}.$$

Thus set $z^pQ(p,k) = Q(i_1,\ldots,i_k)$ where $(i_1,\ldots,i_k)\in\mathbb{I}_p$ and $i_j \le i_k$ for $j<k$. We then have

$$d_{s,t}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{p=0}^{k}\binom{k}{p}Q(p,k)z^p.$$

Now, upon examining (eq:sym-expand 7.6.1) we write, using $k+\ell\mapsto k$ and $\ell\mapsto p$

$$\begin{aligned} &f(x_1,\ldots,x_n) \\ &= -\mathbb{1}_T(x_1)\sum_{k=1}^{2n}\sum_{p=1}^{\min\{n,k\}}(-1)^k\rho^n_{p-1}(\mathbb{1}_T(x_2),\ldots,\mathbb{1}_T(x_n))\rho^n_{k-p}(\mathbb{1}_S(x_1),\ldots,\mathbb{1}_S(x_n)). \end{aligned}$$

So, specifically, we must consider the integral

$$\begin{aligned} I(p,k) := \int_{\mathbb{R}^n} &\mathbb{1}_T(x_1)\rho^n_{p-1}(\mathbb{1}_T(x_2),\ldots,\mathbb{1}_T(x_n)) \\ &\times \rho^n_{k-p}(\mathbb{1}_S(x_1),\ldots,\mathbb{1}_S(x_n)R^{(n)}_n(x_1,\ldots,x_n))\mathrm{d}x_1\cdots\mathrm{d}x_n. \end{aligned} \tag{7.6.3}$$

eq:sym-fun-int

Each of these symmetric functions can be expanded and we must consider integrals of the form

$$\int_{\mathbb{R}^n} \left(\prod_{j=1}^{p} \mathbb{1}_T(x_j) \right) \left(\prod_{\ell=1}^{k-p} \mathbb{1}_S(x_{i_\ell}) \right) R_n^{(n)}(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

Because T and S are disjoint, the only integrals do not vanish occur when $i_\ell > p$ for all ℓ. It will suffice to compute

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\prod_{j=1}^{p} \mathbb{1}_T(x_j) \right) \left(\prod_{\ell=p+1}^{k} \mathbb{1}_S(x_\ell) \right) R_n^{(n)}(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &\quad = (n-k)! \int_{\mathbb{R}^k} \left(\prod_{j=1}^{p} \mathbb{1}_T(x_j) \right) \left(\prod_{\ell=p+1}^{k} \mathbb{1}_S(x_\ell) \right) R_k^{(n)}(x_1, \ldots, x_m) \mathrm{d}x_1 \cdots \mathrm{d}x_k \\ &\quad = (n-k)! Q(p,k) \end{aligned}$$

Counting the number of times this integral occurs in (eq:sym-fun-int 7.6.3) we find

$$I(p,k) = \frac{(n-k)!}{n!} \binom{n-1}{p-1} \binom{n-p}{k-p} Q(p,k) = \frac{p}{k!} \binom{k}{p} Q(p,k)$$

and therefore

$$\mathbb{P}(\text{no eigenvalues in } (s,\infty), \text{exactly one in } (t,s]) = -\sum_{k=1}^{2n} \sum_{p=0}^{\min\{n,k\}} p \frac{(-1)^k}{k!} \binom{k}{p} Q(p,k)$$

Noting that the integral operator with kernel matrix

$$\begin{bmatrix} zK_n & K_n \\ zK_n & K_n \end{bmatrix}$$

has maximal rank $2n$ we find that $Q(p,k) = 0$ if either $k \geq 2n$ or $p \geq k$. So, we write

$$\begin{aligned} &\mathbb{P}(\text{no eigenvalues in } (s,\infty), \text{exactly one in } (t,s]) \\ &= -\sum_{k=1}^{\infty} \sum_{p=0}^{k} p \frac{(-1)^k}{k!} \binom{k}{p} Q(p,k) = -\frac{\mathrm{d}}{\mathrm{d}z} \left(\sum_{k=0}^{\infty} \sum_{p=0}^{k} z^p \frac{(-1)^k}{k!} \binom{k}{p} Q(p,k) \right) \Bigg|_{z=1} \\ &= -d'_{s,t}(1). \end{aligned}$$

From this and the calculations in Section sec:AiryScale 7.4 we arrive at the following theorem.

t:joint **Theorem 59.** *For* $s, t \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}(\lambda_n \leq 2\sqrt{n} + sn^{-1/6}, \lambda_{n-1} \leq 2\sqrt{n} + tn^{-1/6})$$
$$= \begin{cases} \det(\mathbf{1} - K_{\mathsf{Airy}}|_{L^2((s,\infty))}) & s \leq t \\ \det(\mathbf{1} - K_{\mathsf{Airy}}|_{L^2((t,\infty))}) & \\ \quad - \frac{\mathrm{d}}{\mathrm{d}z} \det\left(\mathbf{1} - \begin{bmatrix} zK_{\mathsf{Airy}} & K_{\mathsf{Airy}} \\ zK_{\mathsf{Airy}} & K_{\mathsf{Airy}} \end{bmatrix}\Bigg|_{L^2((t,s])\oplus L^2((s,\infty))}\right)\Bigg|_{z=1} & t < s. \end{cases}$$

Remark 60. It should be clear from this discussion that one can consider the joint distribution of the top k eigenvalues with k fixed as $n \to \infty$. This has a (complicated) Fredholm determinant representation but it can be shown that if $\Lambda_k = [\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n-k+1}]^T$ is the vector of the top k largest eigenvalues of GUE(n) then

$$F_{\Lambda_k}(2\sqrt{n} + n^{-1/6}t_1, \ldots, 2\sqrt{n} + n^{-1/6}t_k) \stackrel{n\to\infty}{\longrightarrow} F_{2,k}(t_1, \ldots, t_k)$$

for every $t_1, \ldots, t_k \in \mathbb{R}$. Here $F_{2,k}$ is expressible in terms of Fredholm determinants that only involve the Airy kernel.

Should we be more explicit?

If $I_1, \ldots, I_N$ are disjoint intervals and $m_j \in \mathbb{N}$ for $j \in [\![1, N]\!]$ then

$$\mathbb{P}(\text{exactly } m_1 \text{ eigenvalues in } I_1, \ldots, \text{exactly } m_N \text{ eigenvalues in } I_N)$$
$$= \frac{(-1)^{\sum_{j=1}^N m_j}}{\prod_{j=1}^N m_j!} \prod_{j=1}^N \frac{\partial^{m_j}}{\partial z_j^{m_j}} d_{I_1,\ldots,I_N}(z_1, \ldots, z_N),$$

where

If we want to include this, it needs to be confirmed

$$d_{I_1,\ldots,I_N}(z_1, \ldots, z_N) = \det\left(\mathbf{1} - [z_k K_n]_{1\leq j,k\leq N}\Big|_{\bigoplus_{j=1}^N L^2(I_j)}\right).$$

7.6.2 Tightness

We have shown that the rescaled distribution function for λ_n and the rescaled joint distribution function for $[\lambda_n, \lambda_{n-1}]^T$ converge pointwise. For this to imply convergence in distribution we need to establish that the function it converges to is itself a distribution function.

We concentrate on

Unify Fredholm determinant notation. Need to make it work with the matrix case.

$$F_2(t) = \det(\mathbf{1} - K_{\mathsf{Airy}}|_{L^2(t,\infty)})$$

We use two estimates to show, using elementary means, that this is a distribution function. Since we have shown that there exists a sequence of random variables X_n such that that $F_{X_n}(t) \to F_2(t)$ pointwise, it suffices to show that the sequence $\{X_n\}$ is tight. The full estimates are based on two fundamental estimates

- If the eigenvalues $(\lambda_j)_{j\geq 1}$ of a trace-class operator A satisfy $\lambda_j \leq 1$ then Somethignhere [?]

$$\det(\mathbf{1}-A) \leq \mathrm{e}^{-\operatorname{Tr} A}.$$

- For a trace-class operator A

$$|1-\det(\mathbf{1}-A)| \leq \|A\|_{\mathrm{Tr}} \mathrm{e}^{1+\|A\|_{\mathrm{Tr}}}.$$

We first verify the hypotheses of the first statement for K_n. Using the notation $\|\cdot\| = \|\cdot\|_{L^2([t,\infty))}$

$$\begin{aligned}
\|K_n|_{L^2([t,\infty))}\|^2 &= \sup_{\|f\|=1} \int_t^\infty \left(\int_t^\infty K_n(x,y) f(y) \mathrm{d}y \right)^2 \mathrm{d}x \\
&= \sup_{\|f\|=1} \int_t^\infty \left(\sum_{k=0}^{n-1} \psi_k(x) \int_t^\infty \psi_k(y) f(y) \mathrm{d}y \right)^2 \mathrm{d}x \\
&\leq \sup_{\|f\|=1} \int_{\mathbb{R}} \left(\sum_{k=0}^{n-1} \psi_k(x) \int_t^\infty \psi_k(y) f(y) \mathrm{d}y \right)^2 \mathrm{d}x \\
&= \sup_{\|f\|=1} \sum_{k=0}^{n-1} \left(\int_t^\infty \psi_k(y) f(y) \mathrm{d}y \right)^2 \\
&= \sup_{\|f\|=1} \sum_{k=0}^{n-1} \left(\int_{\mathbb{R}} \psi_k(y) f(y) \mathbb{1}_{[t,\infty)}(y) \mathrm{d}y \right)^2 \leq 1.
\end{aligned}$$

Therefore, all the eigenvalues are contained in $[-1,1]$. Define

$$\hat{K}_n(x,y) = \frac{1}{n^{\frac{1}{6}}} K_n\left(2\sqrt{n} + \frac{x}{n^{\frac{1}{6}}}, 2\sqrt{n} + \frac{y}{n^{\frac{1}{6}}}\right),$$

and we can then estimate

$$\mathbb{P}(\lambda_n \leq t) = \det(1 - \hat{K}_n \mathbb{1}_{[t,\infty)}) \leq \mathrm{e}^{-\int_t^\infty \hat{K}_n(x,x)\mathrm{d}x}.$$

Then fix $\epsilon > 0$, set $t < 0$ so that

$$\int_s^0 K_{\mathsf{Airy}}(x,x)\mathrm{d}x = \int_s^0 \left[[\mathrm{Ai}'(x)]^2 - x \mathrm{Ai}(x)^2 \right] \mathrm{d}x \geq \log 4\epsilon^{-1},$$

for all $s \leq t$. Such a t exists because (see (eq:airy8 C.3.2))

$$[\mathrm{Ai}'(x)]^2 - x\mathrm{Ai}(x)^2 \sim \frac{|x|^{1/2}}{\sqrt{\pi}}, \quad x \to -\infty.$$

Thus for sufficiently large n, $n \geq N$, $\det(I - \hat{K}_n \mathbb{1}_{[t,\infty)}) < \epsilon/2$ because $\hat{K}_n(\cdot,\cdot) \to K_{\mathsf{Airy}}(\cdot,\cdot)$ uniformly on $[t,0]$ by Lemma le:hermite-airy 19. Then set $T_\epsilon > 0$, so that $-T_\epsilon \leq t$ and $\det(I - \hat{K}_n \mathbb{1}_{[T_\epsilon,\infty)}) < \epsilon/2$ for $n = 1, 2, \ldots, N-1$.

By the second estimate, and the fact that $\|K_n\|_{\mathrm{Tr}} = \int K_n(x,x)\mathrm{d}x$, using a similar argument, after possibly increasing T_ϵ, we find that

$$\mathbb{P}(|\lambda_n - 2\sqrt{n}|n^{1/6} > T_\epsilon) < \epsilon.$$

and therefore the sequence of random variables $((\lambda_n - 2\sqrt{n})n^{1/6})_{n\geq 1}$ is tight. Therefore $F_2(t)$ is indeed a distribution function.

Remark 61. We do not take up this issue for the limiting joint distribution functions $G_k(t_1, \ldots, t_k)$ but it does indeed follow that these are bonafide distribution functions .

Reference?

7.7 Circular law for complex Ginbre matrices

We now describe the global eigenvalue distribution for $\mathrm{Gin}_{\mathbb{C}}(n)$ as $n \to \infty$. We have the following distribution on the (unordered) eigenvalues $Z = (z_1, z_2, \ldots, z_n)$ from (4.1.4) eq:Weyl-Ginc

$$\hat{P}^{(n)}(z_1, \ldots, z_n)\mathrm{D}z = \frac{1}{Z_n}|\triangle(Z)|^2 \mathrm{e}^{-\sum_{j=1}^n |z_j|^2} \prod_{j=1}^n \mathrm{dRe}\, z_j \mathrm{dIm}\, z_j.$$

Owing to the calculations that result in Theorem 53 thm:det-formula we have

$$\begin{aligned}
\hat{P}^{(n)}(z_1, \ldots, z_n) &= \frac{1}{n!}\det(\hat{K}_n(z_j, z_k)_{1\leq j,k\leq n}), \\
\hat{R}_m^{(n)}(z_1, \ldots, z_m) &= \det(\hat{K}_n(z_j, z_k)_{1\leq j,k\leq m}), \quad 1 \leq m \leq n, \\
\hat{K}_n(z, w) &= \sum_{j=0}^{n-1} c_j \Phi_j(z)\overline{\Phi_j(w)}, \quad \Phi_j(z) = c_j z^j \mathrm{e}^{-\frac{1}{2}|z|^2}.
\end{aligned}$$

where $\hat{R}_m^{(n)}$ is the m-point correlation function defined by (6.1.5) eq:det-form4 with $\hat{P}^{(n)}$ instead of $P^{(n)}$ and $\mathrm{dRe}\, z_j \mathrm{dIm}\, z_j$ instead of $\mathrm{d}x_j$. To show that this is the correct choice for $\hat{K}_n$ and to determine c_j we need to show that $\{\Phi_j\}_{j=0}^{n-1}$ are orthogonal and choose $c_j > 0$ to normalize the functions. Consider for $j < k$

$$\begin{aligned}
\int_{\mathbb{C}} \Phi_j(z)\overline{\Phi_k(z)}\, \mathrm{dRe} z\, \mathrm{dIm} z &= c_j \bar{c}_k \int_{\mathbb{C}} \bar{z}^{k-j}|z|^{2j}\mathrm{e}^{-|z|^2}\mathrm{dRe} z\, \mathrm{dIm} z \\
&= c_j \bar{c}_k \int_0^\infty \left(\int_0^{2\pi} (\cos\theta + \mathrm{i}\sin\theta)^{k-j}\mathrm{d}\theta\right) r^{k+j+1}\mathrm{e}^{-|r|^2}\mathrm{dRe} z\, \mathrm{dIm} z = 0.
\end{aligned}$$

If $j = k$ we find

$$\int_{\mathbb{C}} |\Phi_j(z)|^2\, \mathrm{dRe} z\, \mathrm{dIm} z = |c_j|^2 \int_{\mathbb{C}} |z|^{2j}\mathrm{e}^{-|z|^2}\mathrm{dRe} z\, \mathrm{dIm} z,$$

and using $r = \sqrt{s}$

$$\int_{\mathbb{C}} |z|^{2j} \mathrm{e}^{-|z|^2} \mathrm{d}\mathsf{Re}z \, \mathrm{d}\mathsf{Im}z = 2\pi \int_0^\infty r^{2j+1} \mathrm{e}^{-\frac{1}{2}r^2} \mathrm{d}r = \pi \int_0^\infty s^j \mathrm{e}^{-s} \mathrm{d}s$$
$$= \pi \Gamma(j+1) = \pi \, j!$$

so

$$c_j = \frac{1}{\sqrt{\pi j!}}, \quad c_j \frac{1}{\sqrt{j+1}} = c_{j+1}.$$

So, we find a simple two-term recurrence formula

$$\Phi_{j+1}(z) = \frac{z}{\sqrt{j+1}} \Phi_j(z), \quad \Phi_0(z) = \frac{1}{\sqrt{\pi}}.$$

The corresponding Christoffel-Darboux-type formula is

$$\hat{K}_n(z,w) = \frac{\mathrm{e}^{z\bar{w}}}{\pi} \frac{\Gamma(n, z\bar{w})}{(n-1)!} \mathrm{e}^{-\frac{1}{2}(|z|^2+|w|^2)}.$$

where $\Gamma(n,z) = \int_z^\infty t^{n-1} \mathrm{e}^{-t} \mathrm{d}t$ is the incomplete Gamma function. To see this let $f_n(z) = \mathrm{e}^z \Gamma(n,z)$, and we find

$$f_n^{(j)}(0) = (n-1)!, \quad j = 0, 1, 2, \ldots, n-1,$$
$$f_n^{(j)}(0) = 0, \quad j \geq n,$$

so that

$$f_n(z) = \sum_{j=0}^{n-1} \frac{(n-1)!}{j!} z^j.$$

Define the rescaled empirical spectral measure

$$\hat{L}_n(\mathrm{D}z) = \frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k/\sqrt{n}}(\mathrm{D}z), \quad \mathrm{D}z = \mathrm{d}\mathsf{Re}z \, \mathrm{d}\mathsf{Im}z.$$

It then follows that for $f \in C_0(\mathbb{C})$ by (eq:det-form6 6.1.4)

$$\mathbb{E}\left(\int f(z) \hat{L}_n(\mathrm{D}z) \right) =: \int f(z) \mathbb{E} L_n(\mathrm{D}z) = \int f(z) \hat{K}_n(z\sqrt{n}, z\sqrt{n}) \mathrm{D}z.$$

We then perform the asymptotic analysis of this density. Consider

$$\Gamma(n, z\bar{z}) \xrightarrow{z \mapsto z\sqrt{n}} \int_{n|z|^2}^\infty t^{n-1} \mathrm{e}^{-t} \mathrm{d}t.$$

Then

$$\int_{n|z|^2}^\infty t^{n-1} \mathrm{e}^{-t} \mathrm{d}t = n^n \int_{|z|^2}^\infty t^{n-1} \mathrm{e}^{-nt} \mathrm{d}t = n^n \int_{|z|^2}^\infty t^{-1} \mathrm{e}^{-ng(t)} \mathrm{d}t,$$

where $g(t) = t - \log t$. The stationary phase point here is $t = 1$, $g'(1) = 0$ and $g''(1) = 1$. So, if $|z| \leq 1 - \epsilon$, the stationary phase point is in the interval of integration and

$$n^n \int_{|z|^2}^{\infty} t^{-1} \mathrm{e}^{-ng(t)} \mathrm{d}t = \mathrm{e}^{-n} n^{n-1/2} \sqrt{2\pi}(1 + O(n^{-1})) = \mathrm{e}^{-n} n^{n-1} \sqrt{2\pi n}(1 + O(n^{-1}))$$
$$= (n-1)!(1 + O(n^{-1}))$$

uniformly as $n \to \infty$ by Stirling's approximation. Then for $|z| \geq 1 + \epsilon$, by integrating by parts

$$I_n(z) := \int_{|z|^2}^{\infty} t^{n-1} \mathrm{e}^{-nt} \mathrm{d}t = \frac{1}{n} |z|^{2n-2} \mathrm{e}^{-n|z|^2} + \frac{n-1}{n} \int_{|z|^2}^{\infty} t^{n-2} \mathrm{e}^{-nt} \mathrm{d}t$$
$$\leq \frac{1}{n} |z|^{2n-2} \mathrm{e}^{-n|z|^2} + \frac{n-1}{n} I_n(z).$$

Therefore

$$I_n(z) \leq |z|^{2n-2} \mathrm{e}^{-n|z|^2}.$$

From these estimates, the following lemma follows.

Lemma 21. *Fix $0 < \epsilon < 1$. As $n \to \infty$, for $|z| \leq 1 - \epsilon$*

$$\hat{K}_n(z\sqrt{n}, z\sqrt{n}) = \frac{1}{\pi} + O(n^{-1}),$$

uniformly. As $n \to \infty$, for $|z| \geq 1 + \epsilon$

$$\hat{K}_n(z\sqrt{n}, z\sqrt{n}) = O(n^{-1}),$$

uniformly.

This shows that (see Exercise ex:au 7.9)

$$\mathbb{E}\hat{L}_n(\mathrm{D}z) \to \frac{1}{\pi} \mathbb{1}_{\{|z| \leq 1\}} \mathrm{D}z$$

weakly. This is the averaged circular law.

7.8 LUE Scaling limits I: Marchenko–Pastur law

Again, consider $X \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$ or $Y \sim \mathrm{Gin}_{\mathbb{R}}(m,n)$, $m \geq n$, and consider the sample covariance matrices X^*X/m and Y^TY/m, and let $x_1, \ldots, x_n$ be their unordered eigenvalues. Define the empirical spectral measure

$$\check{L}_n^{(\alpha)}(\mathrm{d}x) = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j}(\mathrm{d}x), \quad \alpha = m - n.$$

Assume further that $\mathfrak{d} := n/m \to d \in (0,1]$. The Marchenko–Pastur law states that

Theorem 62.

$$\mathbb{E}L_n(\mathrm{d}x) \to p_{\mathrm{MP}}(x;d)\mathrm{d}x := \frac{1}{2\pi d}\sqrt{\frac{|(\lambda_+ - x)(x-\lambda_-)|}{x^2}}\mathbb{1}_{[\lambda_-,\lambda_+]}(x)\mathrm{d}x,$$
$$\lambda_\pm(d) = (1\pm\sqrt{d})^2,$$

weakly as $n\to\infty$. If $d=0$, then the limiting law is $\delta_1(\mathrm{d}x)$.

The case of $d=0$ follows by the Law of Large Numbers.

In most of the calculations that follow it is convenient to use $\lambda_\pm = \lambda_\pm(\mathfrak{d})$. We now establish the theorem for $Y \sim \mathrm{Gin}_{\mathbb{C}}(m,n)$, i.e., $\beta = 2$. Define the Christoffel-Darboux kernel ($\alpha = m-n$)

$$\check{K}_n^{(\alpha)}(x,y) = \sum_{k=0}^{n-1} \frac{k!m^2}{(k+\alpha)!} L_k^{(\alpha)}(my)\, L_k^{(\alpha)}(mx)\,(mx)^{\alpha/2}\,(my)^{\alpha/2}\,\mathrm{e}^{-\frac{m}{2}(x+y)}$$
$$= \frac{n!\,m}{(n-1+\alpha)!}\frac{L_{n-1}^{(\alpha)}(mx)L_n^{(\alpha)}(my) - L_{n-1}^{(\alpha)}(my)L_n^{(\alpha)}(mx)}{x-y}\left(m^2xy\right)^{\alpha/2}\mathrm{e}^{-\frac{m}{2}(x+y)}.$$

Then using Theorem thm:det-formula 53 applied to eq:scm (5.0.1), we have the following expression for the joint marginal density for the eigenvalues of X^*X/m

$$\frac{1}{n!}\det\left(\check{K}_n^{(\alpha)}(x_j,x_k)_{1\le j,k\le n}\right).$$

It follows (see eq:det-form6 (6.1.4)) that the density of $\mathbb{E}\check{L}_n^{(\alpha)}$ is given by

$$\frac{1}{n}\check{K}_n^{(\alpha)}(x,x).$$

To now perform the asymptotics for this density, we refer to Appendix app:orthpoly E, specifically Section s:laguerre E.2. In that notation, we have

$$\check{K}_n^{(\alpha)}(x,y) = \frac{(n+\alpha)!}{(n-1)!}\frac{\mathfrak{l}_{n,-1}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right)\mathfrak{l}_{n,0}^{(\alpha)}\left(\frac{y}{4\mathfrak{d}}\right) - \mathfrak{l}_{n,-1}^{(\alpha)}\left(\frac{y}{4\mathfrak{d}}\right)\mathfrak{l}_{n,0}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right)}{x-y}\mathrm{e}^{-\frac{n}{2\mathfrak{d}}(x+y)}\left(\frac{n^2xy}{\mathfrak{d}^2}\right)^{-\alpha/2},$$

giving

$$\frac{1}{n}\check{K}_n^{(\alpha)}(x,x)$$
$$= \frac{(n+\alpha)!}{n!}\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{l}_{n,-1}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right)\mathfrak{l}_{n,0}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right) - \mathfrak{l}_{n,-1}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right)\frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{l}_{n,0}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right)\right]\mathrm{e}^{-\frac{n}{\mathfrak{d}}x}\left(\frac{nx}{\mathfrak{d}}\right)^{-\alpha},$$

We then compute using Stirling's approximation

$$\frac{(n+\alpha)!}{n!}\frac{2}{\pi n}\left(\frac{n}{\mathfrak{d}}\right)\frac{\mathfrak{d}^{3/2}(\sqrt{x})^{2\alpha-1}(\sqrt{\mathfrak{d}})^{2n-1}\mathrm{e}^{n\frac{x+1-\mathfrak{d}}{\mathfrak{d}}}}{[(x-\lambda_-)(\lambda_+-x)]^{1/2}}\mathrm{e}^{-\frac{n}{\mathfrak{d}}x}\left(\frac{nx}{\mathfrak{d}}\right)^{-\alpha}$$
$$= \frac{2}{\pi}\frac{(\sqrt{x})^{-1}(\sqrt{\mathfrak{d}})^{-1}}{[(x-\lambda_-)(\lambda_+-x)]^{1/2}}\left(1+O(n^{-1})\right)$$

We then consider the following combination

$$\begin{aligned}
&2\sin\theta_1\sin(\theta_1+2\theta_2-\theta_3)-2\sin(\theta_1+\theta_2-\theta_3)\sin(\theta_1+\theta_2)\\
&\qquad=\cos(2\theta_2-\theta_3)-\cos(2\theta_1+2\theta_2-\theta_3)-\cos\theta_3+\cos(2\theta_1+2\theta_2-\theta_3)\\
&\qquad=\cos(2\theta_2-\theta_3)-\cos\theta_3=2\sin\theta_2\sin(\theta_3-\theta_2).
\end{aligned}$$

If one chooses

$$\begin{aligned}
\theta_1&=k/\mathfrak{d}\arccos\frac{x+1-\mathfrak{d}}{2\sqrt{x}}+(k+1)\arccos\frac{1-x-\mathfrak{d}}{2\sqrt{\mathfrak{d}x}}+\frac{k}{2\mathfrak{d}}\sqrt{(\lambda_+-x)(x-\lambda_-)}\\
&\quad+\frac{1}{2}\arcsin\left(\frac{1}{\sqrt{\mathfrak{d}}}\frac{(x+1-\mathfrak{d})(1-\mathfrak{d})-2x}{2x}\right),\\
\theta_2&=\arccos\frac{x+1-\mathfrak{d}}{2\sqrt{x}},\\
\theta_3&=\arccos\frac{1-x-\mathfrak{d}}{2\sqrt{\mathfrak{d}x}},
\end{aligned}$$

and uses the expansion (E.2.8), we have for $0<d\leq 1$

$$\frac{1}{n}\check{K}_n^{(\alpha)}(x,x)=\frac{1}{2\pi\mathfrak{d}}\frac{\sqrt{(\lambda_+-x)(\lambda_--x)}}{x}(1+O(n^{-1})),$$

uniformly on compact subsets of $((1-\sqrt{d})^2,(1+\sqrt{d})^2)$. Note that the same statement holds with $\mathfrak{d}$ replaced with d.

7.9 The estimates of Silverstein

There is an important shortcoming of what the Marchenko–Pastur law demonstrates. It states that the vast majority of the eigenvalues of $\mathrm{LOE}(m,n)$, $\mathrm{LUE}(m,n)$ lie near the interval

$$[(1-\sqrt{\mathfrak{d}})^2,(1+\sqrt{\mathfrak{d}})^2],\quad \mathfrak{d}=n/m.$$

But it does not preclude the appearance of eigenvalues at a significant distance from this interval. The limit laws for the smallest and largest eigenvalues that will follow do preclude this, but in an asymptotic sense.

Silverstein [Sil85] applied the Gershgorin circle theorem to (9.4.1) to estimate the smallest eigenvalue of $W\sim\mathrm{LOE}(m,n),\mathrm{LUE}(m,n)$, $m\geq n$. We repeat this analysis to obtain a non-asymptotic bound on the spectrum of W. More explicitly, one find that there exists a unitary matrix U such that

$$U^*WU\sim\frac{1}{\beta m}\begin{bmatrix}
\chi^2_{\beta m} & \chi_{\beta(n-1)}\chi_{\beta m} & & \\
\chi_{\beta(n-1)}\chi_{\beta m} & \chi^2_{\beta(m-1)}+\chi^2_{\beta(n-1)} & \chi_{\beta(n-2)}\chi_{\beta(m-1)} & \\
 & \chi_{\beta(n-2)}\chi_{\beta(m-1)} & \chi^2_{\beta(m-2)}+\chi^2_{\beta(n-2)} & \ddots\\
 & & \ddots & \ddots
\end{bmatrix}$$

Since the matrix is symmetric, of course, all Gershgorin disks can be reduced to intervals. The first Gershgorin disk is given by $z \in \mathbb{R}$ such that

$$\left|\beta m z - \chi^2_{\beta m}\right| \leq \chi_{\beta m}\chi_{\beta(n-1)}.$$

And the $(j-1)$th disk is given by $z \in \mathbb{R}$ such that

$$\left|\beta m z - \left[\chi^2_{\beta(m-j)} + \chi^2_{\beta(n-j)}\right]\right| \leq \chi_{\beta(m-j)}\chi_{\beta(n-j-1)} + \chi_{\beta(m-j+1)}\chi_{\beta(n-j)} \tag{7.9.1}$$

eq:Gersh-int

The heuristics of the estimates is that because the χ-distributions concentrate so heavily, we can replace them with their asymptotic means $\chi_k \approx \sqrt{k}$. And even in the case where $j \approx n$ and $\chi_{\beta(n-j)}$ would not concentrate heavily, we expect its contribution to be negligible anyhow and the replacement is still valid. For the first disk,

$$|z-1| \leq \frac{\sqrt{n-1}}{\sqrt{m}} \leq \sqrt{\frac{n}{m}}.$$

And for the others

$$|mz - (m-j+n-j)| \leq \sqrt{m-j}\sqrt{n-j-1} + \sqrt{m-j+1}\sqrt{n-j}.$$

To analyze this inequality, set $a = m-j$ and $b = n-j$ and we see that

$$mz \in \left[a+b-\sqrt{a}\sqrt{b-1}-\sqrt{a+1}\sqrt{b}, a+b+\sqrt{a}\sqrt{b-1}+\sqrt{a+1}\sqrt{b}\right]. \tag{7.9.2}$$

eq:mz-int

Then write

$$\begin{aligned} &a+b-\sqrt{a}\sqrt{b-1}-\sqrt{a+1}\sqrt{b} \\ &= (\sqrt{a}-\sqrt{b})^2 + \sqrt{a}\sqrt{b} - \sqrt{a}\sqrt{b-1} + \sqrt{a}\sqrt{b} - \sqrt{a+1}\sqrt{b}. \end{aligned}$$

By the mean-value theorem

$$\begin{aligned} \sqrt{b}-\sqrt{b-1} &\geq \frac{1}{2\sqrt{b}}, \\ \sqrt{a+1}-\sqrt{a} &\leq \frac{1}{2\sqrt{a}}. \end{aligned}$$

This gives

$$\begin{aligned} a+b-\sqrt{a}\sqrt{b-1}-\sqrt{a+1}\sqrt{b} &\geq (\sqrt{a}-\sqrt{b})^2 + \frac{1}{2}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right) \\ &\geq (\sqrt{a}-\sqrt{b})^2. \end{aligned}$$

A similar calculation holds for the upper bound. And we find that for $j = 0, 1, 2, \ldots, n-1$, the bounds on the Gershgorin intervals are summarized by

$$(\sqrt{m-j} - \sqrt{n-j})^2 \leq mz \leq (\sqrt{m-j} + \sqrt{n-j})^2.$$

It is clear that this upper bound is a monotonically decreasing function of j. What is less clear is that the lower bound is a non-decreasing function of j. Indeed, the claim is clear if $n = m$. If $m > n$ then

$$\sqrt{m-j} - \sqrt{n-j} - (\sqrt{m-j-1} - \sqrt{n-j-1}) \leq \frac{1}{2\sqrt{m-j}} - \frac{1}{2\sqrt{n-j-1}} \leq 0,$$

and, the claim follows. We then figure that the interval

$$[(1-\sqrt{\mathfrak{d}})^2, (1+\sqrt{\mathfrak{d}})^2],$$

should indeed be a good estimate for an interval that contains the entire spectrum of W.

To make this precise, we of course, need to add allow for some small deviations and incorporate estimates on χ random variables. We do this using the moment-generating function for χ-squared random variables:

$$\mathbb{E}\left[\mathrm{e}^{t\chi_k^2}\right] = (1-2t)^{-k/2}, \quad t < 1/2.$$

This can be directly checked using the density

$$\frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} \mathrm{e}^{-x/2} \mathbb{1}_{(0,\infty)}(x)$$

Then we obtain, for $0 < t < 1/2$

$$\mathbb{P}(\chi_k^2 > k + \epsilon) = \mathbb{P}(\mathrm{e}^{t\chi_k^2} > \mathrm{e}^{tk+t\epsilon}).$$

We then apply Markov's inequality (Theorem t:markov 99) to find

$$\mathbb{P}(\chi_k^2 > k + \epsilon) \leq \inf_{0<t<1/2} \left[(1-2t)^{-k/2} \mathrm{e}^{-tk-t\epsilon}\right].$$

Elementary calculus allows one to then minimize this expression, obtaining that the minimum is attained at

$$t = \frac{\epsilon}{2\epsilon + 2k}, \tag{7.9.3}$$

with minimum value

$$\mathbb{P}(\chi_k^2 > k + \epsilon) \leq \mathrm{e}^{-\epsilon/2}\left(1 + \frac{\epsilon}{k}\right)^{k/2} = \exp\left(-\frac{\epsilon}{2} + \frac{k}{2}\log\left(1 + \frac{\epsilon}{k}\right)\right).$$

It will be convenient to replace ϵ with ϵm in this bound to find

$$\mathbb{P}\left(\frac{\chi_k^2}{m} > \frac{k}{m} + \epsilon\right) \leq \exp\left(-\frac{k}{2}\left[\sigma - \log(1+\sigma)\right]\right), \quad \sigma = \frac{\epsilon m}{k}.$$

A reasonable upper bound can then be obtained via

$$\sigma - \log(1+\sigma) = \int_0^\sigma \left(1 - \frac{1}{1+\sigma'}\right) \mathrm{d}\sigma' = \int_0^\sigma \left(\frac{\sigma'}{1+\sigma'}\right) \mathrm{d}\sigma' \geq \frac{\sigma^2}{2+2\sigma}.$$

And since $\sigma \geq \epsilon$,

$$\mathbb{P}\left(\frac{\chi_k^2}{m} > \frac{k}{m} + \epsilon\right) \leq \exp\left(-\frac{m}{4}\frac{\epsilon^2}{1+\epsilon}\right).$$

We will need a matching lower bound. For $t < 0$, again using Theorem 99,

$$\begin{aligned}
\mathbb{P}(\chi_k^2 < k - \epsilon) &= \mathbb{P}(t\chi_k^2 > tk - t\epsilon) \\
&= \mathbb{P}(\mathrm{e}^{t\chi_k^2} > \mathrm{e}^{tk - t\epsilon}) \\
&\leq \inf_{t<0} \left[(1-2t)^{-k/2}\mathrm{e}^{-tk+t\epsilon}\right] \\
&= \exp\left(\frac{\epsilon}{2} - \frac{k}{2}\log\left(\frac{k}{k-\epsilon}\right)\right), \quad \epsilon < k.
\end{aligned}$$

Similarly, replacing ϵ by ϵm we obtain

$$\mathbb{P}\left(\frac{\chi_k^2}{m} < \frac{k}{m} - \epsilon\right) \leq \exp\left(\frac{k}{2}\left[\sigma + \log(1-\sigma)\right]\right), \quad \epsilon < k/m, \quad \sigma = \frac{\epsilon m}{k}.$$

Here we note that if $\epsilon \geq k/m$ the same bound holds trivially. And then one can bound

$$\sigma + \log(1-\sigma) \leq -\frac{\sigma^2}{2(1-\sigma)}.$$

And again because $\sigma \geq \epsilon$ we obtain

$$\mathbb{P}\left(\frac{\chi_k^2}{m} < \frac{k}{m} - \epsilon\right) \leq \exp\left(-\frac{m}{4}\frac{\epsilon^2}{1-\epsilon}\right).$$

We are now ready to analyze, rigorously, what (7.9.1) implies about z. Fix $\epsilon > 0$. With exponentially large probability (with respect to m), we find a perturbed version of (7.9.2). Set $\check{a} = a - m\epsilon$, $\check{b} = b - m\epsilon$ and $\hat{a} = a + m\epsilon$, $\hat{b} = b + m\epsilon$:

$$mz \in \left[\check{a} + \check{b} - \sqrt{\hat{a}}\sqrt{\hat{b}-1} - \sqrt{\hat{a}+1}\sqrt{\hat{b}}, \hat{a} + \hat{b} + \sqrt{\hat{a}}\sqrt{\hat{b}-1} + \sqrt{\hat{a}+1}\sqrt{\hat{b}}\right].$$

Then write

$$\begin{aligned}
\check{a} + \check{b} - \sqrt{\hat{a}}\sqrt{\hat{b}-1} - \sqrt{\hat{a}+1}\sqrt{\hat{b}} &= \hat{a} + \hat{b} - \sqrt{\hat{a}}\sqrt{\hat{b}-1} - \sqrt{\hat{a}+1}\sqrt{\hat{b}} - 4m\epsilon \\
&\geq (\sqrt{\check{a}} - \sqrt{\check{b}})^2 - 4m\epsilon.
\end{aligned}$$

Repeating for the upper bound, and using a union bound over all the (independent) entries of the matrix, we find the non-asymptotic bound.

Theorem 63. *Suppose $W \sim \mathrm{LOE}(m,n), \mathrm{LUE}(m,n)$ ($\beta = 1,2$), then*

$$\mathbb{P}\left(\textit{eigenvalues of } \frac{W}{m} \subset \left[(\sqrt{1+\epsilon} - \sqrt{\mathfrak{d}+\epsilon})^2 - 4\epsilon, (\sqrt{1+\epsilon} + \sqrt{\mathfrak{d}+\epsilon})^2\right]\right)$$
$$\geq 1 - (2n-1)\left[\exp\left(-\frac{\beta m}{4}\frac{\epsilon^2}{1-\epsilon}\right) + \exp\left(-\frac{\beta m}{4}\frac{\epsilon^2}{1+\epsilon}\right)\right].$$

cor:gin_cond **Corollary 2.** *Suppose $X \sim \mathrm{Gin}_{\mathbb{R}}(m,n), \mathrm{Gin}_{\mathbb{C}}(m,n)$, $m \geq n$. For $\epsilon > 0$, with overwhelming probability*

$$1 - \sqrt{\frac{n}{m}} - \epsilon \leq \min_{x \in \mathbb{R}^n,\ x \neq 0} \frac{\|Xx\|_2}{\sqrt{m}\|x\|_2} \leq \max_{x \in \mathbb{R}^n,\ x \neq 0} \frac{\|Xx\|_2}{\sqrt{m}\|x\|_2} \leq 1 + \sqrt{\frac{n}{m}} + \epsilon.$$

7.9.1 An application to sketched least-squares problems

As a small aside, we now include a small discussion of the application of these estimates to the solution of large, over-determinded least-squares problems. Consider minimizing

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2, \quad A \in \mathbb{R}^{m \times n}, \quad m \gg n.$$

Suppose $\operatorname{rank} A = n$.

The idea of sketching is to replace this least-squares problem with a representative one of smaller dimension. So, let $X \sim \mathrm{Gin}_{\mathbb{R}}(k,m), k > m$ and consider minimizing

$$\min_{x \in \mathbb{R}^n} \|XAx - Xb\|_2.$$

One then wishes to first verify that $\operatorname{rank} XA = n$. Let $A = U\Sigma V^*$ be a singular value decomposition of A. Then

$$XA = (XU)\Sigma V.$$

By invariance of X, XU has the same distribution as X. So, we see that XAV^* is a column-scaled version of XU and therefore has the same rank as A. This also follows from Corollary 2.

So, set $x^* = \arg\min_{x \in \mathbb{R}^n} \|XAx - Xb\|_2$, $x^{**} = \arg\min_{x \in \mathbb{R}^n} \|Ax - b\|_2$. We apply Corollary 2. With overwhelming probability, we have

$$\|XAx^* - Xb\|_2 \leq \|XAx^{**} - Xb\|_2 \leq \sqrt{k}\left(1 + \sqrt{\frac{n}{k}} + \epsilon\right)\min_{x \in \mathbb{R}^n} \|Ax - b\|_2.$$

Similarly,

$$\|XAx^* - Xb\|_2 \geq \sqrt{k}\left(1 - \sqrt{\frac{n}{k}} - \epsilon\right)\|Ax^* - b\|_2.$$

Therefore, for $\epsilon > 0$ such that $1 - \sqrt{\frac{n}{k}} - \epsilon > 0$,

$$\|Ax^* - b\|_2 \leq \left[\frac{1 + \sqrt{\frac{n}{k}} + \epsilon}{1 - \sqrt{\frac{n}{k}} - \epsilon}\right]\min_{x \in \mathbb{R}^n} \|Ax - b\|_2,$$

with overwhelming probability.

7.10 LUE scaling limits II: the sine kernel

Now, let $x = r + \frac{\rho s}{n}$, $y = r + \frac{\rho y}{n}$ where $r \in (\lambda_-(d), \lambda_+(d))$. We consider this scaling limit of

$$\frac{\rho}{n}\check{K}_n^{(\alpha)}(x,y).$$

eq:trig-increase

Using (E.2.8)

$$\frac{\rho}{n}\check{K}_n^{(\alpha)}(x,y) = \frac{2(n+\alpha)!}{\pi n!(s-t)}\left(\frac{\mathfrak{d}}{n}\right)^{\alpha}\frac{\mathfrak{d}^{3/2}(\sqrt{\mathfrak{d}})^{2n-1}\mathrm{e}^{n\frac{1-\mathfrak{d}}{\mathfrak{d}}}}{[(x-\lambda_-)(\lambda_+-x)]^{1/4}[(y-\lambda_-)(\lambda_+-y)]^{1/4}}R_n(x,y)$$

where

$$\begin{aligned} R_n(x,y) &= \sin\left(n\phi_1(x)+\phi_2(x)+\phi_3(x)\right)\sin\left(n\phi_1(y)+\phi_2(y)\right) \\ &\quad -\sin\left(n\phi_1(y)+\phi_2(y)+\phi_3(y)\right)\sin\left(n\phi_1(y)+\phi_2(y)\right) \end{aligned}$$

and

$$\begin{aligned} \phi_1(x) &= -\frac{1}{\mathfrak{d}}\arccos\frac{x+1-\mathfrak{d}}{2\sqrt{x}} + \arccos\frac{1-x-\mathfrak{d}}{2\sqrt{\mathfrak{d}x}} + \frac{1}{2\mathfrak{d}}\sqrt{(\lambda_+-x)(x-\lambda_-)} \\ \phi_2(x) &= \arccos\frac{1-x-\mathfrak{d}}{2\sqrt{\mathfrak{d}x}} + \frac{1}{2}\arcsin\left(\frac{1}{\sqrt{\mathfrak{d}}}\frac{(x+1-\mathfrak{d})(1-\mathfrak{d})-2x}{2x}\right) \\ \phi_3(x) &= \arccos\frac{x+1-\mathfrak{d}}{2\sqrt{x}} - \arccos\frac{1-x-\mathfrak{d}}{2\sqrt{\mathfrak{d}x}}. \end{aligned}$$

We then write

$$\begin{aligned} n\phi_1(x)+\phi_2(x) &= n\phi_1(r)+n\phi_1'(r)\rho s+\phi_2(r)+O(n^{-1}), \\ \phi_3(x) &= \phi_3(r)+O(n^{-1}) \end{aligned}$$

to find

$$\begin{aligned} R_n(x,y) &= \sin\left(\phi_1'(r)s\rho+n\phi_1(r)+\phi_2(r)+\phi_3(r)\right)\sin\left(n\phi_1(r)+\phi'(r)t\rho+\phi_2(r)\right) \\ &\quad -\sin\left(\phi'(r)t\rho+n\phi_1(r)+\phi_2(r)+\phi_3(r)\right)\sin\left(n\phi_1(r)+\phi_1'(r)s\rho+\phi_2(r)\right)+O(n^{-1}). \end{aligned}$$

Then

$$\begin{aligned} &2\sin\left(\phi_1'(r)s\rho+n\phi_1(r)+\phi_2(r)+\phi_3(r)\right)\sin\left(n\phi_1(r)+\phi'(r)t\rho+\phi_2(r)\right) \\ &\quad = \cos(\phi'(r)\rho(s-t)+\phi_3(r)) - \cos(\phi'(r)\rho(s+t)+2n\phi_1(r)+2\phi_2(r)+\phi_3(r)) \end{aligned}$$

so that the expression for $R_n(x,y)$ simplifies to

$$\begin{aligned} R_n(x,y) &= \frac{1}{2}\left(\cos(\phi_1'(r)\rho(s-t)+\phi_3(r)) - \cos(\phi_1'(r)\rho(t-s)+\phi_3(r))\right)+O(n^{-1}) \\ &= -\sin(\phi_3(r))\sin\left(\phi_1'(r)\rho(s-t)\right)+O(n^{-1}). \end{aligned}$$

We make the choice $\rho = \pi/\phi_1'(r)$. It remains to simplify

$$-\frac{2\sin\phi_3(r)(n+\alpha)!}{\pi n!(s-t)}\left(\frac{\mathfrak{d}}{n}\right)^{\alpha}\frac{\mathfrak{d}^{3/2}(\sqrt{\mathfrak{d}})^{2n-1}\mathrm{e}^{n\frac{1-\mathfrak{d}}{\mathfrak{d}}}}{[(x-\lambda_-)(\lambda_+-x)]^{1/4}[(y-\lambda_-)(\lambda_+-y)]^{1/4}}$$
$$= \frac{(n+\alpha)!}{\pi n!(s-t)}\left(\frac{\mathfrak{d}}{n}\right)^{\alpha}(\sqrt{\mathfrak{d}})^{2n+1}\mathrm{e}^{n\frac{1-\mathfrak{d}}{\mathfrak{d}}}(1+O(n^{-1})),$$
$$= \frac{1}{\pi(s-t)}(1+O(n^{-1})),$$

where we used Stirling's approximation and that

$$\sin\phi_3(x) = -\frac{1}{2\sqrt{\mathfrak{d}}}\sqrt{(\lambda_+-x)(x-\lambda_-)}.$$

And then compute

$$\phi_1'(x) = \frac{\sqrt{(x-\lambda_-)(\lambda_+-x)}}{2\mathfrak{d}x},$$

so that $\rho = 1/p_{\mathrm{MP}}(r;\mathfrak{d})$. We have established the following[1]:

le:laguerre-sine

Lemma 22. *Let S be a bounded measurable set and $0 < d \leq 1$. Suppose $\mathfrak{d} = n/m \to d$ as $n \to \infty$. Then for $r \in [\lambda_-(d), \lambda_+(d)]$*

$$\lim_{n\to\infty}\sup_{x,y\in S}\left|\frac{1}{np_{\mathrm{MP}}(r;\mathfrak{d})}\check{K}_n^{(m-n)}\left(r+\frac{s}{np_{\mathrm{MP}}(r;\mathfrak{d})}, r+\frac{t}{np_{\mathrm{MP}}(r;\mathfrak{d})}\right) - K_{\mathsf{sine}}(x,y)\right| = 0. \tag{7.10.1}$$

eq:sine-conv1-mp

From this lemma and the results of Section sec:fdconv D.2 we have (recall the proof of Theorem thm:dyson2 56):

thm:dyson2-mp

Theorem 64. *Suppose $n/m \to d \in (0,1]$ as $n \to \infty$. For each integer $k = 0, 1, 2, \ldots,$ bounded, Borel set S and $r \in (\lambda_-(d), \lambda_+(d))$*

$$\lim_{n\to\infty}\mathbb{P}\left(W \sim \mathrm{LUE}(m,n) \text{ has } k \text{ eigenvalues in } \left(r+\frac{S}{np_{\mathrm{MP}}(r;d)}\right)\right)$$
$$= \frac{1}{k!}\left(-\frac{d}{dz}\right)^k \det\left(I - zK_{\mathsf{sine}}\mathbb{1}_S\right)\Big|_{z=1}. \tag{7.10.2}$$

eq:rsc4

7.11 LUE scaling limits III(a): the Bessel kernel at the hard edge

Consider the case $m = n + \alpha$ for $\alpha \in \mathbb{N}$. Then the smallest eigenvalues of a matrix distributed according to LUE(m,n) tends to zero as $n \to \infty$. But the

[1]To truly establish this theorem, one needs to deal with the case $s \approx t$, see the proof of Lemma le:hermite-sine 17

matrix is positive (semi-)definite. Therefore the eigenvalues tend to "build up" at this "hard edge". Consider the kernel $\check{K}_n^{(\alpha)}(x,y)$. Rescale it according to $\sqrt{x} = \sqrt{s}/n$, $\sqrt{y} = \sqrt{t}/n$, i.e.,

$$nx = \frac{s}{n}, \quad ny = \frac{t}{n}.$$

Then use (eq:bessel E.2.7) for s bounded away from t

$$\frac{1}{n^2}\check{K}_n^{(\alpha)}(x,y) = \frac{(n+\alpha)!}{(n-1)!}\frac{1}{2n}\left[J_{\alpha+1}(\sqrt{s})J_\alpha(\sqrt{t}) - J_{\alpha+1}(\sqrt{t})J_\alpha(\sqrt{s}) + O(n^{-1})\right]\frac{\mathrm{e}^{-\frac{s+t}{2n}}\left(\frac{1}{n^2}\right)^{-\alpha/2}}{s-t}$$
$$= \frac{\sqrt{s}J_{\alpha+1}(\sqrt{s})J_\alpha(\sqrt{t}) - \sqrt{t}J_{\alpha+1}(\sqrt{t})J_\alpha(\sqrt{s})}{2(s-t)} + O(n^{-1})$$

This is convergence is uniform[2] for s,t in compact subsets of $[0,\infty)$. We have mostly established the following.

l:bessel-limit **Lemma 23.** *Let S be a bounded measurable set and $\alpha \in \mathbb{N}$ is fixed. Then*

$$\lim_{n\to\infty}\sup_{x,y\in S}\left|\frac{1}{n^2}\check{K}_n^{(\alpha)}\left(\frac{x}{n^2},\frac{y}{n^2}\right) - K_{\mathsf{Bessel}}^{(\alpha)}(x,y)\right| = 0, \tag{7.11.1}$$ eq:sine-conv1-mp

where

$$K_{\mathsf{Bessel}}^{(\alpha)}(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J_\alpha'(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J_\alpha'(\sqrt{x})}{2(x-y)}.$$

Proof. We only have to establish that

$$J_\alpha(\sqrt{x})\sqrt{y}J_\alpha'(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J_\alpha'(\sqrt{x})$$
$$= \sqrt{x}J_{\alpha+1}(\sqrt{x})J_\alpha(\sqrt{y}) - \sqrt{y}J_{\alpha+1}(\sqrt{y})J_\alpha(\sqrt{x}).$$

This follows from the identity [DLMF OLBC10][Section 10.2.(ii)]

$$J_\alpha'(z) = -J_{\alpha+1}(z) + \frac{\alpha}{z}J_\alpha(z).$$

□

Theorem 65. *Fix $\alpha\in\mathbb{N}$ and let S be a bounded measureable subset of $[0,\infty)$. Then*

$$\lim_{n\to\infty}\mathbb{P}\left(W \sim \mathrm{LUE}(n+\alpha,n) \text{ has } k \text{ eigenvalues in } \frac{S}{n^2}\right)$$
$$= \frac{1}{k!}\left(-\frac{d}{dz}\right)^k \det\left(I - zK_{\mathsf{Bessel}}^{(\alpha)}\mathbb{1}_S\right)\Big|_{z=1}. \tag{7.11.2}$$ eq:rsc4

Corollary 3. *Let x_1 be the smallest eigenvalue of an $\mathrm{LUE}(n+\alpha,n)$ matrix for $\alpha\in\mathbb{N}$ fixed. Then*

$$\lim_{n\to\infty}\mathbb{P}(x_1 > t/n^2) = \det\left(I - K_{\mathsf{Bessel}}^{(\alpha)}\mathbb{1}_{[0,t)}\right).$$

In other words, n^2x_1 converges in distribution to $1 - \det\left(I - K_{\mathsf{Bessel}}^{(\alpha)}\mathbb{1}_{[0,t)}\right)$.

[2]Again, the case $s\approx t$ needs to be treated using the methodology in Lemma le:hermite-sine 17.

7.12 LUE scaling limits III(b): the Airy kernel at the softened hard edge

Now, if $n/m \to d$ for $d \in (0,1)$ the smallest eigenvalues of an LUE(m,n) matrix tend to lie away from zero. The "hard edge" effect is diminished and the Tracy–Widom distribution reappears. From a special function perspective, this can be understood using the fact that, in an appropriate scaling region, Bessel functions J_α are well-approximated by Airy functions as $\alpha \to \infty$. This scaling, though complicated, can be exploited to treat the $\alpha \to \infty$ but $\alpha/n \to 0$ case Deift2015,Menon2016 [DMT16, MT16]. Here, we do not take this approach but rather use direct asymptotics because $\alpha/n \to 1-d > 0$.

The scaling here is informed by (eq:left-rescale E.2.14):

$$x = \lambda_+ - \frac{s}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_-^{2/3}, \quad y = \lambda_+ - \frac{t}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_-^{2/3}.$$

Then, we find, using (eq:laguerre-left E.2.15): TODO: Modify this

le:laguerre-right-airy **Lemma 24.** *Suppose $\mathfrak{d} \to d \in (0,1]$. For $s \neq t$, uniformly on bounded sets,*

$$\lim_{n\to\infty} \left| \frac{\sqrt{\mathfrak{d}}\lambda_-^{2/3}}{n^{\frac{2}{3}}} \check{K}_n^{(\alpha)}\left(\lambda_- - \frac{s}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_-^{2/3}, \lambda_- - \frac{t}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_-^{2/3}\right) - K_{\mathsf{Airy}}(x,y)\right| = 0 \tag{7.12.1}$$

eq:airy-conv-lag

Unify the statement of all the kernel limits lemmas

and there exists a function $G(x,y) \in L^2([C,\infty)^2)$ for all $C \in \mathbb{R}$ such that

$$\left| \frac{\sqrt{\mathfrak{d}}\lambda_-^{2/3}}{n^{\frac{2}{3}}} \check{K}_n^{(\alpha)}\left(\lambda_- - \frac{s}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_-^{2/3}, \lambda_- - \frac{t}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_-^{2/3}\right)\right| \leq G(s,t). \tag{7.12.2}$$

t:laguerre-left **Theorem 66.** *Suppose $\mathfrak{d} \to d \in (0,1]$ and let S be a measureable subset of $\mathbb{R}$ that is bounded from below. Then*

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(W \sim \text{LUE}(n+\alpha,n) \text{ has } k \text{ eigenvalues in } \lambda_- - \frac{S}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_-^{2/3}\right)\\ = \frac{1}{k!}\left(-\frac{d}{dz}\right)^k \det\left(I - zK_{\mathsf{Airy}}\mathbb{1}_S\right)\big|_{z=1}.\end{aligned} \tag{7.12.3}$$

eq:rsc41

7.13 LUE scaling limits III(c): the Airy kernel at the soft edge

The requisite rescaling is informed by (eq:right-rescale E.2.11). So, set

$$x = \lambda_+ + \frac{s}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_+^{2/3}, \quad y = \lambda_+ + \frac{t}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_+^{2/3}.$$

Then, we find, using (eq:laguerre-right E.2.13):

e:laguerre-right-airy **Lemma 25.** *Suppose $\mathfrak{d} \to d \in (0,1]$. For $s \neq t$, uniformly on bounded sets,*

$$\lim_{n\to\infty} \left| \frac{\sqrt{\mathfrak{d}}\lambda_+^{2/3}}{n^{\frac{2}{3}}} \check{K}_n^{(\alpha)} \left(\lambda_+ + \frac{s}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_+^{2/3}, \lambda_+ + \frac{t}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_+^{2/3} \right) - K_{\mathsf{Airy}}(x,y) \right| = 0 \tag{7.13.1}$$

eq:airy-conv-lag

and there exists a function $G(x,y) \in L^2([C,\infty)^2)$ for all $C \in \mathbb{R}$ such that

$$\left| \frac{\sqrt{\mathfrak{d}}\lambda_+^{2/3}}{n^{\frac{2}{3}}} \check{K}_n^{(\alpha)} \left(\lambda_+ + \frac{s}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_+^{2/3}, \lambda_+ + \frac{t}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_+^{2/3} \right) \right| \leq G(s,t). \tag{7.13.2}$$

t:laguerre-right **Theorem 67.** *Suppose $\mathfrak{d} \to d \in (0,1]$ and let S be a measureable subset of $\mathbb{R}$ that is bounded from below. Then*

$$\lim_{n\to\infty} \mathbb{P}\left(W \sim \mathrm{LUE}(n+\alpha,n) \text{ has } k \text{ eigenvalues in } \lambda_+ + \frac{S}{n^{2/3}}\sqrt{\mathfrak{d}}\lambda_+^{2/3} \right)$$
$$= \frac{1}{k!}\left(-\frac{d}{dz}\right)^k \det\left(I - zK_{\mathsf{Airy}}\mathbb{1}_S\right)\big|_{z=1}. \tag{7.13.3}$$

eq:rsc4

7.14 Notes on universality and generalizations

In this section we discuss two generalizations:

- Scaling limits for $\beta = 1, 4$.
- The case of non-Gaussian entries.

The calculations behind these results are the topic of a more advanced text such as Deift-universality,TerenceTao2011 [DG09, Tao11].

7.14.1 Limit theorems for $\beta = 1, 4$

When $\beta \neq 2$, the determinantal structure is lost. For $\beta = 1$ the correct technology to use is Pffafians. For $\beta = 4$, one uses . In either case, Lemma le:scaling-semicircle 16 holds. add this But the the other two scaling limits have different limits. Define (see TracyWidomII [TW96]

$$F_1(t) = \exp\left(-\frac{1}{2}\int_t^\infty q(s)\mathrm{d}s\right)(F_2(t))^{1/2},$$
$$F_4(t/\sqrt{2}) = \cosh\left(\frac{1}{2}\int_t^\infty q(s)\mathrm{d}s\right)(F_2(t))^{1/2}.$$

and How much detail to provide here?

$$H_1(t) = ...$$
$$H_4(t) = ...$$

Then Confirm the scaling of t

$$\lim_{n\to\infty} \mathbb{P}\left(M \sim \mathrm{GOE}(n),\ \lambda_n \le 2\sqrt{n} + n^{-1/6}t\right) = F_1(t),$$
$$\lim_{n\to\infty} \mathbb{P}\left(M \sim \mathrm{GSE}(n),\ \lambda_n \le 2\sqrt{n} + n^{-1/6}t\right) = F_4(t),$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(M \sim \mathrm{GOE}(n) \text{ has no eigenvalues in } \sqrt{n}\left(r + \frac{(-\frac{t}{2}, \frac{t}{2})}{np_{\mathrm{sc}}(r)}\right)\right) = H_1(t),$$
$$\lim_{n\to\infty} \mathbb{P}\left(M \sim \mathrm{GSE}(n) \text{ has no eigenvalues in } \sqrt{n}\left(r + \frac{(-\frac{t}{2}, \frac{t}{2})}{np_{\mathrm{sc}}(r)}\right)\right) = H_4(t).$$

7.14.2 Universality theorems

The most basic universality theorem is the central limit theorem, Theorem t:clt 102. It states that the sample average of n iid random variables, with finite variance, after rescaling, converges in distribution to a normal random variable. It is also important to note that in the central limit theorem, the identically distributed assumption can be relaxed if one includes the so-called Lindeberg condition Durrett [?, Need page]. So, beyond the first two moments which are used to define the rescaling, the fluctuations are asymptotically universal — independent of the finer details of the distribution. Similar phenomena is pervasive throughout random matrix theory. To state these results, whose proofs are beyond the scope this the current text, we have to be more precise in our definition of a Wigner ensemble and describe the so-called invariant ensembles. The reader should note that GOE and GUE are the only ensembles that lie in the intersection of both invariant and Wigner ensembles. We only make universality precise in the $\beta = 1, 2$ cases.

Generalized Wigner ensembles

Definition 68 (Erdos2012a [EYY12]). Consider a semi-infinite matrix $\mathcal{M} = (M_{jk})_{j,k\ge 1}$ of real ($\beta = 1$) or complex ($\beta = 2$) random variables such that $M_{jk} = \overline{M_{kj}}$ for all j, k $(M_{jk})_{j\le k}$ are independent random variables with

$$\mathbb{E}[M_{jk}] = 0, \quad \mathrm{Var}(M_{jk}) = \sigma_{jk}^2/N$$

such that

1. For any k fixed
$$\sum_{j=1}^{n} \sigma_{jk}^2 = 1.$$

2. There is a constant $C \ge 1$ such that for all j, k
$$\frac{1}{C} \le \sigma_{jk}^2 \le C.$$

3. In the complex case

$$\mathbb{E}[M_{jk}^2] = 0.$$

Then we say that $M \in \mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$) is from a real or complex generalized Wigner ensemble if M is the upper-left $n \times n$ subblock of $\mathcal{M}$.

The following theorem gives universality for the largest k eigenvalues.

Theorem 69. Erdos2012a *[EYY12] Suppose $\Lambda_k(\beta) = \Lambda_k = [\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n-k+1}]^T$ is the vector of the top k largest eigenvalues a* $\mathrm{GOE}(n)$ *($\beta = 1$) or a* $\mathrm{GUE}(n)$ *($\beta = 2$) distributed matrix. Suppose $\hat{\Lambda}_k(\beta)$ is the same random vector a real or complex generalized Wigner ensemble with $\sigma_{jk} = 1$. Define the rescaled vectors*

$$\Gamma_k(\beta) = (\Lambda_k(\beta) - 2\sqrt{n})n^{1/6}, \quad \hat{\Gamma}_k(\beta) = (\hat{\Lambda}_k(\beta) - 2\sqrt{n})n^{1/6}.$$

Then there is an $\epsilon > 0$ and $\delta > 0$ such that for fixed k and N sufficiently large

$$\begin{aligned} F_{\Lambda_k(\beta)}(t_1 - N^{-\epsilon}, \ldots, t_k - N^{-\epsilon}) - N^{-\delta} \\ \leq F_{\hat{\Lambda}_k(\beta)}(t_1, \ldots, t_k) \leq \\ F_{\Lambda_k(\beta)}(t_1 + N^{-\epsilon}, \ldots, t_k + N^{-\epsilon}) + N^{-\delta}. \end{aligned}$$

If the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are almost deterministic and are distributed according to $p_{\mathrm{sc}}(x)\mathrm{d}x$ then, one would expect

$$\lambda_j \approx \gamma_j, \quad \int_{-\infty}^{\gamma_j} p_{\mathrm{sc}}(x)\mathrm{d}x = \frac{j}{n}, \quad \gamma_j \leq 2.$$

Furthermore, the error in this approximation should be on the same order as the distance between successive γ_j's. This heuristic gives

$$|\lambda_j - \gamma_j| = O\left(\min_{k\neq j} |\gamma_k - \gamma_j|\right).$$

Then, one can estimate, for $j < n$, by the mean-value theorem

$$\begin{aligned} \frac{1}{n} &= \int_{\gamma_j}^{\gamma_{j+1}} p_{\mathrm{sc}}(x)\mathrm{d}x = p_{\mathrm{sc}}(\xi)(\gamma_{j+1} - \gamma_j), \\ \min_{k\neq j} |\gamma_k - \gamma_j| &= \frac{1}{np_{\mathrm{sc}}(\xi)} \approx \frac{1}{np_{\mathrm{sc}}(\gamma_j)} \end{aligned}$$

where ξ some point between γ_j and its nearest neighbor. This is, in turn, only useful if we can find a decent way to estimate $p_{\mathrm{sc}}(\gamma_j)$. For $-2 \leq x \leq 0$ we estimate

$$\frac{1}{\sqrt{2}\pi}\sqrt{x+2} \leq p_{\mathrm{sc}}(x) \leq \frac{1}{\pi}\sqrt{x+2}.$$

Then we define $\tilde{\gamma}_j$ and $\hat{\gamma}_j$, for $j/n \le \frac{4}{3\pi}$

$$\int_{-2}^{\tilde{\gamma}_j} \frac{1}{\sqrt{2}\pi}\sqrt{x+2}\,\mathrm{d}x = \frac{j}{n}, \quad \int_{-2}^{\hat{\gamma}_j} \frac{1}{\pi}\sqrt{x+2}\,\mathrm{d}x = \frac{j}{n}.$$

The condition $j/n \le \frac{4}{3\pi}$ guarantees that $\gamma_j, \tilde{\gamma}_j, \hat{\gamma}_j \le 0$ on which we have the estimate. We then know

$$\hat{\gamma}_j \le \gamma_j \le \tilde{\gamma}_j.$$

Or more specifically,

$$-2 + \left(\frac{3\pi j}{\sqrt{2}n}\right)^{2/3} \le \gamma_j \le -2 + \left(\frac{3\pi j}{n}\right)^{2/3}.$$

Thus for such a γ_j

$$\frac{1}{\sqrt{2}\pi}\left(\frac{3\pi j}{\sqrt{2}n}\right)^{1/3} \le p_{\mathrm{sc}}(\gamma_j) \le \frac{1}{\pi}\left(\frac{3\pi j}{n}\right)^{1/3}.$$

Then, for γ_j where $\frac{4}{3\pi} < \frac{j}{n} \le 1/2$, we know that $p_{\mathrm{sc}}(\gamma_j)$ is bounded away from zero. Thus, by symmetry for $1 > j/n \ge 1/2$, we have a uniform estimate: For some $C \ge 1$

$$\frac{1}{C}\left(\frac{d_n(j)}{n}\right)^{1/3} \le p_{\mathrm{sc}}(\gamma_j) \le C\left(\frac{d_n(j)}{n}\right)^{1/3},$$
$$d_n(j) = \min\{|j|, |n-j+1|\}.$$

A key ingredient in the proof of this result is the following rigidity estimate and the heursitcs

Theorem 70. Erdos2012a *[EYY12] Suppose $\lambda_1 \le \cdots \lambda_n$ are the eigenvalues of a generalized Wigner matrix. Then for every $\epsilon > 0$ and $L > 0$ there exists $N = N(\epsilon, L) > 0$ such that for $n \ge N$*

$$\mathbb{P}\left(|\lambda_j - \gamma_j| \le n^{-2/3} d_n(j)^{-1/3} n^{\epsilon} \text{ for all } j\right) \ge 1 - n^{-L}.$$

This theorem says that on a set of nearly full probability, the eigenvalues stick to their typical locations γ_j. One can also read this as a statement that the probability that any eigenvalue, say λ_j, deviates from γ_j so much so that

$$\left|\int_{-\infty}^{\lambda_j} p_{\mathrm{sc}}(x)\mathrm{d}x - \frac{j}{n}\right| \ge Cn^{\epsilon-1}.$$

is effectively exponentially small.

Something that, for good reason, we have left undiscussed is the question of what the eigenvectors look like. This is because we know that in the cases of

GUE, GOE and GSE, they can effectively be taken to be uniformly distributed on the associated group under whose action the distribution is invariant. But for generalized Wigner ensembles there is, in principle, no invariance. If the entries in $\mathcal{M}$ are iid, we know that the marginal distribution on each eigenvector, modulo normalizations, is the same. This is because the distribution is invariant under conjugations by permutation matrices. But generalized Wigner ensembles do not need to have iid entries. Nonetheless, universality tells us that for large enough n the eigenvectors should behave similarly. But what "behave similarly" actual means is more ambiguous for the eigenvectors than it is for the eigenvalues. We collect two properties of matrices in $\mathsf{O}(n)$ as n becomes large.

t:On-est **Theorem 71.** *Suppose Q is distributed uniformly on $\mathsf{O}(n)$ according to Haar measure.*

1. *Let $S = \{j_1, \ldots, j_\ell\} \subset [\![1, n]\!]$ with $\#S = \ell$ and ℓ fixed. Then for k fixed as $n \to \infty$*

$$n^{1/2} \begin{bmatrix} Q_{j_1 k} \\ \vdots \\ Q_{j_\ell k} \end{bmatrix} \xrightarrow{dist.} \mathcal{N}(0, \mathbf{1}_\ell).$$

2. *Complete delocalization holds: If $a_n \to \infty$ as $n \to \infty$ then*

$$\mathbb{P}\left(\sup_{1 \le j,k \le n} |Q_{jk}| \ge n^{-1/2} a_n \log n \right) = O\left(\frac{1}{a_n}\right).$$

Proof. 1. Because the distribution Q is right invariant (under the action of a permutation matrix), it suffices to set $k = 1$. Then because of left invariance, again, under the action of permutation matrices, it suffices to take $(j_1, \ldots, j_\ell) = (1, 2, \ldots, \ell)$. Then, using left invariance again it follows that the distribution of the first column is uniform on S^{n-1}. This implies

$$\begin{bmatrix} Q_{11} \\ \vdots \\ Q_{n1} \end{bmatrix} \sim \begin{bmatrix} \frac{X_1}{\|X\|_2} \\ \vdots \\ \frac{X_n}{\|X\|_2} \end{bmatrix},$$

where $(X_j)_{j \ge 1}$ is a collection of iid $\mathcal{N}(0,1)$ random variables. The claim then follows from from Lemma 35 and the weak law of large numbers (Theorem 101) applied to the sum $\sum_j X_j^2$.

2. This claim concerns the maximum of n^2 correlated, but compactly supported, random variables. Furthermore, because of the invariance of Haar measure they are all identically distributed. Our main tool is the exponential moment generating function: For $s > 0$

$$\mathbb{E}\left[\sup_{1 \le j,k \le n} |Q_{jk}| \right] = s\mathbb{E}\left[s^{-1} \sup_{1 \le j,k \le n} |Q_{jk}| \right] = s \log \exp \mathbb{E}\left[s^{-1} \sup_{1 \le j,k \le n} |Q_{jk}| \right],$$

which, by Jensen's inequality (Theorem t:jensen 98), implies

$$\mathbb{E}\left[\sup_{1\le j,k\le n}|Q_{jk}|\right] \le s\log\mathbb{E}\left[\exp\left(s^{-1}\sup_{1\le j,k\le n}|Q_{jk}|\right)\right].$$

Then we estimate

$$\mathbb{E}\left[\sup_{1\le j,k\le n}|Q_{jk}|\right] \le s\log\mathbb{E}\left[\sum_{j,k}\exp(s^{-1}|Q_{jk}|)\right] = s\log n^2\mathbb{E}\left[\exp(s^{-1}|Q_{11}|)\right].$$

So, we set $s = n^{-1/2}$ and we need to estimate

$$\mathbb{E}\left[\exp(n^{1/2}|Q_{11}|)\right], \tag{7.14.1}$$

eq:Q11

and it is reasonable, in light of the previous result, to expect this to converge to $\mathbb{E}[\exp(|X|)]$ where $X \sim \mathcal{N}(0,1)$. Indeed this is true because $|Q_{11}|^2$ is distributed as the first component of a 1-Dirichlet random vector – it is beta distributed. Therefore, for $n \ge 2$,

$$\mathbb{E}\left[\exp(n^{1/2}|Q_{11}|)\right] = \frac{1}{\mathrm{B}(1/2,(n-1)/2)}\int_0^1 \mathrm{e}^{n^{1/2}\sqrt{x}}x^{-1/2}(1-x)^{n/2-3/2}\mathrm{d}x$$

$$= \frac{2}{\sqrt{n}\mathrm{B}(1/2,(n-1)/2)}\int_0^\infty \mathrm{e}^{y}\left(1-\frac{y^2}{n}\right)^{n/2-3/2}\mathbb{1}_{[0,\sqrt{n}]}(y)\mathrm{d}y.$$

Stirling's formula gives that $\sqrt{n}\mathrm{B}(1/2,(n-1)/2) \to \sqrt{\pi}$ and the limit can be passed inside the integral because

$$\left(1-\frac{y^2}{n}\right)^{n/2-3/2}\mathbb{1}_{[0,\sqrt{n}]}(y) \le \mathrm{e}^{-y^2/2},$$

and the dominated convergence theorem applies. Here one uses the inequality $\log(1-x) \le -x$ for $x < 1$. The point is that (eq:Q11 7.14.1) is bounded, independent of n by a constant C. And this gives the inequality

$$\mathbb{E}\left[\sup_{1\le j,k\le n}|Q_{jk}|\right] \le n^{-1/2}\log n^2 C$$

Then Markov's inequality produces

$$\mathbb{P}\left(\sup_{1\le j,k\le n}|Q_{jk}| \ge n^{-1/2}x\log n\right) \le \frac{\log n^2 C}{x\log n}.$$

So, provided that $x \to \infty$ with n, this probability decays.

□

Theorem 71 is now compared with a theorem about the eigenvectors for generalized Wigner matrices. Define $\mathbb{T}_{n,\delta} = [\![1, n^{1/4}]\!] \cup [\![n^{1-\delta}, n - n^{1-\delta}]\!] \cup [\![N - N^{1/4}, N]\!]$. This result is summarized from results in [BY17, EYY12]

Theorem 72. *Suppose $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of a generalized Wigner matrix. Suppose further that $q_1, \ldots, q_n$ are the associated orthonormal eigenvectors and set $Q = [q_1, \ldots, q_n]$. For fixed ℓ, there exists $\delta > 0$ such that for any $k \in \mathbb{T}_{n,\delta}$ and $S = (j_1, \ldots, j_\ell) \subset [\![1, n]\!]$, $\#J = \ell$, as $n \to \infty$*

$$\sqrt{n} \begin{bmatrix} |Q_{j_1 k}| \\ \vdots \\ |Q_{j_\ell k}| \end{bmatrix} \xrightarrow{dist.} \begin{bmatrix} |X_1| \\ \vdots \\ |X_\ell| \end{bmatrix} \quad (\beta = 1),$$
$$\sqrt{2n} \begin{bmatrix} |Q_{j_1 k}| \\ \vdots \\ |Q_{j_\ell k}| \end{bmatrix} \xrightarrow{dist.} \begin{bmatrix} |X_1^{(1)} + \mathrm{i} X_1^{(2)}| \\ \vdots \\ |X_\ell^{(1)} + \mathrm{i} X_\ell^{(2)}| \end{bmatrix} \quad (\beta = 2).$$

where $X_j, X_j^{(1)}, X_j^{(2)}$ for $j = 1, 2, \ldots, \ell$ are iid $\mathcal{N}(0,1)$ random variables. Furthermore, delocalization holds: For some $C > 0$

$$\mathbb{P}\left(\max_{1 \leq j,k \leq n} |Q_{jk}| \leq n^{1/2} (\log n)^{C \log\log n} \right) \to 0,$$

as $n \to \infty$.

The reader should take note of the additional technicalities required to make these statements as a harbinger of difficulties that arise in the proofs.

Invariant ensembles

The invariant ensembles are distributions on self-adjoint real, complex or quarterion matrices that are described by a potential function $V : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x \to \pm\infty} \frac{V(x)}{\log(1+|x|)} = +\infty$. The density is then given by

$$\frac{1}{\tilde{Z}_{n,V}(\beta)} \mathrm{e}^{-n \frac{\beta}{4} \operatorname{Tr} V(M)} \partial M.$$

Using Weyl's formula we obtain a marginal joint density for the eigenvalues

$$\frac{1}{Z_{n,V}(\beta)} |\Delta(\Lambda)|^\beta \mathrm{e}^{-n \frac{\beta}{4} \operatorname{Tr} V(\Lambda)} \mathrm{D}\Lambda.$$

For $\beta = 2$, in a natural way, one is led to analyze orthogonal polynomials with respect to the varying weight

$$\omega_n(x)\mathrm{d}x = \mathrm{e}^{-\frac{n}{2} V(x)} \mathrm{d}x.$$

When $V(x)$ is not a quadratic polynomial, correlations between the entries on and above the diagonal are introduced. This destroys $p_{\mathrm{sc}}(x)\mathrm{d}x$ as the global limit of the spectrum. The discussion in Chapter 8 gives the methodology to determine the new limit.

So, one might think that because the global picture is different, universality fails. But, remarkably, the local picture is still the same. The orthogonal polynomials with respect to $\omega_n(x)\mathrm{d}x$ can be analyzed asymptotically with the help of steepest descent Riemann–Hilbert analysis[3] [Dei00] and the sine kernel is found as a scaling limit of gap probabilities in the bulk and the Airy kernel is found (generically) as the scaling limit at the edges. This is true, at least when V is convex. If V is not convex more exotic behavior can occur.

Exercises

7.1. Prove the Christoffel-Darboux identity (E.1.8) for Hermite polynomials. (This is a standard relation and it is easy to find a proof in many texts, but try to do it on your own.)

ex:finiterank 7.2. Show that

$$\int_{\mathbb{R}^k} \det[K(x_p, x_q)]_{1\le p,q\le k}\,\mathrm{d}x_1 \cdots \mathrm{d}x_k = 0, \tag{7.14.2}$$

for $k > n$, if K is of the form

$$K(x,y) = \sum_{j=0}^{n-1} g_j(y) f_j(x), \quad f_j, g_j \in L^2(\mathbb{R}), \quad j = 0, 1, \ldots, n-1. \tag{7.14.3}$$

ex:G 7.3. Finish the proof of Lemma 19 by constructing a function $G(x, y)$.

7.4. Establish (7.5.1).

7.5. Use the method of steepest descent to establish the asymptotic formula (C.3.1) for the Airy function. This is an easy application of the method of steepest descent.

7.6. In order to appreciate the power of the Plancherel-Rotach asymptotics, some numerical calculations will help.

(a) Develop a numerical scheme to compute all the roots of the n-th Hermite polynomial $\mathfrak{h}_n$. Plot the empirical distribution of roots for $n = 100$. Can you determine the limiting density of suitably rescaled roots?

(b) Numerically compute the Hermite wave functions for large n, say $n = 100$, and compare the rescaled wave function with the Plancherel-Rotach asymptotic formulas in all three regions (oscillatory, decaying and transition).

7.7. Use the method of steepest descent to establish the Plancherel-Rotach asymptotics in the region of exponential decay (equation (??)). This requires more care than Q.2.

[3] In general, there will be no contour integral representation of the polynomials to exploit.

7.8. Establish the following a priori bound on the Airy kernel. For any $a \in \mathbb{R}$,

$$\sup_{x,y} e^{x+y} |K_{\mathsf{Airy}}(x,y)| < \infty. \tag{7.14.4}$$

eq:airy-apriori

Let S be the semi-infinite interval (a, ∞). Use the above estimate to establish that the Fredholm determinant $\det(I - zK_{\mathsf{Airy}}\mathbb{1}_S)$ is an entire function.

ex:au 7.9. Let $\rho_n(x)$, $n = 1, 2, \dots$ be probability densities on $\mathbb{R}$ that converge almost uniformly to $\rho(x)$ with respect to Lebesgue measure on $\mathbb{R}$. Assume ρ has compact support. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\rho_n(x)\mathrm{d}x = \int_{\mathbb{R}} f(x)\rho(x)\mathrm{d}x$$

for every continuous function f with compact support.

Chapter 8

The equilibrium measure

ch:eqbm

In this section we establish properties of the equilibrium measure for general invariant ensembles. We also relate the equilibrium measure to the classical theory of orthogonal polynomials and Fekete points.

8.1 The log-gas

sec:log-gas

Let $V : \mathbb{R} \to \mathbb{R}$ denote a potential such that $V(x) \to \infty$ sufficiently rapidly as $|x| \to \infty$. The log-gas with size n and potential nV is a system of n identical charged particles constrained to the line interacting via pairwise Coulomb repulsion and the potential nV (we have scaled the potential V by n in order to ensure a scaling limit). The total energy of the system in any configuration $x \in \mathbb{R}^n$ is given by

$$E(x) = n \sum_{j=1}^{n} V(x_j) + \frac{1}{2} \sum_{j \neq k} \log \frac{1}{|x_j - x_k|}. \tag{8.1.1}$$

eq:eqm1

A fundamental postulate of equilibrium statistical mechanics is that the probability density of finding the system in a state x at inverse temperature $\beta > 0$ is

$$\frac{1}{Z_{n,V}(\beta)} e^{-\beta E(x)}, \tag{8.1.2}$$

eq:eqm2

where $Z_{n,V}$ is the partition function

$$Z_{n,V}(\beta) = \int_{\mathbb{R}^n} e^{-\beta E(x)}\, Dx. \tag{8.1.3}$$

eq:eqm3

The log-gas provides us with a physical caricature of eigenvalue repulsion. On one hand, we see that the energy $E(x)$ has two complementary terms: the logarithmic potential drives charges apart, but the potential V confines them in space. On the other hand, let V define an invariant probability measure of the form (eq:inv-ensemble 1.1.3) on $\mathsf{Symm}(n)$, $\mathsf{Her}(n)$ or $\mathsf{Quart}(n)$. As a consequence of Weyl's formula (Theorem 16), the equilibrium density (8.1.2) is precisely the joint law of the eigenvalues for these ensembles at $\beta = 1$, 2 and 4 respectively. It is in this sense that the 'eigenvalues repel'.

We have scaled the energy V with n in (8.1.1) in order to obtain a simple description of the scaling limit when $n \to \infty$. In order to study this limit, we view the energy function as a functional of the empirical measure, L_n, rather than a configuration $x \in \mathbb{R}^n$. For $(r, s) \in \mathbb{R}^2$ let

$$e(r,s) = \frac{1}{2}V(r) + \frac{1}{2}V(s) + \log\frac{1}{|r-s|}, \tag{8.1.4}$$

eq:eqm4

and given a probability measure μ on the line, define the functional

$$I[\mu] = \int_{\mathbb{R}}\int_{\mathbb{R}} e(r,s)\,\mu(dr)\,\mu(ds). \tag{8.1.5}$$

eq:eqm5

Observe that if L_n is the empirical measure associated to $x \in \mathbb{R}^n$, then

$$E(x) = n^2\left(\frac{1}{n}\sum_{j=1}^{n} V(x_j) + \frac{1}{n^2}\sum_{j\neq k}\log\frac{1}{|x_j - x_k|}\right) = n^2 \tilde{I}[L_n], \tag{8.1.6}$$

eq:eqm6

and we may rewrite the partition function in the form

$$Z_{n,V}(\beta) = \int_{\mathbb{R}^n} e^{-n^2\beta\tilde{I}[L_n]}\,Dx. \tag{8.1.7}$$

eq:eqm7

Here $\tilde{I}[L_n]$ denotes the renormalized functional

$$\tilde{I}[\mu] = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{r\neq s} e(r,s)\,\mu(dr)\,\mu(ds), \tag{8.1.8}$$

eq:fekete2

that takes into account all interaction terms in $I[\mu]$, except the singular self-interaction term from $I[\mu]$. The logarithmic singularity in $e(r,s)$ is integrable if $\mu(ds)$ has an absolutely continuous density. Thus, if the particles in the log-gas spread out sufficiently as $n \to \infty$, we expect that μ has a smooth density, and

$$\lim_{n\to\infty}\frac{1}{n^2}\log Z_{n,V}(\beta) = \min_{\mu} I[\mu]. \tag{8.1.9}$$

eq:eqm8

In order to establish this relation, it is first necessary to obtain a precise analytical understanding of this minimization problem. We first prove such results under the formal assumption that there exists an $R > 0$ such that $V(x) = +\infty$ for $|x| > R$. This simply means that we first restrict attention to measures with support within the interval $[-R, R]$. Once the ideas are clear in this setting, we turn to measures with support on the line.

8.2 Energy minimization for the log-gas

sec:energy-min

8.2.1 Case 1: bounded support

Let $\mathcal{P}_R$ denote the set of probability measures on the interval $[-R, R]$. Recall that the natural topology on $\mathcal{P}_R$ is the weak topology (we adopt the probabilists convention for what is conventionally termed the weak-$*$ topology). A sequence of measures $\{\mu_k\}_{k=1}^{\infty} \in \mathcal{P}_R$ converges weakly to $\mu \in \mathcal{P}_R$ if

$$\lim_{n\to\infty} \langle \mu_n, f\rangle = \langle \mu, f\rangle, \tag{8.2.1}$$

eq:eqm9

for every function $f \in C(\mathbb{R})$. This topology is natural, because it yields compactness by Helly's theorem: Each sequence $\{\mu_k\}_{k=1}^{\infty} \in \mathcal{P}_R$ has a subsequence that converges weakly to a measure in $\mathcal{P}_R$.

thm:eqbm-measure **Theorem 73.** *Assume V is a continuous function on $[-R, R]$. There exists a unique probability measure $\mu_* \in \mathcal{P}_R$ such that*

$$I[\mu_*] = \min_{\mu\in\mathcal{P}_R} I[\mu]. \tag{8.2.2}$$

eq:eqm9

The proof of Theorem 73 is a demonstration of the classical method of the calculus of variations. There are two distinct ideas at work: existence follows from the fact that the functional $I[\mu]$ is weakly lower semicontinuous; uniqueness follows from the fact that $I[\mu]$ is a strictly convex function on $\mathcal{P}_R$.

le:wlsc **Lemma 26.** *Suppose the sequence $\{\mu_n\}_{n=1}^{\infty} \in \mathcal{P}_R$ converges weakly to $\mu \in \mathcal{P}_R$. Then*

$$I[\mu] \leq \liminf_{n\to\infty} I[\mu_n]. \tag{8.2.3}$$

eq:eqm10

le:conv **Lemma 27.** *Let $\mu_0 \neq \mu_1$ be two measures in $\mathcal{P}_R$ and let $\mu_\theta = (1-\theta)\mu_0 + \theta\mu_1$ denote their convex combination for each $\theta \in (0, 1)$. Then*

$$I[\mu_\theta] < (1-\theta)I[\mu_0] + \theta I[\mu_1]. \tag{8.2.4}$$

eq:eqm11

Proof of Theorem 73. Existence. Since V is bounded, the function $e(x, y)$ is bounded below on $[-R, R]$. Therefore, $\inf_{\mu\in\mathcal{P}_R} I[\mu] > -\infty$. Further, since the logarithmic singularity is integrable, $I[\mu] < \infty$ for any measure that is absolutely continuous. Thus, we may assume that there is a sequence of measures $\{\mu_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} I[\mu_k] = \inf_{\mu\in\mathcal{P}_R} I[\mu] < infty. \tag{8.2.5}$$

eq:eqm12

Since $\mathcal{P}_R$ is compact in the weak topology, we may extract a convergent subsequence, also labeled $\{\mu_k\}_{k=1}^{\infty}$ for simplicity. Let μ_* denote the weak limit of this subsequence. We then use Lemma 26 to obtain the chain of inequalities

$$\inf_{\mu\in\mathcal{P}_R} I[\mu] \leq I[\mu_*] \leq \liminf_{k\to\infty} I[\mu_k] = \inf_{\mu\in\mathcal{P}_R} I[\mu]. \tag{8.2.6}$$

Thus, μ_* is a minimizer.

Uniqueness. Assume μ_* and ν_* are two distinct minimizers. We apply Lemma le:conv 27 to their convex combination with $\theta = 1/2$ to obtain the contradiction

$$\inf_{\mu\in\mathcal{P}_R} I[\mu] \leq I[\frac{1}{2}\mu_* + \frac{1}{2}\nu_*] < \frac{1}{2}\left(I[\mu_*] + I[\nu_*]\right) = \inf_{\mu\in\mathcal{P}_R} I[\mu]. \tag{8.2.7}$$

□

8.2.2 Weak lower semicontinuity

We now turn to the proof of Lemma le:wlsc 26. We first observe that for each monomial $r^j s^k$ in the variables r and s, the quadratic functional

$$\mu \mapsto \int_{-R}^{R}\int_{-R}^{R} r^j s^k \,\mu(dr)\,\mu(ds) = \left(\int_{-R}^{R} r^j \mu(dr)\right)\left(\int_{-R}^{R} s^k \mu(ds)\right)$$

is weakly continuous since it is the product of two bounded linear functionals on $\mathcal{P}_R$. Since each polynomial $p(r,s)$ in the variables (r,s) is a finite sum of monomials, the functional

$$\mu \mapsto \int_{-R}^{R}\int_{-R}^{R} p(r,s)\mu(dr)\,\mu(ds)$$

is also weakly continuous. Finally, since each continuous function $f \in C([-R,R]^2)$ may be uniformly approximated by polynomials, the quadratic functional

$$\mu \mapsto \int_{-R}^{R}\int_{-R}^{R} f(r,s)\mu(dr)\,\mu(ds)$$

is weakly continuous.

The function $e(s,t)$ defined in (eq:eqm4 8.1.4) is not continuous on $[-R,R]^2$ since the logarithmic term is unbounded on the diagonal $s=t$. However, for any $M>0$, the truncated function $e_M(r,s) = \min(e(r,s), M)$ is continuous. Thus, given a weakly convergent sequence of measures $\{\mu_k\}_{k=1}^{\infty}$ with limit $\mu \in \mathcal{P}_R$ we find

$$\begin{aligned}\int_{-R}^{R}\int_{-R}^{R} e_M(r,s)\mu(dr)\,\mu(ds) &= \lim_{k\to\infty}\int_{-R}^{R}\int_{-R}^{R} e_M(r,s)\mu_k(dr)\,\mu_k(ds)\\ &\leq \liminf_{k\to\infty}\int_{-R}^{R}\int_{-R}^{R} e(r,s)\mu_k(ds)\mu_k(ds) = \liminf_{k\to\infty} I[\mu_k].\end{aligned}$$

We let $M\to\infty$ on the left hand side and use the monotone convergence theorem to obtain (eq:eqm10 8.2.3).

8.2.3 Strict convexity

Lemma le:conv 27 is a particular consequence of a general fact in potential theory. The essential idea is to recognize that the function $z \mapsto -\log|z|$ is the fundamental solution to Laplace's equation in $\mathbb{C} \cong \mathbb{R}^2$. More precisely, given a signed measure μ with a smooth density $\rho(z)$, supported in the ball $B_R \subset \mathbb{C}$ the unique solution to Poisson's equation with Dirichlet boundary condition

$$-\triangle\psi = \mu, \quad z \in \mathbb{C}\backslash\Omega, \quad \psi(z) = 0, \quad |z| = R, \tag{8.2.8}$$

eq:poisson1

is given by the integral formula

$$\psi(z) = \int_{B_R} G(z,w)\rho(w)\, Dw, \quad z \in B_R, \tag{8.2.9}$$

eq:poisson2

where Dw denotes the two-dimensional area element in $\mathbb{C}$ and $G(z,w)$ is the Green's function for Poisson's equation in the ball B_R with Dirichlet boundary conditions,

$$G(z,w) = \frac{1}{2\pi}\log\left(\frac{|w|}{R}\frac{|z-w^R|}{|z-w|}\right), \quad w^R = \frac{R^2 w}{|w|^2}, \quad z,w \in B_R. \tag{8.2.10}$$

eq:poisson3

The function $G(z,w)$ is obtained by the method of images: the image point w^R is the reflection of the point $w \in B_R$ in the circle ∂B_R [Joh91, §4.1]. What matters here is that the dominant term in the Green's function is the logarithmic term $-\log|z-w|$, just as in equation (8.1.5), and the positivity of

$$\int_{B_R}\int_{B_R} G(z,w)\,\mu(dz)\,\mu(dw) = -\int_{B_R}\psi(w)\triangle\psi(w)\,ds = \int_{B_R}|\nabla\psi(w)|^2\,Dw > 0. \tag{8.2.11}$$

eq:poisson4

However, in contrast with (8.1.5) here we have assumed that $\mu(dw)$ has a smooth density $\rho(w)$, whereas the measures of interest in (8.1.5) are concentrated on an interval, and may have no regularity. Thus, some care is needed in formulating and proving a theorem on positivity analogous to (8.2.11).

Recall that a signed Borel measure μ on the line may be uniquely decomposed into two positive measures $\mu_\pm$ respectively such that $\mu = \mu_+ - \mu_-$. The Fourier transform of a measure is defined by

$$\hat{\mu}(u) = \int_{\mathbb{R}} e^{-ius}\,\mu(ds), \quad u \in \mathbb{R}. \tag{8.2.12}$$

eq:poisson5

The Fourier transform is a well-defined distribution. If $\mu_\pm$ are finite measures on $[-R,R]$, the Fourier transform is a continuous function of u that decays to zero as $|u| \to \infty$ by the Riemann-Lebesgue lemma.

le:poisson **Lemma 28.** *Assume $\mu = \mu_+ - \mu_-$ is a signed measure on $[-R,R]$ such that*

$$\int_{-R}^{R}\mu_+(dr) = \int_{-R}^{R}\mu_-(dr) < \infty. \tag{8.2.13}$$

eq:poisson5

Then we have the identity

$$\begin{aligned}\int_{-R}^{R}\int_{-R}^{R}\log\frac{1}{|r-s|}&\left(\mu_+(dr)\mu_+(ds) + \mu_-(dr)\mu_-(ds)\right) \qquad (8.2.14)\\ &= \int_{-R}^{R}\int_{-R}^{R}\log\frac{1}{|r-s|}\left(\mu_+(dr)\mu_-(ds) + \mu_-(dr)\mu_+(ds)\right) + \int_0^\infty\frac{|\hat{\mu}(u)|^2}{u}\,du.\end{aligned}$$

In particular, $I[\mu] > 0$ if μ is non-zero and satisfies (8.2.13).

Remark 74. Equation (8.2.14) simply says that

$$\int_{-R}^{R}\int_{-R}^{R} \log\frac{1}{|r-s|}\,\mu(dr)\,\mu(ds) = \int_0^\infty \frac{|\hat{\mu}(u)|^2}{u}\,du. \tag{8.2.15}$$

for a signed measure μ with $\int_{-R}^{R}\mu(ds) = 0$. This identity has been written in the form (8.2.14) in order to ensure that there are no ill-defined terms of the form $\infty - \infty$. It is now clear from (8.1.4) and (8.1.5) that $I[\mu] > 0$ for such measures.

Proof. This proof is from [DLW96, p.142]. We first regularize the logarithm at 0 and use the following integral representation. For any real s and $\varepsilon > 0$

$$\log(s^2+\varepsilon^2) = \log\varepsilon^2 + 2\,\mathsf{Im}\int_0^\infty e^{-\varepsilon u}\frac{e^{isu}-1}{iu}\,du. \tag{8.2.16}$$

We apply this integral representation to the following regularization of $I[\mu]$, and use the fact that $\int_{-R}^{R}\mu(dr) = 0$, to obtain

$$\begin{aligned}
&\int_{-R}^{R}\int_{-R}^{R}\log\left((r-s)^2+\varepsilon^2\right)\mu(dr)\mu(ds)\\
&\quad = 2\,\mathsf{Im}\int_0^\infty e^{-\varepsilon u}\int_{-R}^{R}\int_{-R}^{R}\frac{e^{i(r-s)u}-1}{iu}\,\mu(dr)\mu(ds)\,du\\
&\quad = 2\,\mathsf{Im}\int_0^\infty e^{-\varepsilon u}\frac{|\hat{\mu}(u)|^2}{iu}\,du = -2\int_0^\infty e^{-\varepsilon u}\frac{|\hat{\mu}(u)|^2}{u}\,du.
\end{aligned}$$

We may rewrite this identity in terms of $\mu_\pm$ as follows:

$$\begin{aligned}
&\int_{-R}^{R}\int_{-R}^{R}\log\frac{1}{\sqrt{(r-s)^2+\varepsilon^2}}\left(\mu_+(dr)\mu_+(ds)+\mu_-(dr)\mu_-(ds)\right) \qquad (8.2.17)\\
&= \int_{-R}^{R}\int_{-R}^{R}\log\frac{1}{\sqrt{(r-s)^2+\varepsilon^2}}\left(\mu_+(dr)\mu_-(ds)+\mu_-(dr)\mu_+(ds)\right) + \int_0^\infty e^{-\varepsilon u}\frac{|\hat{\mu}(u)|^2}{u}\,du.
\end{aligned}$$

We now let $\varepsilon \downarrow 0$ and use the monotone convergence theorem to obtain (8.2.14) □

Finally, let us prove Lemma 27. Suppose μ_0 and μ_1 be two measures in $\mathcal{P}_R$ as in (8.2.4). The difference

$$(1-\theta)I[\mu_0]+\theta I[\mu_1]-I[\mu_\theta] = \theta(1-\theta)\int\int\log\frac{1}{|r-s|}\,(\mu_0-\mu_1)(dx)\,(\mu_0-\mu_1)(dx)$$

in the sense of signed measures. Thus, it is strictly positive when $\mu_0 \neq \mu_1$ by Lemma 28.

8.2.4 Case 2: Measures on the line

Having explained the main ideas behind Theorem 73 for finite measures, let us turn to the measures on the line. The proof of uniqueness requires no change, since it is easily verified that Lemma 28 holds for measures in $\mathcal{P}_{\mathbb{R}}$. However, it is necessary to modify the proof of existence to account for a possible loss of compactness: a sequence of measures in $\mathcal{P}_{\mathbb{R}}$ may drift off to infinity (e.g. $\mu_k = \delta_k$, $k \in \mathbb{Z}$). The appropriate condition required for compactness here is the following.

Definition 75. A sequence of measures $\{\mu_k\}_{k=1}^{\infty} \in \mathcal{P}_{\mathbb{R}}$ is tight if for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\sup_{k \geq 1} \mu_k \left(\mathbb{R} \backslash [-M_\varepsilon, M_\varepsilon] \right) < \varepsilon. \tag{8.2.18}$$

eq:tight1

Compactness of measures in $\mathcal{P}_{\mathbb{R}}$ is provided by the Prokhorov-Varadarajan criterion: the sequence $\{\mu_k\}_{k=1}^{\infty} \in \mathcal{P}_{\mathbb{R}}$ has a subsequence that converges to a measure $\mu \in \mathcal{P}_{\mathbb{R}}$ if and only if the sequence $\{\mu_k\}_{k=1}^{\infty}$ is tight [Str10]. In practice, application of this criterion requires a uniform estimate on the tails of the measures $\{\mu_k\}_{k=1}^{\infty}$. Such a bound is possible only if the growth of the confining potential $V(x)$ as $|x| \to \infty$ is faster than the divergence of $\log |x|$ as $|x| \to \infty$. We formalize this requirement as follows. For any $\varepsilon > 0$, observe that

$$|r - s| = |r - 1 - (s-1)| \leq \sqrt{r^2+1}\sqrt{s^2+1}. \tag{8.2.19}$$

eq:tight2

Therefore, we have the lower bound

$$\log \frac{1}{r-s} \geq \frac{1}{2} \left(\log \frac{1}{r^2+1} + \log \frac{1}{s^2+1} \right). \tag{8.2.20}$$

eq:tight2

Let us define the function

$$l(s) = \frac{1}{2} \log \frac{1}{s^2+1} + \frac{1}{2} V(s). \tag{8.2.21}$$

eq:tight3

If $l(s)$ is bounded below, then by adding a constant to V if necessary, we can ensure that $l(s) \geq 0$ for all s. Clearly, this does not change the nature of the minimization problem.

thm:eqbm-line

Theorem 76. *Assume $V(s)$ is a continuous function such that $l(s)$ is bounded below and $l(s) \to \infty$ as $|s| \to \infty$.*

(a) There exists a unique probability measure $\mu_* \in \mathcal{P}_{\mathbb{R}}$ such that

$$I[\mu_*] \leq \min_{\mu \in \mathcal{P}_{\mathbb{R}}} I[\mu]. \tag{8.2.22}$$

eq:tight4

(b) The support of the measure μ_* is contained within a finite interval.

Proof. (a) Since V is bounded below and the addition of a constant to V does not change the minimization problem, we may assume that $l(s) \geq 0$. Then

$$e(r,s) = \log \frac{1}{|r-s|} + \frac{1}{2}V(r) + \frac{1}{2}V(s) \geq l(r) + l(s) \geq 0, \tag{8.2.23}$$

and $c := \inf_{\mu \in \mathcal{P}_{\mathbb{R}}} I[\mu] \geq 0$. Suppose $\mu_k{}_{k=1}^{\infty}$ is an infimizing sequence: i.e. $\lim_{k\to\infty} I[\mu_k] = c$. Without loss of generality, we may assume that $I[\mu_k] \leq c+1$ for all k. Tightness of the sequence $\{\mu_k\}_{k=1}^{\infty}$ follows from the following (Chebyshev) inequality. For any $M > 0$,

$$\begin{aligned} c+1 \geq I[\mu_k] &= \int_{\mathbb{R}} \int_{\mathbb{R}} e(r,s)\mu_k(dr)\mu_k(ds) \\ &\geq 2\int_{\mathbb{R}} l(s)\mu_k(ds) \geq 2l_M \int_{|s|>M} \mu_k(ds) = 2l_M \mu_k(\mathbb{R}\backslash[-M,M]), \end{aligned} \tag{8.2.24}$$

where $l_M = \inf_{|s|\geq M} l(s)$. Since $\lim_{|s|\to\infty} l(s) = \infty$, $l_M \to \infty$ as $M \to \infty$. Thus, for any $\varepsilon > 0$, we may choose $M = M_\varepsilon$ large enough so that (8.2.18) holds. The rest of the proof of part (a) follows that of Theorem 73.

(b) For any $M > 0$, let S_M denote the set $(-\infty, M) \cup (M, \infty)$. We will show that $\mu_*(S_M) = 0$ if M is large enough. The proof relies on varying the measure μ_* by adding more mass proportional to μ_* in the set S_M. More precisely, let ν denote the restriction of μ_* to the set S_M, and for any $t \in (-1,1)$, define the measures

$$\mu_t = \frac{\mu_* + t\nu}{1 + t\nu(S_M)}. \tag{8.2.25}$$

We then find that $I[\mu_t]$ is a differentiable function of t, with

$$0 = \left.\frac{dI[\mu_t]}{dt}\right|_{t=0} = 2\int_{S_M} \nu(ds) \int_{\mathbb{R}} \mu_*(dr)e(r,s) - 2\nu(S_M)I[\mu_*]. \tag{8.2.26}$$

The estimate (8.2.23) and positivity of l yields the lower bound

$$\begin{aligned} &2\int_{S_M} \nu(ds) \int_{\mathbb{R}} \mu_*(dr)e(r,s) \\ &\geq \int_{S_M} l(s)\nu(ds) + \int_{\mathbb{R}} l(r)\mu_*(dr) \geq \int_{S_M} l(s)\nu(ds) \geq l_M \nu(S_M). \end{aligned} \tag{8.2.27}$$

As in part (a), $l_M \to \infty$ as $M \to \infty$. Thus, for M sufficiently large, we have $l_M - I[\mu_*]) > 0$ and since ν is a positive measure, we have the (trivial) estimate

$$2(l_M - I[\mu_*])\nu(S_M) \geq 0. \tag{8.2.28}$$

On the other hand, the inequalities (8.2.26) and (8.2.27) yield the opposite inequality

$$2(l_M - I[\mu_*])\nu(S_M) \leq 0. \tag{8.2.29}$$

Thus, $\nu(S_M) = 0$ for all M such that $l_M > I[\mu_*]$. $\square$

8.3 Fekete points

A second approach to the energy minimization problem relies on a study of the minimizers of the function $E(x)$ defined in (8.1.1) for $x \in \mathbb{R}^n$, and a potential V that satisfies the assumptions of Theorem 76. For any such potential, $0 \leq E(x) < \infty$ for any $x \in \mathbb{R}^n$ such that $x_j \neq x_k$, $j \neq k$. Thus, for each n, there exists a set of points $F_n \subset \mathbb{R}^n$, such that

$$E(x_*) = \min_{x \in \mathbb{R}^n} E(x), \quad x_* \in F_n. \tag{8.3.1}$$

eq:fekete1

The set F_n is called the set of n-Fekete points. The Fekete points are naturally connected to the minimization problem for the functional $I[\mu]$ through the modified functional $H[L_n]$, where $L_n(x)$ is the empirical measure associated to a point $x \in \mathbb{R}^n$. Let δ_n denote the rescaled energy of Fekete points

$$\delta_n = \frac{1}{n(n-1)} E(x^{(n)}). \tag{8.3.2}$$

eq:fekete2

The main result is then the following

thm:fekete **Theorem 77.** *Assume V satisfies the assumptions of Theorem 76. Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a sequence of points $x^{(n)} \in F_n$ and Then*

(a) The rescaled energy of Fekete points increases monotonically to $I[\mu_*]$.

$$0 \leq \delta_n \leq \delta_{n+1} \leq I[\mu_*]. \tag{8.3.3}$$

eq:fekete2

(b) The empirical measures $L(x^{(n)})$ converge weakly to μ_*.

Proof of (a). We first prove the estimates (8.3.3). The uniform upper bound on $E(x^{(n)})$ is obtained as follows. Fix a positive integer n and a point $x^{(n)} \in F_n$. By definition, for any $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$,

$$E(x^{(n)}) \leq E(s) = \frac{1}{2} \sum_{j,k=1}^{n} (V(s_j) + V(s_k)) + \sum_{j \neq k = 1}^{n} \log \frac{1}{|s_j - s_k|}. \tag{8.3.4}$$

eq:fekete3

Let $\mu(ds)$ be any probability measure on the line. We integrate (8.3.4) with respect to the n-fold tensorized probability measure $\mu \otimes \mu \cdots \otimes \mu$ on $\mathbb{R}^n$ to obtain

$$\begin{aligned} &E(x^{(n)}) \qquad (8.3.5) \\ &\leq \int_{\mathbb{R}^n} \left[\frac{1}{2} \sum_{j,k=1}^{n} (V(s_j) + V(s_k)) + \sum_{j \neq k=1}^{n} \log \frac{1}{|s_j - s_k|} \right] \mu(ds_1)\mu(ds_2) \cdots \mu(ds_n) \\ &= n(n-1) \int_{\mathbb{R}} \int_{R} e(r,s) \mu(ds) \mu(dr) = I[\mu], \end{aligned}$$

since for each value of the indices j and k only the integrals over $\mu(ds_j)$ and $\mu(ds_k)$ give contributions that are not unity and there are $n(n-1)$ possible unordered pairings of j and k. In particular, $E(x^{(n)}) \leq n(n-1) I[\mu_*]$.

The monotonicity of δ_n follows from the following argument. Suppose $x^{(n+1)} = (x_1, \ldots, x_{n+1})$ is point in the Fekete set F_{n+1}. We fix an index m, $1 \leq m \leq n+1$ and use the definition of E in (8.1.1) to obtain

$$\begin{aligned}
e^{-\frac{1}{n(n+1)}E(x^{(n+1)})} &= \left(\prod_{1 \leq j \neq k \leq n+1} |x_j - x_k| \, e^{-\frac{V(x_j)}{2}} e^{-\frac{V(x_k)}{2}} \right)^{\frac{1}{n(n+1)}} \qquad (8.3.6)\\
&= \left(\prod_{j \neq m} |x_j - x_m| \, e^{-\frac{V(x_j)}{2} - \frac{V(x_m)}{2}} \right)^{\frac{2}{n(n+1)}} \left(\prod_{j,k \neq m} |x_j - x_k| \, e^{-\frac{V(x_j)}{2}} e^{-\frac{V(x_k)}{2}} \right)^{\frac{1}{n(n+1)}} \\
&\leq \left(\prod_{j \neq m} |x_j - x_m| \, e^{-\frac{V(x_j)}{2}} e^{-\frac{V(x_m)}{2}} \right)^{\frac{2}{n(n+1)}} e^{-\delta_n \frac{n-1}{n+1}}
\end{aligned}$$

since the second term is the energy $E(\hat{x})$ of the point $\hat{x} \in \mathbb{R}^n$ obtained from $x^{(n)}$ by projecting out the coordinate x_m.

Since m is arbitrary, we take the product over $1 \leq m \leq n+1$ to obtain

$$\begin{aligned}
e^{-\frac{1}{n}E(x^{(n+1)})} &\leq e^{-(n-1)\delta_n} \left(\prod_{1 \leq m \leq n+1} \prod_{1 \leq j \leq n+1, j \neq m} |x_j - x_m| \, e^{-\frac{V(x_j)}{2} - \frac{V(x_m)}{2}} \right)^{\frac{2}{n(n+1)}} \\
&= e^{-(n-1)\delta_n} e^{-\frac{2}{n(n+1)}E(x^{(n+1)})}. \qquad (8.3.7)
\end{aligned}$$

This inequality simplifies to $\delta_n \leq \delta_{n+1}$. □

Proof of (b). While the self-energy of all the Fekete points is infinite, inequality (8.3.3) shows that a suitably renormalized energy is finite, and bounded above by $I[\mu_*]$. This inequality, in combination with an easy modification of the Chebyshev inequality (8.2.24) also shows that the empirical measures $L(x^{(n)})$ are tight. Thus, there exists a convergent subsequence and a limiting probability measure $\nu \in \mathcal{P}_{\mathbb{R}}$ such that the empirical measures $L^{(n)}$ defined by the Fekete points $x^{(n)}$ converge weakly to ν as $n \to \infty$.

For any $M > 0$, we introduce the cut-off energy $e_M(r,s) = \min(M, e(r,s))$ and observe that

$$\begin{aligned}
\delta_n = \frac{1}{n(n-1)}E(x^{(n)}) &= \frac{n^2}{n(n-1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{r \neq s} e(r,s) L^{(n)}(dr) L^{(n)}(ds) \\
&\geq \frac{n^2}{n(n-1)} \int_{\mathbb{R}} \int_{\mathbb{R}} e_M(r,s) L^{(n)}(dr) L^{(n)}(ds) - \frac{M}{n-1}.
\end{aligned}$$

Since the function $e_M(r,s)$ is continuous and $0 \leq e_M(r,s) \leq M$, we may interchange limits as $n \to \infty$, and use Theorem 77(a) to obtain

$$I[\mu_*] \geq \liminf_{n \to \infty} \delta_n \geq \int_{\mathbb{R}} \int_{\mathbb{R}} e_M(r,s) \nu(dr) \nu(ds). \qquad (8.3.8)$$

We now let $M \to \infty$ and use the monotone convergence theorem and the fact that μ_* is a minimizer to obtain

$$I[\mu_*] \geq I[\mu] \geq I[\mu_*]. \tag{8.3.9}$$

Since μ_* is unique, it follows that $\mu_* = \nu$.

This argument proves that every subsequential limit of $L^{(n)}$ is μ_*. Thus, the entire sequence converges to μ_*. $\square$

8.4 Exercises

The first three questions are related. The goal is to formulate and analyze the equation for the equilibrium measure μ_* associated to the potential $V(x)$. In order to simplify your calculations, assume that μ_* has a continuous density ψ, in all the problems below. The last two questions discuss enumeration problems related to the Catalan numbers.

1. Basics of the Hilbert transform. Let $G(z)$ denote the Stieltjes transform

$$G(z) = \int_{-\infty}^{\infty} \frac{1}{s-z} \mu_*(ds) = \int_{-\infty}^{\infty} \frac{1}{s-z} \psi(s)(ds), \quad z \in \mathbb{C} \backslash \mathrm{supp}(\mu_*). \tag{8.4.1}$$

eq:stieltes-eqm1

The Hilbert transform of ψ is the limit of the Stieltjes transform as $z \to x \in \mathbb{R}$. The Hilbert transform also differs from the Stieltjes transform by the inclusion of a factor of π (since this makes the Fourier transform of the operator $\mathcal{H}$ particularly simple). That is, given μ_* as above, we set

$$\mathcal{H}\psi(x) = \frac{1}{\pi} \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{\psi(s)}{x-s}\, ds := \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{x-s}{(x-s)^2+\varepsilon^2}\, \psi(s)\, ds. \tag{8.4.2}$$

eq:hilbert1

(a) Show that $\mathcal{H}\psi$ is a bounded function when $\psi(x)$ is continuous.

(b) Show that μ_* may be recovered from G by evaluating the jump in the imaginary part of G across the support of μ_*:

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \left(G(x+i\varepsilon) - G(x-i\varepsilon) \right) = \psi(x). \tag{8.4.3}$$

eq:stieltes-eqm2

(c) Compute the Hilbert transform of the following functions to obtain a feel for it (answers are on wikipedia):

$$e^{ix}, \quad \delta_0(x), \quad \mathbb{1}_{[a,b]}(x).$$

2. Integral equation for ψ. Assume V is differentiable and satisfies the assumptions of Theorem thm:eqbm-line 76 so that μ_* has compact support. Show that if μ_* has a density ψ as above, then it satisfies the integral equation

$$\mathcal{H}\psi(x) = \frac{1}{2\pi} V'(x) \quad \text{on} \quad \mathrm{supp}(\mu_*). \tag{8.4.4}$$

eq:hilbert2

3. Fixed point equation for the resolvent. One solution to (eq:hilbert2 8.4.4) uses the Stieltjes transform $G(z)$. Assume that $V(x)$ is a polynomial of degree $d \geq 2$.

(a) Show that G satisfies the quadratic equation

$$G^2(z) + V'(z)G(z) + P(z) = 0, \tag{8.4.5}$$

eq:hilbert3

where $P(z)$ is a polynomial of degree $d-2$ whose coefficients are determined by the moments of μ_* of degree lower than d. The solution branch is determined by the requirement that $G(z) \sim -1/z$ as $z \to \infty$ which is immediate from (eq:stieltes-eqm1 8.4.1).

(b) Equation (eq:hilbert3 8.4.5) may be solved by making further assumptions on the form of μ_*. In particular, assume that $V(z)$ is even, that the support of μ_* is a single interval $[-2a, 2a]$, and show that (eq:hilbert3 8.4.5) simplifies to

$$G(z) = Q(z)\sqrt{z^2 - 4a^2} - \frac{1}{2}V'(z) \tag{8.4.6}$$

eq:hilbert4

where $Q(z)$ is a polynomial of degree $d-2$ whose coefficients are determined by the condition that $G(z) \sim -1/z$ as $z \to \infty$.

(c) Apply these ideas to compute the equilibrium measure for the quartic potential

$$V(x) = \frac{1}{2}x^2 + \frac{g}{4}x^4. \tag{8.4.7}$$

eq:hilbert5

Show that

$$G(z) = \left(\frac{1}{2} + \frac{g}{2}x^2 + ga^2\right)\sqrt{x^2 - 4a^2} - \frac{1}{2}\left(x + gx^3\right), \tag{8.4.8}$$

eq:hilbert6

where a^2 solves the quadratic equation

$$3ga^4 + a^2 - 1 = 0. \tag{8.4.9}$$

eq:hilbert7

(d) Compute the associated density $\psi(x)$ and plot it as g varies.

4. Establish the identity (eq:catalan1 1.4.11).

5. Show that the Catalan numbers enumerate the number of Dyck paths as discussed below equation (eq:catalan2 1.4.12).

Chapter 9

Iterative methods and flows

ch:iterative

9.1 Lanczos iteration

In this section, we will consider applying the Lanczos iteration to a random matrix M. In the end, we will consider $M \sim \mathrm{GOE}(n), \mathrm{GUE}(n), \mathrm{LOE}(m,n)$, or $\mathrm{LOE}(m,n)$. The Lanczos iteration is the following:

Algorithm 1: Lanczos Iteration a:lanczos

1. q_1 is the initial vector. Suppose $\|q_1\|_2^2 = q_1^* q_1 = 1$
2. Set $b_0 = -1$, $q_0 = 0$
3. For $k = 1, 2, \ldots, n$
   (a) Compute $a_k = (Mq_k - b_{k-1}q_{k-1})^* q_k$.
   (b) Set $v_k = Mq_k - a_k q_k - b_{k-1}q_{k-1}$.
   (c) Compute $b_k = \|v_k\|_2$ and if $b_{k-1} \neq 0$, set $q_{k+1} = v_k/b_k$. Otherwise terminate.

Before considering random matrices, we establish properties of this iteration. Define

$$Q_k = \begin{bmatrix} q_1 & q_2 & \cdots & q_k \end{bmatrix}, \quad T_k = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & & \\ & & & & b_{k-1} \\ & & & b_{k-1} & a_k \end{bmatrix}.$$

It follows directly from step 3(b) that

$$MQ_k = Q_k T_k + b_k q_{k+1} e_k^T. \tag{9.1.1}$$

We make an important distinction between these calculations and that in Section 3.4. Here T_k will end up denoting the upper-left $k \times k$ submatrix of a larger Jacobi matrix whereas in Section 3.4 it represented the lower-right $k \times k$ submatrix.

Lemma 29. *Suppose M is a symmetric matrix. And suppose that the Lanczos iteration does not terminate before $k = n$. For $k = 2, 3, \ldots, n$,*

$$q_1, \ldots, q_k$$

is an orthonormal basis for the Krylov subspace $\mathcal{K}_k = \operatorname{span}\{q_1, Mq_1, \ldots, M^{k-1}q_1\}$.

Proof. It is clear that step 3(c) enforces that $q_1, \ldots, q_k$ are all unit vectors. Suppose that $q_1, \ldots, q_{k-1}$ satisfy

$$\delta_{ij} = q_i^* q_j, \quad 1 \le i, j \le k-1.$$

Then consider

$$q_i^* q_k = \frac{1}{b_{k-1}} q_i^* \left(M q_{k-1} - a_{k-1} q_{k-1} - b_{k-2} q_{k-2} \right).$$

For $i < k-2$ we have

$$q_i^* q_k = \frac{1}{b_{k-1}} q_i^* M q_{k-1} = \frac{1}{b_{k-1}} (M q_i)^* q_{k-1}.$$

From (9.1.1) Mq_i is a linear combination of $\{q_{i-1}, q_i, q_{i+1}\}$. And therefore $q_i^* q_k = 0$. Next, for $i = k-2$

$$q_i^* q_k = \frac{1}{b_{k-1}} q_{k-2}^* \left(M q_{k-1} - b_{k-2} q_{k-2} \right) = \frac{1}{b_{k-1}} \left((M q_{k-2})^* q_{k-1} - b_{k-2} \right).$$

Again, from (9.1.1)

$$M q_{k-2} = a_{k-2} q_{k-2} + b_{k-2} q_{k-1} + b_{k-3} q_{k-3}.$$

This shows that

$$(M q_{k-2})^* q_{k-1} = b_{k-2}$$

and $q_i^* q_k = 0$. Lastly, for $i = k-1$

$$q_i^* q_k = \frac{1}{b_{k-1}} q_{k-1}^* \left(M q_{k-1} - a_{k-1} q_{k-1} \right) = \frac{1}{b_{k-1}} \left(q_{k-1}^* M q_{k-1} - a_{k-1} \right).$$

We then need to use that

$$a_{k-1} = (M q_{k-1} - b_{k-2} q_{k-2})^* q_{k-1} = q_{k-1}^* M q_{k-1},$$

to determine that $q_i^* q_k = 0$. This concludes the proof. ∎

The following gives the distribution of T_k throughout the Lanczos iteration.

t:lanczos **Theorem 78.** *Suppose* $M \sim \mathrm{GOE}(n), \mathrm{GUE}(n), \mathrm{LOE}(m,n), \mathrm{LUE}(m,n)$. *For any given* $q_1 \in \mathbb{R}^n$ *(or* $\mathbb{C}^n$ *for* GUE, LUE*) with probability one, the Lanczos iteration does not terminate before* $k = n$. *And the distribution on* a_k, b_k, $k = 1, 2, \ldots, n$ *does not depend on* q_1. *In a distributional sense it suffices to take* $q_1 = e_1$ *and therefore the distribution is determined by the Householder tridigaonalization of* M.

Proof. To establish the first claim it suffices to establish the linear independence of

$$q_1, Mq_1, \ldots, M^{n-1}q_1.$$

Then, diagonalize $M = U\Lambda U^*$:

$$\begin{aligned}\begin{bmatrix} q_1 & Mq_1 & \cdots & M^{n-1}q_1 \end{bmatrix} &\mapsto U \begin{bmatrix} U^*q_1 & \Lambda U^* q_1 \cdots \Lambda^{n-1}U^*q_1 \end{bmatrix} \\ &= U \mathrm{diag}(U^*q_1) \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}\end{aligned}$$

Here $x_1, \ldots, x_n$ are the eigenvalues of M. Therefore the determinant of this matrix is non-zero provided that $(x_1, \ldots, x_n)$ are distinct and no component of U^*b vanishes. By Lemma le:jacobbidiag ??, (5.3.1) (for LOE, LUE) and eq:tridiag2u (3.3.3) (for GOE, GUE) it follows that the eigenvalues are distinct with probability one. Then because the eigenvectors of M can be take to be Haar distributed on $\mathsf{U}(n)$ or $\mathsf{O}(n)$, U^*b is uniformly distributed on the sphere in $\mathbb{R}^n$ (or $\mathbb{C}^n$). So, then then $|U^*b|^2$, taken componentwise, is β-Dirichlet distributed. And therefore, no component vanishes, with probability one. To establish the second claim consider the QR factorization

$$\begin{bmatrix} q_1 & Mq_1 & \cdots & M^{n-1}q_1 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} R.$$

Let Q_0 be an orthogonal (or unitary) matrix so that $q_1 = Q_0 e_1$. Then

$$\begin{aligned}&Q_0^* \begin{bmatrix} Q_0e_1 & MQ_0e_1 & \cdots & M^{n-1}Q_0e_1 \end{bmatrix} = \\ &\begin{bmatrix} e_1 & Q_0^*MQ_0e_1 & \cdots & Q_0^*M^{n-1}Q_0e_1 \end{bmatrix} = Q_0^* \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} R.\end{aligned}$$

The invariance of M implies that $M \sim Q_0^*MQ_0$. So, the Lanczos iteration applied to the matrix M with starting vector q_1 gives

$$T = Q^*MQ, \quad Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}.$$

And then the Lanczos iteration applied to the matrix $Q_0^*MQ_0$ with starting vector e_1 gives

$$\tilde{T} = \tilde{Q}^*Q_0^*MQ_0\tilde{Q}, \quad \tilde{Q} = Q_0^* \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}.$$

We see that $\tilde{T} = T$. And then $\tilde{T}$ is equal in distribution to that which is found apply the Lanczos iteration to M with staring vector $q_1 = e_1$. From the scheme that is reflected in Figure 3.4.1, the spectral measure of a Jacobi matrix T, (3.4.5), encodes T itself. That is, T is determined by its eigenvalues and the absolute values of the first components of its normalized eigenvectors. Because the first column of Q_n is e_1, it is clear that the first components of the eigenvectors of T coincide with those of M. The same is true of tridiagonalization by Householder reflections. Thus Lanczos with q_1, run to completion, coincides with Householder tridiagonalization. □

The behavior of this algorithm in floating-point/finite-precision arithmetic is an extremely important topic. In this text, however, we ignore this, and suppose exact arithmetic.

9.2 An alternate proof of the semicircle law

sec:alternate

To give an alternate proof the the semicircle law, we use the method of moments. The following two theorems give the foundations for the method.

Theorem 79 (Hamburger moment problem)**.** *Let $(m_k)_{k\geq 0}$ be a sequence of real numbers such that the upper-left $\ell \times \ell$ subblocks of the Hankel matrix*

$$\begin{bmatrix} m_0 & m_1 & m_2 & \cdots \\ m_1 & m_2 & m_3 & \cdots \\ m_2 & m_3 & m_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

are positive definite for every $\ell \geq 1$. Then there exists a (positive) Borel measure μ on $\mathbb{R}$ such that

$$m_k = \int x^k \mu(\mathrm{d}x).$$

Furthermore, if there exists constants $C, D > 0$ such that

$$|m_k| \leq C D^k k!$$

then μ is unique.

The most important portion of the previous theorem for the following developments is the uniqueness. The existence, in our case, will be evident.

Theorem 80 (Weak convergence via moments)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of random variables. Suppose further that that there is a random variable X with $\mathbb{E}[|X|^k] < \infty$ for all $k \geq 0$ and*

$$\lim_{n\to\infty} \mathbb{E}[X_n^k] = \mathbb{E}[X^k] =: m_k.$$

Lastly, suppose that there is only one Borel measure μ on $\mathbb{R}$ with moment sequence $(m_k)_{k\geq 0}$. Then

$$X_n \xrightarrow{dist.} X,$$

as $n \to \infty$.

Proof. First, it follows immediately from the $k = 2$ moment condition that $(X_n)_{n\geq 1}$ is tight. So, every subsequence contains a further subsequence that converges in distribution to a probability measure μ. To avoid a flood of subscripts, just suppose $(X_n)_{n\geq 1}$ is the first subsequence. Then let t be a point of continuity for $\mu((\infty, t])$. Then for the further subsequence

$$\lim_{\ell\to\infty} F_{X_{n(\ell)}}(t) = \mu((\infty, t]).$$

We can assume, without loss of generality, that $X_{n(k)} \to Y$ a.s. where $F_Y(t) =$ Reference Theorem 3.2.2 in Durrett, or add it here?
$\mu((\infty, t])$ Then for k fixed, consider

$$\limsup_{\ell\to\infty} \mathbb{E}[X^k_{n(\ell)}] \leq \limsup_{\ell\to\infty} \mathbb{E}[f_R(X_{n(\ell)})] + \limsup_{\ell\to\infty} \mathbb{E}[X^k_{n(\ell)} - f_R(X_{n(\ell)})]$$

where $f_R(x) = x^k$ if $|x| \leq R$ and is equal to zero otherwise. The dominated convergence theorem implies $\lim_{\ell\to\infty} \mathbb{E}[f_R(X_{n(\ell)})] = \mathbb{E}[f_R(Y)]$. Then for $R \geq 1$

$$\limsup_{\ell\to\infty} \mathbb{E}[X^k_{n(\ell)}] \leq \mathbb{E}[f_R(Y)] + R^{-k} m_{2k}.$$

Sending $R \to \infty$ gives $\limsup_{\ell\to\infty} \mathbb{E}[X^k_{n(\ell)}] \leq \mathbb{E}[Y^k]$. The same approach with a lim inf gives the reverse inequality. This shows that the measure μ is the same for every subsequence (and the further subsequence). This then implies, by Lemma l:subseq 33, that $X_n \xrightarrow{dist.} X$ as $n \to \infty$. $\square$

The last fact we use concerns numerical integration.

t:Gaussquad **Theorem 81** (Gaussian quadrature)**.** *Let T be a $n\times n$ Jacobi matrix for $n \leq \infty$ with spectral measure[1] μ_T. For $k < n$, let T_k be the upper-left $k \times k$ subblock of T. Then*

$$\int x^j \mu_{T_k}(\mathrm{d}x) = e_1^T T_k^j e_1 = e_1^T T^j e_1 = \int x^j \mu_T(\mathrm{d}x), \quad 0 \leq j \leq 2k-1,$$

where μ_{T_k} is the spectral measure for T_k.

Proof. The first equality follows from the definition of μ_{T_k}. The second equality is seen by noting that for any Jacobi matrix T (with diagonal entries $a_1, a_2, \ldots$, and off diagonal entries $b_1, b_2, \ldots$)

$$e_1^T T^j e_1$$

[1]The spectral measure of a finite Jacobi matrix is defined (eq:spec-map2 3.4.5). For a semi-infinite Jacobi matrix T the spectral measure is the measure with respect to which the polynomials defined by the three-term recurrence given by T are orthogonal.

depends only on $a_1, b_1, a_2, b_2, \ldots, a_{\frac{j+1}{2}}$ if j is odd and $a_1, b_1, a_2, b_2, \ldots, a_{\frac{j}{2}}, b_{\frac{j}{2}}$ if j is even. So, if $(j+1)/2 \leq k$ (j is odd) and $j/2 \leq k-1$ (j is even)

$$e_1^T T_k^j e_1 = e_1^T T^j e_1.$$

The conditions on j are equivalent to $j \leq 2k-1$ in both cases.

The last equality follows from applying eq:moments2 (3.4.22) to T_k for $j \leq 2k-1$. $\square$

Let $(x_1, x_2, \ldots, x_n)$ be the eigenvalues of $M \sim \mathrm{GOE}(n)$ ($\beta = 1$) or $M \sim \mathrm{GUE}(n)$ ($\beta = 2$). Define the scaled empirical spectral measure

$$S_n(\mathrm{d}x) = \frac{1}{n} \sum_{k=1} \delta_{x_k/\sqrt{n}}(\mathrm{d}x).$$

An inconsistency exists in the test with weak vs. distributional convergence notation, etc.

Recall that the averaged semicircle law states that

$$\mathbb{E} S_n(\mathrm{d}x) \xrightarrow{dist.} p_{\mathrm{sc}}(x)\mathrm{d}x, \tag{9.2.1}$$

eq:sc_law

as $n \to \infty$. By the method of moments, it suffices to show that

$$\int x^k \mathbb{E} S_n(\mathrm{d}x) = \mathbb{E} \frac{1}{n} \sum_{j=1}^k \left(\frac{x_j}{\sqrt{n}} \right)^k \to \int x^k p_{\mathrm{sc}}(x)\mathrm{d}x.$$

Note that because $p_{\mathrm{sc}}(x)\mathrm{d}x$ has compact support it is clear that it is uniquely determined by its moments. The classical method of moments for GOE or GUE works by noting that

$$\mathbb{E} \frac{1}{n} \sum_{j=1}^k \left(\frac{x_j}{\sqrt{n}} \right)^k = \mathbb{E} \frac{1}{n} \operatorname{Tr} \left(\frac{M}{\sqrt{n}} \right)^k,$$

and using combinatorial methods to enumerate the contributions to this trace, to leading order. This is where our approach deviates. Let e_1 be the first standard basis vector and consider the modified moments

$$\mathbb{E}\, e_1^T \left(\frac{M}{\sqrt{n}} \right)^k e_1.$$

To connect this to a measure, we employ the Lanczos iteration, Algorithm a:lanczos 1 with $q_1 = e_1$. Then we have, from eq:Tk (9.1.1), with the convention that $b_n, q_{n+1} = 0$,

$$\left(\frac{M}{\sqrt{n}} \right) Q_k = Q_k T_k + b_k q_{k+1} e_k^T, \quad T_k = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & & b_{k-1} \\ & & & b_{k-1} & a_k \end{bmatrix}.$$

And we are led to consider the distribution of the entries. By Theorem t:lanczos 78 $T = T_n$ coincides with the tridiagonalization found by Householder reflections. So,

$$\begin{aligned} \sqrt{\beta n}\, a_j &\sim \mathcal{N}(0,2), \\ \sqrt{\beta n}\, b_j &\sim \chi_{\beta(n-j)}. \end{aligned} \tag{9.2.2}$$

eq:ajbjs

for $j = 1, 2, \ldots, n$, where these variables are jointly independent.

With the additional convention that $b_0 = 0$, $p_{-1} = 0$, $p_0 = 1$, we define a sequence of polynomials via

$$(a_j - x)p_{j-1}(x) + b_j p_j(x) + b_{k-2}p_j(x) = 0, \quad j = 1, 2, \ldots, k-1. \tag{9.2.3}$$

eq:newp

By Theorem thm:Jacobi-spec 29, or more precisely the constructive proof described in Figure fig:specmap 3.4.1, and Remark r:ONP 32, these polynomials p_j, $j = 0, 1, 2, \ldots, k-1$ are orthonormal with respect to the spectral measure for T_k, and also with respect to the spectral measure for T_ℓ, $\ell > k$.

Next, we consider

$$T_j \sim \begin{bmatrix} \mathcal{N}(0,1/n) & \frac{1}{\sqrt{\beta n}}\chi_{\beta(n-1)} & & & \\ \frac{1}{\sqrt{\beta n}}\chi_{\beta(n-1)} & \mathcal{N}(0,1/n) & \frac{1}{\sqrt{\beta n}}\chi_{\beta(n-2)} & & \\ & \frac{1}{\sqrt{\beta n}}\chi_{\beta(n-2)} & \ddots & \ddots & \\ & & \ddots & & \frac{1}{\sqrt{\beta n}}\chi_{\beta(n-j+1)} \\ & & & \frac{1}{\sqrt{\beta n}}\chi_{\beta(n-j+1)} & \mathcal{N}(0,1/n) \end{bmatrix} \tag{9.2.4}$$

eq:T_dist

Then the central limit theorem (Theorem t:clt 102) implies

$$\sqrt{k}\left(\frac{1}{k}\chi_k^2 - 1\right) \xrightarrow{dist.} \mathcal{N}(0,2),$$

as $k \to \infty$, and therefore

$$\chi_k - \sqrt{k} \xrightarrow{dist.} \mathcal{N}(0,1/2), \tag{9.2.5}$$

eq:chik

as $k \to \infty$.

This implies that for any fixed j

$$\lim_{n\to\infty} \mathbb{E}\left[T_j\right] = \mathbb{T}_j := \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

Beyond that, suppose F is a continuous function of a $m \times m$ Jacobi matrix in a neighborhood of $\mathbb{T}_j$ and that $F(T) \leq C(1 + \|T\|)^q$ for some $C, q > 0$ then prove this?

$$\lim_{n\to\infty} \mathbb{E}F\left(T_j\right) = F(\mathbb{T}_j).$$

Then we choose $F(T_j) = e_1^T T_j^j e_1$. And, then using Theorem 81 we have shown

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\, e_1^T \left(T_j^j\right) e_1 &= \lim_{n\to\infty} \mathbb{E}\, e_1^T \left(T_j^j\right) e_1 \\ &= \lim_{n\to\infty} \int x^j \mathbb{E}\mu_T(\mathrm{d}x) = e_1^T \mathbb{T}_j^j e_1.\end{aligned}$$

We now must identify the sequence $e_1^T \mathbb{T}_j^j e_1$, $j = 1, 2, \ldots$ with the moment sequence of the semicircle distribution. The second-kind Chebyshev polynomials $(U_n)_{n\geq 0}$ are orthonormal polynomials on $[-1, 1]$ with respect to the measure $\frac{2\sqrt{1-x^2}}{\pi}\mathrm{d}x$ satisfying

$$\begin{aligned}U_0(x) &= 1, \\ U_1(x) &= 2x, \\ \frac{1}{2}U_{k+1}(x) - xU_k(x) + \frac{1}{2}U_{k-1}(x) &= 0, \quad k \geq 1.\end{aligned}$$

Define $\hat{U}_k(x) = U_k(x/2)$ and then the sequence $(\hat{U}_n)_{n\geq 0}$ are orthogonal polynomials on $[-2, 2]$ with respect to the semicircle law p_{sc}, satisfying the recurrence

$$\begin{aligned}\hat{U}_0(x) &= 1, \\ \hat{U}_1(x) &= x, \\ \hat{U}_{k+1}(x) - x\hat{U}_k(x) + \hat{U}_{k-1}(x) &= 0, \quad k \geq 1.\end{aligned}$$

We then conclude from Theorem 81 that

$$e_1^T \mathbb{T}_j^j e_1 = \int_{-2}^{2} x^j p_{\mathrm{sc}}(x)\mathrm{d}x.$$

This shows that

$$\mathbb{E}\mu_T(\mathrm{d}x) \xrightarrow{dist.} p_{\mathrm{sc}}(x)\mathrm{d}x.$$

Our last task is to compare $\mathbb{E}\mu_T(\mathrm{d}x)$ and $\mathbb{E}S_N(\mathrm{d}x)$. We recall from Weyl's formula (2.0.1) that one can take the eigenvectors to be distributed according to Haar measure on $\mathsf{O}(n)$ or $\mathsf{U}(n)$ depending on $\beta = 1, 2$. And then we know from Theorem 37 that the first column of such a random matrix is distributed uniformly on the sphere in $\mathbb{R}^n$ or $\mathbb{C}^n$ (again, depending on $\beta = 1, 2$). As $\mathsf{O}(n)$ and $\mathsf{U}(n)$ are compact groups they are unimodular [Fol99] and therefore if Q is Haar distributed on $\mathsf{O}(n)$ then so is $Q^{-1} = Q^T$ (similarly if U is Haar distributed on $\mathsf{U}(n)$ for $U^{-1} = U^*$). So, we find that the first components of the eigenvectors have the β-Dirichlet distribution and it is elementary that each component has mean $1/n$.

The following lemma completes the proof of (9.2.1).

l:KSdist **Lemma 30.** *Suppose* $M \sim \mathrm{GOE}(n)$ *(*$\beta = 1$*) or* $M \sim \mathrm{GUE}(n)$ *(*$\beta = 2$*). Then*

$$\mathbb{E} S_n = \mathbb{E}\mu_T.$$

Proof. We check the action on polynomials. By the independence of eigenvalues and eigenvectors

$$\int x^j \mathbb{E}\mu_T(\mathrm{d}x) = \mathbb{E}\left[\sum_{j=1}^n u_j^2 \left(\frac{\lambda_k}{\sqrt{\beta n}}\right)^j\right] = \sum_{j=1}^n \mathbb{E}[u_j^2]\mathbb{E}\left[\left(\frac{\lambda_k}{\sqrt{\beta n}}\right)^j\right] = \int x^j \mathbb{E} S_n(\mathrm{d}x).$$

$\square$

The method employed above to determine the limit is to (1) establish that the matrix moments

$$e_1^T T_j^k e_1,$$

converge to that of a deterministic, semi-infinite tridiagonal matrix $\mathbb{T}$. Then (2) the orthogonal polynomials that satisfy that the associated three-term recurrence are constructed and their orthogonality measure is deduced. Then the matrix moments must converge to the moments of this measure, and the limit is identified. There is an alternate way to complete (2): The orthogonality measure μ for the orthogonal polynomials satisfies

$$\int_{-\infty}^{\infty} \frac{\mu(\mathrm{d}s)}{s - z} = e_1^T(\mathbb{T} - zI)^{-1} e_1, \quad z \in \mathbb{C} \setminus \mathrm{supp}(\mu).$$

eq:stieltes-eqm2

And then (8.4.3), can be used to compute μ provided it has a continuous density. Residue calculations can also be employed to find point masses. It can be shown directly that

$$G(z) = \int_{-\infty}^{\infty} \frac{p_{\mathrm{sc}}(s)\mathrm{d}s}{s - z} = -\frac{1}{2}\left(z - \sqrt{z-2}\sqrt{z+2}\right),$$

where there square root is the principle branch. One can show directly that

$$(\mathbb{T} - zI)\begin{bmatrix} G(z) \\ G(z)^2 \\ G(z)^3 \\ \vdots \end{bmatrix} = e_1.$$

Discuss upgrading this to a weak, almost sure convergence and discuss limitations?

9.3 The conjugate gradient algorithm

The conjugate gradient algorithm (CGA) for the solution of $Mx = b$ is given by the following:

Algorithm 2: Conjugate Gradient Algorithm a:cga

1. x_0 is the initial guess.
2. Set $r_0 = b - Mx_0$, $p_0 = r_0$.
3. For $k = 1, 2, \ldots, n$
   (a) Compute $a_k = \dfrac{r_{k-1}^* r_{k-1}}{r_{k-1}^* M p_{k-1}}$.
   (b) Set $x_k = x_{k-1} + a_k p_{k-1}$.
   (c) Set $r_k = r_{k-1} - a_k M p_{k-1}$. If $r_k = 0$, terminate.
   (d) Compute $b_k = -\dfrac{r_k^* r_k}{r_{k-1}^* r_{k-1}}$.
   (e) Set $p_k = r_k - b_k p_{k-1}$.

It is important to note that the a_k's and b_k's that are generated here are not the same as those generated in the Lanczos iteration, although there is a bijection.

All properties of the CGA are now developed directly from this iteration. It is important to note that the historical development is not as we present it[2]. We also use $x_0 = 0$ in all our calculations. Throughout what follows we set $\ell = \max\{k : r_k \neq 0\} \leq n$.

Lemma 31. *For $k = 0, 1, 2, \ldots, \ell + 1$*

$$r_k = b - Mx_k.$$

Proof. The proof is by induction. Suppose that $r_{k-1} = b - Mx_{k-1}$. We first compute

$$a_k M p_{k-1} = M(x_k - x_{k-1}).$$

Then

$$r_k = b - Mx_{k-1} - a_k M p_{k-1} = b - Mx_k.$$

□

We then examine additional properties of the vectors r_k and p_k.

l:rk_pk **Lemma 32.** *For $k < \ell$, a_k is well-defined and the vectors $r_0, \ldots, r_k$ form an orthogonal basis of $\mathcal{K}_{k+1} = \operatorname{span}\{b, Mb, \ldots, M^k b\}$ with respect to the inner product $\langle x, y\rangle_2 = x^* y$. The vectors $p_0, \ldots, p_k$ form an orthogonal basis of $\mathcal{K}_{k+1}$ with respect to the inner product $\langle x, y\rangle_M = x^* M y$.*

[2]To derive Algorithm 2 (a:cga) from first principles, one uses the Lanczos iteration to find a square, tridiagonal approximation of the linear system and then solves that by Gaussian elimination. This can be one iteratively, reusing the previously derived LU factorization at each step.

Proof. This is also established by induction. Since the claim is clearly true for $k = 0$, suppose the claim is true for $k+1$ replaced with $k \geq 0$ and we show it holds for $k+1$. To see that the algorithm is well-defined one notes that, based on the induction hypothesis,

$$r_{k-1}^* M p_{k-1} = (p_{k-1} + b_{k-2} p_{k-2})^* M p_{k-1} = p_{k-1}^* M p_{k-1} \neq 0. \tag{9.3.1}$$ eq:rk_pk

From the relation,

$$p_j = r_j - b_j p_{j-1},$$

it follows that $p_k \in \operatorname{span}\{r_0, r_1, \ldots, r_k\}$ and

$$r_k \in \operatorname{span}\{p_{k-1}, p_k\} \subset \operatorname{span}\{p_0, p_1, \ldots, p_k\}.$$

It also follows that the spans of these two collections of vectors are both subspaces of $\mathcal{K}_k$. Since orthogonal vectors can only fail to be linearly independent if they vanish, we show none vanish: Since $k \leq \ell$, it follows by assumption that $r_k \neq 0$ and that $\operatorname{span}\{r_0, r_1, \ldots, r_k\} = \mathcal{K}_{k+1}$. From step 3(e), if $p_k = 0$, then $r_k \in \operatorname{span}\{p_0, p_1, \ldots, p_{k-1}\} \subset \mathcal{K}_k = \operatorname{span}\{r_0, r_1, \ldots, r_{k-1}\}$, contradicting the linear independence of $\{r_0, r_1, \ldots, r_k\}$. Thus $\operatorname{span}\{p_0, p_1, \ldots, p_k\} = \mathcal{K}_k$.

Then, using the definition of a_k

$$r_{k-1}^* r_k = r_{k-1}^* r_{k-1} - a_k r_{k-1}^* M p_{k-1} = 0.$$

For $j \leq k-2$

$$r_j^* r_k = -a_k r_j^* M p_{k-1}.$$

Then the fact that $r_j \in \operatorname{span}\{p_{j-1}, p_j\}$ and the orthogonality of the p_j's implies that $r_j^* r_k = 0$. Now, for $j < k$, we consider

$$(Mp_k)^* p_j = r_k^* M p_j - b_k p_{k-1}^* p_j.$$

For $j \leq k-2$ we have

$$(Mp_k)^* p_j = r_k^* M p_j.$$

Then $Mp_j \in \mathcal{K}_{j+2} = \operatorname{span}\{r_0, r_1, \ldots, r_{j+1}\}$ implying that $r_k^* M p_j = 0$. Then for $j = k-1$

$$\begin{aligned}
(Mp_k)^* p_j &= r_k^* M p_{k-1} - b_k p_{k-1}^* M p_{k-1}, \\
&= \frac{1}{a_k} r_k^* \left(r_{k-1} - r_k\right) + \frac{r_k^* r_k}{r_{k-1}^* r_{k-1}} p_{k-1}^* M p_{k-1} \\
&= r_k^* r_k \left(-\frac{1}{a_k} + \frac{p_{k-1}^* M p_{k-1}}{r_{k-1}^* r_{k-1}} \right) = 0,
\end{aligned}$$

by (eq:rk_pk 9.3.1). This completes the proof. $\square$

Remark 82. Replacing b with $b - Mx_0$ in the previous statement gives its extension to the case of non-zero initial guess x_0.

Remark 83. In the notation of Lemma l:rk_pk 32, it is clear that $M^{-1}b \in \mathcal{K}_{\ell+1}$ since $r_{\ell+1} = 0$.

Theorem 84. *For* $k \leq \ell + 1$

$$\begin{aligned} x_k &= \operatorname{argmin}_{y \in \mathcal{K}_k} \|x - y\|_W, \quad \|\cdot\|_W^2 = \langle \cdot, \cdot \rangle_W, \\ \mathcal{K}_k &= \operatorname{span}\{b, Mb, \ldots, M^{k-1}b\}. \end{aligned}$$

Proof. The vectors $p_1, \ldots, p_{k-1}$ are the correct orthogonal basis of $\mathcal{K}_k$ to consider. Since $r_{\ell+1} = 0$ implying that $x_{\ell+1} = x = M^{-1}b$ we know

$$x = \sum_{j=1}^{\ell+1} a_j p_{j-1}, \quad y = \sum_{j=1}^{k} \alpha_j p_{j-1}.$$

Then by orthogonality

$$\|x - y\|_W^2 = \sum_{j=1}^{k} |\alpha_j - a_j|^2 \|p_{j-1}\|_W^2 + \sum_{j=k+1}^{\ell+1} |a_j|^2 \|p_{j-1}\|_W^2,$$

This expression is minimized choosing $\alpha_j = a_j$ for $j \leq k$, i.e., choosing $y = x_k$. $\square$

Working directly with this optimization problem we see that

$$y = \sum_{j=1}^{k} \alpha_j p_{j-1} = \sum_{j=0}^{k-1} c_j W^j b,$$

for some choice of coefficients c_j. Then using that $b = Wx$ we find that

$$x - y = \sum_{j=0}^{k} \tilde{c}_j W^j x, \quad c_0 = 1.$$

Theorem 85. *For* $k \leq \ell + 1$

$$\begin{aligned} \|x - x_k\|_W = \min_{p \in \mathbb{P}_k} \|p(W)x\|_W, \quad \|\cdot\|_W^2 = \langle \cdot, \cdot \rangle_W, \\ \mathbb{P}_k = \{p : p \text{ is a polynomial of degree} \leq k, \quad p(0) = 1\}. \end{aligned}$$

This theorem allows one to then characterize the minimizing polynomial. For an arbitrary polynomial, after diagonalizing $M = U\Lambda U^*$ and using $x = M^{-1}b$ we find

$$\|p(W)x\|_W^2 = \sum_{j=1}^{\infty} \frac{|p(\lambda_j)|^2}{\lambda_j} |u_j|^2 = \int p(\lambda) \frac{\mu(\mathrm{d}\lambda)}{\lambda}$$

where

$$\mu = \sum_{j=1}^{n} |u_j|^2 \delta_{\lambda_j}, \quad u = U^* b.$$

Let $\check{p}_k \in \mathbb{P}_k$ be the polynomial that minimizes $\|p(W)x\|_W$. Let δp be a polynomial of degree at most k that preserves the normalization, i.e., $\delta p(0) = 0$. Then

$$\|p(W)x\|_W^2 = \int |\check{p}_k(\lambda)|^2 \frac{\mu(\mathrm{d}\lambda)}{\lambda} + 2\mathsf{Re} \int \check{p}_k(\lambda)\overline{\delta p(\lambda)} \frac{\mu(\mathrm{d}\lambda)}{\lambda} + \int |\delta p(\lambda)|^2 \frac{\mu(\mathrm{d}\lambda)}{\lambda}.$$

For $\check{p}_k$ to indeed be the minimizers, all "directional derivatives" need to vanish, and so

$$\int \check{p}_k(\lambda)\overline{\delta p(\lambda)} \frac{\mu(\mathrm{d}\lambda)}{\lambda} = 0$$

for all such δp. This implies that $\check{p}_k$ must be proportional to the k orthogonal polynomial with respect to μ.

To make the most important connection of the CGA to the Lanczos iteration, we compute the first components of the eigenvectors of the matrix T obtained by applying Algorithm t:lanczos 78 to M with starting vector b:

$$T = Q^* M Q, \quad Q = \begin{bmatrix} b & q_2 & \cdots & q_n \end{bmatrix}.$$

Here we suppose that $\|b\|_2 = 1$. Then

$$T = V \Lambda V^*, \quad V = Q^* U.$$

The (complex conjugates of the) first components of the eigenvectors of T are given by

$$(Q^* U)^* e_1 = U^* Q e_1 = U^* b.$$

The spectral measure μ_T for T coincides with μ. So,

$$\check{p}_k(\lambda) = \frac{\pi_k(\lambda)}{\pi_k(0)} = \frac{p_k(\lambda)}{p_k(0)},$$

where π_k (resp., p_k) is the kth monic (resp., normalized) orthogonal polynomial with respect to $\mu = \mu_T$. One can then verify that

$$\pi_k(\lambda) = \det(\lambda I - T_k)$$

and this gives the well-known expression

$$\check{p}_k(\lambda) = \frac{\det(\lambda I - T_k)}{\det T_k},$$

that appears in Greenbaum1989 [Gre89], for example.

It will turn out that this polynomial is best analyzed after we have preformed a Cholesky decomposition of T_k, i.e., write

$$T_k = H_k H_k^T, \quad H_k = \begin{bmatrix} \alpha_1 & & & & \\ \beta_1 & \alpha_2 & & & \\ & \beta_2 & \alpha_3 & & \\ & & \ddots & \ddots & \\ & & & \beta_{k-1} & \alpha_k \end{bmatrix}, \tag{9.3.2}$$

eq:tri_cholesky

This is correct. But do we want to add Cholesky to the text?

where all entries are non-negative, because the same is true of T_k. Then we note that

$$a_1 = \alpha_1^2, \quad a_j = \alpha_j^2 + \beta_{j-1}^2, \quad j > 1,$$
$$b_j = \alpha_j \beta_j, \quad j \geq 1.$$

We immediately see that $\pi_k(0) = \prod_{j=1}^k \alpha_j^4$. But more is true, because of Remark r:ONP 32, we see that

$$\int \pi_k(\lambda)^2 \mu(\mathrm{d}\lambda) = \prod_{j=1}^k \beta_j^2 \alpha_j^2.$$

This gives the rather remarkable relation

$$\|b - Mx\|_2^2 = \int \check{p}_k(\lambda)^2 \mu(\mathrm{d}\lambda) = \prod_{j=1}^k \frac{\beta_j^2}{\alpha_j^2}. \tag{9.3.3}$$

eq:res_prod

To obtain an analogous expression for $\|x - x_k\|_M^2$ one has to work a bit harder. Define the Stieltjes transforms of the monic polynomials

$$c_k(z) = \int \frac{\pi_k(\lambda)}{\lambda - z} \mu(\mathrm{d}\lambda).$$

It is straightforward to show that these $c_k(z)$ satisfy the same three-term recurrence as $\pi_k(z)$ with initial conditions $c_{-1}(z) = -1$ and

$$c_0(z) = \int \frac{\mu(\mathrm{d}\lambda)}{\lambda - z}.$$

Recall in Algorithm a:lanczos 1 we use the convention that $b_0 = -1$. This is necessary here. One more definition is needed. Define $\tilde{\pi}_k(x)$ to be a sequence of polynomials defined by the same three-term recurrence as $\pi_k(x)$ but with initial conditions $\tilde{\pi}_{-1}(x) = 1$, $\tilde{\pi}_0(x) = 0$. This implies that

$$c_k(z) = c_0(z)\pi_k(z) - \tilde{\pi}_k(z).$$

Compute

$$\begin{aligned}\|x-x_k\|_W^2 &= \int \check{p}_k(\lambda)^2 \frac{\mu(\mathrm{d}\lambda)}{\lambda} = \int \frac{\pi_k(\lambda)^2}{\pi_k(0)^2}\frac{\mu(\mathrm{d}\lambda)}{\lambda} \\ &= \frac{1}{\pi_k(0)^2}\int \pi_k(\lambda)\left[\frac{\pi_k(0)}{\lambda}+\ell_k(\lambda)\right]\mu(\mathrm{d}\lambda) \\ &= \frac{c_k(0)}{\pi_k(0)}\end{aligned}$$

because ℓ_k is ,at most, a degree $k-1$ polynomial and it is therefore orthogonal to π_k. The final expression becomes

$$\|x-x_k\|_W^2 = c_0(0) - \frac{\tilde{\pi}_k(0)}{\pi_k(0)}.$$

9.4 Asymptotic analysis of the CGA applied to LOE, LUE

We now consider the solution of

$$Mx = b, \quad M \sim \mathrm{LOE}(m,n), \mathrm{LUE}(m,n),$$

as $n \to \infty$ and $b = b_n$ is any sequence of unit vectors. We also assume that $m = \lceil \frac{n}{d} \rceil$ for $0 < d < 1$.

The key asymptotic fact in doing the following analysis is (9.2.5). This is because if we take a matrix M distributed according to $\mathrm{LOE}(m,n)$ or $\mathrm{LUE}(m,n)$, $m \geq n$, apply the Lanczos iteration (Algorithm 1) with starting vector b, obtain a tridiagonal matrix $T = T_n$, and then compute its Cholesky decomposition $T = H_n H_n^T$, by Theorem 78, we know that H_n has the same distribution as that of the first n columns of the bidiagonalization of $\mathrm{Gin}_{\mathbb{R}}(n,m)/\sqrt{m}$ or $\mathrm{Gin}_{\mathbb{C}}(n,m)/\sqrt{m}$. See Section 5.3. So, for $1 \leq k \leq n$

$$T_k = H_k H_k^T,$$

$$H_k \sim \frac{1}{\sqrt{\beta m}}\begin{bmatrix} \chi_{\beta m} & & & & & \\ \chi_{\beta(n-1)} & \chi_{\beta(m-1)} & & & & \\ & \chi_{\beta(n-2)} & \chi_{\beta(m-2)} & & & \\ & & \ddots & \ddots & & \\ & & & & \chi_{\beta(n-k+1)} & \chi_{\beta(m-k+1)} \end{bmatrix}. \tag{9.4.1}$$

where all entries are jointly independent. We then obtain, in the notation of (9.3.2)

$$\sqrt{2\beta m}\left(\frac{\alpha_j}{\beta_j} - \sqrt{d}\right) \xrightarrow{dist.} Z_1 - \sqrt{d} Z_2$$

as $n \to \infty$ where $Z_1, Z_2 \sim \mathcal{N}(0,1)$ are independent. Let $(Z_j)_{j\geq 1}$ be a process of iid $\mathcal{N}(0,1)$ random variables. We then see that in a distributional sense

$$\prod_{j=1}^{k} \frac{\alpha_j}{\beta_j} \approx \prod_{j=1}^{k} \left(\sqrt{d} + \frac{Z_{2j-1}}{\sqrt{2\beta m}} - \sqrt{d}\frac{Z_{2j-2}}{\sqrt{2\beta m}} \right)$$
$$= d^{k/2} + \frac{d^{\frac{k-1}{2}}}{\sqrt{2\beta m}} \sum_{j=1}^{k} (Z_{2j-1} - \sqrt{d} Z_{2j-2}) + O(m^{-1}).$$

This calculation implies

$$\sqrt{2\beta m}\left(\|b - Mx_k\|_2 - d^{k/2} \right) \xrightarrow{dist.} d^{\frac{k-1}{2}} \sum_{j=1}^{k} (Z_{2j-1} - \sqrt{d} Z_{2j-2}).$$

From this characterization the asymptotic covariance of $\|b - Mx_k\|_2, \|b - Mx_\ell\|_2$ can easily be obtained. And, in particular,

$$\sqrt{2\beta m}\left(\|b - Mx_k\|_2 - d^{k/2} \right) \xrightarrow{dist.} \mathcal{N}\left(0, k d^{k-1}(1+d)\right)$$

as $n \to \infty$.

Exact, non-asymptotic expressions such as

$$\mathbb{E}\|b - Mx_k\|^2 = \prod_{j=1}^{k} \frac{\beta(N-j)}{\beta(M-j+1)-2},$$

can also be obtained from this formulation. Here one just has to use independence and that

$$\mathbb{E}\left[\chi_k^2\right] = k, \quad k \geq 0, \quad \mathbb{E}\left[\frac{1}{\chi_k^2}\right] = \frac{1}{k-2}, \quad k > 2.$$

We can connect these to the Marchenko–Pastur distribution directly. From (9.4.1), we see that

$$T_k \to \mathbb{T}_k, \quad \mathbb{T} = \mathbb{T}(d) = \begin{bmatrix} 1 & \sqrt{d} & & \\ \sqrt{d} & 1+d & \sqrt{d} & \\ & \sqrt{d} & 1+d & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

where $\mathbb{T}_k$ is the upper-left principle subblock of $\mathbb{T}$. We compute $e_1^T(\mathbb{T}(d) - zI)e_1$ by first computing

$$e_1^T(\mathbb{T}(d) + d e_1^T e_1 - zI)^{-1} e_1 = \sqrt{d} e_1^T \left(\check{\mathbb{T}} - \frac{z-1-d}{\sqrt{d}} I \right)^{-1} e_1, \mathbb{T} = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}.$$

It follows that for $G(z) = -\frac{1}{2}\left(z - \sqrt{z-2}\sqrt{z+2}\right)$, we have

$$\begin{aligned} e_1^T(\mathbb{T}(d) + de_1e_1^T - zI)^{-1}e_1 &= \frac{1}{\sqrt{d}}G\left(\frac{z-1-d}{\sqrt{d}}\right) \\ &= \frac{1+d-z}{2d} + \frac{1}{2d}\sqrt{z-(1-\sqrt{d})^2}\sqrt{z-(1+\sqrt{d})^2} \\ &:= S(z). \end{aligned}$$

Then to compute $e_1^T(\mathbb{T}(d) - zI)^{-1}e_1$ we use the Woodbury matrix identity

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}.$$

Then set $A = \mathbb{T}(d) + de_1^Te_1 - zI$, $C = -d$, $U = e_1$, and $V = e_1^T$ to find

$$M(z) := e_1^T(\mathbb{T}(d) - zI)^{-1}e_1 = S(z) - S(z)(S(z) - d^{-1})^{-1}S(z)$$

Then, we compute

$$\frac{1}{2\pi \mathrm{i}} \lim_{\epsilon \downarrow 0}(M(x + \mathrm{i}\epsilon) - M(x - \mathrm{i}\epsilon)) =$$

9.5 Gradient descent

Let $X \in \mathbb{C}^{m\times n}$ and consider the iterative solution of the least-squares problem

$$\min_{y\in\mathbb{C}^n} \frac{1}{\sqrt{2}}\|Xy - b\|_2.$$

That is, the minimization of the functional

$$F(y) = \frac{1}{2}(Xy - b)^*(Xy - b) = \frac{1}{2}\left(y^*X^*Xy - y^*X^*b - b^*Xy + b^*b\right).$$

In linearizing the functional, one finds the gradient

$$\begin{aligned} F(y_0 + \epsilon y) - F(y_0) &= \epsilon\nabla F(y_0)\cdot y + O(\epsilon^2), \\ &= \frac{\epsilon}{2}\left(y^*X^*Xy_0 + y_0^*X^*Xy - y^*X^*b - b^*Xy\right) + O(\epsilon^2), \\ \nabla F(y_0) &= X^*Xy_0 - X^*b. \end{aligned}$$

Then the gradient descent iteration is given by

$$y_{n+1} = y_n - \gamma_n F(y_n),$$

for a sequence of step-sizes $\{\gamma_n\}$. Then for $c = X^*b$, $M = X^*X$, we compute

$$\begin{aligned} y_0 &= \text{given}, \\ y_1 &= y_0 - \gamma_1(My_0 - c), \\ y_2 &= y_1 - \gamma_1(My_2 - c), \\ &\vdots \end{aligned}$$

This iteration can be simplified by defining $e_j = y_j - x$, $x = (X^*X)^{-1}X^*b = M^{-1}c$ to find

$$\begin{aligned} e_0 &= y_0 - x, \\ e_1 &= e_0 - \gamma_1 e_0 = (I - \gamma_1 M)e_0, \\ e_2 &= (I - \gamma_2 M)e_1 = (I - \gamma_1 M)(I - \gamma_2 M)e_0, \\ &\vdots \end{aligned}$$

Clearly, one obtains

$$e_k = \prod_{j=1}^{k} (I - \gamma_k M) r_0 = P_k(M) e_0.$$

We then consider

$$r_k = X e_k = X P_k(M) e_0 = X P_k(X^*X) e_0 = P_k(XX^*) r_0.$$

Then, decompose $X = U\Sigma V^*$ in a singular-value decomposition. Suppose $m \geq n$ and $\Sigma \in \mathbb{R}^{m\times n}$: For $\tilde{M} = XX^*$

$$\|r_k\|_2^2 = r_0^* P_k(\tilde{M})^2 r_0 = e_0^* V P_k(\Sigma^*\Sigma)^2 V^* e_0$$

To simplify the analysis, take $y_0 = 0$ so that

$$V^* e_0 = -V^*(X^*X)^{-1}X^*b = -(\Sigma^*\Sigma)^{-1}\Sigma^* U^* b.$$

So, write

$$\Sigma^*\Sigma = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \quad w_j = (U^*b)_j, \quad j = 1, \ldots, m,$$

to find

$$\|r_k\|_2^2 = b^* U P_k(\Sigma\Sigma^*)^2 U^* b = \sum_{j=1}^{n} w_j^2 P_k(\lambda_j)^2.$$

There is always an issue using e_k for error vectors and the standard basis vectors.

To analyze the distribution of this quantity, we consider the Lanczos iteration applied to the pair (XX^*, b). By invariance, it suffices to suppose b is the first standard basis vector f_1 and therefore we can use Golub-Kahan bidiagonalization to obtain for the tridiagonalization T_k for $k \leq n$. There exists unitary matrices U_1, U_2 such that

$$U_1 X U_2 \sim \begin{bmatrix} \chi_{\beta n} & & & & \\ \chi_{\beta(m-1)} & \chi_{\beta(n-1)} & & & \\ & \chi_{\beta(m-2)} & \chi_{\beta(n-2)} & & \\ & & \ddots & \ddots & \\ & & & \chi_{\beta(m-n+1)} & \chi_\beta \\ & & & & \chi_{\beta(m-n)} \\ 0 & & \cdots & & 0 \\ \vdots & & & & \vdots \\ 0 & & \cdots & & 0 \end{bmatrix}.$$

Therefore

$$T_k \sim H_k H_k^T, \quad H_k \sim \begin{bmatrix} \chi_{\beta n} & & & & & \\ \chi_{\beta(m-1)} & \chi_{\beta(n-1)} & & & & \\ & \chi_{\beta(m-2)} & \chi_{\beta(n-2)} & & & \\ & & \ddots & \ddots & & \\ & & & & \chi_{\beta(m-k+1)} & \chi_{\beta(n-k)} \end{bmatrix}.$$

Then, for $j \leq 2k-1$

$$b^* U (\Sigma \Sigma^*)^j U^* b \sim f_1 T_k^j f_1.$$

This immediately implies the concentration of $\|r_k\|_2^2$ as $m, n \to \infty$. But we can do more to clarify the limit.

Chapter 10

Numerical methods for random matrix theory

ch:numerics

In this text, we have seen how random matrices have applications to numerical analysis by giving a class of distribution on which to test algorithms and determine an average-case behavior. In this chapter we explore the use of tools from numerical analysis to gain insight into random matrix theory. The distributions that arise in random matrix theory are transcendental functions and computing them is a non-trivial matter. Furthermore, there are many distributions from random matrix theory that one wants to be able to draw samples from.

10.1 Computing Fredholm determinants

sec:det-numerics

The expression for $D(z) = \det(\mathbf{1} - zK)$ as given in (D.1.8) is not a useful expression from a numerical analysis perspective — we begin with the problem computing the determinant of an integral operator acting in one spatial dimension and we have to then compute integrals over increasingly higher-dimensional spaces. What is more fruitful is going back to the original motivation of Fredholm and considering the discretization (D.1.4) of (D.1.1) and modifying it to use a more effective quadrature routine.

The underpinnings of developing highly-accurate quadrature rules was already introduced in Section 9.2, see Theorem 81, and it can be summarized as follows: Given a sequence of orthgonal polynomials with respect to a probability measure μ, take the upper-left $k \times k$ subblock of the associated Jacobi matrix and use the eigenvalues as quadrature nodes $\check{x}_1 = \check{x}_1(k), \ldots, \check{x}_k = \check{x}_k(k)$ and use the squared modulus of the first components of the normalized eigenvectors as weights $w_1 = w_1(k), \ldots, w_k(k)$. Theorem 81 then implies that

$$\sum_{j=1}^{k} p(\check{x}_j) w_j = \int p(x) \mu(\mathrm{d}x),$$

for all polynomials $\mathcal{P}_{2k-1}$ of degree $\leq 2k-1$. For general functions f, the error estimate takes the form

$$\left|\sum_{j=1}^{k} f(\check{x}_j)w_j - \int f(x)\mu(\mathrm{d}x)\right| \leq \left|\sum_{j=1}^{k}(f(\check{x}_j) - p(\check{x}_j))w_j\right| + \left|\int (f(x) - p(x))\mu(\mathrm{d}x)\right|$$

where $p \in \mathcal{P}_{2k-1}$. Then in the case that the support of μ is an interval, it follows that the eigenvalues of T_k all must lie in this interval and therefore

$$\left|\sum_{j=1}^{k} f(\check{x}_j)w_j - \int f(x)\mu(\mathrm{d}x)\right| \leq 2 \min_{p\in\mathcal{P}_{2k-1}} \sup_{x\in\operatorname{supp}\mu} |p(x) - f(x)|.$$

If f is continuous on $\operatorname{supp}\mu$ it can be approximated uniformly with polynomials. This tells us that

$$\sum_{j=1}^{k} f(\check{x}_j)w_j \to \int f(x)\mu(\mathrm{d}x),$$

for all continuous functions f on $\operatorname{supp}\mu$. Of course, the convergence can be much better and we will establish just how much better it can be in a bit. We will conjecture the existence of a second measure σ that is absolutely continuous with respect to Lebesgue measure with a bounded density so that

$$\frac{1}{k}\sum_{j=1}^{k} f(\check{x}_j) \to \int f(x)\sigma(\mathrm{d}x),$$

for all continuous functions f on $\operatorname{supp}\mu$.

To then get a handle on how small $\sup_{x\in\operatorname{supp}\mu}|p(x) - f(x)|$ can be, we discuss interpolation. And while this expression is independent of the choice of interpolation nodes, one can get an upper bound on the minimum of $\mathcal{P}_{2k-1}$ using (any) prescribed interpolation nodes. For the sake of simplicity, we will assume that f is analytic interior to a simple smooth curve Γ that encircles the interval $[a, b]$. Let $a \leq \check{x}_1 < \check{x}_2 < \cdots < \check{x}_k \leq b$ be a interpolation nodes, we wish to construct a polynomial p of degree at most k such that $p(\check{x}_j) = f(x_j)$. One such way to do this is by using a formula due to Hermite:

$$f(x) - p(x) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\nu_k(x)}{\nu_k(z)}\frac{f(z)}{z-x}\mathrm{d}z, \quad \nu_k = \prod_{j=1}^{k}(x - \check{x}_j), \quad x \in [a, b].$$

To verify this formula it is easy to first check that the right-hand side vanishes at $\check{x}_j$ for each j. And then a residue calculation confirms that the right-hand side is indeed equal to $f(x)$ plus a polynomial. One then finds that

$$|f(x) - p(x)| \leq \frac{\|f\|_{L^1(\Gamma)}}{2\pi}\frac{\max_{x\in[a,b]}|\nu_k(x)|}{\min_{z\in\Gamma}|\nu_k(z)|}. \qquad (10.1.1)$$ eq:interp-est

To begin to estmate the remaining terms here, we note that

$$|\nu_k(z)| = \exp\left(-k\int \log\frac{1}{|z-t|}\mu_k(\mathrm{d}t)\right), \quad \mu_k = \frac{1}{k}\sum_{j=1}^{k}\delta_{\tilde{x}_j}.$$

If $z \notin [a,b]$ then $\log\frac{1}{|z-t|}$ is a smooth function of t we see that as $k\to\infty$

$$|\nu_k(z)| = \exp\left(-k\left(\int \log\frac{1}{|z-t|}\sigma(\mathrm{d}t) + o(1)\right)\right). \tag{10.1.2}$$ eq:nuk

Furthermore,

$$\left(\int \log\frac{1}{|z-t|}\mu_k(\mathrm{d}t)\right)_{k\geq 1},$$

is an equicontinuous family for $z\in\Gamma$, implying that it converges uniformly, and therefore the error term in (eq:nuk 10.1.2) can be taken to be uniform in $z\in\Gamma$.

Then for $x \notin [a,b]$, the function $t\mapsto \log\frac{1}{|x-t|}$ is lower semicontinuous. This implies that there exists a sequence of increasing continuous functions $(\ell_j(x,t))_{j\geq 1}$ with this function as its pointwise limit. Indeed, these functions are easily constructed by "cutting off" the singularity of the logarithm. We compute

$$\begin{aligned}
g_k(x) &:= \int \log\frac{1}{|x-t|}\sigma(\mathrm{d}t) - \int \log\frac{1}{|x-t|}\mu_k(\mathrm{d}t)\\
&= \int \ell_j(x,t)\left(\sigma(\mathrm{d}t)-\mu_k(\mathrm{d}t)\right) + \int\left(\log\frac{1}{|x-t|} - \ell_j(x,t)\right)\sigma(\mathrm{d}t)\\
&+ \int\left(\ell_j(x,t) - \log\frac{1}{|x-t|}\right)\sigma(\mathrm{d}t).
\end{aligned}$$

The last term here is non-positive. The second term is k-independent and tends to zero as $j\to\infty$. We arrive at the upper bound

$$g_k(x) \leq \int \ell_j(x,t)\left(\sigma(\mathrm{d}t)-\mu_k(\mathrm{d}t)\right) + o(1),$$

where this error term is uniform in x owing to the fact that σ has a bounded density. From here, one can send $k\to\infty$, then $j\to\infty$, to see that $\limsup_{k\to\infty} g_k(x) \leq 0$. But next, we see that for fixed j the family of functions

$$\left(\int \ell_j(x,t)\left(\sigma(\mathrm{d}t)-\mu_k(\mathrm{d}t)\right)\right)_{k\geq 1}$$

is equicontinuous and this imples that pointwise convergence is upgraded to uniform, in x, convergence. For fixed j, this implies

$$\limsup_{k\to\infty}\max_{x\in[a,b]} g_k(x) \leq o(1), \quad j\to\infty.$$

And sending $j \to \infty$, we find

$$\limsup_{k\to\infty} \max_{x\in[a,b]} g_k(x) \leq 0.$$

What we have shown is that for every $\epsilon > 0$ there exists $K = K(\epsilon)$ so that for $k \geq K(\epsilon)$

$$\begin{aligned}
&\int \log \frac{1}{|x-t|}\sigma(\mathrm{d}t) - \int \log \frac{1}{|x-t|}\mu_k(\mathrm{d}t) \leq \epsilon, \quad \text{for all} \quad x \in [a,b], \\
&\int \log \frac{1}{|z-t|}\sigma(\mathrm{d}t) - \int \log \frac{1}{|z-t|}\mu_k(\mathrm{d}t) \geq \epsilon, \quad \text{for all} \quad z \in \Gamma.
\end{aligned}$$

This, in turn, implies

$$\begin{aligned}
\max_{x\in[a,b]} |\nu_k(x)| &\leq \exp\left(-k\left[\min_{x\in[a,b]} \int \log \frac{1}{|x-t|}\sigma(\mathrm{d}t) - \epsilon\right]\right), \\
\min_{z\in\Gamma} |\nu_k(z)| &\geq \exp\left(-k\left[\max_{z\in\Gamma} \int \log \frac{1}{|x-t|}\sigma(\mathrm{d}t) + \epsilon\right]\right),
\end{aligned}$$

giving the important estimate

$$\frac{\max_{x\in[a,b]} |\nu_k(x)|}{\min_{z\in\Gamma} |\nu_k(z)|} \leq \mathrm{e}^{k(2\epsilon + \Delta(\Gamma;\sigma))}, \tag{10.1.3}$$

eq:Delta

$$\Delta(\Gamma;\sigma) := \max_{z\in\Gamma} \int \log \frac{1}{|x-t|}\sigma(\mathrm{d}t) - \min_{x\in[a,b]} \int \log \frac{1}{|x-t|}\sigma(\mathrm{d}t). \tag{10.1.4}$$

And we conclude that the interpolant converges at a geometric rate if $\Delta(\Gamma;\sigma) < 0$.

10.1.1 Gauss-Legendre quadrature

Arguably, the most important case is where μ is uniform on $[-1,1]$ and the associated polynomials $(p_k)_{k\geq 0}$ are the normalized Legendre polynomials. The three-term recurrence for these polynomials is encapsulated in the Jacobi matrix J_k^{Leg} of the form (eq:defjac 3.1.1) with

$$a_j = 0, \quad b_j = \frac{j}{\sqrt{(2j-1)(2j+1)}}, \quad j = 1, 2, \ldots.$$

The quadrature rule that results is called the Gauss-Legendre rule. To establish the asymptotic distribution of the associated nodes we consider the Chebyshev polynomials of the first kind, associated with the Jacobi matrix J_k^{Cheb} with

$$a_j = 0, \quad b_j = \begin{cases} 1/\sqrt{2} & j = 1, \\ 1/2 & j > 1 \end{cases}, \quad j = 1, 2, \ldots.$$

It is straighforward to see that there exists an absolute constant $C > 0$ such that $\|J_k^{\text{Leg}} - J_k^{\text{Cheb}}\| \leq C$ where $\|\cdot\|$ is the Hilbert-Schmidt norm. Define $\check{x}_1 < \check{x}_2 < \cdots < \check{x}_k$ to be the eigenvalues of J_k^{Leg} and define $\check{y}_1 < \check{y}_2 < \cdots < \check{y}_k$ to be the eigenvalues of J_k^{Cheb}. We leave it as an exercise to first show that

$$\check{y}_j = \cos\left(\frac{2(k-j+1)-1}{k}\pi\right),$$

and then to show that for any function f, continuous on $[-1,1]$

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} f(\check{y}_j) = \frac{1}{\pi}\int_{-1}^{1} f(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}},$$

i.e., $\frac{1}{k}\sum_j \delta_{\check{y}_j}(\mathrm{d}t) \to \frac{1}{\pi}\frac{\mathrm{d}t}{\sqrt{1-t^2}}\mathbb{1}_{[-1,1]}(t)$ weakly.

Now, suppose that f is uniformly Lipschitz continuous on $[-1,1]$ so that there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x-y|$. And consider

$$\left|\frac{1}{k}\sum_{j=1}^{k} f(\check{y}_j) - \frac{1}{k}\sum_{j=1}^{k} f(\check{x}_j)\right| \leq \frac{L}{k}\sqrt{k}\|J_k^{\text{Leg}} - J_k^{\text{Cheb}}\|,$$

where we used the Cauchy-Schwarz inequality and Lemma l:hoffman 4. This vanishes as $k \to \infty$. Then using the fact that general continuous functions can be approximated by polynomials (which are, of course, uniformly Lipschitz) we conclude that $\frac{1}{k}\sum_j \delta_{\check{x}_j}(\mathrm{d}t) \to \frac{1}{\pi}\frac{\mathrm{d}t}{\sqrt{1-t^2}}\mathbb{1}_{[-1,1]}(t)$ weakly.

In the notation of the previous section we have identified

$$\sigma(\mathrm{d}t) = \frac{1}{\pi}\frac{\mathrm{d}t}{\sqrt{1-t^2}}\mathbb{1}_{[-1,1]}(t).$$

We then claim that

$$\int \log\frac{1}{|t-z|}\sigma(\mathrm{d}t) = -\mathsf{Re}\log\left(\frac{z+\sqrt{z^2-1}}{2}\right). \tag{10.1.5}$$ eq:log-form

Here $\sqrt{z^2-1}$ has its branch cut on $[-1,1]$ and it tends to z as $|z| \to \infty$. If $-1 \leq z \leq 1$ then $|z + \sqrt{z^2-1}|^2 = z^2 + z^2 - 1 = -1$ and $\mathsf{Re}\log\left(\frac{z+\sqrt{z^2-1}}{2}\right)$ is constant in z and equal to $\log 2$.

An easy way to understand what the function $\eta(z) = z + \sqrt{z^2-1}$ looks like off $[-1,1]$ is to understand it as an inverse of

$$z = z(\eta) = \frac{1}{2}\left(\eta + \frac{1}{\eta}\right).$$

This map is called the Joukowsky map and it maps the unit circle in the η-plane to the interval $[-1,1]$ in the z-plane. Our previous calculation confirmed this. One inverse of the Joukowsky map will map $\mathbb{C} \setminus [-1,1]$ in the z-plane to the interior of the unit circle in the η plane and the other inverse will map $\mathbb{C} \setminus [-1,1]$ to the exterior of the unit circle in the η plane. So, it remains to determine which inverse we are dealing with. Indeed, as $|z| \to \infty$ we can see that $z + \sqrt{z^2-1}$ blows up indicating that $|z + \sqrt{z^2-1}| > 1$ for $z \notin [1,1]$. All of this implies

$$\begin{aligned}\Delta(\Gamma,\sigma) &= \max_{z\in\Gamma}\left(-\log\left|\frac{z+\sqrt{z^2-1}}{2}\right|\right) - \log 2,\\ &= -\min_{z\in\Gamma}\log\left|z+\sqrt{z^2-1}\right| < 0,\end{aligned}$$

provided that $\Gamma \cap [-1,1] = 0$. The following theorem gives a general geometric rate of convergence for analytic functions.

Theorem 86. *Suppose f is analytic in an open set Ω that contains the unit interval $[-1,1]$. Let $p_k(x)$ be the polynomial interpolant of f at $-1 \leq \check{x}_1 < \check{x}_2 < \cdots < \check{x}_k \leq 1$ where $(\check{x}_j)_{j=1}^k = (\check{x}_j(k))_{j=1}^k$ are the eigenvalues of a sequence of Jacobi matrices J_k satisfying*

$$\sup_k \|J_k - J_k^{\mathrm{Cheb}}\| < \infty.$$

Then

$$\limsup_{k\to\infty}\frac{1}{k}\log\max_{x\in[-1,1]}|f(x)-p_k(x)| < 0.$$

Furthermore, if Ω contains the Bernstein ellipse

$$\mathcal{B}_\gamma = \left\{\frac{1}{2}\left(\eta+\frac{1}{\eta}\right)\middle|\ \eta = r\mathrm{e}^{\mathrm{i}\theta},\quad 1\leq r\leq\gamma,\quad 0\leq\theta<2\pi\right\},\quad \gamma>1,$$

then

$$\limsup_{k\to\infty}\frac{1}{k}\log\max_{x\in[-1,1]}|f(x)-p_k(x)| \leq -\log\gamma.$$

Corollary 4. *Suppose J_k is as in the previous theorem with $(\check{x}_j)_{j=1}^k = (\check{x}_j(k))_{j=1}^k$ being the eigenvalues of J_k and $w_1,\ldots,w_k$ being the squared modulus of the first components of the normalized eigenvectors of J_k. Suppose further f is analytic in $\mathcal{B}_\gamma$ that J_k is the upper-left $k\times k$ subblock of a semi-infinite Jacobi matrix whose entries for the three-term recurrence for a sequence of polynomials orthogonal with respect to μ which is supported on $[-1,1]$. Then*

$$\limsup_{k\to\infty}\frac{1}{k}\log\left|\sum_{j=1}^k f(\check{x}_j)w_j - \int f(x)\mu(\mathrm{d}x)\right| \leq -2\log\gamma.$$

Consider a self-adoint operator $\mathcal{K}$ with kernel $K : [-1,1]^2 \to \mathbb{R}$, $K(x,y) = K(y,x)$. Since $p_\ell(x)p_m(y)$ for $\ell, m = 0,1,2,\ldots$ forms an orthonormal basis for $L^2([-1,1]^2)$. We first suppose that $K(x,y)$ can be expressed as a finite linear combination of this basis:

$$K(x,y) = \sum_{\ell=0}^{k-1}\sum_{m=0}^{k-1} \alpha_{\ell,m} p_\ell(x) p_m(y). \tag{10.1.6}$$

eq:kapprox

And, for simplicity, we write this as

$$K(x,y) = \sum_{\ell=0}^{k-1} q_\ell(x) p_\ell(y), \quad q_\ell \in \mathcal{P}_\ell.$$

We then consider the series (eq:def-fred1 D.1.8). Since this kernel induces a finite-rank operator the series truncates at $k-1$ terms. The first term is given by

$$\begin{aligned}\int_{-1}^1 K(x_1,x_1)\mathrm{d}x_1 &= \int_{-1}^1 \sum_{\ell=0}^{k-1} q_\ell(x_1)p_\ell(x_1)\mathrm{d}x_1 = \sum_{j_1=1}^{k}\sum_{\ell=0}^{k-1} q_\ell(\check{x}_{j_1})p_\ell(\check{x}_{j_1})w_{j_1}\\ &= \sum_{j_1=1}^{k} K(\check{x}_{j_1},\check{x}_{j_1})w_{j_1}\end{aligned}$$

where the x_{j_1}'s and the w_{j_1}'s are nodes and weights associated to the Gauss-Legendre quadrature rule. Note that because $\int_{-1}^1 \mathrm{d}x = 2$ the weights should be normalized to $\sum_j w_j = 2$. And so, on hopes that all integrals in (eq:def-fred1 D.1.8) can be turned into sums using the quadrature rule. To see this is true in general, use a cofactor expansion down the last column

$$\det(K(x_p,x_q))_{1\le p,q\le n+1} = (-1)^{n-1}\sum_{i=1}^{n+1}(-1)^i K(x_i,x_{n+1})\det(K(x_p,x_q))_{\substack{1\le p\le n+1, p\neq i,\\ 1\le q\le n}}$$

For the determinants in the first n terms, perform a cofactor expansion across the nth row. This gives

$$\begin{aligned}\det(K(x_p,x_q))_{1\le p,q\le n+1} &= K(x_{n+1},x_{n+1})\det(K(x_p,x_q))_{1\le p,q\le n}\\ &+ \sum_{i=1}^{n} K(x_i,x_{n+1})\sum_{j=1}^{n}(-1)^{j+i-1}K(x_{n+1},x_j)\det(K(x_p,x_q))_{\substack{1\le p\le n, p\neq i,\\ 1\le q\le n, q\neq j}}.\end{aligned}$$

From this it is apparent that this determinant is a polynomial of degree at most $2k-2$ in x_{n+1}. The same holds for any other x_j. Therefore

$$\begin{aligned}&\int_{[-1,1]^n}\det(K(x_p,x_q))_{1\le p,q\le n}\mathrm{d}x_1\ldots\mathrm{d}x_n\\ &\qquad = \sum_{j_1,\ldots,j_n=1}^{k}\det(K(\check{x}_{j_p},\check{x}_{j_q}))_{1\le p,q\le n} w_{j_1}\ldots w_{j_n}.\end{aligned}$$

Furthermore, it follows that

$$\det(\mathbf{1} - zK) = \det(\delta_{pq} - zK(\check{x}_p, \check{x}_q)w_q)_{1\le p,q\le k}.$$

In the general case, let $K_k(x, y)$ be a polynomial interpolant of the form (10.1.6) of a general continuous kernel $K(x, y)$ such that $K_k(\check{x}_p, \check{x}_q) = K(\check{x}_p, \check{x}_q)$ for all choices of (p, q), $1 \le p, q \le k$ and denote the associated operator by $\mathcal{K}_k$. Then

$$\begin{aligned}
&|\det(\delta_{pq} - zK(\check{x}_p, \check{x}_q)w_q)_{1\le p,q\le k} - \det(\mathbf{1} - z\mathcal{K})| \\
&\le \sum_{\ell=1}^{k} \frac{|z|^\ell}{\ell!} \int_{[-1,1]^\ell} |\det(K(x_p, x_q))_{1\le p,q\le \ell} - \det(K_k(x_p, x_q))_{1\le p,q\le \ell}| \mathrm{d}x_1 \cdots \mathrm{d}x_\ell \\
&+ \sum_{\ell=k+1}^{\infty} \frac{|z|^\ell}{\ell!} \int_{[-1,1]^\ell} |\det(K(x_p, x_q))_{1\le p,q\le \ell}| \mathrm{d}x_1 \cdots \mathrm{d}x_\ell.
\end{aligned}$$

Now, suppose that

$$|K(x, y)| \le M(x), \quad |K(x, y) - K_k(x, y)| \le N(x, k).$$

Then the basic estimates are:

$$|\det(K(x_p, x_q))_{1\le p,q\le \ell}| \le \ell^{\ell/2} M(x_1) \cdots M(x_\ell).$$

and

$$\begin{aligned}
&|\det(K(x_p, x_q))_{1\le p,q\le \ell} - \det(K_k(x_p, x_q))_{1\le p,q\le \ell}| \\
&\quad \le \sum_{j=1}^{\ell} N(x_j, k)\ell^{\ell/2} \left(\prod_{q=1}^{j-1} M(x_q) \right) \left(\prod_{q=j+1}^{\ell} [M(x_q) + N(x_q, k)] \right).
\end{aligned}$$

To see how to establish the latter, let $A_0 = (K(x_p, x_q))_{1\le p,q\le \ell}$ and $A_\ell = (K_k(x_p, x_q))_{1\le p,q\le \ell}$ and for $j = 1, 2, \ldots, \ell - 1$ define A_j to be the matrix that has its first j columns coincide with A_0 and its last $\ell - j$ columns coincide with A_ℓ. Then

$$A_0 - A_\ell = \sum_{j=0}^{\ell-1} (A_j - A_{j+1}).$$

Then the difference $A_j - A_{j+1}$ is estimated by

$$\begin{aligned}
|A_j - A_{j+1}| \le\ & N(x_j, k)\ell^{\ell/2} \left(M(x_1) \cdots M(x_{j-1})\right) \\
&\times \left((M(x_{j+1}) + N(x_{j+1}; k)) \cdots (M(x_\ell) + N(x_\ell; k))\right).
\end{aligned}$$

This is found by performing a cofactor expansion down the jth column and accounting for the first $j - 1$ columns and then for the last $\ell - j - 1$ columns.

From these basic estimates, one obtains

$$\int_{[-1,1]^\ell} |\det(K(x_p,x_q))_{1\le p,q\le \ell} - \det(K_k(x_p,x_q))_{1\le p,q\le\ell}|\mathrm{d}x_1\cdots\mathrm{d}x_\ell \le \ell^{\ell/2+1}c_k(C+c_k)^\ell,$$
$$\int_{[-1,1]^\ell} |\det(K(x_p,x_q))_{1\le p,q\le\ell}|\mathrm{d}x_1\cdots\mathrm{d}x_\ell \le \ell^{\ell/2}C^\ell,$$

where $C = \int M(x)\mathrm{d}x$, $c_k = \int N(x,k)\mathrm{d}x$. It is then important to use Stirling's formula to estimate

$$\frac{\ell^{\ell/2}}{\ell!} = \frac{\gamma_\ell}{\sqrt{2\pi\ell}}\frac{\mathrm{e}^\ell}{\ell^{\ell/2}}, \quad \gamma_\ell = 1+o(1), \quad \ell\to\infty.$$

This, of course, decays to zero super-exponentially. It can also be established that $\gamma_\ell \le 1$. To find a final estimate for the error, define the, in particular continuous, function

$$G(c) = \sum_{j=1}^\infty \left(\frac{c}{\sqrt{j}}\right)^j, \quad c\ge 0.$$

This then gives the estimate

$$\begin{aligned}
&|\det(\delta_{pq} - zK(\check{x}_p,\check{x}_q)w_q)_{1\le p,q\le k} - \det(\mathbf{1}-z\mathcal{K})| \\
&\le c_k\sum_{\ell=1}^k \ell\frac{|z|^\ell}{\ell^{\ell/2}\sqrt{2\pi\ell}}(\mathrm{e}(C+c_k))^\ell + \sum_{\ell=k+1}^\infty \frac{|z|^\ell}{\ell^{\ell/2}\sqrt{2\pi\ell}}(\mathrm{e}C)^\ell, \\
&\le c_k\sqrt{\frac{k}{2\pi}}G(|z|\mathrm{e}(C+c_k)) + \frac{1}{\sqrt{2\pi k}}\left(\frac{|z|\mathrm{e}C}{\sqrt{k}}\right)^k G(|z|\mathrm{e}C).
\end{aligned}$$

Since the second term tends to zero so quickly, one can often infer that $\det(\delta_{pq} - zK_k(\check{x}_p,\check{x}_q)w_q)_{1\le p,q\le k}$ tends to $\det(\mathbf{1}-z\mathcal{K})$ at a rate determined by, up to a factor of $\sqrt{k}$, c_k.

Theorem 87. *Suppose $K : [-1,1]^2 \to \mathbb{R}$ is analytic in the sense that it can be represented as*

$$K(x,y) = -\frac{1}{4\pi^2}\int_\Gamma\int_\Gamma \frac{K(z,w)}{(z-x)(w-y)}\mathrm{d}w\mathrm{d}z,$$

where Γ is a simple smooth curve that encircles $[-1,1]$. Then the unique bivariate polynomial interpolant K_k at the eigenvalues $\check{x}_1,\ldots,\check{x}_k$ of J_k^{Leg} satisfies

$$\max_{(x,y)\in[-1,1]^2} |K_k(x,y) - K(x,y)| \le C\mathrm{e}^{-k\delta}$$

for some $\delta > 0$.

Proof. We first construct the interpolant after fixing x:

$$K(x,y) = p_k(y;x) + e_k(y;x).$$

From (eq:interp-est 10.1.1) and (eq:Delta 10.1.3) it follows that $|e_k(y;x)| \leq \frac{\|K(x,\cdot)\|_{L^1(\Gamma)}}{2\pi} \mathrm{e}^{-k\delta}$ for some $\delta > 0$ that is independent of both x and y. We then write the interpolant in the Lagrange form

$$p_k(y;x) = \sum_{j=1}^{k} K(x,\check{x}_j)\ell_j(y),$$

where $\ell_j(\check{x}_i) = \delta_{ij}$. We then apply the same procedure to $K(x,\check{x}_j)$ arriving at

$$K(x,\check{x}_j) = q_k(x;j) + \check{e}_k(x;j), \quad |\check{e}_k(x;j)| \leq \frac{\|K(\check{x}_j,\cdot)\|_{L^1(\Gamma)}}{2\pi} \mathrm{e}^{-k\delta}.$$

The interpolation formula takes the form

$$K(x,y) = \sum_{j=1}^{k} \left[q_k(x;j) + \check{e}_k(x;j)\right] \ell_j(y) + e_k(y;x),$$

where

$$\sum_{j=1}^{k} q_k(x;j)\ell_j(y)$$

interpolates $K(x,y)$. There exists a constant $C > 0$ such that $e_k, \check{e}_k \leq C\mathrm{e}^{-k\delta}$. It then remains to estimate

$$\Lambda_k \max_{y\in[-1,1]} \sum_{j=1}^{k} |\ell_j(y)|.$$

This is the so-called Lebesgue constant. Estimating this is not simple. The classic book by Szegő demonstrates that $\Lambda_k = O(n^{1/2})$ for Legendre polynomials. The theorem then holds by replacing δ any smaller value.

$\square$

We conclude that

$$\det(\delta_{pq} - zK(\check{x}_p,\check{x}_q)w_q)_{1\leq p,q\leq k} \to \det\left(\mathbf{1} - zK\right) \tag{10.1.7}$$ eq:detlim

at a geometric rate.

10.1.2 An implementation in Julia

Here we use the Julia programming language to compute Fredholm determinant and thus compute the limiting distributions that have appear previously.

The following code constructs the nodes xj and weights wj for Gauss-Legendre quadrature on an interval $[a, b]$.

```
using LinearAlgebra

function Jacobi(a,b,k) # creates k x k Jacobi matrix
   SymTridiagonal([a(i) for i in 1:k],[b(i) for i in 1:k-1])
end

aj = j ->  0.0; bj = j ->  j/sqrt((2j+1)*(2j-1))

L = Jacobi(aj,bj,10) |> eigen
xj = (a+b)/2 .+ (b-a)/2*L.values
wj = abs2.(L.vectors[1,:])*(b-a)
```

In some situations it maybe be advisable to use orthogonal polynomials on a semi-infinite interval $[a, \infty)$, such as Laguerre polynomials, to compute Fredholm determinants. We leave this as an exercise. For our examples, the only time we will compute the Fredholm determinant for an operator posed on an unbounded domain will be for the Airy kernel. In this case we have clear decay estimates for the kernel and the truncation of the "infinite" Fredholm determinant to one on a finite interval will be straightforward, with accuracy guarantees.

The sine kernel determinant

The following simple code now evaluates the sine kernel determinant to within nearly 16 digits.

```
function Ksine(x,y)
    return sinc(x-y)
end

function SineDet()
    # k is the number of quadrature nodes
    k = 30; L = Jacobi(aj,bj,k) |> eigen
    function freddet(a,b) # a < b
        xj = (a+b)/2 .+ (b-a)/2*L.values
        wj = abs2.(L.vectors[1,:])*(b-a)
        X = repeat(xj,1,k)
        Kmat = map(Ksine,X,X |> transpose)*diagm(sqrt.(wj))
        return I - diagm(sqrt.(wj))*Kmat |> det
```

```
    end
end

SD = SineDet(); SD(-0.1,0.1)
```

Also of note is the fact that we actually evaluate

$$\det(\delta_{pq} - z\sqrt{w_p}K(\check{x}_p, \check{x}_q)\sqrt{w_q})_{1\le p,q\le k}$$

which, of course, is the same as the desired determinant (eq:detlim 10.1.7) but it involves the determinant of a symmetric matrix when $K(x,y) = K(y,x)$. The execution of `F2(2.0)` takes on the order of 10^{-3} seconds on a laptop.

The Airy kernel determinant

The following simple code evaluates F_2 with nearly 16 digits of accuracy. If the density is desired, one can differentiate the distribution function using the expression $F_2'(t) \approx \mathsf{Im}\left(\frac{F_2(t+ih)}{h}\right)$ for h small. Note that the code below is written to avoid any complex conjugation `X |> transpose` instead of `X'` to allow for the code to produce an analytic function of t. With $h = 10^{-6}$ the density can be approximated to around 12 digits of accuracy using this method.

The only additional complications one encounters in this code is the truca- tion of the domain (the `l` parameter takes care of this) and the definition of the kernel itself.

```
Ai = x -> airyai(x); DAi = x -> airyaiprime(x)

function Kairy(x,y)
    if x == y
        return  DAi(x)^2 - x*Ai(x)^2
    else
        return (Ai(x)*DAi(y)-Ai(y)*DAi(x))/(x-y)
    end
end

function TracyWidom()
    # k is the number of quadrature nodes
    # l is the trucation paramater,
    # the upper bound on the interval length
    k = 30; l = 8; L = Jacobi(aj,bj,k) |> eigen
    function freddet(t)
        if real(t) > real(l)
            return 1
        else
```

```
                xj = (t+1)/2 .+ (1-t)/2*L.values
                wj = abs2.(L.vectors[1,:])*(1-t)
                X = repeat(xj,1,k)
                Kmat = map(Kairy,X,X |> transpose)
                Kmat = Kmat*diagm(sqrt.(wj))
                return I - diagm(sqrt.(wj))*Kmat |> det
            end
        end
end

F2 = TracyWidom(); F2(-2.0)
```

The Bessel kernel determinant

The Bessel kernel $K_{\mathsf{Bessel}}^{(\alpha)}(x,y)$ introduces some additional complications. First, it is not clear if and when the kernel is smooth when $x, y \approx 0$. And secondly, it is rather involved to compute $K_{\mathsf{Bessel}}^{(\alpha)}(x,x)$. We use the relations DLMF [OLBC10, 10.6.2]

$$
\begin{aligned}
J_\alpha'(z) &= -J_{\alpha+1}(z) + \frac{\alpha}{z} J_\alpha(z),\\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} J_\alpha(\sqrt{x}) &= \frac{1}{2\sqrt{x}} \left(-J_{\alpha+1}(\sqrt{x}) + \frac{\alpha}{\sqrt{x}} J_\alpha(\sqrt{x}) \right),\\
J_\alpha'(z) &= J_{\alpha-1}(z) - \frac{\alpha}{z} J_\alpha(z),\\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} \sqrt{x} J_{\alpha+1}(\sqrt{x}) &= \frac{1}{2\sqrt{x}} \left(\sqrt{x} J_\alpha(\sqrt{x}) - \alpha J_{\alpha+1}(\sqrt{x}) \right),
\end{aligned}
$$

and from the proof of Lemma l:bessel-limit 23 we write

$$
K_{\mathsf{Bessel}}^{(\alpha)}(x,y) = \frac{\sqrt{x} J_{\alpha+1}(\sqrt{x}) J_\alpha(\sqrt{y}) - \sqrt{y} J_{\alpha+1}(\sqrt{y}) J_\alpha(\sqrt{x})}{2(x-y)}.
$$

Then

$$
K_{\mathsf{Bessel}}^{(\alpha)}(x,x) = \frac{1}{4} \left(J_{\alpha+1}(\sqrt{x})^2 + J_\alpha(\sqrt{x})^2 - \frac{2\alpha}{\sqrt{x}} J_{\alpha+1}(\sqrt{x}) J_\alpha(\sqrt{x}) \right)
$$

10.2 Sampling determinantal point processes

Three approaches: DBM, Poulson, sampling marginals

10.3 Brownian bridges and non-intersecting Brownian paths

Include this?

10.4 Implications for randomized numerical linear algebra

Probably not...

10.5 Exercises

1. Chebyshev nodes 2. Distribution of Chebyshev nodes 3. Logarithmic potential of arcsine law with hint: Write it as as the real part of log transform, differentiate, solve with Cauchy transform. Reference hilbert transform from previous chapter. 4. Fredhold det with Laguerre polynomials

Part III

Appendices and background material

Appendix A

Elementary probability theory

app:prob

A.1 Axioms of probability

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space. Here $\mathcal{B}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a measure on $\mathcal{B}$ with total mass one, $\mathbb{P}(\Omega) = 1$. Thus we have the axioms of the measure

- $\mathbb{P}(\varnothing) = 0$, and
- if $E_1, E_2, \ldots$ is a sequence of disjoint sets from $\mathcal{B}$

$$\mathbb{P}\left(\bigcup_j E_j\right) = \sum_j \mathbb{P}(E_j).$$

These axioms give the following properties.

t:mprop

Theorem 88. *1. If $E_1, E_2 \in \mathcal{B}$, $E_1 \subset E_2$ then $\mathbb{P}(E_1) \leq \mathbb{P}(E_2)$.*

2. For sets $E_1, E_2, \ldots$ from $\mathcal{B}$ that are not disjoint, we have the union bound

$$\mathbb{P}\left(\bigcup_j E_j\right) \leq \sum_j \mathbb{P}(E_j).$$

3. For sets $E_1, E_2, \ldots$ from $\mathcal{B}$ with $E_j \subset E_{j+1}$ for all j

$$\mathbb{P}\left(\bigcup_j E_j\right) = \lim_{j\to\infty} \mathbb{P}(E_j).$$

4. *For sets $E_1, E_2, \ldots$ from $\mathcal{B}$ with $E_j \supset E_{j+1}$ for all j*

$$\mathbb{P}\left(\bigcap_j E_j\right) = \lim_{j\to\infty} \mathbb{P}(E_j).$$

A random variable X is a measurable function, defined on Ω, taking values in a measurable space $(\Sigma, \mathcal{S})$, $X : \Omega \to \Sigma$. The only properties of X we are allowed to take into account is its distribution. That is, we can observe

$$\mathbb{P}(X \in R) = \mathbb{P}(X^{-1}(R)) \in [0, 1],$$

for each $R \in \mathcal{S}$. We do not characterize the exact value of $X(\omega)$ for $\omega \in \Omega$. To emphasize this fact we will often introduce X using the notation $X \in \Sigma$ to indicate the codomain of X, ignoring its domain. In every case we consider, Σ can be identified with a subset of $\mathbb{R}^k$ for some k, and $\mathcal{S} = \mathcal{S}_k$ will be the Borel σ-algebra generated by open sets, in the relative topology.

Now consider $f : \Sigma \to \Sigma'$, measureable, where $(\Sigma', \mathcal{S}')$ is another measurable space. Define $Y = f(X)$. The distribution of Y, given by,

$$\mathbb{P}(Y \in R') = \mathbb{P}(f(X) \in R') = \mathbb{P}(X \in f^{-1}(R'))$$

is called the pushforward of X under f. In the case where $\Sigma = \mathbb{R}$, define the (cumulative) distribution function for X by

$$F_X(t) = \mathbb{P}(X \leq t).$$

From Theorem t:mprop 88, we have that $F_X(t) \leq F_X(s)$ if $t \leq s$. But more is true. Let $t_j \geq t$ with $t_j \to t$ monotonically. Then

$$\mathbb{P}(X \leq t) = \mathbb{P}\left(\bigcap_j \{X \leq t_j\}\right) = \lim_{j\to\infty} \mathbb{P}(X \leq t_j).$$

Now, assume $t_j \geq t$, $t_j \to t$, without any monotonicity assumption. Recall the following fact.

l:subseq **Lemma 33.** *Let $(s_n)_{n\geq 1}$ be a sequence in a metric space S, and let $s \in S$. If every subsequence of $(s_n)_{n\geq 1}$ has a further subsequence that converges to s, then $\lim_{n\to\infty} s_n = s$.*

Now, every subsequence of $(t_j)_{j\geq 0}$ has a monotonically converging, further subsequence. Then $\mathbb{P}(X \leq t_j)$, along this further subsequence, has a limit of $\mathbb{P}(X \leq t)$. Therefore $\lim_{j\to\infty} \mathbb{P}(X \leq t_j) = \mathbb{P}(X \leq t)$. This shows that $F_X(t)$ is right-continuous. Any function F that satisfies these properties

- F is right-continuous, and
- $F(t) - \mathbb{1}_{[0,\infty)}(t)$ decays for large $|t|$

is called a distribution function.

A.2 Multivariate distributions and independence

Now, consider the case where $\Sigma = \mathbb{R}^k$ for some k, i.e, X is vector valued. Let X_j, $1 \le j \le k$, be the components of X. Then the (cumulative) distribution function for X is given by

$$F_X(t_1, \dots, t_k) = \mathbb{P}(X_1 \le t_1, \dots, X_k \le t_k).$$

The distribution function has three main properties:

1. $0 \le F(t_1, \dots, t_k) \le 1$ for all $t_1, t_2, \dots, t_k \in \mathbb{R}$,

2. For fixed t_ℓ, $\ell \neq j$, $F(t_1, \dots, t_{j-1}, \cdot, t_{j+1}, \dots, t_k)$ is a distribution function, and

3. $F(t_1, \dots, t_k) - 1 + (1 - \prod_{j=1}^k \mathbb{1}_{[0,\infty)}(t_j)$ decays to zero as $\max_j |t_j| \to \infty$.

A point $(t_1, \dots, t_k)$ is a continuity point of F if F is continuous at $(t_1, \dots, t_k)$.

The components of X are said to be independent if for every choice of sets $B_i \in S_1$, $1 \le i \le k$,

$$\mathbb{P}(X_1 \in B_1, \dots X_k \in B_k) = \prod_{j=1}^k \mathbb{P}(X_j \in B_j).$$

An important tool in establishing that two random variables are independent is the conditional expectation. First, for a random variable Y define

$$\sigma(Y) := \{Y^{-1}(B) | B \in \mathcal{S}\}.$$

Definition 89 (Conditional expectation)**.** The conditional expectation of X given Y, denoted $\mathbb{E}\left[X|Y\right]$ is any random variable Z such that

1. Z is $\sigma(Y)$-measurable, and

2. for all $S \in \sigma(Y)$, $\mathbb{E}[X\mathbb{1}_S] = \mathbb{E}[Z\mathbb{1}_S]$.

Importantly, it turns out that the conditional expextation exists and is unique. The main way in which we use the conditional expectation is in the context of the following lemma.

l:phi **Lemma 34.** *Suppose X and Y are independent random variables with*[1] $\Sigma = \mathbb{R}^n$. *Let $\varphi : \Sigma \times \Sigma \to \mathbb{C}$ be so that $\mathbb{E}\left[|\varphi(X,Y)|\right] < \infty$. Then*

$$\mathbb{E}\left[\varphi(X,Y)|Y\right] = G(Y), \quad G(y) = \mathbb{E}\left[\varphi(X,y)\right].$$

[1] Here it is not required that n is the same for X and Y. Furthermore, X and Y could be matrix valued.

Proof. It suffices to show that for any $S \in \sigma(Y)$ we have

$$\mathbb{E}[G(Y)\mathbb{1}_S] = \mathbb{E}[\varphi(X,Y)\mathbb{1}_S].$$

The important observation is that $S = Y^{-1}(B)$ for some $B \in \mathcal{S}$ implying that $\mathbb{1}_S = \mathbb{1}_B(Y)$ and then

$$\mathbb{E}[G(Y)\mathbb{1}_S] = \int g(y)\mathbb{1}_B(y)\nu(\mathrm{d}y)$$

where ν is the distribution of Y. Then

$$g(y) = \int \varphi(x,y)\mu(\mathrm{d}x)$$

where μ is the distribution of X. By independence joint distribution of (X,Y) is the product measure $\mu(\mathrm{d}x)\nu(\mathrm{d}y)$ so that

$$\int g(y)\mathbb{1}_B(y)\nu(\mathrm{d}y) = \int \mathbb{1}_B(y)\varphi(x,y)\nu(\mathrm{d}y)\mu(\mathrm{d}x) = \mathbb{E}[\varphi(X,Y)\mathbb{1}_S].$$

□

A.2.1 Integration and L^p spaces

The expectation of a complex-valued random variable X is defined by

$$\mathbb{E}\left[X\right] = \int_\Omega X \mathrm{d}\mathbb{P},$$

provided that this integral exists. When $\Omega \subset \mathbb{R}^k$ for some k we often use the notation

$$\mathbb{E}\left[X\right] = \int_\Omega X(\omega)\mathbb{P}(\mathrm{d}\omega).$$

We say that $X \in L^p(\Omega, \mathcal{B}, \mathbb{P}) = L^p(\Omega)$ for $0 \leq p < \infty$ if

$$\mathbb{E}\left[|X|^p\right] < \infty.$$

A.2.2 Modes of convergence

We first discuss the convergence of scalar-valued random variables. Let $(X_n)_{n\geq 1}$, X be a sequence of random variables defined on a common probability space $(\Omega, \mathcal{B}, \mathbb{P})$[2]. We have four types of convergence:

1. <u>Almost sure convergence:</u> $X_n \to X$ almost surely ($X_n \xrightarrow{a.s.} X$) if

$$\mathbb{P}\left(\limsup_n X_n = \liminf_n X_n = X\right) = 1.$$

[2]Kolmogorov extension theorem....

2. $\underline{L^p \text{ convergence:}}$ Suppose $X_n \in L^p(\Omega)$ for every n and $X \in L^p(\Omega)$. Then $X_n \to X$ in $L^p(\Omega)$ ($X_n \xrightarrow{L^p(\Omega)} X$) if
$$\mathbb{E}\left[|X_n - X|^p\right] \xrightarrow{n\to\infty} 0.$$

3. Convergence in probability: $X_n \to X$ in probability ($X_n \xrightarrow{prob.} X$) if for every $\epsilon > 0$
$$\mathbb{P}\left(|X_n - X| > \epsilon\right) \xrightarrow{n\to\infty} 0.$$

4. Convergence in distribution: $X_n \to X$ in distribution ($X_n \xrightarrow{dist.} X$) if for every t at which $F_X(t)$ is continuous
$$F_{X_n}(t) \xrightarrow{n\to\infty} F_X(t).$$

To check convergence in distribution one needs to see that $F_{X_n}(t) \to F(t)$ for some function F (at all its points of continuity) and that F is itself a distribution function.

Include proofs and/or references.

Definition 90. A sequence of random variables $(X_n)_{n\geq 1}$ is tight if for every $\epsilon > 0$ there exists R such that
$$\limsup_{n\to\infty} \mathbb{P}(|X_n| > R) < \epsilon.$$

Definition 91. Suppose for some Borel measure μ, a sequence of random variables $(X_n)_{n\geq 1}$ satisfies
$$F_{X_n}(t) \xrightarrow{n\to\infty} F(t) := \mu((-\infty, t])$$
at all the continuity points of F. Then F_{X_n} is said to converge to F vaguely. Alternatively, we say that $X_n \xrightarrow{v} \mu$ as $n \to \infty$.

It is important to note that μ is not necessarily a probability measure. If it is then we have convergence in distribution. Tightness guarantees this.

Theorem 92. *A sequence random variables $(X_n)_{n\geq 1}$ is tight if and only if every vaguely convergent subsequence converges to a probability measure.*

t:Helly **Theorem 93** (Helly's selection theorem)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables. Then there exists a subsequence $(X_{n(k)})_{k\geq 1}$ and a measure μ such that $X_{n(k)} \xrightarrow{v} \mu$ as $k \to \infty$.*

An important consequence is that if $(X_n)_{n\geq 1}$ is a tight sequence of random variables then there exists a subsequence that converges in distribution to a probability measure μ (i.e., a random variable X with distribution μ).

All of these notion extend to the multi-variate case by replacing the absolute value with a norm $\|\cdot\|$ and replacing (t) with $(t_1, \ldots, t_k)$ in the definition of convergence in distribution. The following theorem is not needed often in the text, but it is necessary to understand the statement of many results in the literature. Another consequence, is that if, for example $F_{X_n}(t) \to F(t)$ for every $t \in \mathbb{R}$, F is continous, and $(X_n)_{n\geq 1}$ is tight, then F is a distribution function.

t:weak-conv **Theorem 94.** *Let $(X_n)_{n\geq 1}$ be a sequence of vector-valued random variables taking values in $\mathbb{C}^n$ (or $\mathbb{R}^n$). Then as $n \to \infty$*

$$X_n \xrightarrow{dist.} X$$

for a random variable X if and only if

$$\mathbb{E}[F(X_n] \to \mathbb{E}[F(X)]$$

for all bounded, infinitely differentiable functions functions $F : \mathbb{C}^n \to \mathbb{R}$ (or $F : \mathbb{R}^n \to \mathbb{R}$).

Another convenient characterization of convergence in distribution is the following.

t:levy **Theorem 95** (Levy continuity theorem)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of vector-valued random variables taking values in $\mathbb{R}^n$. Then as $n \to \infty$*

$$X_n \xrightarrow{dist.} X$$

for a random variable X if and only if

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle t, X_n\rangle}] \to \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle t, X\rangle}] \quad \textit{for all} \quad t \in \mathbb{R}^n.$$

This theorem provides a convenient way to prove the following result.

l:twoconv **Lemma 35.** *Suppose $(Y_n)_{n\geq 1}$ is a sequence of real-valued random variables and suppose $(X_n)_{n\geq 1}$ is a sequence of vector-valued random variables taking values in $\mathbb{R}^n$. Suppose that as $n \to \infty$, $Y_n \xrightarrow{prob.} c \in \mathbb{R}$ and $X_n \xrightarrow{dist.} X$ then $Y_n X_n \xrightarrow{dist.} cX$.*

Proof. Fix $\epsilon > 0$. Because the sequence $(X_n)_{n\geq 1}$ converges in distribution, it is tight. This means that there exits $R > 0$ such that $\mathbb{P}(\|X_n\|_2 \geq R) \leq \epsilon$ for all n. Then for n sufficiently large, we know that

$$\mathbb{P}(|Y_n - c| \geq \epsilon/R) \leq \epsilon.$$

With this in hand we estimate

$$\begin{aligned} \left|\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle t, Y_n X_n\rangle}] - \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle t, cX_n\rangle}]\right| &= \left|\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle t, cX_n\rangle}(\mathrm{e}^{\mathrm{i}\langle t, Y_n X_n\rangle - \mathrm{i}\langle t, cX_n\rangle} - 1)\right]\right| \\ &\leq \mathbb{E}\left[\left|\mathrm{e}^{\mathrm{i}\langle t, (Y_n - c)X_n\rangle} - 1\right|\right] \end{aligned}$$

Then write

$$\begin{aligned}\left|\mathrm{e}^{\mathrm{i}\langle t,(Y_n-c)X_n\rangle}-1\right| &= \left|\mathrm{e}^{\mathrm{i}\langle t,(Y_n-c)X_n\rangle}-1\right| \mathbb{1}_{\{\|X_n\|_2<R\}\cap\{|Y_n-c|\le\epsilon/R\}} \\ &\quad + \left|\mathrm{e}^{\mathrm{i}\langle t,(Y_n-c)X_n\rangle}-1\right| \mathbb{1}_{\{\|X_n\|_2\ge R\}\cup\{|Y_n-c|>\epsilon/R\}}.\end{aligned}$$

We use the elementary inequality for $x \in \mathbb{R}$

$$|\mathrm{e}^{\mathrm{i}x}-1| \le |x|,$$

to bound, on the set $\{\|X_n\|_2 < R\} \cap \{|Y_n - c| \le \epsilon/R\}$,

$$\left|\mathrm{e}^{\mathrm{i}\langle t,(Y_n-c)X_n\rangle}-1\right| \le |\langle t,(Y_n-c)X_n\rangle| \le \|t\|_2|Y_n-c|\|X_n\| \le \epsilon\|t\|_2.$$

Then by the union bound, for n sufficiently large

$$\begin{aligned}&\mathbb{E}\left[\left|\mathrm{e}^{\mathrm{i}\langle t,(Y_n-c)X_n\rangle}-1\right| \mathbb{1}_{\{\|X_n\|_2\ge R\}\cup\{|Y_n-c|>\epsilon/R\}}\right] \\ &\quad \le 2\mathbb{P}\left(\{\|X_n\|_2 \ge R\} \cap \{|Y_n-c| > \epsilon/R\}\right) \le 4\epsilon.\end{aligned}$$

As ϵ is arbitrary we conclude that

$$\left|\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle t,Y_nX_n\rangle}] - \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle t,cX_n\rangle}]\right| \to 0,$$

and the proof is finished by simply observing that $\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle t,cX_n\rangle} \to \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle t,cX\rangle}]$. □

A.3 Classical distributions

We now describe some classical distributions that will play a significant role in what follows. First the abbreviation iid is short for independent and identically distribution. So, a collection of random variables $(X_n)_{n\ge1}$ is iid if it is independent and

$$F_{X_j} = F_{X_1} \quad \text{for all } j.$$

1. Normal (Gaussian) random variable: $X \in \mathbb{R}$ is normally distributed with mean μ and standard deviation $\sigma > 0$ ($X \sim \mathcal{N}(\mu,\sigma^2)$) if

$$F_X(t) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{t} \mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}\mathrm{d}x.$$

2. Multi-variate normal random variable: $X \in \mathbb{R}^n$ is normally distributed with mean $\mu \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n\times n}$, $\Sigma > 0$ ($X \sim \mathcal{N}(\mu,\Sigma)$) if

$$F_X(t_1,\ldots,t_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det\Sigma}}\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_n} \mathrm{e}^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}\mathrm{D}x.$$

3. Standard complex normal random variable: $X \in \mathbb{C}$ has a standard complex normal distribution if

$$X \sim Y^T \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}, \quad Y \sim \mathcal{N}\left(0, \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\right).$$

4. Exponential distribution: $X \in \mathbb{R}_+$ is exponentially distributed with rate $\lambda > 0$ (mean $1/\lambda$) if

$$F_X(t) = 1 - \mathrm{e}^{-\lambda t}.$$

5. Chi-squared distribution: $X \in \mathbb{R}_+$ has a chi-squared distribution with m degrees of freedom ($X \sim \chi_m^2$) if

$$X \sim \sum_{j=1}^{m} X_j, \quad (X_j)_{j=1}^m \text{ iid}, \quad X_1 \sim \mathcal{N}(0,1).$$

In other words,

$$F_X(t) = \frac{1}{2^{m/2}\Gamma(m/2)} \int_0^t x^{m/2-1} \mathrm{e}^{-x/2} \mathrm{d}x,$$

where $\Gamma(\cdot)$ is the Gamma function. Note that this second expression can be use to extend the definition to non-integer m.

6. Chi distribution: $X \in \mathbb{R}_+$ has a chi distribution with m degrees ($X \sim \chi_m$) of freedom if

$$X \sim \sqrt{Y}, \quad Y \sim \chi_m^2.$$

It follows that

$$F_X(t) = \frac{1}{2^{m/2-1}\Gamma(m/2)} \int_0^t x^{m-1} \mathrm{e}^{-x^2/2} \mathrm{d}x.$$

7. β-Dirichlet distribution: $X \in \mathbb{R}^n$ has the β-Dirichlet distribution[3] if

$$X \sim Y/\|Y\|_1, \quad Y = \begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix}^T, \quad (Y_j)_{j=1}^n \text{ iid}, \quad Y_1 \sim \chi_\beta^2.$$

8. Beta distribution: $X \in [0,1]$ is beta distributed with parameters (α, β) if

$$F_X(t) = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^t x^{\alpha-1}(1-x)^{\beta-1} \mathrm{d}x, \quad \mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

It should be noted that the marginal distribution of each component of a β-Dirichlet distribution is the beta distribution with parameters $(\beta/2, (n-1)\beta/2)$.

[3]This is note the classical definition of the Dirichlet distribution but it is convenient for our purposes.

A.4 Other measure-theoretic facts

Lemma 36. *Suppose $P(x_1, \ldots, x_k)$ is a polynomial,*

$$P(x_1, \ldots, x_k) = \sum_{d=0}^{n} \sum_{\substack{(i_1, \ldots, i_d) \in [\![1,k]\!]^d \\ i_1 \le i_2 \le \cdots \le i_d}} c_{(i_1,\ldots,i_d)} x_{i_1} \cdots x_{i_d}.$$

If P vanishes on set of positive k-dimensional Lebesgue measure, then $P \equiv 0$.

Proof. We prove this by induction on k. For $k = 1$ the claim follows because an analytic function that vanishes on a set with a limit point must vanish identically. Assume the lemma for $k-1$. Suppose R is chosen such that $\{x \in \mathbb{R}^k : P(x) = 0\} \cap B(0,R)$ has positive Lebesgue measure. Under the hypotheses of the lemma, there exists an infinite number of distinct values a_j, $j \ge 0$, so that the polynomial in $k-1$ variables

$$P(x_1, \ldots, x_{k-1}, a_j) = \sum_{d=0}^{n} p_d(x_1, \ldots, x_{k-1}) a_j^d \tag{A.4.1}$$ eq:pol-res

vanishes on set of positive $(k-1)$-dimesional Lebesgue measure. To see this, let $F(x) = \mathbb{1}_{\{P=0\}\cap B(0,R)}(x)$ and consider

$$0 < \int_{\mathbb{R}^k} F(x) \mathrm{D}x = \int_{-R}^{R} \left(\int_{\mathbb{R}^{k-1}} F(x, x_k) \mathrm{D}x \right) \mathrm{d}x_k.$$

Therefore there must exists an infinite number of points x_k where $\int_{\mathbb{R}^{k-1}} F(x, x_k)\mathrm{D}x$ does not vanish. By the induction hypothesis (eq:pol-res A.4.1) vanishes on $\mathbb{R}^{k-1}$ for each j. We obtain a system of $d+1$ linear equations

$$\sum_{d=0}^{n} p_d(x_1, \ldots, x_{k-1}) a_j^d = 0, \quad j = 0, 1, \ldots, d, \quad (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1}.$$

This is a Vandermonde system (see eq:vdm 2.6.6) and it implies that $p_d = 0$ for each d. □

t:domconv **Theorem 96.** *Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Let f_k be a sequence of integrable functions such that $f_k \to f$ almost everywhere. Further, suppose that $|f_k| \le g$ for all k and $\int g \,\mathrm{d}\mu < \infty$. Then*

$$\lim_{k\to\infty} \int f_k \mathrm{d}\mu = \int f \mathrm{d}\mu.$$

Theorem 97 (Borel-Cantelli, if needed?)**.**

t:jensen **Theorem 98** (Jensen's inequality)**.**

t:markov **Theorem 99** (Markov's inequality)**.** *Suppose X is a non-negative random variable and $t > 0$. Then*

$$\mathbb{P}(X \ge t) \le \frac{\mathbb{E}[X]}{t}.$$

A.5 Classical limit theorems

In order to put limit theorems from random matrix theory in their correct context, we briefly review the two main limit theorems from classical probability. The strong law of large numbers states the following.

t:lln **Theorem 100** (Strong law of large numbers)**.** *let $(X_n)_{n\geq 1}$ be a sequence of iid real random variables with $\mu = \mathbb{E}[X_1] < \infty$. Consider the sample average*

$$S_n = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

Then

$$\mathbb{P}(\lim_{n\to\infty} S_n \neq \mu) = 0,$$

that is, $S_n \xrightarrow{a.s.} \mu$.

The strong law of large numbers implies the weak law of large numbers.

t:wlln **Theorem 101** (Weak law of large numbers)**.** *let $(X_n)_{n\geq 1}$ be a sequence of iid real random variables with $\mu = \mathbb{E}[X_1] < \infty$. Consider the sample average*

$$S_n = \frac{1}{n}\sum_{j=1}^{n} X_j.$$

Then $S_n \xrightarrow{prob.} \mu$.

The central limit theorem concerns the correction term ξ_n in the expansion $S_n = \mu + \xi_n$.

t:clt **Theorem 102** (Central limit theorem)**.** *In the setting of the previous theorem, suppose, in addition, that $\sigma^2 = \mathrm{Var}(X_1) < \infty$. Then for all $t \in \mathbb{R}$*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{S_n - \mu}{\sigma/\sqrt{n}} \leq t\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} \mathrm{e}^{-x^2/2}\mathrm{d}x.$$

In other words, if $Y \sim \mathcal{N}(0,1)$ then

$$\frac{S_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{dist.} Y,$$

as $n \to \infty$.

In a distributional sense, the central limit theorem states

$$S_n \approx \mu + \frac{\sigma Y}{\sqrt{n}} \quad \text{or} \quad S_n \approx \mathcal{N}(\mu, \sigma^2/n).$$

A.6 Point processes on $\mathbb{R}$

The simplest point processes on $\mathbb{R}$ comes from a finite-dimensional distribution. For example, if $X \sim \mathcal{N}(\mu, \Sigma)$ is a n-dimensional normal random variable we can consider the point process $\{X_1, X_2, \ldots, X_n\} \subset \mathbb{R}$ or represent the process as a random measure

$$\mu_X := \sum_{j=1}^{n} \delta_{X_j}.$$

It turns out that this representation of the point process is often convenient. In this notation, the distribution of the largest component of X would be given by

$$\mathbb{P}(\max_j X_j \leq t) = \mathbb{P}\left(\mu_X((t,\infty)) = 0\right).$$

Furthermore, the random measure interpretation makes it clearer why this is indeed a stochastic process: Define a stochastic process $(Z_t)_{t\in\mathbb{R}}$ by

$$Z_t = \mu_X((-\infty, t]).$$

A.6.1 Homogeneous Poisson point process

The most famous instance of a point process is the Poisson point process. Here we only discuss its homogeneous incarnation on $\mathbb{R}$. Let $\lambda > 0$ be the rate for the process. The the Poisson point process is given by a Borel random measure μ_λ that satisfies

1. $\mu_\lambda(B) \in \mathbb{N}$ for any Borel set B,
2. $\mu_\lambda(\{a\}) = 0$ almost surely, for some $a \in \mathbb{R}$,
3. for any $t_0 < t_1 < \cdots < t_m$ the increment random variables
$$I_j = \mu_\lambda((t_j, t_{j+1}]), \quad 0 \leq j \leq m-1,$$
are independent, and
4. for $a < b$
$$\mathbb{P}(\mu_\lambda((a,b]) = k) = \frac{(\lambda(b-a))^k}{k!}\mathrm{e}^{-\lambda(b-a)}.$$

The last property implies exponentially-distributed gap probabilities

$$\mathbb{P}(\mu_\lambda((a,b]) = 0) = \mathrm{e}^{-\lambda(b-a)}.$$

Then compute, for $\epsilon\lambda \leq 1$,

$$\mathbb{P}(\mu_\lambda([c, c+\epsilon)) \geq 1) = \sum_{k=1}^{\infty} \frac{(\lambda\epsilon)^k}{k!}\mathrm{e}^{-\lambda\epsilon} \leq \epsilon\lambda\mathrm{e} \xrightarrow{\epsilon\downarrow 0} 0,$$

giving that $\mathbb{P}(\mu_\lambda(\{c\}) \geq 1) = 0$.

Next, consider the spacing statistics in an interval $(a, b]$. Since $N := \mu_\lambda((a, b])$ is almost surely finite, define $X_1, \dots, X_N$, to be all the points in $(a, b]$ such that

$$\mu_\lambda(\{X_j\}) \neq 0,$$

ordered in weakly increasing order, counted according to multiplicity. For example, if $\mathbb{P}(\mu_\lambda(\{X_1\}) = j$ then $X_1 = X_2 = \cdots = X_j$. For $t_1, t_2 \geq 0$, by independence

$$\begin{aligned}
\mathbb{P}(X_1 \leq t_1, X_2 - X_1 \leq t_2) &= \mathbb{P}\left(\mu_\lambda((a, t_1]) \geq 1, \mu_\lambda([t_1, t_1 + t_2]) \geq 1\right) \\
&= \mathbb{P}\left(\mu_\lambda((a, t_1]) \geq 1, \mu_\lambda((t_1, t_1 + t_2]) \geq 1\right) \\
&= (1 - \mathrm{e}^{-\lambda(t_1 - a)})(1 - \mathrm{e}^{-\lambda t_2}).
\end{aligned}$$

So, we conclude that X_1 and the increment $X_2 - X_1$ are independent. Furthermore, $\mathbb{P}(X_1 = X_2) = 0$. Thus the Poisson point process is <u>simple</u>: $\mu_\lambda(\{c\}) \in \{0, 1\}$ almost surely.

A.6.2 Characterization of a point process

A set $A \subset \mathbb{R}$ is said to be a <u>locally finite point configuration</u> if $|A \cap B| < \infty$ whenever B is bounded. Here $|\cdot|$ denotes the cardinality of the set. Define P to set of all locally finite point configurations. Then let $\mathcal{P}$ be the smallest σ-algebra generated by the sets

$$\{A \in P : |A \cap B| = m\}, \quad B \in \mathcal{B}, \quad m \in \mathbb{N}.$$

def:pp **Definition 103.** A <u>point process</u> X is a measurable mapping from a probability space to P. The distribution of X is the function $F_{\mathrm{pp},X}(Z) = \mathbb{P}(X \in Z)$ defined on $\mathcal{P}$. The induced random measure is given by $\mu_X(B) = |X \cap B|$ for $B \in \mathcal{B}$.

Two point processes X_1 and X_1 are equal in distribution if $F_{\mathrm{pp},X_1}(Z) = F_{\mathrm{pp},X_2}(Z)$ for all $Z \in \mathcal{P}$. It is an important fact that the distribution of a simple point process is completely determined by its gap probabilities.

Theorem 104 (ref here, supposedly Renyi 1967 but that doesn't completely narrow it down)**.** *The distribution of a simple point process*[4] *X on $\mathbb{R}$ is uniquely determined by its gap probabilities*

$$\mathbb{P}(\mu_X(B) = 0), \quad B \in \mathcal{B}, \quad B \text{ bounded.}$$

[4]This holds more generally on metric spaces.

Appendix B

Elementary differential geometry

app:diff

Differential geometry, and more specifically, integration on manifolds, plays a fairly significant role in this text. We only treat manifolds that are subsets of real or complex vector spaces.

Definition 105. Suppose $\mathbb{S}$ is a normed vector space over $\mathbb{R}$ (or $\mathbb{C}$). A smooth manifold in $\mathbb{S}$ is a subset $\mathbb{M} \subset \mathbb{S}$ equipped with a countable collection of continuous, injective mappings $\varphi_j : \Omega_j \to \mathbb{R}^p$, $j = 1, 2, \ldots$ such that

- $0 \in \varphi_j(\Omega_j)$,
- Ω_j is relatively open for each j,
- $\displaystyle\bigcup_j \Omega_j = \mathbb{M}$, and
- $T_{\ell,j} := \varphi_\ell \circ \varphi_j^{-1}|_{\varphi_j(\Omega_j \cap \Omega_\ell)}$ is a C^∞ transformation whenever $\varphi_j(\Omega_j \cap \Omega_\ell)$ is non-empty.

The collection $((\Omega_j, \varphi_j))_{j \geq 1}$ is called an atlas for $\mathbb{M}$ and p is the dimension of $\mathbb{M}$.

The main tool to allow us to integrate on manifolds is the idea of a partition of unity.

Definition 106. Let $(U_\alpha)_{\alpha \in I}$ be an open cover of a metric space Ω. A partition of unity subordinate to this open cover is a collection of continuous functions $(f_\alpha)_{\alpha \in I}$ such that $\mathrm{supp} f_\alpha \subset U_\alpha$, $f_\alpha(x) \in [0, 1]$ and for each $x \in \Omega$ there exists a neighborhood V of x such that $|\{\alpha : f_\alpha(V) \neq \{0\}\}| < \infty$ and

$$\sum_{\alpha \in I} f_\alpha(x) = 1.$$

A general fact (reference?) is that a smooth manifold $\mathbb{M}$ with atlas $((\Omega_j, \varphi_j))_{j\geq 1}$ possesses partition of unity $(f_j)_{j\geq 1}$ subordinate to $(U_j)_{j\geq 1}$ such that $f_j \circ \varphi_j^{-1}$ is a C^∞ function on $\varphi_j(\Omega_j)$.

B.1 The tangent space

Let $\mathbb{M}$ be a smooth manifold of dimension p. Fix a point $M \in \mathbb{M}$, where $M \in \Omega_j$. Let $\gamma : (-\epsilon, \epsilon) \to \mathbb{M}$ be a curve that passes through M, $\gamma(0) = M$. The curve is said to be differentiable if $\varphi_j \circ \gamma$ is differentiable in the classical sense. Because $\mathbb{S}$ is a normed vector space we can just impose that[1] there exists $\gamma'(t) \in \mathbb{S}$ such that

$$\gamma(s) = \gamma(t) + \gamma'(t)(s-t) + o(|s-t|).$$

Recall

$$g(s) = o(|s|) \quad \text{if} \quad \lim_{s\to 0} \frac{\|g(s)\|}{|s|} = 0,$$

where $\|\cdot\|$ is the norm on $\mathbb{S}$. Two such curves γ_1, γ_2 are equivalent if $\gamma_1'(0) = \gamma_2'(0)$. The tangent space $T_M(\mathbb{M})$ at M is the set of all equivalence classes of differentiable curves passing through M.

Lemma 37.

$$T_M(\mathbb{M}) \cong \{x \in \mathbb{R}^p : \gamma'(0) = x, \quad \gamma \textit{ is a differentiable curve passing through } M\ \},$$

and the latter is a subspace of $\mathbb{R}^p$.

Proof. The congruence is clear. Now, suppose $x, y \in \mathbb{T}_M(\mathbb{M})$ and let $c_1, c_2 \in \mathbb{R}$. Then we have γ_x and γ_y such that $\gamma_x'(0) = x$ and $\gamma_y'(0) = y$. Consider

$$\tilde\gamma(t) = \gamma_x(c_1 t) + \gamma_y(c_2 t),$$

so that $\tilde\gamma'(0) = c_1 x + c_2 y \in T_M(\mathbb{M})$. $\square$

B.2 Metric tensors, Jacobians and integration

There is a natural volume form on each finite-dimensional inner-product space of dimension p. For example, on $\mathbb{R}^p$, the standard inner product defines the metric with infinitesimal length element $\mathrm{d}s^2 = \sum_{j=1}^p \mathrm{d}x_j^2$ and the volume form $Dx = \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_p$ (we follow the notation of [Zub12] for volume forms). More generally, each $g \in \mathsf{Symm}_+(p)$ defines an inner-product and metric on $\mathbb{R}^p$:

$$\langle x, y\rangle_g = \sum_{j,k=1}^p g_{jk} x_j y_k, \quad \mathrm{d}s^2 = \sum_{j,k=1}^p g_{jk} \mathrm{d}x_j \mathrm{d}x_k. \tag{B.2.1}$$ eq:inp

[1] If $\mathbb{S}$ was not a normed vector space we would need to consider the derivative of $\varphi_j \circ \gamma$.

The associated p-dimensional volume form is

$$\mathrm{D}x = \sqrt{\det(g)}\,\mathrm{d}x_1 \ldots \mathrm{d}x_p. \tag{B.2.2}$$

eq:inp2

A smooth manifold $\mathbb{M}$ inherits the topology of the ambient metric space $\mathbb{S}$. Therefore we have a natural Borel σ-algebra $\mathcal{B}_{\mathbb{M}}$ generated by the (relatively) open sets of $\mathbb{M}$. Suppose we have a measure space $(\mathbb{M}, \mathcal{B}_{\mathbb{M}}, \mu)$, then for example, continuous functions $F : \mathbb{M} \to \mathbb{R}$ are measurable and we can define

$$\int_{\mathbb{M}} F \mathrm{d}\mu$$

in the standard measure-theoretic manner. But we wish to do two additional operations:

- Show the existence of such a measure μ using integrals on $\mathbb{R}^p$.
- Express this integral in terms of integrals on $\mathbb{R}^p$.

For a general manifold $\mathbb{M}$ a metric tensor $\mathrm{d}s^2$ is a non-negative bilinear form g_M on $T_M(\mathbb{M}) \times T_M(\mathbb{M})$. Fix a basis $X_1, \ldots, X_p$ for $T_M(\mathbb{M})$, and define, with some abuse of notation, the matrix $g(M) = (g_{jk}(M))_{j,k}$, $g_{jk}(M) = g_M(X_j, X_k)$. This parameterization gives a natural invertible mapping ϕ from $\mathbb{R}^p$ onto $T_M(\mathbb{M})$ given by

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} \overset{\phi}{\mapsto} \sum_j x_j X_j = X.$$

The metric tensor is then written in this basis

$$\mathrm{d}s^2 = \sum_{j,k} g_{jk}(M) \mathrm{d}x_j \mathrm{d}x_k = \langle \mathrm{d}x, \mathrm{d}x \rangle_{g(M)}.$$

Now, let U be a (relatively) open subset of $\mathbb{M}$. Let $\psi : V \to U$ be a bijective diffeomorphism on an open set $V \subset \mathbb{R}^p$. Then if $M = \psi(y)$, the Jacobian $\mathrm{D}\psi(y)$ is an invertible linear transformation from $\mathbb{R}^p$ onto $T_M(\mathbb{M})$ and we write

$$\mathrm{d}s^2 = \langle \mathrm{D}_\phi \psi(y) \mathrm{d}y, \mathrm{D}_\phi \psi(y) \mathrm{d}y \rangle_{g(\psi(y))}. \tag{B.2.3}$$

eq:ydep

where we use $\mathrm{D}_\phi \psi(y)$ to denote the matrix for the linear transformation $\phi^{-1} \circ \mathrm{D}\psi(y) : \mathbb{R}^p \to \mathbb{R}^p$ in the standard basis.

Then (B.2.3) induces the volume form

$$\sqrt{\det(g(\psi(y)))} |\det \mathrm{D}_\phi \psi(y)| \mathrm{d}y_1 \cdots \mathrm{d}y_p.$$

As a consistency check, suppose a different basis $(Z_j)_{j \geq 1}$ is chosen, $X_j = \sum_k \ell_{kj} Z_k$. Then

$$X = \sum_j x_j X_j = \sum_k x_k \sum_j \ell_{jk} Z_j = \sum_j \left(\sum_k \ell_{jk} x_k \right) Z_j = \sum_j z_j Z_j.$$

And then it follows that $z = Lx$ where $L = (\ell_{jk})_{j,k}$. And then

$$\mathrm{d}s^2 = \langle L^{-1}\mathrm{d}z, L^{-1}\mathrm{d}z\rangle_{g(M)} = \langle \mathrm{d}z, \mathrm{d}z\rangle_{L^{-T}g(M)L^{-1}}.$$

We need to change basis for the Jacobian $D\psi(y)$ and consider the matrix for $L\phi^{-1} \circ \mathrm{D}\psi(y)$ and we find

$$\begin{aligned} \mathrm{d}s^2 &= \langle \mathrm{D}_{\phi\circ L^{-1}}\psi(y)\mathrm{d}y, \mathrm{D}_{\phi\circ L^{-1}}\psi(y)\mathrm{d}y\rangle_{L^{-T}g(\psi(y))L^{-1}} \\ &= \langle L\mathrm{D}_{\phi}\psi(y)\mathrm{d}y, L\mathrm{D}_{\phi}\psi(y)\mathrm{d}y\rangle_{L^{-T}g(\psi(y))L^{-1}}. \end{aligned}$$

This produces the same volume form as (B.2.3) eq:ydep.

Let $f : \mathbb{M} \to \mathbb{R}$ be supported in U, we then define

$$\int f\mathrm{d}\mu_g := \int_V f \circ \psi(y)\sqrt{\det(g(\psi(y)))}|\det \mathrm{D}\psi(y)|\mathrm{d}y_1 \cdots \mathrm{d}y_p. \tag{B.2.4}$$

eq:mug

This is well-defined (i.e., independent of the choice of parameterization ψ) because of the standard change-of-variables formula on $\mathbb{R}^p$. By the Riesz Representation Theorem μ_g extends to a measure on the measureable space $(\mathbb{M}, \mathcal{B}_{\mathbb{M}})$.

For the second task we use a partition of unity $(f_j)_{j\geq 1}$ subordinate to an atlas $(\Omega_j, \varphi_j)_{j\geq 1}$ and, just for a matter of simplicity, we assume the atlas is finite[2], $j \in [\![1, q]\!]$. For a measurable function $F : \mathbb{M} \to \mathbb{R}$,

$$F = \sum_{j=1}^{q} F f_j,$$

so that

$$\int_{\mathbb{M}} F\mathrm{d}\mu = \sum_{j=1}^{q} \int_{\Omega_j} F f_j d\mu.$$

Then, define the pushforward of $\mu|_{\Omega_j}$ under φ_j: $\nu_j(B) := \mu(\varphi_j^{-1}(B))$ for a Borel measurable $B \in \varphi_j(\Omega_j) \subset \mathbb{R}^p$. Then

$$\int_{\Omega_j} F f_j d\mu = \int_{\varphi_j(\Omega_j)} F(\varphi_j^{-1}(x)) f_j(\varphi_j^{-1}(x))\nu_j(\mathrm{d}x).$$

B.3 Implicit function theorem

add this, analytic version too

B.4 Lie groups and Haar measure

s:Haar

A Lie group $\mathbb{G}$ is a group that is also a smooth manifold. Furthermore, the group operations must be smooth. That is:

$$g * h \quad g^{-1}h$$

[2] In what follows we will be largely concerned with compact manifolds and this restriction will suffice.

must depend smoothly on $(g, h) \in \mathbb{G} \times G$. In general, definition of smoothness needs to be understood via the manifold's differential structure. Because the Lie groups we consider can be identified with subsets $\mathbb{R}^p$ this extra step is unnecessary.

The associated lie algebra $\mathfrak{g}$ is the tangent space at the identity $\mathfrak{g} := T_{\mathrm{id}}(\mathbb{G})$. Additional topics related to Lie groups and Lie algebras are not needed for the developments in this text. But, due to the importance of Lie groups in the wider mathematical literature we use this notation.

Given a topological group, such as a Lie group, Haar measure is a natural measure determined by the group structure itself. The simplest example is viewing $\mathbb{R}$ as the additive group and then Haar measure coincides with Lebesgue measure. The existence and properties are described in the following theorem originally due to Weil Weil1951 [Wei51] but translated in Nachbin1976 [Nac76].

Theorem 107 (Weil1951 [Wei51])**.** *Let G be a locally compact Hausdorff topological group. Then, up to a unique multiplicative constant, there exists a unique non-trivial Borel measure μ such that*

- *$\mu(gS) = \mu(S)$ for all $g \in G$ and S a Borel set,*
- *μ is countably additive,*
- *$\mu(K) < \infty$ for K compact,*
- *μ is inner regular on open sets and outer regular on Borel sets*[3]*.*

B.5 Examples

ex:polar **Example 108.** The canonical example of a manifold is the sphere in n dimensions, denoted $S^{n-1} \subset \mathbb{R}^n$ consisting of all unit vectors. And, generalized spherical coordinates give $\mathbb{R}^n = \mathbb{R}_+ \times S^{n-1}$. Natural parameterizations are given in standard multivariable calculus courses for S^1 and S^2. We demonstrate the use of metric tensors to compute the change of variables formula for spherical coordinates. The first calculation we perform holds for all n while the actual parametrization we use clearly holds only for $n = 3$. Set

$$x = r\omega, \quad r > 0,\ \omega \in S^{n-1}.$$

Then

$$\mathrm{d}x = \mathrm{d}r\omega + r\mathrm{d}\omega.$$

Using the standard metric on $\mathbb{R}^n$ we have

$$\mathrm{d}s^2 = \sum_j \mathrm{d}x_j^2 = \mathrm{d}r^2 \sum_j \omega_j^2 + 2r\mathrm{d}r \sum_j \omega_j \mathrm{d}\omega_j + r^2 \sum_j \mathrm{d}\omega_j^2.$$

[3] Inner regularity states that any set can be approximated, in measure, by compact subsets. Outer regularity states the same approximation by open supersets.

This is then simplified noting that $\sum_j \omega_j^2 = 1$ and therefore $\sum_j \omega_j \mathrm{d}\omega_j = 0$, giving

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \sum_j \mathrm{d}\omega_j^2.$$

So, we find

$$T_{(r,\omega)} = \mathbb{R} \times \{y \in \mathbb{R}^n : \langle \omega, y \rangle = 0\}.$$

Now for $n = 3$ we parameterize for $0 \leq \theta < 2\pi$, $0 \leq \varphi < \pi$:

$$\begin{aligned}
\omega_1 &= \cos\theta \sin\varphi, \\
\omega_2 &= \sin\theta \sin\varphi, \\
\omega_3 &= \cos\varphi.
\end{aligned}$$

This provides the linear transformation $J(\theta, \varphi)$ onto the tangent space

$$\mathrm{d}\omega = J(\theta,\varphi) \begin{bmatrix} \mathrm{d}\theta \\ \mathrm{d}\varphi \end{bmatrix} = \begin{bmatrix} -\sin\theta\sin\varphi & \cos\theta\cos\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ 0 & -\sin\varphi \end{bmatrix} \begin{bmatrix} \mathrm{d}\theta \\ \mathrm{d}\varphi \end{bmatrix}.$$

The metric tensor becomes

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \left\langle J(\theta,\varphi) \begin{bmatrix} \mathrm{d}\theta \\ \mathrm{d}\varphi \end{bmatrix}, J(\theta,\varphi) \begin{bmatrix} \mathrm{d}\theta \\ \mathrm{d}\varphi \end{bmatrix} \right\rangle.$$

Then using

$$\det J(\theta,\varphi)^T J(\theta,\varphi) = \sin^2\varphi,$$

we find the volume form

$$r^2 \sin\varphi \, \mathrm{d}r \mathrm{d}\theta \mathrm{d}\varphi,$$

as is expected.

Example 109. The orthogonal group, denoted $\mathsf{O}(n) \subset \mathbb{R}^{n\times n} \equiv \mathbb{R}^{n^2}$, is the subset of real $n \times n$ matrices O that satisfy $O^T O = I$. This is clearly a closed, compact subgroup of all invertible matrices and it is also a Lie group. So, there is exists a unique measure left-invariant Haar measure μ normalized such that $\mu(\mathsf{O}(n)) = 1$. Compactness guarantees that this measure is also right-invariant [Fol99].

The tangent space $\mathfrak{o}(n)$ at the identity, i.e., the Lie algebra, is computed in Chapter 2. It is given by the subspace of all $n \times n$ skew-symmetric matrices

$$\mathfrak{o}(n) \cong \{A \in \mathbb{R}^{n\times n} \mid A^T = -A\}.$$

Furthermore, for $O \in \mathsf{O}(n)$

$$T_O \mathsf{O}(n) \cong O\mathfrak{o}(n).$$

One can compute the associated metric tensor by restricting the standard metric on $\mathbb{R}^{n^2}$: For $X = (x_{jk}) \in \mathbb{R}^{n\times n}$ we map it to $\mathsf{O}(n)$ by performing Gram-Schmidt on the columns, i.e. the QR factorization of X. The standard metric can be written as

$$\mathrm{d}s^2 = \sum_{j,k} \mathrm{d}x_{j,k}^2 = \mathrm{Tr}\left(\mathrm{d}X^T \mathrm{d}X\right).$$

Then we write $X = OR$ which implies $\mathrm{d}X = \mathrm{d}OR + O\mathrm{d}R = O\mathrm{d}AR + O\mathrm{d}R = O(\mathrm{d}AR + \mathrm{d}R)$. We then artificially set $R = I$ and $\mathrm{d}R = 0$ in the same way one sets $r = 1$ and $\mathrm{d}r = 0$ to find the measure induced on S^{n-1} in the previous example, up to a possible normalization constant. So,

$$\mathrm{d}s^2 \propto \mathrm{Tr}\,\mathrm{d}A^T\mathrm{d}A = -\,\mathrm{Tr}\,\mathrm{d}A^2.$$

This gives the metric tensor on $\mathsf{O}(n)$ in induced from Lebesgue measure on $\mathbb{R}^{n^2}$. It turns out that this gives the correct volume form for Haar measure on $\mathsf{O}(n)$ using (eq:mug B.2.4).

ex:dirichlet **Example 110.** Consider the problem of determining the density of $Z = \sqrt{X}$ where $X \in \mathbb{R}^n$ has the β-Dirichlet distribution. We consider the entrywise square root of X so that $Z \in S^{n-1}$. We use the Gaussian distribution on $\mathbb{R}^n$ as a proxy to compute this density. We say in Example ex:polar 108 that the standard metric tensor on $\mathbb{R}^n$ induces a volume form on S^{n-1} with associated probability measure σ_{n-1}. Transforming the n-dimensional standard normal density

Is $\mathrm{D}\sigma_{n-1}$ really the notation we want to use here?

$$\frac{1}{(2\pi)^{n/2}}\mathrm{e}^{-\frac{1}{2}\sum_j x_j^2}\mathrm{D}x = \frac{1}{2^{n/2-1}\Gamma\left(\frac{n}{2}\right)} r^{n-1}\mathrm{e}^{-\frac{r^2}{2}}\mathrm{d}r\mathrm{D}\sigma_{n-1}(\omega).$$

Often, in this text, we write $\mathrm{D}\omega$ in place of $\mathrm{D}\sigma_{n-1}(\omega)$ when its meaning is clear from context. Next, we consider what the transformation $\omega \mapsto |\omega| := \tau(\omega)$, where the absolute value is applied elementwise, does to the measure σ_{n-1}. The induced measure σ_{n-1}^+ is just the pushforward of σ_{n-1}^+ under τ: $\sigma_{n-1}^+(S) = \sigma_{n-1}(\tau^{-1}(S)) = 2^n\sigma_{n-1}(S)$ for $S \subset S^{n-1}$, contained in the positive orthant S_+^{n-1} of S^{n-1}. We refer to σ_{n-1}^+ as uniform measure on S_+^{n-1}. As with σ_{n-1} sometimes we will write $\mathrm{D}\omega$ in place of $\mathrm{D}\sigma_{n-1}^-(\omega)$. Then a vector $Y \in \mathbb{R}^n$ of independent χ_β random variables has joint density

$$\frac{1}{2^{(n\beta)/2-n}\Gamma\left(\frac{\beta}{2}\right)^n}\left(\prod_{j=1}^n y_j^{\beta-1}\right)\mathrm{e}^{-\frac{1}{2}\sum_j y_j^2}\mathbb{1}_{\{y_j>0 \text{ for all } j\}}.$$

Define a new vector X by $X_j = \delta_j Y_j$ where the iid random variables δ_j, independent of Y, satisfy $\delta_j = \pm 1$ each with probability $1/2$. Then X has joint density

$$\frac{1}{2^{(n\beta)/2}\Gamma\left(\frac{\beta}{2}\right)^n}\left(\prod_{j=1}^n |x_j|^{\beta-1}\right)\mathrm{e}^{-\frac{1}{2}\sum_j x_j^2}$$

giving the relation ($x_j = r\omega_j$)

$$\begin{aligned}
&\frac{1}{2^{(n\beta)/2}\Gamma\left(\frac{\beta}{2}\right)^n}\left(\prod_{j=1}^n |x_j|^{\beta-1}\right)\mathrm{e}^{-\frac{1}{2}\sum_j x_j^2}\mathrm{D}x \\
&= \frac{(2\pi)^{n/2}}{2^{(n\beta)/2}\Gamma\left(\frac{\beta}{2}\right)^n 2^{n/2-1}\Gamma\left(\frac{n}{2}\right)} r^{\beta n-1}\mathrm{e}^{-\frac{1}{2}r^2}\left(\prod_{j=1}^n |\omega_j|^{\beta-1}\right)\mathrm{d}r\mathrm{D}\sigma_{n-1}(\omega).
\end{aligned}$$

And then the distribution of $Z \sim |Y|/\|Y\|_2$ is given by the joint density (with respect to σ_{n-1}) of $|\omega|$. The following gives the joint density of $|Y|$ in polar coordinates

$$\frac{\pi^{n/2}}{2^{(n\beta)/2-n-1}\Gamma\left(\frac{\beta}{2}\right)^n\Gamma\left(\frac{n}{2}\right)} r^{\beta n-1}\mathrm{e}^{-\frac{1}{2}r^2}\left(\prod_{j=1}^n \omega_j^{\beta-1}\right)\mathrm{d}r\mathrm{D}\sigma_{n-1}(\omega)\mathbb{1}_{\{\omega_j>0 \text{ for all } j\}}.$$

Integrating out the r variable we obtain the joint density for Z

$$\frac{2^n\pi^{n/2}\Gamma\left(\frac{n\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)^n\Gamma\left(\frac{n}{2}\right)}\left(\prod_{j=1}^n \omega_j^{\beta-1}\right)\mathrm{D}\sigma_{n-1}(\omega)\mathbb{1}_{\{\omega_j>0 \text{ for all } j\}}.$$

Appendix C

The Airy function

app:airy

C.1 Integral representation

There are several different conventions for the definition of the Airy function. The standardization adopted here follows [AS72]. The Airy function, $\mathrm{Ai}(x)$ is defined as the oscillatory integral

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\,\mathrm{d}t = \frac{1}{\pi}\lim_{b\to\infty}\int_0^b \cos\left(\frac{t^3}{3} + xt\right)\,\mathrm{d}t. \tag{C.1.1}$$

eq:airy1

This is an improper integral, that is, the integral converges conditionally, not absolutely. In order to obtain an absolutely convergent integral, it is necessary to work in the complex plane. Let C denote a contour in the complex plane that starts and ends at the point at infinity, and is asymptotically tangent to the rays $e^{-i\pi/3}$ and $e^{+i\pi/3}$ respectively. Then first setting $t = -iz$ and then deforming the contour, we have

$$\mathrm{Ai}(x) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\left(\frac{z^3}{3} - xz\right)}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{e}^{\mathrm{i}\left(\frac{z^3}{3} - xz\right)}\,\mathrm{d}z. \tag{C.1.2}$$

eq:airy2

The integral is absolutely convergent for every $x \in \mathbb{C}$ on the contour C. Indeed, with $z = re^{i\theta}$,

$$\left|\mathrm{e}^{\mathrm{i}\left(\frac{z^3}{3} - xz\right)}\right| \leq \mathrm{e}^{|x|r}\mathrm{e}^{-r^3\cos(3\theta)/3} \sim \mathrm{e}^{-r^3/3}\mathrm{e}^{r|x|} \tag{C.1.3}$$

eq:airy3

as $z \to \infty$ along the rays $\theta = \pm\pi/3$. Thus, $\mathrm{Ai}(x)$ is an entire function.

C.2 Differential equation

We differentiate under the integral sign (justified by (eq:airy3 C.1.3)) and integrate by parts to obtain

$$\begin{aligned}\mathrm{Ai}''(x) &= \frac{1}{2\pi i}\int_C z^2 e^{\left(\frac{z^3}{3}-xz\right)}\,dz \qquad \text{(C.2.1)}\\ &= \frac{1}{2\pi i}\int_C \frac{d}{dz}e^{\frac{z^3}{3}}\,e^{-xz}\,dz = -\frac{1}{2\pi i}\int_C e^{\frac{z^3}{3}}\,\frac{d}{dz}e^{-xz}\,dz = x\mathrm{Ai}(x).\end{aligned}$$

Thus, $\mathrm{Ai}(x)$ satisfies the Airy differential equation

$$y'' = xy, \quad x \in \mathbb{C}. \tag{C.2.2}$$

eq:airy5

This differential equation has a scaling invariance: if $y(x)$ is a solution, so are $y(\omega x)$ and $y(\omega^2 x)$ where $\omega = e^{2\pi i/3}$ is a cube root of unity. Thus, both $\mathrm{Ai}(\omega x)$ and $\mathrm{Ai}(\omega^2 x)$ solve (eq:airy5 C.2.2). Each of these solutions is linearly independent of $\mathrm{Ai}(x)$. A solution to (eq:airy5 C.2.2) that is real when x is real, and is linearly independent from $\mathrm{Ai}(x)$, is obtained from the linear combination

$$\mathrm{Bi}(x) = e^{\pi i/6}\mathrm{Ai}(\omega x) + e^{\pi - i/6}\mathrm{Ai}(\omega^2 x). \tag{C.2.3}$$

eq:airy6

C.3 Asymptotics

The functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ have the following asymptotic properties.

Asymptotics as $x \to \infty$.

$$\begin{gathered}\zeta = \frac{2}{3}x^{\frac{3}{2}}, \quad \mathrm{Ai}(x) \sim \frac{e^{-\zeta}}{2x^{\frac{1}{4}}\sqrt{\pi}}, \quad \mathrm{Bi}(x) \sim \frac{1}{x^{\frac{1}{4}}\sqrt{\pi}}e^{\zeta},\\ \mathrm{Ai}'(x) \sim -\frac{x^{\frac{1}{4}}e^{-\zeta}}{2\sqrt{\pi}}, \quad \mathrm{Bi}'(x) \sim \frac{x^{\frac{1}{4}}}{\sqrt{\pi}}e^{\zeta}\end{gathered} \tag{C.3.1}$$

eq:airy7

Asymptotics as $x \to -\infty$.

$$\begin{gathered}\zeta = \frac{2}{3}(-x)^{\frac{3}{2}}, \quad \mathrm{Ai}(x) \sim \frac{1}{(-x)^{\frac{1}{4}}\sqrt{\pi}}\sin\left(\zeta + \frac{\pi}{4}\right), \quad \mathrm{Bi}(x) \sim \frac{1}{(-x)^{\frac{1}{4}}\sqrt{\pi}}\cos\left(\zeta + \frac{\pi}{4}\right),\\ \mathrm{Ai}'(x) \sim \frac{x^{\frac{1}{4}}}{\sqrt{\pi}}\cos\left(\zeta + \frac{\pi}{4}\right), \quad \mathrm{Bi}'(x) \sim -\frac{x^{\frac{1}{4}}}{\sqrt{\pi}}\sin\left(\zeta + \frac{\pi}{4}\right).\end{gathered} \tag{C.3.2}$$

eq:airy8

Appendix D

Fredholm determinants

app:fredholm

D.1 Definitions

Our purpose in this section is to explain the notion of a Fredholm determinant and resolvent in a simple and concrete setting. The ideas presented here originated in Fredholm's attempt to find a solution formula akin to Cramer's rule for linear integral equations. The notion of a determinant for an infinite-dimensional linear operator is, of course, of independent interest and has attracted the interest of many mathematicians. Simon's book provides an excellent overview of current knowledge [Sim05].

Assume a given continuous kernel $K : [0,1] \times [0,1] \to \mathbb{R}$ and a continuous function $h : [0,1] \to \mathbb{R}$. Fix a spectral parameter $z \in \mathbb{C}$ and consider the linear integral equation

$$\varphi(x) - z \int_0^1 K(x,y)\varphi(y)\,\mathrm{d}y = h(x), \quad x \in [0,1]. \tag{D.1.1}$$

eq:ie-fred1

The integral equation (D.1.1) may be written in the more compact form

$$(\mathbf{1} - zK)\varphi = h, \tag{D.1.2}$$

eq:ie-fred2

where $I - zK$ denotes the bounded linear operator on $L^2([a,b])$ defined by

$$\varphi \mapsto (\mathbf{1} - zK)\varphi, \quad (\mathbf{1} - zK)\varphi(x) = \varphi(x) - z \int_a^b K(x,y)\varphi(y)\,\mathrm{d}y \quad x \in [a,b]. \tag{D.1.3}$$

eq:ie-fred3

Integral equations such as (D.1.1) may naturally be viewed as continuum limits of linear equations. More precisely, we fix a positive integer n, consider a uniform grid $x_j = j/n$, with uniform weights $w_j = 1/n$, define the vector $h_j^{(n)} = h(x_j)$, matrix $K_{j,k}^{(n)} = w_j K(x_j, x_k)$, $1 \le j,k \le n$ and discretize (D.1.1) by the linear equation

$$\varphi_j^{(n)} - z \sum_{k=1}^{n} K_{j,k}^{(n)} \varphi_k^{(n)} = h_j^{(n)}, \quad 1 \le j \le n. \tag{D.1.4}$$

eq:ie-fred4

Equation (eq:ie-fred4 D.1.4) has a unique solution if and only if $\det(I_n - zK^{(n)}) \neq 0$. By linearity, the solution for arbitrary $h^{(n)}$ is determined by the resolvent $R^{(n)} = (\mathbf{1}_n - zK^{(n)})^{-1}$, which is given by Cramer's rule.

Remark 111. If one wants to compute a Fredholm determinant numerically and K is a smooth function, quadrature rules (such as Gaussian quadrature or Clenshaw–Curtis quadrature) can be used to choose x_j and w_j. See, for example, [BornemannFredholm Bor10].

$$R^{(n)}_{j,k} = (-1)^{j+k}\frac{\det(M_{jk})}{\det(\mathbf{1}_n - zK^{(n)})}, \tag{D.1.5}$$ eq:ie-fred7

where M_{jk} denotes the matrix obtained from $I_n - zK^{(n)}$ by removing the j-th row and k-th column. Further, if z_j, $j = 1, \dots n$, denote the zeros of the polynomial $\det(I_n - zK^{(n)})$, the eigenvalues of $K^{(n)}$ are given by $1/z_j$. Both these notions may be extended to (eq:ie-fred1 D.1.1) via the Fredholm determinant. The basic observation that allows passage to the limit is the identity

$$\det(\mathbf{1}_n - zK^{(n)}) = \tag{D.1.6}$$
$$1 - \frac{z}{n}\sum_{j_1=1}^{n} K(x_{j_1}, x_{j_1}) + \frac{z^2}{2!}\frac{1}{n^2}\sum_{j_1,j_2=1}^{n}\begin{vmatrix} K(x_{j_1}, x_{j_1}) & K(x_{j_1}, x_{j_2}) \\ K(x_{j_2}, x_{j_1}) & K(x_{j_2}, x_{j_2})\end{vmatrix} + \dots$$

The coefficient of z^k in the expansion above may be computed by differentiating the left hand side k times with respect to z, and setting $z = 0$. Since K is continuous, as $n \to \infty$, the k-th term in the sum above converges to the integral

$$\frac{(-z)^k}{k!}\int_{[0,1]^k} \det\left(K(x_p, x_q)_{1\le p,q\le k}\right) \mathrm{d}x_1 \dots \mathrm{d}x_k. \tag{D.1.7}$$ eq:ie-fred6

thm:def-fred **Definition-Theorem 112.** The Fredholm determinant of the operator $\mathbf{1} - zK$ is the entire function of z defined by the convergent series

$$D(z) = \det\left(\mathbf{1} - zK\right) := 1 + \sum_{k=1}^{\infty}\frac{(-z)^k}{k!}\int_{[0,1]^k}\left(\det(K(x_p, x_q)_{1\le p,q\le k})\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k. \tag{D.1.8}$$ eq:def-fred1

Proof. It is only necessary to show that the series(eq:ie-fred6 D.1.7) is convergent for all $z \in \mathbb{C}$. The determinant of a $k \times k$ matrix A with columns $a_1, \dots, a_k$ is the (signed) volume of the parallelopiped spanned by the vectors $a_1, \dots, a_k$. Therefore,

$$|\det(A)| \le |a_1||a_2|\cdots|a_k| \le \left(\max_{1\le j\le k}|a_j|\right)^k. \tag{D.1.9}$$ eq:ie-fred8

We have assumed that K is bounded on $[0,1]\times[0,1]$, say $\max|K| \le M < \infty$. By the inequality above,

$$|(\det(K(x_p, x_q)_{1\le p,q\le k}))| \le k^{k/2}M^k. \tag{D.1.10}$$ eq:ie-fred9

Thus, the k-term in the series (D.1.8) is dominated by

$$\left|\frac{(-z)^k}{k!}\int_{[0,1]^k}\det\left[K(x_p,x_q)\right]_{1\le p,q\le k}\,\mathrm{d}x_1\cdots\mathrm{d}x_k\right|$$
$$\le (|z|M)^k\frac{k^{k/2}}{k!}\sim\frac{1}{\sqrt{2\pi}}(|z|Me)^k\frac{1}{k^{\frac{k+1}{2}}},$$

where we have used Stirling's approximation in the last step. □

Remark 113. If $[0,1]$ is replaced by a general Borel set S, we assume

$$|K(x,y)|\le M(x),$$

where $M\in L^1(S)$. The same statements about the determinant follow.

Since $D(z)$ is entire, we may differentiate term-by-term to obtain

$$\left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\det(\mathbf{1}-zK) \tag{D.1.11}$$
$$=\sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\int_{[0,1]^{m+k}}\det\left[K(x_p,x_q)\right]_{1\le p,q\le m+k}\,\mathrm{d}x_1\cdots\mathrm{d}x_{m+k}$$

for $m\ge 1$. Recall that the zeros of a non-zero entire function form a discrete, countable set. The entire function $\det(\mathbf{1}-\lambda^{-1}K)$ is an infinite-dimensional generalization of the characteristic polynomial of the matrix $K^{(n)}$ in the following sense:

thm:invertibility

Theorem 114 (Eigenvalues of K)**.** *Assume that K is a continuous kernel. The complex number λ is an eigenvalue of K if and only if $D(\lambda^{-1})=0$.*

In certain situations it is convenient to use the equivalent notation

$$\det\left(\mathbf{1}-zK|_{L^2(S)}\right):=\det\left(\mathbf{1}-zK\mathbb{1}_S\right)=\det\left(\mathbf{1}-zK\right).$$

Here the last determinant is only unambiguous if $K:S\times S\to\mathbb{R}$ and S is clear from context. This presents a dichotomy in the theory of Fredholm determinants – a kernel-focused theory and an operator focused theory. The kernel-focused theory defines the determinant using the kernel function as in Definition-Theorem 112. The operator-focused approach defines the determinant in terms of the operator $\mathcal{K}=K|_{L^2(S)}$ defined by $\mathcal{K}f(x)=\int_S K(\cdot,y)f(y)\mathrm{d}y$ posed on $L^2(S)$. By making appropriate definitions one can avoid referring to the kernel function, see [Sim05]. The theories are equivalent when they both apply.

We will need to extend the notion of a determinant to systems of integral equations and we do this with the operator-focused notation. Consider the integral operator on $\bigoplus_{\ell=1}^N L^2(I_\ell)$ for intervals $I_\ell\subset\mathbb{R}$

$$\mathcal{K}\begin{bmatrix}f_1\\ \vdots\\ f_N\end{bmatrix}=\begin{bmatrix}\sum_{\ell=1}^N\mathcal{K}_{1,\ell}f_\ell\\ \vdots\\ \sum_{\ell=1}^N\mathcal{K}_{N,\ell}f_\ell\end{bmatrix}$$

where

$$\mathcal{K}_{j,\ell} g = \int_{I_\ell} K_{j,\ell}(x,y) g(y) \mathrm{d}y$$

for a continuous kernel $K_{j,\ell} : I_j \times I_\ell \to \mathbb{R}$, $g \in L^2(I_\ell)$. It is then clear that $\mathcal{K}$ is completely determined by its kernel matrix

$$\mathcal{K} \leftrightarrow K := \begin{bmatrix} K_{1,1} & \cdots & K_{1,N} \\ \vdots & \ddots & \vdots \\ K_{N,1} & \cdots & K_{N,N} \end{bmatrix}.$$

Then define Bornemann [?] (or another reference?)

$$\det(\mathbf{1} - \mathcal{K}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{i_1,\ldots,i_k=1}^{N} \int_{I_{i_1} \times \cdots \times I_{i_k}} \det \left[K_{i_j, i_\ell}(x_j, x_\ell) \right]_{1 \le j, \ell \le k} \mathrm{d}x_1 \cdots \mathrm{d}x_k. \tag{D.1.12}$$

eq:mat-fred

We remark that this description can be captured in the following notation

$$\det(\mathbf{1} - K|_{\bigoplus_{\ell=1}^{N} L^2(I_\ell)}) = \det(I - \mathcal{K}).$$

For more on Fredholm determinants, see Lax [Lax02, Ch.24] and Simon [Sim05].

D.2 Convergence

sec:fdconv

Suppose a kernel $K_n(x,y) \to K_\infty(x,y)$, $(x,y) \in S^2$, pointwise. One needs the additional convergence criteria to conclude

$$\det(\mathbf{1} - K_n) \to \det(\mathbf{1} - K_\infty). \tag{D.2.1}$$

The following are from Simon [Sim05] using the operator-focused approach. Let $\mathcal{K}_n$ and $\mathcal{K}_\infty$ be the operators on $L^2(S)$ with kernels K_n and K_∞, respectively. Then the trace norm of an operator $\mathcal{K}$ is given by

$$\|\mathcal{K}\|_{\mathrm{Tr}} = \mathrm{Tr}\, \sqrt{\mathcal{K}^* \mathcal{K}}, \tag{D.2.2}$$

where $\mathcal{K}^*$ is the adjoint of $\mathcal{K}$. The general definition of $\sqrt{\mathcal{K}^* \mathcal{K}}$ for general operators is unimportant for us and an operator with finite trace norm is said to be trace class. But, for example, if $\mathcal{K}$ is a non-negative self-adjoint operator with continuous kernel K then

$$\|\mathcal{K}\|_{\mathrm{Tr}} = \int_S K(x,x) \mathrm{d}x. \tag{D.2.3}$$

Alternatively, one can use the definition that for a self-adoint operator $\mathcal{K}$ on $L^2(S)$,

$$\mathrm{Tr}\, \mathcal{K} = \sum_k \langle \mathcal{K} e_k, e_k \rangle, \tag{D.2.4}$$

eq:trace-basis

for any orthonormal basis $(e_k)_{k \ge 1}$ of $L^2(S)$.

Theorem 115. *The map $\mathcal{K} \mapsto \det(\mathbf{1}+\mathcal{K})$ is a continuous function on the space of trace-class operators (i.e. operators with $\|\mathcal{K}\|_{\mathrm{Tr}} < \infty$) and*

$$|\det(\mathbf{1}+\mathcal{K}) - \det(\mathbf{1}+\mathcal{L})| \le \|\mathcal{K}-\mathcal{L}\|_{\mathrm{Tr}} \exp(\|\mathcal{K}\|_{\mathrm{Tr}} + \|\mathcal{L}\|_{\mathrm{Tr}} + 1). \tag{D.2.5}$$

Theorem 116. *Suppose $\mathcal{K}_n, \mathcal{K}$ are trace class. If $\mathcal{K}_n \to \mathcal{K}, |\mathcal{K}_n| \to |\mathcal{K}|$ and $|\mathcal{K}_n^*| \to |\mathcal{K}^*|$ all weakly, then $\|\mathcal{K}_n - \mathcal{K}\|_{\mathrm{Tr}} \to 0$.*

In our cases, $|\mathcal{K}_n| = \mathcal{K}_n = |\mathcal{K}_n^*|$, so to show that $\det(I-K_n) \to \det(I-K)$ it suffices to show for each $f, g \in L^2(S)$ that

$$\int_S \int_S K_n(x,y) f(x) g(y) \mathrm{d}x\mathrm{d}y \to \int_S \int_S K(x,y) f(x) g(y) \mathrm{d}x\mathrm{d}y. \tag{D.2.6}$$

Two such conditions for this to occur are

1. If S is bounded then
$$\sup_{x,y\in S} |K_n(x,y) - K(x,y)| \to 0. \tag{D.2.7}$$

2. If S is unbounded then we require
$$K_n(x,y) \to K(x,y), \tag{D.2.8}$$
for each $x, y \in S$ and there exists $G(x,y) \in L^2(S^2)$ such that $|K_n(x,y)| \le G(x,y)$. This allows one to use the dominated convergence theorem.

D.2.1 Change of variables and kernel extension

sec:cov

Let $K : S^2 \to \mathbb{R}$ be a kernel. Let $x = r(t)$ and $y = r(s)$ for $s, t \in T$ where r' exists, is continuous and does not vanish. Define

$$\hat{K}(s,t) = \frac{1}{r'(s)} K\left(r(s), r(t)\right), \quad s, t \in T^2. \tag{D.2.9}$$

Then

$$\det(\mathbf{1} - K) = \det(\mathbf{1} - \hat{K}). \tag{D.2.10}$$

D.3 Separable kernels

Appendix E

Classical orthogonal polynomials

app:orthpoly
app:orthpoly

In this appendix we consider detailed properties of Hermite and Laguerre polynomials.

E.1 Hermite polynomials

In this section, μ denotes the measure

$$\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}\, \mathrm{d}x. \tag{E.1.1}$$ eq:hermite0

The (probablilists') Hermite polynomials $\{\mathfrak{h}_k\}_{k=0}^{\infty}$ are the monic family of polynomials of degree k orthogonal with respect to the weight μ.

E.1.1 Basic formulas

$$\mathfrak{h}_k(x) = \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^k e^{\frac{-x^2}{2}}. \tag{E.1.2}$$ eq:hermite1

$$\mathfrak{h}_k(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-\mathrm{i}\xi)^k \mathrm{e}^{-\frac{1}{2}(\xi - \mathrm{i}x)^2}\, \mathrm{d}\xi. \tag{E.1.3}$$ eq:hermite-int

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathfrak{h}_k(x)\mathfrak{h}_l(x) e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi} k! \delta_{kl}. \tag{E.1.4}$$ eq:hermite2

$$x\mathfrak{h}_k(x) = \mathfrak{h}_{k+1}(x) + k\mathfrak{h}_{k-1}(x), \quad k \geq 1. \tag{E.1.5}$$ eq:hermite3

$$\mathfrak{h}_k'(x) = k\mathfrak{h}_{k-1}(x). \tag{E.1.6}$$ eq:hermite4

$$\mathfrak{h}_k''(x) - x\mathfrak{h}_k'(x) + k\mathfrak{h}_k(x) = 0. \tag{E.1.7}$$

eq:hermite5

$$\sum_{j=0}^{k-1} \frac{1}{j!}\mathfrak{h}_j(x)\mathfrak{h}_j(y) = \frac{(\mathfrak{h}_k(x)\mathfrak{h}_{k-1}(y) - \mathfrak{h}_{k-1}(x)\mathfrak{h}_k(y))}{(k-1)!(x-y)}, \quad x \neq y. \tag{E.1.8}$$

eq:hermite6

Relation (E.1.2) may be treated as an alternate definition of the Hermite polynomials. On the other hand, since we have defined the Hermite polynomials as the monic orthogonal polynomials obtained by applying the Gram-Schmidt procedure to the set $\{1, x, x^2, \ldots\}$ in $L^2(\mathbb{R}, \mu)$, equation (E.1.2) may be verified as follows. First, it is clear from (E.1.2) that $\mathfrak{h}_k(x)$ is a monic polynomial of degree k and that $\mathfrak{h}_0(x) = 1$, $\mathfrak{h}_1(x) = x$. By induction, if it has been established that property (E.1.2) defines the Hermite polynomials for $j \leq k-1$, then it is only necessary to show that the monic polynomial

$$P_k(x) = \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \mathrm{e}^{-\frac{x^2}{2}},$$

is the same as $\mathfrak{h}_k$. The polynomial P_k is orthogonal to $\mathfrak{h}_j$, $0 \leq j \leq k-1$ because, using integration by parts,

$$\int_{\mathbb{R}} P_k(x)\mathfrak{h}_j(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \mathfrak{h}_j(x)\mu(\mathrm{d}x) = 0,$$

since H_j has degree less than k. Since P_k is monic, it must be $\mathfrak{h}_k$. The same calculation serves to establish (E.1.4).

The integral representation (E.1.3) follows from the formula for the Fourier transform of a Gaussian

$$\mathrm{e}^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi x} \mathrm{e}^{-\frac{\xi^2}{2}}\, \mathrm{d}\xi, \tag{E.1.9}$$

eq:hermite-int2

and the identity (E.1.2).

The two-term recurrence relation follows from (3.4.18) and (E.1.4) (see also Remark 30). The coefficient a_k vanishes because equation (E.1.2) shows that $\mathfrak{h}_k^2$ is an even polynomial for all k. The coefficient b_k^2 may be rewritten

$$b_k^2 = \frac{\int x\mathfrak{h}_{k-1}(x)\mathfrak{h}_k(x)\mu(\mathrm{d}x)}{\int \mathfrak{h}_{k-1}^2\mu(\mathrm{d}x)} = \frac{\int x\mathfrak{h}_{k-1}(x)\mathfrak{h}_k(x)\mu(\mathrm{d}x)}{\int \mathfrak{h}_k^2\mu(\mathrm{d}x)} \frac{\int \mathfrak{h}_k^2(x)\mu(\mathrm{d}x)}{\int \mathfrak{h}_{k-1}^2\mu(\mathrm{d}x)} = 1 \cdot k^2, \tag{E.1.10}$$

eq:hermite7

by (E.1.4).

The differential equation (E.1.6) is obtained by rewriting (E.1.2) in the form

$$\mathrm{e}^{-\frac{x^2}{2}}\mathfrak{h}_k(x) = (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \mathrm{e}^{-\frac{x^2}{2}},$$

differentiating both sides, and then multiplying by $\mathrm{e}^{\frac{x^2}{2}}$. Equation (E.1.7) is obtained by differentiating (E.1.6) and using (E.1.5). The proof of the Christoffel-Darboux identity is left as an exercise to the reader.

E.1.2 Hermite wave functions

The Hermite wave functions $\{\psi\}_{k=0}^{\infty}$ are defined by

$$\psi_k(x) = \frac{1}{\sqrt{k!}} \frac{e^{-x^2/4}}{(2\pi)^{1/4}} \mathfrak{h}_k(x), \quad k = 0, 1, 2, \ldots \tag{E.1.11}$$

eq:hwave1

The following properties of the Hermite wave-functions follow immediately from the corresponding properties of the Hermite polynomials.

$$\int_{\mathbb{R}} \psi_k(x)\psi_l(x)\, dx = \delta_{kl}. \tag{E.1.12}$$

eq:hwave2

$$x\psi_k(x) = \sqrt{k+1}\psi_{k+1}(x) + \sqrt{k}\psi_{k-1}(x). \tag{E.1.13}$$

eq:hwave3

$$\psi_k'(x) = -\frac{x}{2}\psi_k(x) + \sqrt{k}\psi_{k-1}(x). \tag{E.1.14}$$

eq:hwave4

$$\psi_k''(x) + \left(k + \frac{1}{2} - \frac{x^2}{4}\right)\psi_k(x) = 0. \tag{E.1.15}$$

eq:hwave4b

$$\sum_{k=0}^{n-1} \psi_k(x)\psi_k(y) = \sqrt{n}\frac{(\psi_n(x)\psi_{n-1}(y) - \psi_{n-1}(x)\psi_n(x))}{x - y}. \tag{E.1.16}$$

eq:hwave5

E.1.3 Small x asymptotics

The following classical formulas capture the asymptotics of the Hermite polynomials near the origin [AS72, §22.15].

$$\lim_{n\to\infty} \frac{(-1)^n}{2^n}\frac{\sqrt{n}}{n!}\,\mathfrak{h}_{2n}\left(\frac{x}{\sqrt{2n}}\right) = \frac{1}{\sqrt{\pi}}\cos x. \tag{E.1.17}$$

eq:her-asympt1

$$\lim_{n\to\infty} \frac{(-1)^n}{2^n\, n!}\,\mathfrak{h}_{2n+1}\left(\frac{x}{\sqrt{2n}}\right) = \sqrt{\frac{2}{\pi}}\sin x. \tag{E.1.18}$$

eq:her-asympt2

Further, the convergence to the limit is uniform over x in a bounded interval.

In comparing equations (E.1.17) and (E.1.18) with a standard reference such as [AS72], the reader should note that there are two conventions in the definition of Hermite polynomials. The exponential weight in earlier sources was chosen to be e^{-x^2}, which differs from our choice (E.1.1). The relation between the Hermite polynomials, $\{H_n(x)\}$ in [AS72], and those used here are:

$$H_n(x) = 2^{\frac{n}{2}}\,\mathfrak{h}_n(x\sqrt{2}), \quad \mathfrak{h}_n(x) = 2^{-\frac{n}{2}}\,H_n\left(\frac{x}{\sqrt{2}}\right). \tag{E.1.19}$$

eq:hermite-convention

These formulas may be immediately translated into asymptotic formulas for the Hermite wave functions, using Stirling's approximation for the factorial:

$$n! = \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n (1 + O(n^{-1})) \text{ as } n \to \infty. \tag{E.1.20}$$ $\boxed{\texttt{eq:stirling}}$

$$\lim_{n\to\infty} (2n)^{1/4}(-1)^n\, \psi_{2n}\left(\frac{x}{\sqrt{2n}}\right) = \frac{1}{\sqrt{\pi}}\cos x. \tag{E.1.21}$$ $\boxed{\texttt{eq:her-asympt3}}$

$$\lim_{n\to\infty} (2n)^{1/4}(-1)^n\psi_{2n+1}\left(\frac{x}{\sqrt{2n}}\right) = \frac{1}{\sqrt{\pi}}\,\sin x. \tag{E.1.22}$$ $\boxed{\texttt{eq:her-asympt4}}$

The asymptotic formulas (E.1.17) and (E.1.18) are proved by applying Laplace's method to the integral formula (E.1.3). We only explain how to prove (E.1.17) since equation (E.1.18) is similar. Since $(\mathrm{i})^{2n} = (-1)^n$, we take the real part of (E.1.3) to find

$$\begin{aligned}(-1)^{2n}\mathfrak{h}_{2n}\left(\frac{x}{\sqrt{2n}}\right) &= \sqrt{\frac{2}{\pi}}\,\mathrm{e}^{\frac{x^2}{4n}}\int_0^\infty \xi^{2n}\mathrm{e}^{-\frac{\xi^2}{2}}\cos\left(\frac{x\xi}{\sqrt{2n}}\right)\mathrm{d}\xi\\ &= \frac{2^{n+1}n^{n+\frac{1}{2}}}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-n\left(t^2-2\log t\right)}\cos xt\,\mathrm{d}t, \end{aligned}\tag{E.1.23}$$

by rescaling $\xi = \sqrt{n}\,t$. We now apply Laplace's method to the integral above. The function $g(t) = t^2 - 2\log t$ has a single minimum on the interval $(0,\infty)$ at $t = 1$. At this point

$$g(1) = 1, \quad g'(1) = 0, \quad g''(1) = 4. \tag{E.1.24}$$ $\boxed{\texttt{eq:her-asympt6}}$

Laplace's approximation now yields

$$\int_0^\infty \mathrm{e}^{-ng(t)}\cos xt\,\mathrm{d}x \sim \mathrm{e}^{-n}\sqrt{\frac{\pi}{2n}}\cos x, \tag{E.1.25}$$ $\boxed{\texttt{eq:her-asympt7}}$

which when combined with (E.1.23) implies

$$(-1)^{2n}\mathfrak{h}_{2n}\left(\frac{x}{\sqrt{2n}}\right) \sim 2^{n+\frac{1}{2}}n^n\mathrm{e}^{-n}\cos x. \tag{E.1.26}$$ $\boxed{\texttt{eq:her-asympt8}}$

Equation (E.1.26) is equivalent to (E.1.17) by Stirling's approximation (E.1.20). Further, it is easy to check that the error is uniformly small for x in a bounded set.

E.1.4 Steepest descent for integrals

Consider the integral

$$\int_\Gamma f(t)\mathrm{e}^{-n\Phi(t)}\mathrm{d}t \tag{E.1.27}$$

where f and Φ are entire functions. Assume $\Phi(t^*) = 0$, $\Phi'(t^*) = 0$, $\Phi''(t^*) \neq 0$, $\mathsf{Im}\Phi(t) = 0$ for $t \in \Gamma$. Further assume Γ is the path of steepest ascent for Φ, i.e. the path of steepest descent for $-\Phi(t)$. Having $\mathsf{Im}\Phi(t) = 0$ is enough to ensure that Γ is either the path of steepest ascent (locally) or steepest descent: Let $t = x(s) + \mathrm{i}y(s)$ be a smooth local parameterization of Γ, then by the Cauchy–Riemann equations

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\mathsf{Im}\,\Phi(t) = \mathsf{Im}\,\Phi_x(t)\frac{\mathrm{d}x}{\mathrm{d}s} + \mathsf{Im}\,\Phi_y(t)\frac{\mathrm{d}y}{\mathrm{d}s} = -\mathsf{Re}\,\Phi_x(t)\frac{\mathrm{d}x}{\mathrm{d}s} + \mathsf{Re}\,\Phi_x(t)\frac{\mathrm{d}y}{\mathrm{d}s}.$$

This shows that $\nabla\mathsf{Re}\,\Phi$ is orthogonal to the tangent vector $(-y'(s), x'(s))$, implying that Γ is in the direction of greatest increase/decrease for $\mathsf{Re}\,\Phi$.

Performing a Taylor expansion, we have

$$\Phi(t) = \frac{\Phi''(t^*)}{2}(t-t^*)^2(1 + O(|t-t^*|). \tag{E.1.28}$$

The point is that Φ is locally quadratic at t^* and we use this to inform the change of variables. But if we naïvely looked to solve

$$\Phi(t^* + v) = s^2,$$

for v as a function of s, $v(0) = 0$, we would fail. The implicit function theorem fails because we have two solution branches! Instead we consider

$$\frac{\Phi(t^* + sv)}{s^2} - 1 = 0 = \frac{\Phi''(t^*)}{2}v^2 - 1 + O(|sv^3|). \tag{E.1.29}$$ eq:blowup

We can choose $v = \pm R^{-1/2}\mathrm{e}^{-\mathrm{i}\phi/2}$ where $\frac{\Phi''(t^*)}{2} = R\mathrm{e}^{\mathrm{i}\phi}$. For either choice, we can apply the implicit function theorem (the derivative with respect to v, evaluated at $(s,v) = (0, \pm R^{-1/2}\mathrm{e}^{-\mathrm{i}\phi/2})$ does not vanish). We use $v = \pm R^{-1/2}\mathrm{e}^{-\mathrm{i}\phi/2}$ to obtain $v(s)$, and our local parameterization of Γ: $t(s) = t^* + sv(s)$. We use this a change of variables, within a neighborhood $B(t^*, \epsilon)$ on which the implicit function theorem applies (here we assume the orientation of Γ is the same as the induced orientation on $t((-\delta_1, \delta_2)))$

$$\int_{\Gamma\setminus B(t^*,\epsilon)} f(t)\mathrm{e}^{-n\Phi(t)}\mathrm{d}t = \int_{-\delta_1}^{\delta_2} f(t^* + sv(s))\mathrm{e}^{-ns^2}\left(v(s) + sv'(s)\right)\mathrm{d}s, \quad \delta_1, \delta_2 > 0. \tag{E.1.30}$$

Now let $\delta = \min\{\delta_1, \delta_2\}$. It follows that on $\Gamma_\delta = \Gamma \setminus t(-\delta, \delta)$, $\Phi(t) \geq \delta^2$. Then

$$\left|\int_{\Gamma_\delta} f(t)\mathrm{e}^{-n\Phi(t)}\mathrm{d}t\right| \leq \mathrm{e}^{-n\delta^2}\int_{\Gamma_\delta}|f(t)|\mathrm{e}^{-n(\Phi(t)-\delta^2)}\mathrm{d}t. \tag{E.1.31}$$ eq:awayfrom

For $n \geq 1$, we have

$$\int_{\Gamma_\delta}|f(t)|\mathrm{e}^{-n(\Phi(t)-\delta^2)}\mathrm{d}t \leq \int_{\Gamma_\delta}|f(t)|\mathrm{e}^{-(\Phi(t)-\delta^2)}\mathrm{d}t := M. \tag{E.1.32}$$

And therefore (eq:awayfrom E.1.31) is exponentially small in n, less than $M\mathrm{e}^{-n\delta^2}$. Now, consider

$$\int_{-\delta}^{\delta} f(t^* + sv(s))\mathrm{e}^{-ns^2} \left(v(s) + sv'(s)\right) \mathrm{d}s \tag{E.1.33}$$

and we can directly apply Laplace's method. Taylor expand the function

$$f(t^* + sv(s))(v(s) + sv'(s))$$

at $s = 0$, and term by term integration gives an expansion in powers of $n^{-1/2}$ with the leading order term being

$$\begin{aligned}\int_\Gamma f(t)e^{-n\Phi(t)}\mathrm{d}t \\ &= \int_{-\delta}^{\delta} f(t^* + sv(s))\mathrm{e}^{-ns^2}(v(s) + sv'(s))\mathrm{d}s + O(n^{-\alpha}) \\ &= \int_{-\delta}^{\delta} f(t^*)v(0)(1 + O(s))\mathrm{e}^{-ns^2}\mathrm{d}s + O(n^{-\alpha}), \quad \text{for all } \alpha > 0. \end{aligned} \tag{E.1.34}$$

eq:leading-order

Performing a change of variables $s = y/\sqrt{2n}$ we have

$$\begin{aligned}\int_{-\delta}^{\delta} \mathrm{e}^{-ns^2}\mathrm{d}s &= \frac{1}{\sqrt{2n}}\int_{-\sqrt{2n}\delta}^{\sqrt{2n}\delta} \mathrm{e}^{-y^2/2}\mathrm{d}y = \sqrt{\frac{\pi}{n}} + O(n^{-\alpha}), \quad \text{for all } \alpha > 0, \\ \int_{-\delta}^{\delta} |s|\mathrm{e}^{-ns^2}\mathrm{d}s &= \frac{1}{\sqrt{2n}}\int_{-\sqrt{2n}\delta}^{\sqrt{2n}\delta} |y|\mathrm{e}^{-y^2/2}\mathrm{d}y = \frac{C}{n} + O(n^{-\alpha}), \quad \text{for all } \alpha > 0.\end{aligned} \tag{E.1.35}$$

eq:gaussian_int

So, we have

$$\int_\Gamma f(t)e^{-n\Phi(t)}\mathrm{d}t = \sqrt{\frac{2\pi}{n}} f(t^*)|\Phi''(t^*)|^{-1/2}\mathrm{e}^{-\mathrm{i}\phi/2} + O(n^{-1}) \text{ as } n \to \infty. \tag{E.1.36}$$

In our setting, we will want Φ to depend on an additional parameter. So, we need a couple results to allow for uniformity in estimates with respect to this parameter. First, we need to understand the domain on which $v(s)$ is defined by (eq:blowup E.1.29), and derive an upper bound on $v(s)$ over this domain. We follow Chang2003 [?].

We have a theorem that is a more detailed version of what preceeded it. Remove what came before or reword?

Unify "ball" notation

Theorem 117 (Quantitative implicit function theorem)**.** *For $R_1, R_2, \epsilon > 0$, suppose $F(s, v)$ is an analytic function for $(s, v) \in B_{R_1+\epsilon}(0) \times B_{R_2+\epsilon}(0)$. Suppose further that $F(0, 0) = 0$ and $\partial_v F(0, 0) = a \neq 0$ and $|F(s, v)| \leq M$ on $B_{R_1+\epsilon}(0) \times B_{R_2+\epsilon}(0)$. Then there exists an analytic function $z(s)$ such that $F(s, z(s)) = 0$ for*

$$s \in B_{\tilde{R}}(0), \quad \tilde{R} = \frac{R_1|a|r}{M} - \frac{r^2 R_1}{R_2(R_2 - r)}, \quad r < \frac{R_2}{1 + \frac{M}{|a|R_2}},$$

one has $z(s) \in B_r(0)$.

Proof. As a first estimate, we have

$$F(0,v) = az + \sum_{j=2}^{\infty} F^{(j)}(0,0)\frac{z^j}{j!}.$$

By the boundedness $|F(0,v)| \leq M$, $|v| \leq R_2$ and Cauchy's integral formula, we have

$$\frac{F^{(j)}(0,0)}{j!} = \frac{1}{2\pi i}\int_{\partial B_{R_2}(0)} \frac{F(0,z)}{z^{j+1}}\mathrm{d}z,$$
$$\frac{|F^{(j)}|(0,0)}{j!} \leq MR_2^{-j}.$$

This gives

$$|F(0,v)| \geq |az| - M\sum_{j=2}^{\infty}\left(\frac{|z|}{R_2}\right)^j = |az| - M\frac{|z|^2}{R_2^2 - |z|R_2}.$$

This lower bound is positive provided

$$|z| \leq r < \frac{R_2}{1 + \frac{M}{|a|R_2}}.$$

Next, we recall Rouché's theorem which states that if f, g are analytic in a neighborhood of a simple closed contour C and that $|g(v)| > |f(v)|$ on C, then $f(v)$ and $f(v)+g(v)$ have the same number of zeros inside C. So, for each fixed s, we consider

$$g(v) = F(0,v), \quad f(v) = F(0,v) - F(s,v),$$

and choose $C = \partial B_r(0)$.

We can also bound $|\partial_s F(0,v)|$ using Cauchy's integral formula by

$$|\partial_s F(0,v)| \leq \frac{M}{R_1}.$$

This implies that

$$|f(v)| \leq \frac{M}{R_1}|s|.$$

Therefore, we can apply Rouché's theorem if

$$|s| < \tilde{R} := \frac{R_1|a|r}{M} - \frac{r^2R_1}{R_2(R_2 - r)}.$$

So, for each s there exists only one (simple) zero $z(s) \in B_r(0)$ satisfying $F(s, z(s)) = 0$ as s varies in $B_{\tilde{R}}(0)$. The analytic implicit function theorem can be applied in a neighborhood of any of these these points $(s, z(s))$ to verify the analyticity of $z(s)$. $\square$

To make this theorem a bit more convenient to use, we note that if we choose the parameter $\rho = M/(|a|R_2)$ and set

$$r = \alpha \frac{R_2}{1+\rho}, \quad 0 < 1 < \alpha,$$

and we conclude that

$$\tilde{R} = \frac{\alpha R_1}{\rho} \frac{1-\alpha}{1-\alpha+\rho}.$$

Then we note that because $z(s)$ is analytic in an open neighborhood of $B_{\tilde{R}}(0)$, we can estimate derivatives

$$|z^{(j)}(s)| \leq \frac{2^j r}{\tilde{R}^j}, \quad s \in B_{\tilde{R}/2}(0).$$

In particular, this gives

$$|z'(s)| \leq 2 \frac{R_2}{R_1} \frac{\frac{\rho}{1+\rho}}{\frac{1-\alpha}{1-\alpha+\rho}} \leq \frac{R_2}{R_1} \frac{2\rho}{1-\alpha}, \quad s \in B_{\tilde{R}/2}(0). \tag{E.1.37}$$

eq:deriv_est

The following theorem gives sufficient conditions for the error term derived in the method of steepest descent for integrals to be uniformly valid with respect to non-asymptotic parameters.

t:uniform_expansion **Theorem 118.** *Consider the integral*

$$I_{x,n} = \int_{\Gamma^x} f(t) \mathrm{e}^{-n\Phi(t;x)} \mathrm{d}t, \quad n \in \mathbb{N}_+, \quad x \in X, \tag{E.1.38}$$

where f and Φ are analytic functions of t in a neighborhood U_x of Γ^x. Assume $\Phi(t^*;x) = 0$, $\Phi'(t^*;x) = 0$, $\Phi''(t^*;x) \neq 0$, $\mathsf{Im}\Phi(t;x) = 0$ for $t \in \Gamma^x$. Further assume Γ^x is a subset of of a path of steepest ascent for Φ passing through t^* and t^* is not an endpoint of Γ^x. If there exists $C, D, \delta, \epsilon > 0, N \geq 0$, independent of x, and functions $h(x), g(x) \geq 0$ such that

$$B_\epsilon(t^*) \subset U_x, \quad |\Phi''(t^*)| \geq \delta, \quad \sup_{t \in B_\epsilon(t^*)} \left| \frac{\Phi'''(t)}{\Phi''(t^*)} \right| \leq C,$$

$$\int_{\Gamma^x_\gamma} |f(t)| \mathrm{e}^{-N(\Phi(t;x)-\gamma^2)} \leq g(x), \quad \sup_{\Gamma^x \setminus \Gamma^x_\gamma} |f'(t)| \leq h(x),$$

for all $0 \leq \gamma \leq D$, where $\Gamma^x_\gamma = \Gamma^x \cap \{t : \Phi(t;x) > \gamma^2\}$. Then

$$I_{x,n} = \frac{\mathrm{e}^{-\mathrm{i}\phi/2}}{\sqrt{2\pi n |\Phi''(t^*)|}} f(t^*) + E_{x,n},$$

where

$$|E_{x,n}| \leq \mathrm{e}^{-n\delta'^2} g(x) + \frac{1}{n} \left(\sqrt{\frac{8}{\delta}} h(x) + \frac{8M}{\delta\epsilon} \right) \int_{-\sqrt{2n\delta'}}^{\sqrt{2n\delta'}} |y| \mathrm{e}^{-y^2/2} \mathrm{d}y,$$

$M = 3 + \frac{4C\epsilon}{3}$ and δ' is any number satisfying $0 < \delta' < \min\{\frac{\epsilon\sqrt{\delta}}{4\sqrt{2}M}\frac{1}{1+M}, D\}$.

Before we prove this theorem, there are some important points to highlight. First, there is no reason that δ cannot be large, or even increasing. Also, if X is just a singleton set then these estimates are trivial and this gives the standard leading-order behavior by the method of steepest descent.

In the cases we need to analyze, Φ'' will not depend on the non-asymptotic parameter x, but the point t^* will. So, if t^*, as x varies, is bounded away from singularities of Φ'' and f, only finding g is necessary.

The actual size of the error term in the above approximation is $O(n^{-\frac{3}{2}})$ because $\int_{-\delta'}^{\delta'} y\mathrm{e}^{y^2}\mathrm{d}y = 0$. One has to perform a Taylor expansion to one more order than is used in the proof of the theorem. This results in more complicated expressions. We leave this extension as an exercise for the reader as this fact is not needed in the analysis of random matrices.

r:Cblowup **Remark 119.** Suppose $C \leq cn^{1/3}$ and $\delta \geq cn^{-1/3}$. Then $M \leq c_0 n^{1/3}$ and $\delta' \leq c_1 n^{-1/3}$ and while it is not lower-order than the "leading" term, one derives estimates of the form:

$$|I_{x,n}| \leq Cn^{-\frac{1}{3}}|f(t^*)| + \mathrm{e}^{-c_1^2 n^{1/3}} g(x) + 2\left(\sqrt{\frac{8}{c}} n^{-\frac{5}{6}} h(x) + \frac{8c_0 n^{-\frac{1}{3}}}{\epsilon c}\right)$$

Proof of Theorem t:uniform_expansion *118.* To find the (local) change of variables we apply the quantitative implicit function theorem to

$$F(s,v) = \frac{\Phi(t^* + s(v^* + v))}{s^2} - 1,$$

in a neighborhood of $(s,v) = (0,0)$, where $v^* := R^{-1/2}\mathrm{e}^{-\mathrm{i}\phi/2}$ and $\Phi''(t^*)/2 = R\mathrm{e}^{\mathrm{i}\phi}$. The implicit function theorem gives $v = v(s)$. We write

$$F(s,v) = \sum_{j=2}^{\infty} \frac{\Phi^{(j)}(t^*)}{j!} s^{j-2}(v^* + v)^j - 1.$$

This power series is guaranteed to converge provided that $|s(v^* + v)| < \delta$, so we consider $(s,v) \in B_{\epsilon/(2|v^*|)}(0) \times B_{|v^*|}(0)$, and we then need to compute an upper-bound on F. We write

$$\Phi(t) = \Phi(t^*) + \Phi'(t^*)(t - t^*) + \frac{\Phi''(t^*)}{2}(t - t^*)^2 + \int_{t^*}^{t}\int_{t^*}^{t'}\int_{t^*}^{t''} \Phi'''(t''')\mathrm{d}t'''\mathrm{d}t''\mathrm{d}t'$$

were we integrate along the line connecting t to t^*. We find

$$F(s,v) = -1 + \frac{\Phi''(t^*)}{2}(v^* + v)^2 + s^{-2}\int_{t^*}^{t^*+s(v^*+v)}\int_{t^*}^{t'}\int_{t^*}^{t''} \Phi'''(t''')\mathrm{d}t'''\mathrm{d}t''\mathrm{d}t'$$

Then we bound $|\Phi'''(t''')| \leq C|\Phi''(t^*)|$ to find

$$\begin{aligned}
|F(s,v)| &\leq \frac{|\Phi''(t^*)|}{2}(2v^*v + v^2) + C\frac{|\Phi''(t^*)|}{6}|s(v^*+v)^3| \\
&\leq 3 + C\frac{|\Phi''(t^*)|}{6} 8|s||v^*|^3 \\
&\leq 3 + C\frac{2|\Phi''(t^*)|}{3}\epsilon|v^*|^2 \\
&\leq 3 + \frac{4C\epsilon}{3} =: M = M(C,\epsilon)
\end{aligned}$$

double check these bounds

By (E.1.37) [eq:deriv_est], for any choice of $0 < \alpha < 1$

$$|v'(s)| \leq \frac{4M}{\epsilon(1-\alpha)|\Phi''(t^*)|}, \quad \text{for} \quad |s| < \frac{\alpha\epsilon|\Phi''(t^*)|^{1/2}}{\sqrt{8}M}\frac{1-\alpha}{1+\frac{M}{2}-\alpha}.$$

These inequalities are then simplified using the lower bound on $|\Phi''(t^*)|$ and setting $\alpha = 1/2$, for example:

$$|v'(s)| \leq \frac{8M}{\delta\epsilon}, \quad \text{for} \quad |s| < \frac{\epsilon\sqrt{\delta}}{4\sqrt{2}M}\frac{1}{1+M}.$$

Most importantly, this gives us a uniform lower bound on the size the interval over which v is defined. So, we choose δ' to be some number less than $\min\{\frac{\epsilon\sqrt{\delta}}{4\sqrt{2}M}\frac{1}{1+M}, D\}$.

The integral $I_{x,n}$ can be truncated to $\Gamma^x_{\delta'}$ at a cost of an error bounded by $g(x)\mathrm{e}^{-\delta'^2 n}$, see (E.1.31) [eq:awayfrom]. Following (E.1.34) [eq:leading-order] we find

$$\int_{\Gamma^x\setminus\Gamma^x_{\delta'}} f(t)\mathrm{e}^{-n\Phi(t)}\mathrm{d}t = \int_{-\delta'}^{\delta'} f(t^* + sV(s))\mathrm{e}^{-ns^2}(V(s) + sV'(s))\mathrm{d}s,$$
$$V(s) = v^* + v(s),$$

Expanding all but the exponential:

$$f(t^* + sV(s)) = f(t^*) + \int_{t^*}^{t^*+sV(s)} f'(s')\mathrm{d}s',$$
$$V(s) + sV'(s) = V(0) + sV'(s) + \int_0^s V'(s')\mathrm{d}s'.$$

We know that for $|s| \leq \delta'$, $|sV(s)| \leq \epsilon$ and $|V(s)| \leq 2|v^*|$ so that

$$\left|\int_{t^*}^{t^*+sV(s)} f'(s')\mathrm{d}s'\right| \leq 2|s||v^*|h(x) \leq \sqrt{\frac{8}{\delta}}|s|h(x),$$
$$\left|sV'(s) + \int_0^s V'(s')\mathrm{d}s'\right| \leq |s|\frac{8M}{\delta\epsilon}.$$

Then, using the estimates from (E.1.35) [eq:gaussian_int] the theorem follows.

□

E.1.5 Plancherel–Rotach asymptotics

Another asymptotic regime is obtained when we consider $x = O(\sqrt{n})$ and let $n \to \infty$. Plancharel–Rotach asymptotics refer to the asymptotics of polynomials scaled by their largest zero. The limit is oscillatory or exponential depending on the range of x. This is to be expected: for each n, the polynomial $\mathfrak{h}_n(x)$, and thus the wave function $\psi_n(x)$, has n zeros. The largest and smallest of the zeros are approximately $\pm\sqrt{(n+1/2)}$. The oscillatory regime is obtained when $x(n+1/2)^{-1/2}$ lies well within the interval $(-1,1)$. Outside this interval, the Hermite wave function decays exponentially fast. A more delicate calculation, using the Airy function, is required to understand the transition from oscillatory to exponential behavior.

We will prove a weaker version of the Plancherel-Rotach formulas, that suffices for our needs. These formula are as follows.

Case 1. Oscillatory behavior.

$$x = 2\cos\varphi, \qquad 0 < \varphi < \pi. \tag{E.1.39}$$

eq:pr-proof9

$$n^{\frac{1}{4}}\psi_{n+p}\left(x\sqrt{n}\right) = \frac{1}{\sqrt{\pi\sin\varphi}}\left(\cos\left[n\left(\varphi - \frac{1}{2}\sin 2\varphi\right) + \left(p+\frac{1}{2}\right)\varphi - \frac{\pi}{4}\right] + O(n^{-1})\right). \tag{E.1.40}$$

eq:pr-proof8b

The convergence is uniform for φ in a compact closed subset of $(0,\pi)$.

Don't we need this kind of uniformity?

Case 2. Exponential decay.

$$|x| = 2\cosh\varphi, \qquad 0 < \varphi. \tag{E.1.41}$$

eq:pr-proof10

$$\begin{aligned} n^{\frac{1}{4}}\psi_{n+p}\left(x\sqrt{n}\right) &= \frac{(\mathrm{sgn}(x))^{n+p}}{2n^{\frac{1}{4}}}\frac{\mathrm{e}^{(p+1/2)\varphi}\mathrm{e}^{-\frac{n}{2}\left(\frac{x^2}{2}-1-\mathrm{e}^{-2\varphi}-2\varphi\right)}}{\sqrt{\pi\sinh\varphi}}\left(1+O(n^{-1})\right), \\ &= \frac{(\mathrm{sgn}(x))^{n+p}}{2n^{\frac{1}{4}}}\frac{\mathrm{e}^{(p+1/2)\varphi}\mathrm{e}^{-\frac{n}{2}(\sinh(2\varphi)-2\varphi)}}{\sqrt{\pi\sinh\varphi}}\left(1+O(n^{-1})\right). \end{aligned}$$

The convergence is uniform for φ in a closed subset of $(0,\infty)$. Observe that $\sinh(2\varphi) - 2\varphi > 0$ when $\varphi > 0$, ensuring exponential decay.

Case 3. The transition region.

$$x = 2\sqrt{n} + \frac{s}{n^{\frac{1}{6}}} \quad s \in \mathbb{C}, \tag{E.1.42}$$

eq:pr-proof21

$$n^{\frac{1}{12}}\psi_n(x) \sim \mathrm{Ai}(s) + O\left(n^{-\frac{2}{3}}\right). \tag{E.1.43}$$

$$n^{\frac{1}{12}}\psi_{n+p}(x\sqrt{n}) = \mathrm{Ai}(s) + n^{-1/3}\left(\frac{1}{2} - p\right)\mathrm{Ai}'(s) + O(n^{-2/3}). \tag{E.1.44}$$

eq:pr-proof22

The convergence is uniform for s in a compact subset of $\mathbb{C}$.

All three asymptotic relations are obtained by the method of steepest descent for integrals. Assume $x \in \mathbb{R}$. We fix an integer p, use the integral identity eq:hermite-int (E.1.3) with $k = n + p$, and rescale $\xi = nt$ to obtain

$$
\begin{aligned}
\mathfrak{h}_{n+p}\left(x\sqrt{n}\right) &= \left(-\mathrm{i}\sqrt{n}\right)^{n+p}\sqrt{\frac{n}{2\pi}}\int_{-\infty}^{\infty} t^{n+p}\mathrm{e}^{-\frac{n}{2}(t-\mathrm{i}x)^2}\,dt && \text{(E.1.45)}\\
&= \left(-\mathrm{i}\sqrt{n}\right)^{n+p}\sqrt{\frac{n}{2\pi}}\left(\int_0^{\infty} t^{n+p}\mathrm{e}^{-\frac{n}{2}(t-\mathrm{i}x)^2}\,dt + (-1)^{n+p}\int_0^{\infty} t^{n+p}\mathrm{e}^{-\frac{n}{2}(t+\mathrm{i}x)^2}\,dt\right)\\
&:= \left(-\mathrm{i}\sqrt{n}\right)^{n+p}\sqrt{\frac{n}{2\pi}}\left(I_{n,p}(x) + (-1)^{n+p}I_{n,p}(-x)\right). && \text{(E.1.46)}
\end{aligned}
$$

The integral $I_{n,p}(x)$ may be rewritten in the form

$$
I_{n,p}(x) = \int_0^{\infty} t^p \mathrm{e}^{-ng(t)}\,\mathrm{d}t, \quad g(t) = \frac{1}{2}(t-\mathrm{i}x)^2 - \log t. \tag{E.1.47}
$$

eq:pr-proof2

As is usual, the first step is to determine the critical points where $g'(t) = 0$. This reduces to the quadratic equation $t^2 - \mathrm{i}xt - 1 = 0$. The three distinct asymptotic limits arise from the three distinct possibilities for the roots.

(a) $|x| < 2$. The function g has two critical points on the unit circle, given by

$$
t_{\pm} = \frac{\mathrm{i}x \pm \sqrt{4-x^2}}{2} = \mathrm{i}\mathrm{e}^{\mp \mathrm{i}\varphi}, \tag{E.1.48}
$$

eq:pr-proof3

where x and φ are related through eq:pr-proof9 (E.1.39).

(b) $|x| > 2$. The two critical points lie on the imaginary axis, and may be written in the form

$$
t_{\pm} = \mathrm{i}\left(\frac{x \pm \sqrt{x^2-4}}{2}\right) = \mathrm{i}\,\mathrm{sgn}(x)\mathrm{e}^{\pm\varphi}, \tag{E.1.49}
$$

eq:pr-proof3

where each branch of φ is defined through the relation eq:pr-proof10 (E.1.41).

(c) $|x| = 2$. The two critical points coalesce into a single value $t = \mathrm{i}$. A further blow-up is necessary to obtain the Airy asymptotics eq:pr-proof22 (E.1.44).

Let us first consider the integral $I_{n,p}(x)$ in case (a), and let us assume that $x > 0$ to be concrete. We deform the integral over $(0,\infty)$ a contour Γ which is the path of steepest descent that passes through the critical point t_+ as shown in Figure fig:PR1 E.1.1. The existence of such a contour may be deduced by continuity, beginning with the observation that when $x = 0$, Γ is simply the segment $(0,\infty)$ along the real line. While in general, Γ is given by the equation $\mathsf{Im}(g(t)) = \mathsf{Im}(g(t_+))$. It is not important for us to solve for the contour explicitly: all that is required is to understand the phase of $g''(t_+)$, check that $0 \in \Gamma$ and the integral over $(0,\infty)$ can be deformed to an integral over Γ.

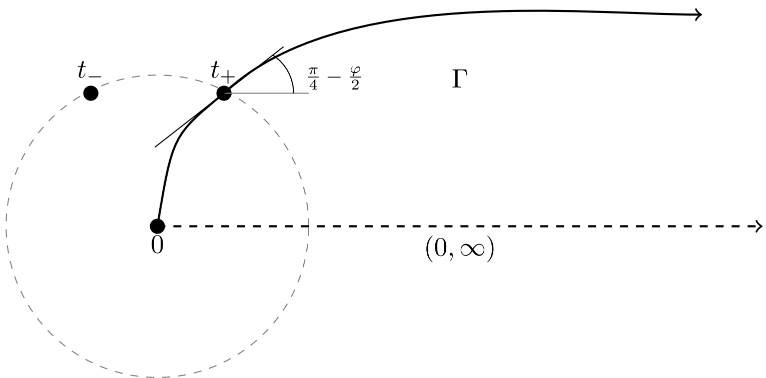

Figure E.1.1: fig:PR1

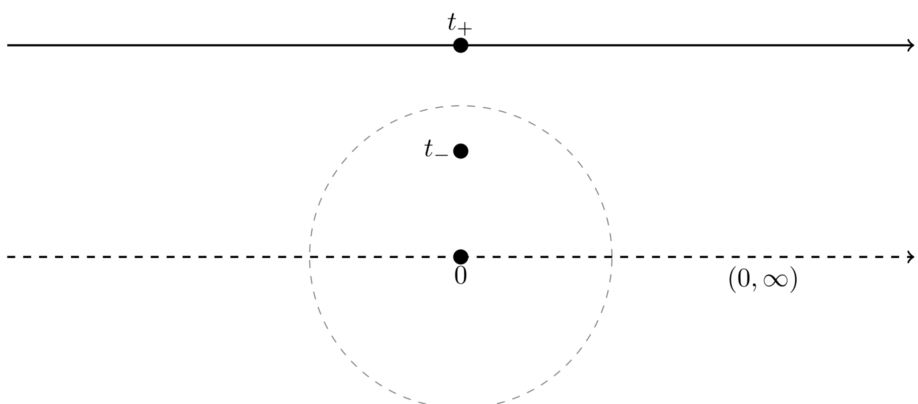

Figure E.1.2: fig:PR2

Lecture Note 7. As Γ is traversed the imaginary part of $g(t)$ is constant and, leaving t_+, the real part of $g(t)$ will only increase from its value of $\frac{1}{2} - \frac{x^2}{4}$. Also, Γ can only fail to be defined at either a singularity, ∞ or another critical point. By checking the imaginary part of g on the positive imaginary axis, and then the negative real axis (noting that it then must approach from below) we can rule out Γ crossing these axes. By checking the real part of g on the unit circle and determining that $\frac{1}{2} - \frac{x^2}{4}$ is its maximum, we see that Γ can only intersect the unit circle at t_+. As $\mathsf{Im}\, g(t)$ is harmonic in $\{|t| < 1 \mid \mathsf{Re}\, t > 0 \text{ or } \mathsf{Im}\, t < 0\}$ it cannot have any closed level curves. Therefore $0 \in \Gamma$. For $t = \alpha + \mathrm{i}\beta$ and α fixed, $\mathsf{Im}\,(g(t) - g(t_+)) = 0$ gives an implicit function for $\beta = \beta(x)$, and $\beta(x)$ has a bounded derivative, uniform in α. Thus Γ must remain in a strip containing the real axis. So, we can deform $(0, \infty)$ to Γ.

It is easy to check that when $|x| < 2$

$$g''(t_+) = 1 + \frac{1}{t_+^2} = 1 - e^{2\mathrm{i}\varphi} = \left(-\mathrm{i}e^{\mathrm{i}\varphi}\right)(2\sin\varphi)\,. \tag{E.1.50}$$

eq:pr-proof4

Thus, we have

$$\begin{aligned} I_{n,p}(x) &= \int_0^\infty t^p \mathrm{e}^{-ng(t)}\,\mathrm{d}t = \mathrm{e}^{-ng(t_+)} \int_\Gamma t^p \mathrm{e}^{-n(g(t)-g(t_+))}\,\mathrm{d}t \\ &= \mathrm{e}^{-ng(t_+)} t_+^p \left.\frac{\mathrm{d}t}{\mathrm{d}s}\right|_{t_+} \int_{-\infty}^\infty \mathrm{e}^{-\frac{n}{2}|g''(t_+)|s^2}\,\mathrm{d}s + O(n^{-1}). \end{aligned} \tag{E.1.51}$$

In the second line, we have used the fact that $\mathsf{Im}(g(t)-g(t_+)) = 0$ on Γ, and we have further approximated the integral over Γ by an integral over the tangent to Γ at t_+. More precisely, the approximation here is

$$g''(t_+)(t-t_+)^2 = |g''(t_+)|s^2,$$

which implies

$$\left.\frac{\mathrm{d}t}{\mathrm{d}s}\right|_{t_+} = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)}. \tag{E.1.52}$$

eq:pr-proof6

We now combine the values

$$t_+ = \mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi}, \quad g(t_+) = -\frac{\mathrm{e}^{2\mathrm{i}\varphi}}{2} + \mathrm{i}\left(\varphi - \frac{\pi}{2}\right),$$

with (E.1.51) and (E.1.52) to obtain

$$I_{n,p}(x) = \mathrm{e}^{\frac{n}{2}\cos 2\varphi}\sqrt{\frac{\pi}{n\sin\varphi}}\left(\mathrm{e}^{\mathrm{i}\left(\frac{n}{2}\sin 2\varphi + (n+p+\frac{1}{2})\left(\frac{\pi}{2}-\varphi\right)\right)} + O(n^{-\frac{1}{2}})\right). \tag{E.1.53}$$

eq:pr-proof7

Finally, since x is real, we have $\overline{I_{n,p}(x)} = I_{n,p}(-x)$. We combine (E.1.46) with (E.1.53) to obtain

$$\begin{aligned} \mathfrak{h}_{n+p}(x\sqrt{n}) &= n^{\frac{n+p}{2}}\sqrt{\frac{2}{\sin\varphi}}\mathrm{e}^{\frac{n}{2}\cos 2\varphi} \\ &\times \left(\cos\left[n\left(\varphi - \frac{1}{2}\sin 2\varphi\right) + \left(p+\frac{1}{2}\right)\varphi - \frac{\pi}{4}\right] + O(n^{-\frac{1}{2}})\right), \end{aligned} \tag{E.1.54}$$

eq:pr-proof8

where x and φ are related via (E.1.39). We now use (E.1.11) and Stirling's approximation (E.1.20) to obtain (E.1.40). The error term is uniform for x in closed subsets of $(-2,2)$ by Theorem 118.

The asymptotics in case (b) are obtained as follows. Since the stationary phase points are on the imaginary axis, it pays to work directly with (E.1.45). The path of steepest descent from t_- extends vertically. The path of steepest descent through t_+ is locally horizontal. And, so we just move the integral in (E.1.45) to a (locally) horizontal contour that passes through t_+, see Figure E.1.2. Using $|x| = 2\sinh\varphi$ this gives

$$\mathfrak{h}_{n+p}(x\sqrt{n}) = (\sqrt{n})^{n+p}(\operatorname{sgn}(x))^{n+p}\frac{\mathrm{e}^{(n+p)\varphi}\mathrm{e}^{\frac{n}{2}(\mathrm{e}^{\varphi}-x)^2}}{\sqrt{2\mathrm{e}^{-\varphi}\sinh\varphi}}(1+O(n^{-1})).$$

Using (E.1.20) one obtains (E.1.41). Those calculations are left to the reader.

The remaining calculations are left to the reader. The final asymptotic relation is

$$\mathfrak{h}_{n+p}(x\sqrt{n}) = n^{\frac{n+p}{2}}\frac{\mathrm{e}^{-\frac{n}{2}}}{\sqrt{\sinh\varphi}}\mathrm{e}^{\left(p+\frac{1}{2}\right)\varphi-\frac{n}{2}(\sinh(2\varphi)-2\varphi)}(1+o(1)), \tag{E.1.55}$$ $\boxed{\texttt{eq:pr-proof13}}$

which combines with (E.1.11) and Stirling's approximation (E.1.20) to yield (??).

We now turn to case (c). We only present the main change of variables that underly the result. We begin with the integral representation(E.1.45) and substitute

$$t = \mathrm{i} + \frac{r}{n^{\frac{1}{3}}}, \quad x = 2\sqrt{n} + \frac{s}{n^{\frac{1}{6}}}, \tag{E.1.56}$$ $\boxed{\texttt{eq:pr-proof14}}$

moving the integral over $\mathbb{R}$ to an integral over the line $\mathrm{i}+\mathbb{R}$, to obtain

$$\mathfrak{h}_n(x\sqrt{n}) = (-\mathrm{i}\sqrt{n})^n\frac{n^{\frac{1}{6}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{nh(r)}\,\mathrm{d}r, \tag{E.1.57}$$ $\boxed{\texttt{eq:pr-proof15}}$

where

$$\begin{aligned} h(r) &= \log\left(\mathrm{i}+\frac{r}{n^{\frac{1}{3}}}\right) - \frac{1}{2}\left(\left(\mathrm{i}+\frac{r}{n^{\frac{1}{3}}}\right) - \mathrm{i}\left(2+\frac{s}{n^{\frac{2}{3}}}\right)\right)^2 \\ &= \frac{1}{2} + \log\mathrm{i} + \frac{s}{n^{\frac{2}{3}}} + \frac{1}{n}\left(\mathrm{i}sr + \frac{\mathrm{i}}{3}r^3\right) + \frac{s^2}{2n^{\frac{4}{3}}} + O(n^{-\frac{4}{3}}r^4), \end{aligned} \tag{E.1.58}$$

using the Taylor expansion of the logarithm. The terms that depend on s may be pulled out of the integral and we are left with

$$\mathfrak{h}_n(x\sqrt{n}) \approx \frac{n^{\frac{n}{2}+\frac{1}{6}}}{\sqrt{2\pi}}\mathrm{e}^{\frac{n}{2}}\mathrm{e}^{sn^{\frac{1}{3}}}\int_{-\infty}^{\infty}e^{\mathrm{i}sr+\frac{\mathrm{i}}{3}r^3}\mathrm{d}r \tag{E.1.59}$$

$$\mathfrak{h}_n(x\sqrt{n}) = \sqrt{2\pi}n^{\frac{n}{2}+\frac{1}{6}}\mathrm{e}^{\frac{n}{2}}\mathrm{e}^{sn^{\frac{1}{3}}}(\mathrm{Ai}(s)+O(n^{-1/3}))$$

We now use the definition (E.1.11) and Stirling's approximation (E.1.20) to obtain (E.1.44). To make this rigorous, and to obtain the next term in the expansion, We take the integral

$$\mathfrak{h}_{n+p}(x) = \sqrt{\frac{n}{2\pi}}(-\mathrm{i}\sqrt{n})^{n+p}\int_{\mathbb{R}}t^p\mathrm{e}^{-n(\frac{1}{2}(t-\mathrm{i}x)^2-\log t)}\mathrm{d}t \tag{E.1.60}$$

and deform to $\mathrm{i}+\mathbb{R}$. Then, let $t = \mathrm{i}+r$ and we arrive at

$$\mathfrak{h}_n(x) = \sqrt{\frac{n}{2\pi}}(-\mathrm{i}\sqrt{n})^{n+p}\int_{\mathbb{R}}(\mathrm{i}+r)^p \mathrm{e}^{-n(\frac{1}{2}(r+\mathrm{i}(1-x))^2-\log(\mathrm{i}+r))}\mathrm{d}r. \tag{E.1.61}$$

Then this can be deformed to a contour $\Gamma = \mathrm{e}^{-\mathrm{i}\pi/6}(-\infty,0]\cup\mathrm{e}^{\mathrm{i}\pi/6}[0,\infty)$.

Then, we perform a Taylor expansion of the logarithm to find, for $H(r) = \frac{1}{2}(r+\mathrm{i}(1-x))^2 - \log(\mathrm{i}+r)$ and $x>1$,

$$\mathrm{e}^{-\frac{n}{2}(1-x)^2}\int_\Gamma(\mathrm{i}+r)^p\mathrm{e}^{-nH(r)}\mathrm{d}r \tag{E.1.62}$$

$$+\,O(\mathrm{e}^{-n(x-1)\delta}) \tag{E.1.63}$$

$$= \mathrm{e}^{\mathrm{i}n\frac{\pi}{2}}\int_{\Gamma\cap B(0,\delta)}\mathrm{e}^{\mathrm{i}n(x-2)r+\mathrm{i}n\frac{r^3}{3}}\left(\mathrm{i}^p + rp\mathrm{i}^{p-1} + O(r^2) - n\mathrm{i}^p\frac{r^4}{4} + nO(r^5)\right)\mathrm{d}r \tag{E.1.64}$$

We compute

$$\int_0^\delta \mathrm{e}^{-n\frac{y^3}{3}}y^\alpha \mathrm{d}y = O(n^{-(\alpha+1)/3}),$$

so that

$$\mathrm{e}^{-\frac{n}{2}(1-x)^2}\int_\Gamma(\mathrm{i}+r)^p\mathrm{e}^{-nH(r)}\mathrm{d}r = \mathrm{e}^{\mathrm{i}n\frac{\pi}{2}}\int_{\Gamma\cap B(0,\delta)}\left(\mathrm{i}^p + p\mathrm{i}^{p-1}r - n\mathrm{i}^p\frac{r^4}{4}\right)\mathrm{e}^{\mathrm{i}n(x-2)r+\mathrm{i}n\frac{r^3}{3}}\mathrm{d}r \\ +O(n^{-1}).$$

Finally, it follows that if $x = 2+sn^{-2/3}$ and setting $r = k/n^{-1/3}$

$$\int_{\Gamma\cap B(0,\delta)}\mathrm{e}^{\mathrm{i}n(x-2)r+\mathrm{i}n\frac{r^3}{3}}r^\gamma\mathrm{d}r = 2\pi n^{-(\gamma+1)/3}(-\mathrm{i})^\gamma\mathrm{Ai}^{(\gamma)}(s) + O(n^{-\alpha}) \quad \text{for all} \quad \alpha>0. \tag{E.1.65}$$

This gives

$$\psi_{n+p}(x\sqrt{n}) = (2\pi)^{1/4}\sqrt{n}\frac{\mathrm{e}^{-n\frac{x^2}{4}}\mathrm{e}^{\frac{n}{2}(1-x)^2}}{\sqrt{(n+p)!}}n^{\frac{n}{2}+\frac{p}{2}}n^{-1/3} \tag{E.1.66}$$

$$\times\left(\mathrm{Ai}(s) + n^{-1/3}\left(-p\mathrm{Ai}'(s) - \frac{1}{4}\mathrm{Ai}^{(4)}(s)\right) + O(n^{-2/3})\right). \tag{E.1.67}$$

We compute

$$\mathrm{e}^{-n\frac{x^2}{4}}\mathrm{e}^{\frac{n}{2}(1-x)^2} = \mathrm{e}^{-\frac{n}{4}(4+4\frac{s}{n^{2/3}}+\frac{s^2}{n^{4/3}})}\mathrm{e}^{\frac{n}{2}(1+2\frac{s}{n^{2/3}}+\frac{s^2}{n^{4/3}})} = \mathrm{e}^{-\frac{n}{2}}\mathrm{e}^{\frac{s^2n^{-1/3}}{4}}, \tag{E.1.68}$$

and use Stirling's approximation to write

$$\begin{aligned}\frac{(2\pi)^{1/4}}{\sqrt{n!}}\mathrm{e}^{-\frac{n}{2}}n^{\frac{n}{2}} &= \frac{(2\pi n)^{1/4}}{\sqrt{n!}}\mathrm{e}^{-\frac{n}{2}}n^{\frac{n}{2}}n^{-1/4} && \text{(E.1.69)}\\ &= n^{-1/4}(1+O(n^{-1})). && \text{(E.1.70)}\end{aligned}$$

Continuing, we obtain

$$\begin{aligned}\psi_{n+p}(x\sqrt{n}) &= n^{-1/12+p/2}\sqrt{\frac{n!}{(n+p)!}}\mathrm{e}^{\frac{s^2n^{1/3}}{4}} && \text{(E.1.71)}\\ &\times\left(\mathrm{Ai}(s)+n^{-1/3}\left(-p\mathrm{Ai}'(s)-\frac{1}{4}\mathrm{Ai}^{(4)}(s)\right)+O(n^{-2/3})\right) && \text{(E.1.72)}\\ &= n^{-1/12+p/2}\sqrt{\frac{n!}{(n+p)!}} && \text{(E.1.73)}\\ &\times\left(\mathrm{Ai}(s)+n^{-1/3}\left(-p\mathrm{Ai}'(s)+\frac{1}{4}(s^2\mathrm{Ai}(s)-\mathrm{Ai}^{(4)}(s))\right)+O(n^{-2/3})\right) && \text{(E.1.74)}\\ &= n^{-1/12+p/2}\sqrt{\frac{n!}{(n+p)!}} && \text{(E.1.75)}\\ &\times\left(\mathrm{Ai}(s)+n^{-1/3}\left(\frac{1}{2}-p\right)\mathrm{Ai}'(s)+O(n^{-2/3})\right), && \text{(E.1.76)}\end{aligned}$$

eq:airyform

where we used $\mathrm{Ai}^{(4)}(s) = s^2\mathrm{Ai}(s)+2\mathrm{Ai}'(s)$ in the last line.

Uniform bounds

We need uniform estimates when $x = 2+sn^{-2/3}$ and $0 \le s \le n^{2/3}$ to allow us to transition into case (b), (E.1.49). We could use asymptotics of Airy functions and Airy-like integrals to extend (E.1.76) to $s = O(n^{2/3})$ with a uniform error term. We take a different approach and work from first principles. Let $\Gamma = \mathrm{e}^{-\mathrm{i}\pi/6}(-\infty,0]\cup\mathrm{e}^{\mathrm{i}\pi/6}[0,\infty)$.

$$\mathrm{e}^{-\frac{n}{2}(1-x)^2}\int_\Gamma \mathrm{e}^{-nH(r)}\mathrm{d}r = \mathrm{e}^{\mathrm{i}n\frac{\pi}{2}}\int_{\Gamma\cap B(0,\delta)}\mathrm{e}^{-nH(r)}\mathrm{d}r + O(\mathrm{e}^{-n(x-1)\delta}).$$

Then, we deform

$$\mathrm{e}^{-\frac{n}{2}(1-x)^2}\int_{\Gamma\cap B(0,\delta)}\mathrm{e}^{-nH(r)}\mathrm{d}r = \mathrm{e}^{-\frac{n}{2}(1-x)^2}\int_C \mathrm{e}^{-nH(r)}\mathrm{d}r \tag{E.1.77}$$

to a horizontal contour connecting its endpoints. Then on this contour,

$$\begin{aligned}&\mathrm{e}^{-\frac{n}{2}(1-x)^2}\int_C(\mathrm{e}^{-nH(r)}-\mathrm{e}^{\frac{n}{2}(1-x)^2})\mathrm{d}r && \text{(E.1.78)}\\ &\quad= \int_C \mathrm{e}^{\mathrm{i}nr(x-2)}\left[\mathrm{e}^{-n\frac{r^2}{2}+\mathrm{i}nr+n\log(\mathrm{i}+r)}-\mathrm{e}^{\mathrm{i}n\frac{r^3}{3}}\right]\mathrm{d}r. && \text{(E.1.79)}\end{aligned}$$

For $r \in C$, $|\mathrm{e}^{\mathrm{i}nr(x-2)}| \le \mathrm{e}^{n\delta/\sqrt{2}(x-2)}$, and we find

$$\left|\mathrm{e}^{-\frac{n}{2}(1-x)^2}\int_C (\mathrm{e}^{-nH(r)} - \mathrm{e}^{\frac{n}{2}(1-x)^2})\mathrm{d}r\right| \le Mn^{-2/3}\mathrm{e}^{n\delta/\sqrt{2}(x-2)}. \tag{E.1.80}$$

Then, define $f_n(s) = \frac{n^{1/3}}{2\pi}\int_C \mathrm{e}^{\mathrm{i}n(x-2)r+\mathrm{i}n\frac{r^3}{3}}\mathrm{d}r$ and we have

$$\left|n^{1/12-p/2}\psi_{n+p}(x\sqrt{n}) - \sqrt{\frac{n!}{(n+p)!}}\mathrm{e}^{\frac{s^2 n^{-1/3}}{4}} f_n(s)\right| \le Mn^{-2/3}\mathrm{e}^{\frac{s^2 n^{1/3}}{4} - n\delta/\sqrt{2}(x-2)}. \tag{E.1.81}$$

Choosing $\delta = \sqrt{2}$, we find $\mathrm{e}^{\frac{s^2 n^{1/3}}{4} - n\delta/\sqrt{2}(x-2)} \le \mathrm{e}^{-s\frac{3}{4}n^{1/3}}$. A similar estimate follows for ψ_{n+p} and we obtain that there exist a constant $M > 0$ such that for $0 \le s \le n^{2/3}$

$$\begin{aligned}\left|n^{1/12}\psi_n(x\sqrt{n}) - \mathrm{e}^{\frac{s^2 n^{-1/3}}{4}} f_n(s)\right| &\le Mn^{-2/3}\mathrm{e}^{-s\frac{3}{4}n^{1/3}},\\ \left|n^{1/12}\psi_{n-1}(x\sqrt{n}) - \mathrm{e}^{\frac{s^2 n^{-1/3}}{4}} f_n(s)\right| &\le Mn^{-2/3}\mathrm{e}^{-s\frac{3}{4}n^{1/3}}.\end{aligned} \tag{E.1.82}$$ eq:smalls

For $s \ge n^{2/3}$, we can use (eq:pr-proof20 ??) to find

$$n^{1/4}(|\psi_n(x\sqrt{n})| + |\psi_{n-1}(x\sqrt{n})|) \le M\mathrm{e}^{-s\frac{3}{4}n^{1/3}}. \tag{E.1.83}$$ eq:larges

E.2 Laguerre polynomials

Include this logical derivation? Or just list the properties like Hermite?

Consider polynomial solutions to the differential equation

$$xy'' + (\alpha + 1 - x)y' + ky = 0. \tag{E.2.1}$$ eq:lag1

Such functions are called generalized Laguerre polynomials. For each choice of $\alpha > -1$ and $n \in \mathbb{N}$ there exists one and only one such solution. This follows from the following fact:

Lemma 38. *Any polynomial solution of* (eq:lag1 E.2.1) *must be of degree k.*

Proof. Suppose y is polynomial of degree m, with non-zero leading coefficient. As $|x| \to \infty$ in (eq:lag1 E.2.1) we have

$$-xy'(x) + ky(x) = O(|x|^{m-1}).$$

Then $xy'(x) = my(x) + O(|x|^{m-1})$ and the condition $k = m$ must be satisfied. □

So, if there existed two linearly independent polynomial solutions of (E.2.1) a linear combination would also be a solution and the combination could be taken so that it is a degree $< k$ polynomial, contradicting the lemma.

One can then check directly that

$$L_k^{(\alpha)}(x) = \frac{x^{-\alpha}\mathrm{e}^{x}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\mathrm{e}^{-x}x^{k+\alpha}\right) \tag{E.2.2}$$

eq:lag2

is the polynomial solution of (E.2.1). And from this it follows that

$$\int_0^\infty L_k^{(\alpha)}(x)L_j^{(\alpha)}(x)x^\alpha \mathrm{e}^{-x}\mathrm{d}x = \delta_{jk}\frac{\Gamma(k+\alpha+1)}{k!}. \tag{E.2.3}$$

eq:lag3

An explicit formula also follows

$$L_k^{(\alpha)}(x) = \sum_{j=0}^{k}(-1)^j\binom{k+\alpha}{k-j}x^j \tag{E.2.4}$$

eq:lag4

and then the Laplace transform can be computed explicitly

$$\int_0^\infty x^\alpha L_k^{(\alpha)}(x)\mathrm{e}^{-sx}\mathrm{d}x = \frac{(k+\alpha)!}{k!}\left(1-\frac{1}{s}\right)^k\frac{1}{s^{\alpha+1}}.$$

Inverting this transform gives the integral representation,

$$x^\alpha L_k^{(\alpha)}(x) = \frac{1}{2\pi\mathrm{i}}\frac{(k+\alpha)!}{k!}\int_{-\mathrm{i}\infty+\epsilon}^{\mathrm{i}\infty+\epsilon}\left(1-\frac{1}{s}\right)^k\frac{\mathrm{e}^{xs}}{s^\alpha}\frac{\mathrm{d}s}{s}, \quad \epsilon > 0. \tag{E.2.5}$$

eq:lag5

As a precursor to asymptotic analysis we compute

$$\left(1-\frac{1}{s}\right)^k\frac{\mathrm{e}^{xs}}{s^\alpha} = \mathrm{e}^{k\ell(s;\alpha)}, \quad \ell(s;\alpha) = \log\left(1-\frac{1}{s}\right) - \frac{\alpha}{k}\log s + \frac{x}{k}s.$$

Then, looking for stationary points, set

$$\ell'(s;\alpha) = \frac{1}{s(s-1)} - \frac{\alpha}{k}\frac{1}{s} + \frac{x}{k} = 0,$$

and the two roots are

$$s_\pm(\alpha) = \frac{\frac{\alpha+x}{k} \pm \sqrt{\left(\frac{\alpha+x}{k}\right)^2 - 4\frac{x}{k}\left(1+\frac{\alpha}{k}\right)}}{2\frac{x}{k}}.$$

The asymptotics will change for values of x where the two roots coincide. So, examine

$$\left(\frac{\alpha+x}{k}\right)^2 - 4\frac{x}{k}\left(1+\frac{\alpha}{k}\right) = \frac{\alpha^2+x^2-2\alpha x-4xk}{k^2} = 0.$$

This gives

$$x_\pm(\alpha) = 2k+\alpha \pm \sqrt{(2k+\alpha)^2-\alpha^2} = (\sqrt{k+\alpha}\pm\sqrt{k})^2.$$

We highlight the α-dependence of $x_\pm$ and suppress the k-dependence for future convenience. Additionally, because (E.2.5) is straightforward to differentiate with respect to x, we will include asymptotics for derivatives.

E.2.1 Fixed $\alpha > -1$

Here we suppose α is fixed and we write eq:lag5 (E.2.5) as

$$x^\alpha L_k^{(\alpha)}(x) = \frac{1}{2\pi \mathrm{i}} \frac{(k+\alpha)!}{k!} \int_{-\mathrm{i}\infty+\epsilon}^{\mathrm{i}\infty+\epsilon} \mathrm{e}^{k\ell(s;0)} \frac{1}{s^{\alpha+1}} \mathrm{d}s, \quad \text{for any } \epsilon > 0.$$

Note that we could absorb the $s^{-\alpha}$ factor in the exponent but this formulation is more convenient. Additionally, for α small this integral is just conditionally convergent and it can assist in the analysis to deform it to a contour

$$\Gamma_\epsilon = \mathrm{e}^{\mathrm{i}\theta}[0,\infty) \cup \mathrm{e}^{-\mathrm{i}\theta}[0,\infty) + \epsilon, \quad -\pi < \theta < -\pi/2$$

oriented with increasing increasing imaginary part. But more can be done. For $x \gg 0$ the contour of integration can be truncated at the cost of an exponentially small error.

Consider the function

$$|\mathrm{e}^{k\ell(s;0)}|, \quad s \in \mathrm{e}^{\mathrm{i}\theta}[0,\infty) + \epsilon.$$

For $|s| \geq R$ and $s = \epsilon + r\mathrm{e}^{\pm \mathrm{i}\theta}$

$$\begin{aligned} |\mathrm{e}^{k\ell(s;0)}| &\leq \exp\left(k \log\left(1 + \frac{1}{R}\right) + x(\epsilon - r|\cos\theta|)\right) \\ &\leq \exp\left(\frac{k}{R} + x(\epsilon - r|\cos\theta|)\right) \end{aligned}$$

Define

$$\Gamma_{\epsilon,R} = \mathrm{e}^{\mathrm{i}\theta}[0,R) \cup \mathrm{e}^{\mathrm{i}\theta}[0,R) + \epsilon.$$

Estimate, for $R > \epsilon$,

$$\begin{aligned} &\int_R^\infty |e^{k\ell(\epsilon + r\mathrm{e}^{\mathrm{i}\theta};0)}| \frac{\mathrm{d}r}{|\epsilon + r\mathrm{e}^{\mathrm{i}\theta}|^{\alpha+1}} \leq \int_R^\infty \exp\left(\frac{k}{r-\epsilon} + x(\epsilon - r|\cos\theta|)\right)(r-\epsilon)^{-\alpha-1}\mathrm{d}r \\ &\quad \leq \frac{\exp\left(\frac{k}{R-\epsilon} + x\epsilon\right)}{(R-\epsilon)^{\alpha+1}} \frac{\mathrm{e}^{-Rx|\cos\theta|}}{x|\cos\theta|}. \end{aligned}$$

To summarize, let $R > \epsilon$ let $g(s)$ be analytic in the region

$$\Delta = \{s \in \mathbb{C} \mid s = \epsilon + r\mathrm{e}^{\pm \mathrm{i}\phi}, \quad r \in (0,\infty), \quad \phi \in (-\pi/2, \theta)\},$$

and bounded on the closure. Then

$$\begin{aligned} &\left| \int_{-\mathrm{i}\infty+\epsilon}^{\mathrm{i}\infty+\epsilon} \mathrm{e}^{k\ell(s;0)} \frac{1}{s^{\alpha+1}} g(s)\mathrm{d}s - \int_{\Gamma_{\epsilon,R}} \mathrm{e}^{k\ell(s;0)} \frac{1}{s^{\alpha+1}} g(s)\mathrm{d}s \right| \\ &\quad \leq \sup_{s\in\Gamma_\epsilon} |g(s)| \frac{\exp\left(\frac{k}{R-\epsilon} + x\epsilon\right)}{(R-\epsilon)^{\alpha+1}} \frac{\mathrm{e}^{-Rx|\cos\theta|}}{2x|\cos\theta|}. \end{aligned} \tag{E.2.6}$$

eq:largex-trunc

We will mainly consider this estimate when $x \geq \delta k$, in which case exponential decay is realized for $R|\cos\theta| > \epsilon + \frac{1}{\delta(R-\epsilon)}$.

Next, the stationary phase points are

$$s_\pm(0) = \frac{1 \pm \sqrt{1 - 4\frac{k}{x}}}{2}$$

with

$$x_\pm(0) = (\sqrt{k} \pm \sqrt{k})^2 = 4k,\ 0.$$

Define

$$\mathfrak{l}_{k,p}^{(\alpha)}(x) = \frac{(k+p)!}{(k+p+\alpha)!}\,(4kx)^\alpha\, L_{k+p}^{(\alpha)}(4kx)$$

so that we consider modifications of $x_\pm$ and $s_\pm$

$$\tilde{s}_\pm(0) = \frac{1 \pm \sqrt{1 - \frac{1}{x}}}{2}, \quad \tilde{x}_\pm(0) = 1,\ 0.$$

Bulk asymptotics

Fix $0 < \delta < 1$. We consider the asymptotics of $\mathfrak{l}_k^{(\alpha)}(x)$ for $\delta < x < 1 - \delta$. The following expansions will be uniform by Theorem t:uniform_expansion 118. Indeed, by Theorem t:uniform_expansion 118

$$\begin{aligned}\mathfrak{l}_{k,0}^{(\alpha)}(x) = {} & \frac{x^{\frac{\alpha+1}{2}} 2^\alpha (-1)^k \mathrm{e}^{2kx}}{\sqrt{2\pi k} x^{3/4} (1-x)^{1/4}} \\ & \times \left(\mathsf{Re}\left[\mathrm{i}\mathrm{e}^{-2\mathrm{i}k \arccos\sqrt{x}} \mathrm{e}^{2\mathrm{i}k\sqrt{x(1-x)}} \mathrm{e}^{-\mathrm{i}(\alpha+1)\arccos\sqrt{x}} \mathrm{e}^{3\mathrm{i}\pi/4} \right] + O\left(\frac{1}{\sqrt{k}}\right) \right)\end{aligned}$$

and therefore

$$\begin{aligned}\mathfrak{l}_{k,0}^{(\alpha)}(x) = {} & \frac{x^{\frac{\alpha+1}{2}} 2^\alpha (-1)^k \mathrm{e}^{2kx}}{\sqrt{2\pi k} x^{3/4} (1-x)^{1/4}} \\ & \times \left(\sin\left[(2k+\alpha+1)\arccos\sqrt{x} - 2k\sqrt{x(1-x)} + \frac{\pi}{4} \right] + O\left(\frac{1}{k}\right) \right).\end{aligned}$$

In this calculation we used that

$$2\tilde{s}_+(0) = 1 + \sqrt{1 - \frac{1}{x}} = \frac{\sqrt{x} + \mathrm{i}\sqrt{1-x}}{\sqrt{x}} = \frac{1}{\sqrt{x}} \mathrm{e}^{\mathrm{i}\arccos\sqrt{x}}.$$

More generally, by the method of steepest descent

$$\begin{aligned}\frac{\mathrm{d}^j}{\mathrm{d}x^j} \mathfrak{l}_{k,p}^{(\alpha)}(x) = {} & (4k)^j \frac{x^{\frac{\alpha+1}{2}} 2^\alpha (-1)^k \mathrm{e}^{2kx}}{\sqrt{2\pi k} x^{3/4} (1-x)^{1/4}} \\ \times & \left(\mathsf{Re}\left[\left(1 - \frac{1}{\tilde{s}_+(0)}\right)^p \tilde{s}_+(0)^j \mathrm{i}\mathrm{e}^{-2\mathrm{i}k\arccos\sqrt{x}} \mathrm{e}^{2\mathrm{i}k\sqrt{x(1-x)}} \mathrm{e}^{-\mathrm{i}(\alpha+1)\arccos\sqrt{x}} \mathrm{e}^{3\mathrm{i}\pi/4} \right] + O\left(\frac{1}{k}\right) \right)\end{aligned}$$

and therefore

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\mathfrak{l}_{k,p}^{(\alpha)}(x) = (4k)^j \frac{x^{\frac{\alpha-j+1}{2}} 2^\alpha (-1)^{k+p} \mathrm{e}^{2kx}}{\sqrt{2\pi k} x^{3/4}(1-x)^{1/4}}$$
$$\times \left(\sin\left[(2k-j+\alpha+2p+1)\arccos\sqrt{x} - 2k\sqrt{x(1-x)} + \frac{\pi}{4}\right] + O\left(\frac{1}{k}\right)\right).$$

The quantity that arises in the global eigenvalue distribution is proportional to

$$x^{-\alpha}\mathrm{e}^{-4kx}\left[\mathfrak{l}_{k,0}^{(\alpha)}(x)\frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{l}_{k,-1}^{(\alpha)}(x) - \mathfrak{l}_{k,-1}^{(\alpha)}(x)\frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{l}_{k,0}^{(\alpha)}(x)\right]$$

$$x^{-\alpha}\mathrm{e}^{-4kx}\mathfrak{l}_{k,0}^{(\alpha)}(x)\frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{l}_{k,-1}^{(\alpha)}(x) \sim 4k\frac{2^\alpha(-1)^{2k-1}2^{2\alpha-1}}{x\sqrt{2\pi k}\sqrt{1-x}}$$
$$\times \sin\left[(2k+\alpha+1)\arccos\sqrt{x} - 2k\sqrt{x(1-x)} + \frac{\pi}{4}\right]$$
$$\times \sin\left[(2k+\alpha-2)\arccos\sqrt{x} - 2k\sqrt{x(1-x)} + \frac{\pi}{4}\right]$$

$$x^{-\alpha}\mathrm{e}^{-4kx}\frac{\mathrm{d}}{\mathrm{d}x}\mathfrak{l}_{k,0}^{(\alpha)}(x)\mathfrak{l}_{k,-1}^{(\alpha)}(x) \sim 4k\frac{2^\alpha(-1)^{2k-1}2^{2\alpha-1}}{x\sqrt{2\pi k}\sqrt{1-x}}$$
$$\times \sin\left[(2k+\alpha-1)\arccos\sqrt{x} - 2k\sqrt{x(1-x)} + \frac{\pi}{4}\right]$$
$$\times \sin\left[(2k+\alpha)\arccos\sqrt{x} - 2k\sqrt{x(1-x)} + \frac{\pi}{4}\right]$$

Then set

$$\Theta = (2k+\alpha+1)\arccos\sqrt{x} - 2k\sqrt{x(1-x)} + \frac{\pi}{4},$$
$$\varphi = -\arccos\sqrt{x}.$$

We need to examine the quantity

$$f(\Theta,\varphi) = \sin(\Theta+3\varphi)\sin\Theta - \sin(\Theta+2\varphi)\sin(\Theta+\varphi)$$

and we claim that

$$f(\Theta,\varphi) = -2\cos(-\varphi)\sin^2(-\varphi).$$

This follows using

$$2\sin y\sin z = \cos(x+y) - \cos(x-y).$$

Airy asymptotics

Define

$$\tilde{\ell}(s;\alpha) = \log\left(1 - \frac{1}{s}\right) - \frac{\alpha}{k}\log s + 4xs$$

As $x \to 1$, the stationary phase points, the zeros of $\tilde{\ell}'$, collapse at $s = 1/2$. To capture the behavior there we expand $\tilde{\ell}(s;0)$ about $s = 1/2$

$$\tilde{\ell}(s;0) = \mathrm{i}\pi + 2x + 4(x-1)\left(s - \frac{1}{2}\right) - \frac{16}{3}\left(s - \frac{1}{2}\right)^3 + O\left(s - \frac{1}{2}\right)^5.$$

Then we choose

$$x = 1 + \frac{y}{2^{\frac{2}{3}}k^{\frac{2}{3}}}, \quad s = \frac{1}{2} + \mathrm{i}\frac{r}{2^{\frac{4}{3}}k^{\frac{1}{3}}}.$$

The exponents of $2/3$ and $1/3$ are chosen so that $k\tilde{\ell}(s;0)$, when expanded in powers of r, will have no k-dependence at order $1, 2$ or 3. Indeed,

$$\tilde{\ell}\left(\frac{1}{2} + \mathrm{i}\frac{r}{2^{\frac{4}{3}}k^{\frac{1}{3}}};0\right) = \mathrm{i}\pi + 2x + \frac{\mathrm{i}y}{k}r + \frac{\mathrm{i}}{3k}r^3 + O\left(r^5k^{-5/3}\right).$$

eq:largex-trunc
Using (E.2.6) with $\epsilon = 1/2$, $\theta = -2\pi/3$ and x replaced with $4kx$ it follows that for $R = 3$

$$\begin{aligned}\mathfrak{l}^{\alpha}_{k,p}(x) = &\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/2,3}} \mathrm{e}^{k\tilde{\ell}(s;0)}\left(1 - \frac{1}{s}\right)^p \frac{\mathrm{d}s}{s^{\alpha+1}}\\ &+ O\left(\frac{e^{\frac{2}{5}k - 4kx}}{kx}\right).\end{aligned}$$

Then we expand

$$\left(1 - \frac{1}{s}\right)^p \frac{1}{s^{\alpha+1}} = \frac{(1-s)^p}{s^{\alpha+p+1}} = (-1)^p 2^{\alpha+1}\left[1 - 2(2p+a+1)\left(s - \frac{1}{2}\right)\right] + O\left(s - \frac{1}{2}\right)^2.$$

Therefore

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1/2,3}} \mathrm{e}^{k\tilde{\ell}(s;0)}\left(1 - \frac{1}{s}\right)^p \frac{\mathrm{d}s}{s^{\alpha+1}}\\ &= \frac{(-1)^{k+p}\mathrm{e}^{2kx}2^{\alpha+1}}{2^{\frac{4}{3}}k^{\frac{1}{3}}2\pi}\int_{\hat{\Gamma}_k} \mathrm{e}^{\mathrm{i}yr + \frac{\mathrm{i}}{3}r^3}\left(1 - 2\mathrm{i}(2p+\alpha+1)\frac{r}{2^{\frac{4}{3}}k^{\frac{1}{3}}} + O((r^2+r^5)k^{-2/3})\right)\mathrm{d}r\\ &= \frac{(-1)^{k+p}\mathrm{e}^{2kx}2^{\alpha+1}}{2^{\frac{4}{3}}k^{\frac{1}{3}}}\left(\mathrm{Ai}(y) - (2p+\alpha+1)\frac{\mathrm{Ai}'(y)}{2^{\frac{1}{3}}k^{\frac{1}{3}}} + \frac{G_k(y)}{k^{\frac{2}{3}}}\right)\mathrm{d}r,\end{aligned}$$

where

$$\hat{\Gamma}_k = \mathrm{e}^{\mathrm{i}5\pi/6}[0, 3\cdot 2^{\frac{4}{3}}k^{\frac{1}{3}}] \cup \mathrm{e}^{\mathrm{i}\pi/6}[0, 3\cdot 2^{\frac{4}{3}}k^{\frac{1}{3}}],$$

and $G_k(y)$ is such that $G(y) := \sup_k |G_k(y)|$ decays exponentially as $y \to +\infty$.

Bessel asymptotics

Next, we consider the case where $x \approx 0$. Note that the stationary phase points head off to infinity as $x \to 0$. So, we need to rescale the integral to pull the stationary phase points back to zero. We first send $s \to s/\sqrt{x}$ so that $\mathrm{d}s \to \mathrm{d}s/\sqrt{x}$ and deform back to the contour:

$$\mathfrak{l}_{k,p}^{(\alpha)}(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+\epsilon}^{\mathrm{i}\infty+\epsilon} \left(1 - \frac{1}{s}\right)^k \mathrm{e}^{4kxs} \frac{\mathrm{d}s}{s^{\alpha+1}} = \frac{(x)^{\frac{\alpha}{2}}}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+\epsilon}^{\mathrm{i}\infty+\epsilon} \left(1 - \frac{\sqrt{x}}{s}\right)^{k+p} \mathrm{e}^{4k\sqrt{x}s} \frac{\mathrm{d}s}{s^{\alpha+1}}$$

Then set $\sqrt{x} = c_1\sqrt{y}k^{\gamma}$ and $s = c_2 t k^{\delta}$. We choose c_1, c_2, δ, γ so that $4k\sqrt{x}s = \frac{\sqrt{y}t}{2}$ and $\frac{\sqrt{x}}{s} = \frac{\sqrt{y}}{2kt}$. We find that $\gamma = -1$, $\delta = 0$, $c_2 = 1/2$ and $c_1 = 1/4$. Then,

$$\mathfrak{l}_{k,p}^{(\alpha)}(x) = \frac{(4x)^{\frac{\alpha}{2}}}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+\epsilon}^{\mathrm{i}\infty+\epsilon} \left(1 - \frac{\sqrt{y}}{2kt}\right)^k \mathrm{e}^{\frac{\sqrt{y}t}{2}} \frac{\mathrm{d}t}{t^{\alpha+1}}.$$

Now, this integral must be deformed to reveal its true asymptotic form. Cauchy's Theorem allows us to deform this to a circle C_0, of any radius, with counter-clockwise orientation, that is centered at the origin for integer α. Then, compute

$$\log\left(1 - \frac{\sqrt{y}}{2kt}\right) = -\frac{\sqrt{y}}{2kt} - \frac{1}{2}\left(\frac{\sqrt{y}}{2kt}\right)^2 + O(k^{-3}).$$

$$\begin{aligned}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\mathfrak{l}_{k,p}^{(\alpha)}(x) &= \frac{(4k)^j (4x)^{\frac{\alpha-j}{2}}}{2\pi \mathrm{i}} \int_{C_0} \mathrm{e}^{\frac{\sqrt{y}}{2}\left(t-\frac{1}{t}\right)} \left(1 - \frac{\sqrt{y}}{4tk}\left(2p + \frac{\sqrt{y}}{2t}\right) + O(k^{-2})\right) \frac{\mathrm{d}t}{t^{\alpha-j+1}} \\ &= (4k)^j (4x)^{\frac{\alpha-j}{2}} \left[J_{\alpha-j}(\sqrt{y}) - \frac{p\sqrt{y}}{2k} J_{\alpha-j+1}(\sqrt{y}) - \frac{y}{8k} J_{\alpha-j+2}(\sqrt{y}) + O(k^{-2})\right].\end{aligned} \tag{E.2.7}$$

eq:bessel

Here $J_\alpha(y)$ is the Bessel function of order α

$$J_\alpha(y) := \frac{1}{2\pi \mathrm{i}} \int_{C_0} \mathrm{e}^{\frac{y}{2}\left(t-\frac{1}{t}\right)} \frac{\mathrm{d}t}{t^{\alpha+1}}.$$

E.2.2 Increasing α

Now, we consider the case where

$$\alpha = \left(\frac{1}{d} - 1\right)k + c_k, \quad 0 < d < 1, \quad \sup_k |c_k| < \infty.$$

For simplicity, we impose that c_k must be chosen so that α is an integer. We write

$$\mathfrak{l}_{k,p}^{\alpha}\left(\frac{x}{4\mathfrak{d}}\right) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+\epsilon}^{\mathrm{i}\infty+\epsilon} \mathrm{e}^{k\hat{\ell}(s;\mathfrak{d})} \left(1 - \frac{1}{s}\right)^p \frac{\mathrm{d}s}{s},$$

where $\mathfrak{d}$ is defined by $\mathfrak{d}^{-1} - 1 = \frac{\alpha}{k}$,

$$\hat{\ell}(s;\mathfrak{d}) = \log\left(1 - \frac{1}{s}\right) - \left(\frac{1}{\mathfrak{d}} - 1\right)\log s + \frac{xs}{\mathfrak{d}},$$

and we see that $\mathfrak{d} \to d$ as $k \to \infty$ and therefore $0 < \mathfrak{d} < 1$ for sufficiently large k. The stationary points are then given by

$$\hat{s}_\pm(\mathfrak{d}) = \frac{x + 1 - \mathfrak{d} \pm \sqrt{(x - \lambda_+)(x - \lambda_-)}}{2x}, \quad \lambda_\pm = \lambda_\pm(\mathfrak{d}) = (1 \pm \sqrt{\mathfrak{d}})^2.$$

Trigonometric asymptotics

Fix $\delta > 0$, sufficiently small, and consider $(1 - \sqrt{d})^2 + \delta \leq x \leq (1 + \sqrt{d})^2 - \delta$. This ensures that for sufficiently large k, the two stationary points are both bounded and bounded away from each other. We then find

$$\hat{\ell}''(s;\mathfrak{d}) = \frac{(1 - \mathfrak{d})s^2 - 2s + 1}{\mathfrak{d}s^2(s-1)^2}.$$

To simplify this expression, when $s = \hat{s}_\pm(\mathfrak{d})$ we note that

$$|\hat{s}_\pm(\mathfrak{d})| = \frac{1}{\sqrt{x}}, \quad |1 - \hat{s}_\pm(\mathfrak{d})| = \frac{\sqrt{\mathfrak{d}}}{\sqrt{x}}.$$

Then

$$\sqrt{x}\hat{s}_\pm(\mathfrak{d}) = \frac{x + 1 - \mathfrak{d} \pm \sqrt{(x - \lambda_+)(x - \lambda_-)}}{2\sqrt{x}},$$

so that

$$\hat{s}_\pm(\mathfrak{d}) = \frac{1}{\sqrt{x}} \mathrm{e}^{\pm \mathrm{i} \arccos \frac{x+1-\mathfrak{d}}{2\sqrt{x}}}$$

Similarly,

$$\hat{s}_\pm(\mathfrak{d}) - 1 = \frac{\sqrt{\mathfrak{d}}}{\sqrt{x}} \mathrm{e}^{\pm \mathrm{i} \arccos \frac{1-x-\mathfrak{d}}{2\sqrt{\mathfrak{d}x}}}.$$

We also compute

$$|(1 - \mathfrak{d})\hat{s}_\pm(\mathfrak{d})^2 - 2s_\pm(\mathfrak{d}) + 1| = \frac{\sqrt{\mathfrak{d}}}{x}\sqrt{(x - \lambda_-)(\lambda_+ - x)}.$$

and

$$\begin{aligned}&(1 - \mathfrak{d})\hat{s}_\pm(\mathfrak{d})^2 - 2s_\pm(\mathfrak{d}) + 1 \\ &\qquad = \frac{\sqrt{\mathfrak{d}}}{x}\sqrt{(x - \lambda_-)(\lambda_+ - x)} \exp\left(\mathrm{i}\pi - \mathrm{i}\arcsin\left(\frac{1}{\sqrt{\mathfrak{d}}}\frac{(x + 1 - \mathfrak{d})(1 - \mathfrak{d}) - 2x}{2x}\right)\right)\end{aligned}$$

Then, by the method of steepest descent

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\mathfrak{l}_{k,p}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right)$$
$$= 2\mathsf{Re}\left[\left(\frac{k}{\mathfrak{d}}\right)^j (\hat{s}_+(\mathfrak{d}))^{j-p-1-k/\mathfrak{d}}(\hat{s}_+(\mathfrak{d})-1)^{p+k}\frac{1}{\mathrm{i}\sqrt{2\pi k}}\frac{\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\frac{\theta}{2}\right)+\frac{kx\hat{s}_+(\mathfrak{d})}{\mathfrak{d}}}}{\sqrt{|\ell''(\hat{s}_+(\mathfrak{d}))|}}\left(1+O\left(\frac{1}{k}\right)\right)\right].$$

where

$$\theta = \pi - \arcsin\left(\frac{1}{\sqrt{\mathfrak{d}}}\frac{(x+1-\mathfrak{d})(1-\mathfrak{d})-2x}{2x}\right) - 2\arccos\frac{1-x-\mathfrak{d}}{2\sqrt{\mathfrak{d}x}} - 2\arccos\frac{x+1-\mathfrak{d}}{2\sqrt{x}}.$$

This, more explicitly, is

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\mathfrak{l}_{k,p}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right) \tag{E.2.8}$$

eq:trig-increase

$$= \sqrt{\frac{2}{\pi k}}\left(\frac{k}{\mathfrak{d}}\right)^j \frac{\mathfrak{d}^{3/4}(\sqrt{x})^{\alpha-j}(\sqrt{\mathfrak{d}})^{p+k}\mathrm{e}^{k\frac{x+1-\mathfrak{d}}{2\mathfrak{d}}}}{[(x-\lambda_-)(\lambda_+-x)]^{1/4}}\left(\sin\left[(j-p-k/\mathfrak{d})\arccos\frac{x+1-\mathfrak{d}}{2\sqrt{x}}\right.\right. \tag{E.2.9}$$

$$\left.\left.+(p+k+1)\arccos\frac{1-x-\mathfrak{d}}{2\sqrt{\mathfrak{d}x}} + \frac{k}{2\mathfrak{d}}\sqrt{(\lambda_+-x)(x-\lambda_-)} + \frac{\hat{\theta}}{2}\right] + O\left(\frac{1}{k}\right)\right) \tag{E.2.10}$$

where

$$\hat{\theta} = \arcsin\left(\frac{1}{\sqrt{\mathfrak{d}}}\frac{(x+1-\mathfrak{d})(1-\mathfrak{d})-2x}{2x}\right).$$

Airy asymptotics: Right edge

For $x \approx \lambda_+$ the stationary phase points are close to $s^* := 1/\sqrt{\lambda_+}$. The Taylor expansion of $\hat{\ell}$ near s^* gives

$$\begin{aligned}
\hat{\ell}(s;\mathfrak{d}) &= \mathrm{i}\pi + \frac{(1-\mathfrak{d})\log\lambda_+}{2\mathfrak{d}} + \frac{x}{\mathfrak{d}\sqrt{\lambda_+}} + \log\left(\sqrt{\lambda_+}-1\right) \\
&+ \left(\frac{(\mathfrak{d}-1)\sqrt{\lambda_+}}{\mathfrak{d}} + \frac{x}{\mathfrak{d}} - \frac{\lambda_+}{\sqrt{\lambda_+}-1}\right)(s-s^*) \\
&+ \frac{1}{2}\lambda_+\left(\frac{1}{\mathfrak{d}} - \frac{1}{\left(\sqrt{\lambda_+}-1\right)^2}\right)(s-s^*)^2 \\
&+ \frac{1}{3}\lambda_+^{3/2}\left(-\frac{1}{\mathfrak{d}} - \frac{1}{\left(\sqrt{\lambda_+}-1\right)^3}\right)(s-s^*)^3 \\
&+ \frac{1}{4}\lambda_+^{2}\left(\frac{1}{\mathfrak{d}} - \frac{1}{\left(\sqrt{\lambda_+}-1\right)^4}\right)(s-s^*)^4 \\
&+ O(s-s^*)^5.
\end{aligned}$$

Using that $\sqrt{\lambda_+} - 1 = \sqrt{\mathfrak{d}}$ this expression is simplified to

$$\begin{aligned}\hat{\ell}(s;\mathfrak{d}) &= \mathrm{i}\pi + \frac{(1-\mathfrak{d})\log\lambda_+}{2\mathfrak{d}} + \frac{x}{\mathfrak{d}\sqrt{\lambda_+}} + \log\left(\sqrt{\lambda_+}-1\right)\\ &+ \left(\frac{x-\lambda_+}{\mathfrak{d}}\right)(s-s^*)\\ &+ \frac{1}{3}\lambda_+^{3/2}\left(-\frac{1}{\mathfrak{d}} - \frac{1}{\mathfrak{d}^{3/2}}\right)(s-s^*)^3\\ &+ \frac{1}{4}\lambda_+^2\left(\frac{1}{\mathfrak{d}} - \frac{1}{\mathfrak{d}^2}\right)(s-s^*)^4\\ &+ O(s-s^*)^5.\end{aligned}$$

The strategy is to shift and scale x and s so that the first and third order terms are on the same order and have simple coefficients. Perform a change of variables

$$x = \lambda_+ + \frac{y}{k^\delta}\sqrt{\mathfrak{d}}\lambda_+^{2/3}, \quad s = s^* + \mathrm{i}\frac{t}{k^\gamma}\frac{\sqrt{\mathfrak{d}}}{\lambda_+^{2/3}}. \tag{E.2.11}$$

eq:right-rescale

with the constraint that $\delta + \gamma = 3\gamma$. And since one must consider $\mathrm{e}^{k\hat{\ell}(s;\mathfrak{d})}$, take $\gamma = 1/3$ implying $\delta = 2/3$. The truncation of the integral formula for $\mathfrak{l}^{\alpha}_{k,p}\left(\frac{x}{4\mathfrak{d}}\right)$ eq:largex-trunc to a small finite contour $\Gamma_{1/\sqrt{\lambda_+},\epsilon}$ of integration can be justified using (E.2.6) at the cost of exponentially small errors. Then using

$$\left(1-\frac{1}{s}\right)^p s^{j-1} = (-1)^p\mathfrak{d}^{p/2}\lambda_+^{\frac{1-j}{2}}\left(1 - \sqrt{\frac{\lambda_+}{\mathfrak{d}}}\left[\sqrt{\mathfrak{d}}(1-j) + \sqrt{\lambda_+}p\right](s-s^*)\right) + O(s-s^*)^2$$

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\mathfrak{l}^{(\alpha)}_{k,p}\left(\frac{x}{4\mathfrak{d}}\right) = \exp\left(k\left(\mathrm{i}\pi + \frac{(1-\mathfrak{d})\log\lambda_+}{2\mathfrak{d}} + \frac{x}{\mathfrak{d}\sqrt{\lambda_+}} + \log\left(\sqrt{\lambda_+}-1\right)\right)\right) \tag{E.2.12}$$

$$\begin{aligned}&\times\left((-1)^p\frac{\mathfrak{d}^{\frac{p+1}{2}-j}\lambda_+^{\frac{1-j}{2}-\frac{2}{3}}}{2\pi k^{1/3}}\int_C \mathrm{e}^{\mathrm{i}yt+\frac{\mathrm{i}}{3}y^3}\left(1 - \mathrm{i}\sqrt{\frac{\lambda_+}{\mathfrak{d}}}\left[\sqrt{\mathfrak{d}}(1-j)+\sqrt{\lambda_+}p\right]t\frac{\sqrt{\mathfrak{d}}}{k^{1/3}\lambda_+^{2/3}}\right.\right.\\ &\left.\left.+\ \frac{1}{4\lambda_+^{2/3}}(d-1)t^4 + O(t^2k^{-2/3})\right)\mathrm{d}t\right)\\ &= (-1)^{p+k}\lambda_+^{\frac{1-j}{2}-\frac{2}{3}}\frac{\mathfrak{d}^{\frac{p+k-2j+1}{2}}}{k^{1/3}}\exp\left(\frac{k(1-\mathfrak{d})\log\lambda_+}{2\mathfrak{d}} + \frac{kx}{\mathfrak{d}\sqrt{\lambda_+}}\right)\left(\mathrm{Ai}(y)\right.\\ &\left.-\frac{1}{k^{\frac{1}{3}}\lambda_+^{\frac{1}{6}}}\left[\sqrt{\mathfrak{d}}(1-j)+p\sqrt{\lambda_+}\right]\mathrm{Ai}'(y) + \frac{1}{4\lambda_+^{2/3}}(d-1)\mathrm{Ai}^{(4)}(y) + \frac{G(y;\mathfrak{d},k,p)}{k^{\frac{2}{3}}}\right)\end{aligned} \tag{E.2.13}$$

eq:laguerre-right

where $C = \left(\mathrm{e}^{-\mathrm{i}\phi}(-\infty,0] \cup \mathrm{e}^{\mathrm{i}\phi}[0,\infty)\right) \frac{\lambda_+^{2/3}}{\sqrt{d}} k^{2/3}$ and $|G(y;\mathfrak{d},k,p))| \leq G_{\mathfrak{d}}(y)$ decays exponentially as $t \to \infty$.

Airy asymptotics: Left edge

Following similar arguments as in the previous section, for $x \approx \lambda_-$ the stationary phase points are close to $s^* := 1/\sqrt{\lambda_-}$. The Taylor expansion of $\hat{\ell}$ near s^* now gives

$$\begin{aligned}\hat{\ell}(s;\mathfrak{d}) = &\frac{(1-\mathfrak{d})\log\lambda_-}{2\mathfrak{d}} + \frac{x}{\mathfrak{d}\sqrt{\lambda_-}} + \log\left(1-\sqrt{\lambda_-}\right)\\ &+ \left(\frac{x-\lambda_-}{\mathfrak{d}}\right)(s-s^*)\\ &+ \frac{1}{3}\lambda_-^{3/2}\left(-\frac{1}{\mathfrak{d}} + \frac{1}{\mathfrak{d}^{3/2}}\right)(s-s^*)^3\\ &+ \frac{1}{4}\lambda_-^2\left(\frac{1}{\mathfrak{d}} - \frac{1}{\mathfrak{d}^2}\right)(s-s^*)^4 + O(s-s^*)^5.\end{aligned}$$

In this case, we set

$$x = \lambda_- + \frac{y}{k^{\frac{2}{3}}}\sqrt{\mathfrak{d}}\lambda_-^{\frac{2}{3}}, \quad s = s^* + \mathrm{i}\frac{t}{k^{\frac{1}{3}}}\frac{\sqrt{\mathfrak{d}}}{\lambda_-^{\frac{2}{3}}}. \tag{E.2.14}$$ $\boxed{\texttt{eq:left-rescale}}$

to find

$$\begin{aligned}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\mathfrak{l}_{k,p}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right) = \lambda_-^{\frac{1-j}{2}-\frac{2}{3}}\frac{\mathfrak{d}^{\frac{p+k-2j+1}{2}}}{k^{1/3}}\exp\left(\frac{k(1-\mathfrak{d})\log\lambda_-}{2\mathfrak{d}} + \frac{kx}{\mathfrak{d}\sqrt{\lambda_-}}\right)\Bigg(\mathrm{Ai}(-y)\\ -\frac{1}{k^{\frac{1}{3}}\lambda_-^{\frac{1}{6}}}\left[\sqrt{\mathfrak{d}}(1-j)+\sqrt{\lambda_-}p\right]\mathrm{Ai}'(-y) + \frac{\mathfrak{d}-1}{4\lambda_-^{2/3}k^{\frac{1}{3}}}\mathrm{Ai}^{(4)}(-y) + \frac{R(y;\mathfrak{d},k,p)}{k^{\frac{2}{3}}}\Bigg)\end{aligned} \tag{E.2.15}$$ $\boxed{\texttt{eq:laguerre-left}}$

where $|R(y;\mathfrak{d},k,p))| \leq R_{\mathfrak{d}}(y)$ decays exponentially as $y \to \infty$.

Decay estimates: Right edge

We now look at the behavior of Laguerre polynomials off the interval $[\lambda_-(d), \lambda_+(d)]$ for $d \in (0,1]$. Suppose $x \geq \lambda_+(d) + \delta$ for some $\delta > 0$. The method of steepest descent using only $\hat{s}_-(\mathfrak{d})$ gives

$$\begin{aligned}\mathfrak{l}_{k,p}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right) = &\frac{1}{\sqrt{2\pi k|\hat{\ell}''(\hat{s}_-(\mathfrak{d});\mathfrak{d})|}}\left(1-\frac{1}{\hat{s}_-(\mathfrak{d})}\right)^{k+p}\\ &\times \hat{s}_-(\mathfrak{d})^{(1-\mathfrak{d}^{-1})k-j-1}\mathrm{e}^{\frac{kx\hat{s}_-(\mathfrak{d})}{\mathfrak{d}}}\left(1+O(k^{-1/2})\right),\end{aligned}$$

where the error term is uniform on closed subsets of $(\lambda_+(d), \infty)$ supposing that $\mathfrak{d} \to d \in (0,1]$ as $n \to \infty$. Here one uses Theorem 118. One has to be careful though because the stationary phase point $\hat{s}_-(\mathfrak{d})$ approaches the origin where the phase function $\hat{\ell}$ is singular. In the language of Theorem 118, for some $c > 0$

$$C \sim x, \quad \delta \sim x^2, \quad \epsilon \sim 1/x.$$

Then

$$M \sim 1, \quad \delta' \sim 1, \quad \delta\epsilon \sim x,$$

and therefore the expansion is valid for all $x > \lambda_+(d)$.

Then, to utilize this expression, consider the combination, using Stirling's approximation

$$\begin{aligned}
&\sqrt{\frac{(k+\alpha)!}{k!}} \mathrm{e}^{k\hat{l}(s_-(\mathfrak{d});\mathfrak{d})} \left(\frac{kx}{\mathfrak{d}}\right)^{-\frac{\alpha}{2}} \mathrm{e}^{-\frac{kx}{2\mathfrak{d}}} \\
&= \mathrm{e}^{k\hat{l}(s_-(\mathfrak{d});\mathfrak{d})} \left(\frac{kx}{\mathfrak{d}}\right)^{-\frac{\alpha}{2}} \mathrm{e}^{-\frac{kx}{2\mathfrak{d}}} \mathfrak{d}^{-\frac{k}{2}} \left(\frac{k}{\mathfrak{d}}\right)^{\frac{\alpha}{2}} \mathrm{e}^{-\frac{\alpha}{2}} \left(1 + O(k^{-1})\right), \\
&= \exp\left(k \left[\log\left[\left(1 - \frac{1}{s_-(\mathfrak{d})}\right)\mathfrak{d}^{-\frac{1}{2}}\right] - \frac{\alpha}{k}\log\left[s_-(\mathfrak{d})\sqrt{x}\right] - \frac{\sqrt{(x-\lambda_+)(x-\lambda_-)}}{2\mathfrak{d}}\right]\right), \\
&=: (-1)^k e^{k\gamma(x;\mathfrak{d})}.
\end{aligned} \tag{E.2.16}$$

eq:lag_decay

It is easy to check that when $x = \lambda_+$ this expression is equal to unity, i.e., $\gamma(\lambda_+;\mathfrak{d}) = 0$. Furthermore, one can readily check that for fixed $\mathfrak{d}$, γ is a strictly decreasing function of x and $\lim_{x\to\infty} \gamma'(x;\mathfrak{d}) = -\frac{1}{2\mathfrak{d}}$. This implies that the leading-order behavior of

$$\sqrt{\frac{(k+\alpha+p)!}{(k+p)!}} \frac{\mathrm{d}^j}{\mathrm{d}x^j} \mathfrak{l}_{k,p}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right)\left(\frac{kx}{\mathfrak{d}}\right)^{-\frac{\alpha}{2}} \mathrm{e}^{-\frac{kx}{2\mathfrak{d}}}$$

is that of exponential decay for $x \geq \lambda_+ + \delta$, as $k \to \infty$. The rate of decay increases as x increases.

The previous estimate can be used, to some effect, down to $x = \lambda_+ + O(n^{-2/3})$. Because then $\hat{\ell}''(s_-(\mathfrak{d}), \mathfrak{d}) = O(n^{-1/3})$. We can then use Remark 119 to state that for $\lambda_+ + Cn^{-2/3} \leq x \leq \lambda_+ + \delta$

$$\left|\sqrt{\frac{(k+\alpha+p)!}{(k+p)!}} \frac{\mathrm{d}^j}{\mathrm{d}x^j} \mathfrak{l}_{k,p}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right)\left(\frac{kx}{\mathfrak{d}}\right)^{-\frac{\alpha}{2}} \mathrm{e}^{-\frac{kx}{2\mathfrak{d}}}\right| \leq \frac{C_{p,j}}{k^{\frac{1}{3}}} \mathrm{e}^{k\gamma(x;\mathfrak{d})}\left(1 - \frac{1}{\hat{s}_-(\mathfrak{d})}\right)^p \hat{s}_-(\mathfrak{d})^{-j-1}. \tag{E.2.17}$$

eq:decay_right

Decay estimates: Left edge

In the case $d \in (0,1)$ we perform an analysis for $x \in (0, \lambda_- - \delta]$ for δ small. Then we extend the calculation, as in the previous section to allow δ to decrease obtaining only bounds.

The method of steepest descent using only $\hat{s}_+(\mathfrak{d})$ gives

$$\mathfrak{l}_{k,p}^{(\alpha)}\left(\frac{x}{4\mathfrak{d}}\right) = \frac{1}{\sqrt{2\pi k|\hat{\ell}''(\hat{s}_+(\mathfrak{d});\mathfrak{d})|}}\left(1-\frac{1}{\hat{s}_+(\mathfrak{d})}\right)^{k+p} \\ \times \hat{s}_+(\mathfrak{d})^{(1-\mathfrak{d}^{-1})k-j-1}\mathrm{e}^{\frac{kx\hat{s}_+(\mathfrak{d})}{\mathfrak{d}}}\left(1+O(k^{-1/2})\right).$$

To establish that the error term is uniform all the way down to $x = 0$, we note that $s_+(\mathfrak{d}) \to +\infty$ as $x \to 0$. Then, again, in the language of Theorem t:uniform_expansion 118, for some $c > 0$

$$C \sim x, \quad \delta \sim x^2, \quad \epsilon \sim 1/x.$$

Then, as before,

$$M \sim 1, \quad \delta' \sim 1, \quad \delta\epsilon \sim x.$$

So, $\delta\epsilon$ has the same order as $\sqrt{\delta}$ and $\sqrt{|\hat{\ell}''(\hat{s}_+(\mathfrak{d});\mathfrak{d})|}$ and this implies that the error term is uniform on $(0, \lambda_- - \delta]$. We leave it as an exercise to follow the prescription in the previous section and derive the analogs of expression (eq:lag_decay E.2.16) and estimate (eq:decay_right E.2.17).

Doing this more carefully is absolutely critical for condition number tail bounds, should we do it?

Index

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