

Statistical theories of turbulence

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Overview

The purpose of these notes is to introduce the reader to some topics of current interest in mathematics, that are related in a broad sense to turbulence in fluids. The fundamental physical example of turbulence relates to the behavior of incompressible fluids at very high Reynolds number. However, very little is known about this problem that is based on first principles. That is, there are few results that connect the underlying partial differential equations (the Euler and Navier-Stokes equations) to the main empirical regularities observed in turbulence, especially isotropic, homogeneous turbulence.

The central theme in these notes is the development of a mathematical understanding of *ensembles of turbulent flows*. This involves three different areas: the analysis of nonlinear partial differential equations; stochastic processes; and statistical physics. The relation between these areas in the study of turbulence, is roughly as follows. First, the study of the equations of fluid mechanics, and reduced models inspired by fluid mechanics, has been a central concern in partial differential equations since the 1930s. The study of stochastic processes is the branch of probability theory devoted to the analysis of random functions. It is of central importance here, since in order to formulate the notion of a ‘solution chosen at random’, we must understand how to associate probabilities to solutions of differential equations. More formally, a *turbulent ensemble* is a probability measure supported on a set of solutions to an underlying family of partial differential equations, such as the Euler or Navier-Stokes equations. Finally, the choice of models, ensembles and questions must be guided by the principles of statistical physics.

These notes are primarily directed at mathematicians. However, I hope they will also be of value to readers in the sciences and engineering. To this end, I have tried to eschew jargon and technicalities, and to provide an informal sense of why mathematicians study problems the way they do. This is particularly important in the study of nonlinear partial differential equations, since often the very notion of what constitutes a solution to the equations requires care. This is not pedantry: the non-existence of solutions typically corresponds to interesting physical singularities such as the formation of shocks, or the blow-up of vorticity.

The notes are structured in a set of self-contained modules to assist readers with varied backgrounds. These modules are based on three distinct approaches to the problem, due to Hopf, Kolmogorov and Onsager. Their work provides a ‘framework’ for the problem, though at present there are many gaps in the

implementation of this framework for three-dimensional fluid flows (i.e. real fluids). For these reasons, the remainder of the notes are devoted to the analysis of model problems, each of which while approximate, possesses some ‘solvability’, challenges, and riches, of its own.

Here are the topics in more detail:

1. Onsager’s approach in 2D. Mean-field equations and their analysis.
2. Dimensional analysis and Kolmogorov scaling in 3D.
3. Onsager’s approach in 3D. Critical regularity and energy conservation.
4. Hopf’s method. Hierarchies of equations for the evolution of moments, the closure problem and positivity criterion.
5. Gaussian ensembles: Wick’s lemma and Feynman diagrams; closure for linear PDE, ‘non-closure’ for nonlinear PDE.
6. 1D: PDE theory for conservation laws; closure and statistical theory.
7. 2D Euler revisited: Arnold’s least action principle. Zeitlin’s model. Brenier’s approach.
8. Kraichnan’s Direct Interaction Approximation (DIA) and linear response theory. The kinetics of spherical spin glasses.

For both the 1D and 2D models, the notes contain a description of the underlying PDE theory before a statistical theory for solutions is developed.

The modular structure here is roughly as follows: the work of Kolmogorov, Hopf and Onsager forms a core set of ideas that motivates everything else. Gaussian ensembles provide the simplest tractable example of random fields – these are quite inconsistent for turbulence, but it is necessary to understand why. The 1D models illustrate an elegant resolution of the closure problem. They also connect to the study of stochastic particle systems and the KPZ equation. The 2D models are important for oceanographic flows, and nicely illustrate the connection with equilibrium statistical mechanics. Finally, Kraichnan’s DIA model was motivated by turbulence, but is also of interest in phase transitions, since it is an early example of Mezard and Parisi’s celebrated replica method in spin glass theory.

Suggested reading

It is impossible to present a self-contained treatment of the topics here. While I will aim to present the essential details in the notes, additional reading is essential. The following sources are very useful as concurrent reading.

1. V. Šverák, *Lectures on fluid mechanics*. Available online at the author's website. This is a wonderful introduction to mathematical problems in fluid mechanics by one of the leading researchers in the subject. The lectures focus on a wide range of fluid phenomena, in parallel with a discussion of mathematical challenges. The lectures contain an excellent presentation of certain aspects of turbulence, in particular the inverse cascade in 2D, the scaling argument underlying Kolmogorov's theory and Hopf's example of a loss of stability through a cascade of bifurcations. It does not include any stochastic models, however, or a description of Hopf's functional equation [16].
2. A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow* [21]. This book originated in lectures by Majda at Princeton and NYU. It contains simple and complete proofs for many basic problems in fluid mechanics, related to the well-posedness theory for the Euler and Navier-Stokes equations, as well as singularity formation. As the title makes clear, the focus is on the role of vorticity.
3. V. Elser, *Three lectures on statistical mechanics*. This recent set of lectures provides a quick introduction to the main ideas of equilibrium statistical mechanics.
4. A.J. Chorin, *Vorticity and turbulence* [9]. This book contains a description of some stochastic models in 2D and 3D turbulence, building up to the statistical mechanics of vortex filaments. While there is not a great deal of overlap with these notes (except the 2D theory), the book contains several interesting ideas.
5. U. Frisch, *Turbulence* [13]. This book presents a phenomenological approach to turbulence from a physicist's perspective. It contains an interesting overview of several questions of interest, without much mathematical detail.

6. G.L. Eyink and K. R. Sreenivasan, *Onsager and the theory of hydrodynamic turbulence* [12]. This is a review paper that expands on Onsager's article, by including several additional documents. It may be viewed as a counterpoint to Frisch's book, which emphasizes Kolmogorov's scaling theory, rather than Onsager's ideas.
7. Finally, most of the primary sources in the subject are available online and well worth reading. These include the papers by Burgers [7, 5, 6], Chandrasekhar [8], Hopf [16], Kolmogorov [18], Onsager [27], and Taylor [29] as well as reviews by Batchelor [3] and von Neumann [31].

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Chapter 1

Phenomena and equations

1.1 Phenomena

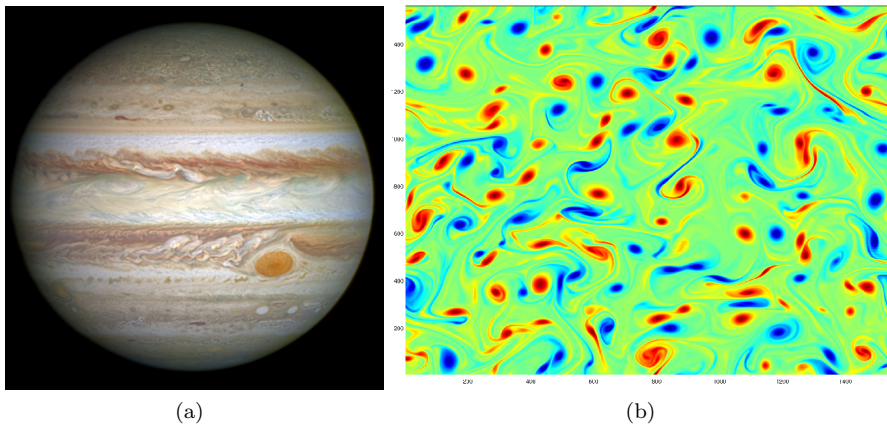


Figure 1.1.1: **2D turbulence**. (a) Turbulence on a thin shell– the Giant Red Spot on Jupiter (Wikimedia commons); (b) A snapshot of the distribution of vorticity in a two dimensional flow with small viscosity. The vortices are observed to coalesce into larger and larger vortices, see [1].

The interplay between order and disorder in turbulent flows is best illustrated with images of fluid flows. The Great Red Spot on Jupiter is an anticyclonic storm that has been raging for at least a few hundred years. The dominant feature of the atmospheric flow in the Red Spot is a giant, stable vortex. It has been speculated that this vortex was formed by the coalescence of many smaller vortices. Such a process is illustrated by numerical computations of two-dimensional (2D) fluid flows with small viscosity in Figure 1.1.1. The structure

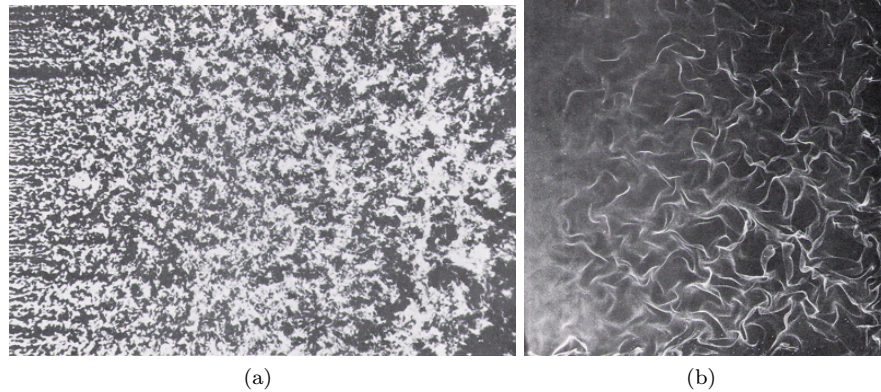


Figure 1.1.2: **Turbulence in incompressible fluids in 3D.** (a) Decaying turbulence behind a grid. Streamlines are visualized by wires that release thin tendrils of smoke into the flow. An initially laminar flow grows into a fully developed turbulent flow. (b) Wrinkling of fluid surfaces. An electric current through a thin, straight platinum wire releases a continuous sheet of hydrogen bubbles that is deformed by the flow. Both figures are taken from Van Dyke's *Album of Fluid Motion* [30].

of turbulent flows in three dimensions (3D) is more subtle. While vorticity continues to play a central role, it is not as obvious visually – the vortices stretch and twist as seen in Figure 1.1.2.

The viewpoint that is adopted in these notes is that the analysis of the Navier-Stokes and Euler equations forms a natural starting point for the study of turbulence. This viewpoint is natural because these equations are firmly founded in classical physics. They are derived in a direct manner from Newton's laws, using the geometry of fluid flows and a constitutive relation between viscosity and shear as explained in books on fluid mechanics [4, 14]. However, it is not known if solutions to these equations form singularities. If so, the resolution of these singularities would cause a paradigm shift in our understanding of fluid flows. A lack of detailed understanding of the Navier-Stokes and Euler equations immediately impacts any attempt to combine the analysis with probability theory to develop interesting turbulent ensembles. For this reason, an important role in the theory is played by simplified partial differential equations in one and two dimensions. These simplified models help illustrate one of the advantages of starting with the partial differential equations: the methods developed to study turbulence have applications well beyond fluid mechanics. Let us illustrate this idea with an example.

The generation and propagation of disorder is of central interest in condensed matter physics, particularly in the study of critical phenomena and phase transitions. Figure 1.1.3 illustrates one such example: grain boundary evolution.

In the simplest setting, the grains are topological polygons in the plane, separated by smooth curves, which meet at perfect 120° degree triple junctions. Each curve evolves by motion by mean curvature – i.e., the normal velocity of a point on the curve is proportional to its curvature. The net effect of this motion is that domains with five sides and fewer vanish, and the grain boundary network coarsens. As we shall see below, one of the simplest tractable models of turbulence introduced by Burgers to describe turbulence in fluids, is a better phenomenological model of domain coarsening, than it is of homogeneous, isotropic turbulence as in Figure 1.1.2.

A common theme in phase transitions and turbulence is the balance between *microstructure* and *empirical scaling laws*. By microstructure, we mean the complexity of individual fluid flows, or the geometric and topological complexity of grain boundaries as shown, for example, in Figures 1.1.1– 1.1.3. It is a remarkable, and fundamental, physical fact that in all these instances the complex microstructure coexists with simple and robust empirical scaling laws. Two examples of such scaling laws in turbulent flows are illustrated in Figure 1.1.4. The first of these is the celebrated Kolmogorov spectrum in isotropic, homogeneous turbulence. The second image is a less well-known spectrum – the distribution of kinetic energy among different modes in the atmosphere.

The coexistence of complex microstructure, but simple scaling laws, is mediated by a statistical viewpoint. Figures 1.1.2– 1.1.3 should be viewed as snapshots of typical representatives of ensembles of fluid flows or material microstructures that satisfy both the governing equations and the empirical scaling laws. The central theme of these notes is to understand the challenges implicit in constructing such ensembles starting from the equations of continuum physics. Our hope is that such a study will shed light on what is at present a vast gap between our understanding of the fundamental laws of fluid mechanics and the empirically observed universal scaling laws of turbulence.

In order to make meaningful progress from such a starting point, one must confront both analytic and probabilistic difficulties. First, it is necessary to understand the underlying partial differential equations. In technical terms, this means that we require well-posedness theorems with minimal regularity of initial data. Second, it is challenging to construct random fields in 2D and 3D such as those shown in Figures 1.1.1–1.1.3. To illustrate these issues, let us consider what is required for a rigorous analysis of Figure 1.1.3. Our tasks include: (a) finding a good ‘coordinate system’ for the combinatorial topology and geometric embedding of the network; (b) a tractable probability measure on such networks; (c) a precise understanding of motion by mean curvature of networks satisfying the Herring boundary condition that includes grain deletion; and finally, (d) an understanding of the flow of probability measures on random networks induced by the deterministic evolution of an initial random network. As stated, this problem is currently out of reach. But there is cause for optimism and plenty of room for questions of intermediate difficulty that are mathematically tractable and provide real physical insight.

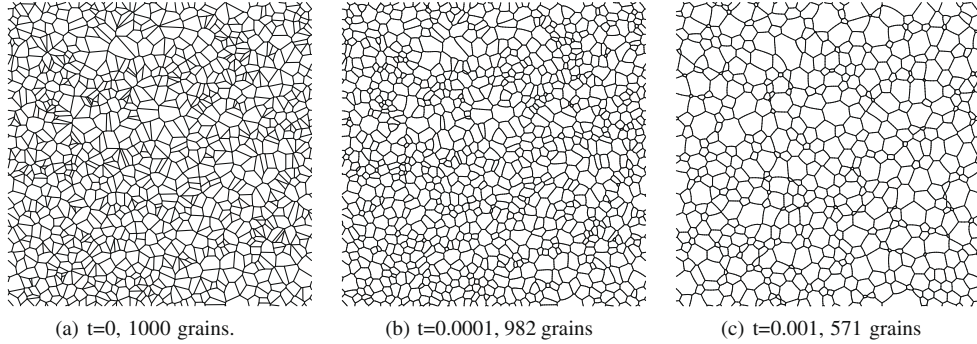


Figure 1.1.3: **Coarsening of a random 2D grain boundary network.** Motion by mean curvature of a network of individual curves with the Herring boundary condition (all curves meet at 120° in triple junctions, except when a polygon vanishes). The initial network is chosen at random. It is observed that the mean area of the grains grows linearly in time, for random initial conditions. Numerical simulations by Lazar, MacPherson, and Srolovitz [19].

1.2 Equations

We now turn to the basic equations of fluid flow for an incompressible fluid with constant density. In all that follows, we assume the fluid occupies a fixed domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with boundary $\partial\Omega$. It is convenient to first ignore the effect of boundaries (though these are very important in practice). A special role is played by the domains $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{T}^d$. In particular, Fourier analysis of the fluid equations is simplest in these domains. The velocity field is a map $\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ for some positive time $T > 0$. The Navier-Stokes equations for a fluid with constant density ρ and viscosity μ are

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \Delta \mathbf{u}, \quad (1.2.1)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (1.2.2)$$

The first equation expresses conservation of momentum (Newton's law). The second equation expresses conservation of mass (incompressibility). The differential expression on the left-hand side of (1.2.1) is the convective derivative which is also denoted

$$\frac{D\mathbf{u}}{Dt} := \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}. \quad (1.2.3)$$

The expression on the right-hand-side of (1.2.1) is the net force on a fluid particle. These include a pressure gradient, ∇p , and viscous shear stresses.

The Navier-Stokes equations may be recast in non-dimensional form as follows. We choose mass, space and time scales M , L and T respectively, and use these to derive the associated velocity, density and force scales L/T , M/L^3 , ML/T^2 . The length scales L and T are typically set by the geometry of the

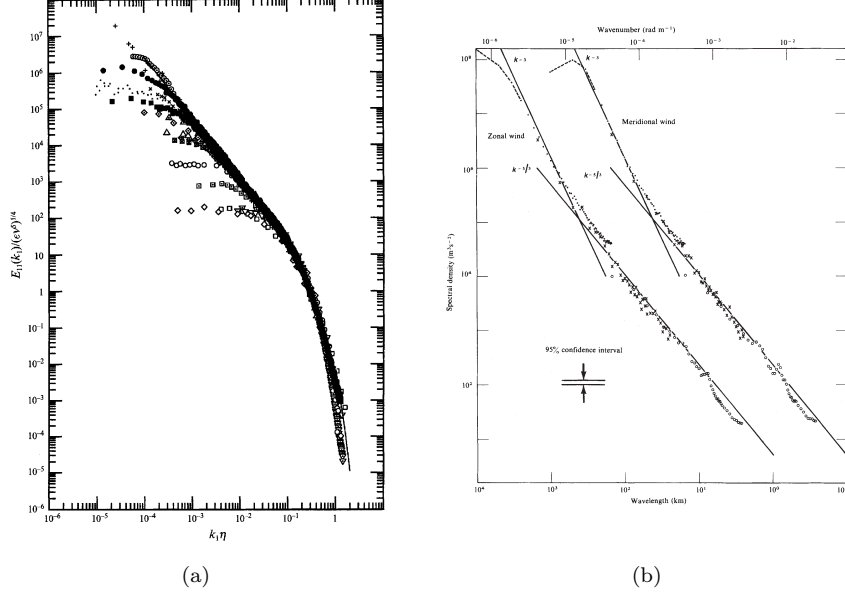


Figure 1.1.4: **Energy spectra of turbulent flows.** (a) The Kolmogorov spectrum – time averaged energy density in Fourier modes as measured in wind-tunnel experiments [28]. Experimental data from many sets of experiments are shown to collapse onto one curve in the inertial range, where the slope is $E(k) \sim k^{-5/3}$; (b) Time averaged energy density of winds in the tropopause [26]. There are two ranges: the one on the left has slope k^{-3} , this is followed by a range $k^{-5/3}$.

flow, and as we will see below, the mass-scale drops out when we assume that the fluid has constant density. We then form the non-dimensional space and time variables, velocity and pressure

$$\tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{\mathbf{u}} = \frac{T}{L}\mathbf{u}, \quad \tilde{p} = \frac{LT^2}{M}p. \quad (1.2.4)$$

The tilde has been introduced for clarity. We now drop the tilde sign and rewrite the Navier-Stokes equations (1.2.1)–(1.2.2) in non-dimensional form

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u}, \quad (1.2.5)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (1.2.6)$$

where the Reynolds number, Re , is the non-dimensional ratio

$$Re = \frac{\rho L^2}{\mu T} = \frac{L^2}{\nu T}, \quad (1.2.7)$$

and the ratio $\nu = \mu/\rho$ is called the kinematic viscosity.

The Reynolds number admits many different interpretations, all of which express the fact that it is the ratio of inertial effects to viscous effects in the flow. For example, in a flow through a pipe or shear flow, it is conventional to define L to be the width of the pipe or channel, and to set $T = L/U$, where U is the mean velocity at the inlet of the pipe or the relative velocity between the plates. We then find

$$Re = \frac{L^2/\nu}{L/U} := \frac{\tau_{\text{viscous}}}{\tau_{\text{inertia}}}, \quad (1.2.8)$$

where the numerator, τ_{viscous} , and the denominator τ_{inertia} , define time-scales on which viscous and inertial effects manifest themselves. The main characteristic of turbulent flows is that we do not see viscous effects for a long time. Thus, our interest lies in the limit $Re \rightarrow \infty$.

An ideal fluid has no viscosity. The equations for an ideal fluid were formulated by Euler, before Navier and Stokes formulated the notion of a constitutive relation for viscous fluids. But these models are closely related, and Euler's equations for an ideal fluid may be obtained from the Navier-Stokes equation by dropping the viscous term

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p, \quad (1.2.9)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (1.2.10)$$

Formally, Euler's equations are obtained by setting $Re = \infty$, but this is a singular limit. In particular, viscous and inviscid flows satisfy fundamentally different boundary conditions. The microscopic mechanisms that create viscosity impose a no-slip boundary condition for viscous flows and the Navier-Stokes equations (1.2.5)–(1.2.6) must be augmented by the boundary condition

$$\mathbf{u}(x, t) = 0, \quad x \in \partial\Omega. \quad (1.2.11)$$

An ideal fluid, however, can slide along a boundary since there is no friction, but it cannot penetrate the boundary. Thus, the Euler equations are augmented with the boundary condition

$$\mathbf{u}(x, t) \cdot \mathbf{n}(x) = 0, \quad x \in \partial\Omega, \quad (1.2.12)$$

where $\mathbf{n}(x)$ denotes the unit normal to the boundary $\partial\Omega$. When the Reynolds number is large, despite the fact that inertial effects may dominate the bulk of the flow, the structure of the flow depends crucially on the behavior of the flow near the boundary. The description of turbulent boundary layers is beyond current mathematical methods (as indeed is much of turbulence without boundaries!). For these reasons, we will not consider the effects of boundaries in these lectures.

1.3 Vorticity

The vorticity, $\boldsymbol{\omega}$, is the curl of the velocity field. For two dimensional flows, the vorticity is a scalar field, and in three dimensions it is a vector field¹. Both the Euler and the Navier-Stokes equations may be expressed completely in terms of the vorticity. This follows from two properties of (1.2.5)–(1.2.6). The first is that the linear system

$$\nabla \times \mathbf{u} = \boldsymbol{\omega}, \quad \nabla \cdot \mathbf{u} = 0, \quad (1.3.1)$$

may be inverted to recover a velocity field from its vorticity. The second is that the curl of (1.2.5) yields a transport equation for the vorticity.

It is easiest to invert (1.3.1) for flows in \mathbb{R}^d (see the Exercises for bounded domains). We take the curl of the first equation in (1.3.1), and use the vector identity $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \Delta \mathbf{u} = -\Delta \mathbf{u}$, to obtain the equation

$$-\Delta \mathbf{u} = \nabla \times \boldsymbol{\omega}. \quad (1.3.2)$$

We solve equation (1.3.2) using the fundamental solution of Poisson's equation, and integrate by parts to obtain the Biot-Savart law (Section 1.4 below).

$$\mathbf{u}(x, t) = \int_{\mathbb{R}^d} K_d(x - y) \boldsymbol{\omega}(y, t) dy, \quad d = 2, 3, \quad (1.3.3)$$

$$K_2(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \quad K_3(x)h = \frac{1}{4\pi} \frac{x \times h}{|x|^3}, \quad x, h \in \mathbb{R}^3, \quad (1.3.4)$$

Here K_2 is a vector and K_3 is a matrix whose action on a vector h is given by the formula above. We have also used the notation $x^\perp = (-x_2, x_1)$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

The evolution of the vorticity field is determined as follows. First, we note that the Navier-Stokes equation (1.2.5) may be rewritten in the form

$$\partial_t \mathbf{u} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \left(p + \frac{|\mathbf{u}|^2}{2} \right) + \frac{1}{Re} \Delta \mathbf{u}, \quad (1.3.5)$$

using the vector identity

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla \mathbf{u}. \quad (1.3.6)$$

We now take the curl of (1.3.5), and use the identities $\nabla \cdot \mathbf{u} = \nabla \cdot \boldsymbol{\omega} = 0$, to obtain the vorticity evolution equation for viscous flows

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \frac{1}{Re} \Delta \boldsymbol{\omega}. \quad (1.3.7)$$

Similarly, the vorticity formulation of the Euler equations is

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}. \quad (1.3.8)$$

¹Though we focus on two and three dimensional flows, the vorticity has an elegant mathematical description in any dimension [2, Ch.1]

In both these equations, the i -th component of the vector $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ is given by

$$(\boldsymbol{\omega} \cdot \nabla \mathbf{u})_i = \sum_{j=1}^3 \omega_j \partial_{x_j} u_i. \quad (1.3.9)$$

In two dimensions, the vorticity is orthogonal to the plane of the fluid flow, and we see immediately that the term $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ vanishes. Thus, the vorticity is simply transported by the flow in two dimensions. This allows a dramatic simplification of the analysis as described below. In three dimensions, however, this term has a subtle effect on the flow, that is best interpreted in light of *Kelvin's circulation theorem*.

Consider a closed material line $\Gamma(t)$ that encloses a surface $S(t)$. The circulation around $\Gamma(t)$ is the line integral

$$C(t) = \oint_{\Gamma(t)} \mathbf{u} \cdot d\mathbf{l} = \int_{S(t)} \boldsymbol{\omega} \cdot d\mathbf{a}. \quad (1.3.10)$$

(Here $d\mathbf{l}$ and $d\mathbf{a}$ denote the infinitesimal oriented line and area element respectively). The rate of change of the circulation is computed from (1.3.7) and Reynolds transport theorem, and we find [4, §5.2]

$$\frac{dC}{dt} = -\frac{1}{Re} \oint (\nabla \times \boldsymbol{\omega}) \cdot d\mathbf{l}. \quad (1.3.11)$$

In particular, the circulation around a closed loop contained within the fluid is conserved for ideal flows: this is Kelvin's theorem.

We see immediately from the expression of the circulation as an area integral, that if a small loop $\Gamma(t)$ shrinks as it is convected by the flow, then the vorticity field must grow in order that the circulation stays constant. It is not known if such 'vortex stretching' is sufficient to cause the formation of singularities in the Euler equations (see [21, Thm 3.6]).

1.4 Point vortices

In this section, we consider two dimensional flows $\mathbf{u} = (u, v)$ on a bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary. The incompressibility condition $\nabla \cdot \mathbf{u} = 0$ is immediately satisfied if we can find a *streamfunction*, ψ , such that

$$u_1 = -\partial_{x_2} \psi, u_2 = \partial_{x_1} \psi, \quad \text{or} \quad \mathbf{u} = \nabla^\perp \psi. \quad (1.4.1)$$

The flow is parallel to the contours of the stream function. The (scalar) vorticity and the stream function are related by Poisson's equation

$$\Delta \psi = \omega, \quad x \in \Omega. \quad (1.4.2)$$

Further, when we rewrite the boundary condition (1.2.12) using (1.4.1), we find that the tangential derivative $\nabla \psi \cdot \mathbf{t}(x)$ vanishes on the boundary $\partial\Omega$. When Ω is simply connected, we may assume without loss of generality that

$$\psi = 0, \quad x \in \partial\Omega. \quad (1.4.3)$$

In general, the boundary may consist of countably many components and ψ may take a different constant value on each component.

Let us now explain the Biot-Savart law in the simplest setting in 2D. Poisson's equation with Dirichlet boundary conditions may be solved by the method of Green's functions. The fundamental solution, denoted $G_\Omega(x, y)$, to Poisson's equation (1.4.2) solves

$$\Delta_x G = \delta_y(x), \quad x, y \in \Omega, \quad (1.4.4)$$

along with the boundary condition $G_\Omega(x, y) = 0$, $x \in \partial\Omega$. Since the vorticity distribution is concentrated at a single point, we call the resulting flow a *point vortex*. The solution to (1.4.2) is then given by

$$\psi(x) = \int_\Omega G_\Omega(x, y) \omega(y) dy. \quad (1.4.5)$$

When $\Omega = \mathbb{R}^2$, the flow field of a point vortex is obtained from the free-space Green's function

$$G_{\mathbb{R}^2}(x, y) = \frac{1}{2\pi} \log|x - y| = \frac{1}{4\pi} \log((x_1 - y_1)^2 + (x_2 - y_2)^2). \quad (1.4.6)$$

We compute the velocity field using (1.4.1) to obtain the Biot-Savart kernel

$$\mathbf{u}(x, y) = \frac{1}{2\pi} \frac{(x - y)^\perp}{|x - y|^2} = K_2(x - y). \quad (1.4.7)$$

For simple domains with some symmetry, such as the disk, the Green's function may be computed by the method of images (see Exercises). While it is not possible to explicitly describe the Green's function on an arbitrary domain, an important aspect is that its leading-order behavior is always described by $G_{\mathbb{R}^2}$. That is, the difference

$$G_\Omega(x, y) - G_{\mathbb{R}^2}(x, y) \quad (1.4.8)$$

is a harmonic function in Ω . In particular, this means that a point-vortex on a bounded domain Ω has infinite *self-energy*. Indeed, the kinetic energy

$$\frac{1}{2} \int_\Omega |\mathbf{u}|^2 dx = \frac{1}{2} \int_\Omega |\nabla_x G_\Omega|^2 dx \quad (1.4.9)$$

differs by only a finite amount from the divergent integral

$$\frac{1}{2} \int_\Omega |\nabla_x G_{\mathbb{R}^2}|^2 dx = \frac{1}{8\pi^2} \int_\Omega \frac{1}{|x - y|^2} dx. \quad (1.4.10)$$

Despite the fact that the velocity field of a point-vortex has infinite self-energy, Kirchhoff observed that the flow-field of a finite number of point vortices has an elegant description. The key observation is that while the velocity field induced by a vortex at y diverges at the rate $1/|x - y|$ as $x \rightarrow y$, the 'eye of the

storm' at $x = y$ is steady. That is, the flow field induced by a point vortex at y , does not cause the vortex itself to move.

Since equation (1.4.2) is linear, one may superpose N point vortices with strengths $\{\kappa_j\}_{j=1}^N$, located at points $\{x_j\}_{j=1}^N \subset \Omega$, to obtain a velocity field

$$\mathbf{u}(x; x_1, \dots, x_N) = \sum_{j=1}^N \kappa_j \nabla_x^\perp G_\Omega(x, x_j), \quad x \in \Omega. \quad (1.4.11)$$

Since the k -th vortex does not move itself, it is advected by the flow field induced by all other vortices

$$\dot{x}_k = \mathbf{u}(x_k; x_1, \dots, x_N) = \sum_{j \neq k} \kappa_j \nabla_x^\perp G_\Omega(x_k, x_j). \quad (1.4.12)$$

Thus, we have obtained a closed system of ordinary differential equations for the evolution of N vortices. These equations may be written as a Hamiltonian system:

$$\kappa_k \dot{x}_k = J \nabla_{x_k} H(x_1, \dots, x_n), \quad k = 1, \dots, N, \quad (1.4.13)$$

where J denotes the standard 2×2 symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (1.4.14)$$

and H denotes the 'renormalized' Hamiltonian

$$H(x_1, \dots, x_n) = -\frac{1}{2} \sum_{1=j < k \leq N} \kappa_j \kappa_k G_\Omega(x_j, x_k). \quad (1.4.15)$$

The negative sign above stems from (formally) integrating equation (1.4.9) by parts:

$$\begin{aligned} H(x_1, \dots, x_n) &\stackrel{?}{=} \frac{1}{2} \int_\Omega |\mathbf{u}|^2 = \frac{1}{2} \int_\Omega |\nabla \psi|^2 \\ &- \int_\Omega \psi \Delta \psi \stackrel{(1.4.2)}{=} - \int_\Omega \psi \omega \stackrel{(1.4.5)}{=} - \int_\Omega \int_\Omega G_\Omega(x-y) \omega(x) \omega(y) dx dy. \end{aligned} \quad (1.4.16)$$

(The question mark in the first equality, reflects the fact that the integrals are not well-defined because of the self-energy of each vortex). Thus, comparing (1.4.16) and (1.4.15) we see that the self-energy of each point vortex has been removed from the kinetic energy, which is why we say that the Hamiltonian H has been 'renormalized'.

Equations (1.4.15) are in the standard form of Hamiltonian dynamics if all the vortex strengths $\{\kappa_k\}_{k=1}^N$ are equal. If not, one may check that the rescaled variables $p_k = \sqrt{\kappa_k} x_k$, $k = 1, \dots, N$, evolve according to a Hamiltonian flow with the Hamiltonian $K(p_1, \dots, p_n) = H(x_1, \dots, x_N)$.²

²First check this under the assumption that all the κ_k are positive. Then modify the argument for negative κ , noting that under the identification $\mathbb{C} = \mathbb{R}^2$, multiplying a complex number $z = x + iy$ by $\sqrt{-1}$ is the same as multiplying the vector $(x, y)^T$ by J .

This system of equations connects fluid mechanics with a class of problems in electromagnetism and gravitation. The motion of classical point charges in a Coulombic potential is described by a system above. Similarly, the classical N -body problem is the study of N point masses in a gravitational field. Thus, the statistical mechanics of N -vortices is naturally linked to the statistical mechanics of galaxies and the statistical mechanics of charge clouds.

1.5 Steady flows in 2D

We say that a velocity field is steady, if it does not depend on time. In this section and the next, we consider steady solutions to the Euler equations on a domain Ω with the boundary condition (1.2.12). These solutions also serve as building blocks for time-dependent solutions to the Navier-Stokes equations (see [21, Ch.2]).

In 2D, the vorticity formulation (1.3.8) reduces to the transport equation

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0. \quad (1.5.1)$$

The flow is steady if and only if

$$0 = \mathbf{u} \cdot \nabla \omega = u_1 \omega_{x_1} + u_2 \omega_{x_2} \quad (1.5.2)$$

Further, since $\mathbf{u} = \nabla^\perp \psi$ and $\omega = \Delta \psi$, we may rewrite (1.5.2) in the form

$$0 = -\psi_{x_2} \Delta \psi_{x_1} + \psi_{x_1} \Delta \psi_{x_2} = \begin{vmatrix} \psi_{x_1} & \psi_{x_2} \\ \Delta \psi_{x_1} & \Delta \psi_{x_2} \end{vmatrix}. \quad (1.5.3)$$

It follows that the vectors $\nabla \psi$ and $\nabla \Delta \psi$ are everywhere linearly dependent. Thus, they have the same level curves: i.e. for each constant c , the set $\{\psi = c\}$ is identical to the set $\{\Delta \psi = \tilde{c}\}$ for a constant \tilde{c} that depends only on c . But this means that there is a functional dependence between ψ and $\Delta \psi$ of the form

$$\Delta \psi = f(\psi). \quad (1.5.4)$$

Conversely, if we assume a dependence of the form (1.5.4) it is clear that the resulting flow is steady. In summary, all steady, ideal flows in 2D are described by partial differential equations of the above form. In Chapter 2 we shall see how Onsager derived an equation of this form to describe a ‘most-likely’ velocity field in 2D turbulence.

1.6 Steady flows in 3D: Beltrami fields

It follows from (1.3.8) that a steady, ideal flow in 3D must satisfy the equation

$$\mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}. \quad (1.6.1)$$

A large class of steady 3D flows may be constructed by seeking velocity fields that are parallel to their curl. That is, assume there exists a (sufficiently smooth) scalar function $\alpha(x)$ and a divergence-free velocity field $\mathbf{u}(x)$ such that

$$\boldsymbol{\omega}(x) = \alpha(x)\mathbf{u}(x). \quad (1.6.2)$$

We call all such velocity fields, *Beltrami fields*. Since $0 = \nabla \cdot \mathbf{u} = \nabla \cdot \boldsymbol{\omega}$, any such scalar field must satisfy the condition

$$\mathbf{u} \cdot \nabla \alpha = 0. \quad (1.6.3)$$

That is, α is constant along the stream lines of \mathbf{u} . We substitute (1.6.2) in (1.6.1), and use (1.6.3) to see that every Beltrami field defines an ideal steady 3D flow.

In order to obtain a deeper understanding of Beltrami fields, we assume the domain Ω is the unit torus \mathbb{T}^3 , and we further assume that $\alpha(x)$ is a constant α . In this setting, Beltrami fields are constructed by solving the eigenvalue problem

$$\nabla \times \mathbf{u} = \alpha \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad x \in \mathbb{T}^3. \quad (1.6.4)$$

This problem may be solved by Fourier analysis. Let us represent the velocity field by its Fourier series

$$\mathbf{u}(x) = \sum_{k \in \mathbb{Z}^3} \mathbf{a}(k) e^{2\pi i k \cdot x}. \quad (1.6.5)$$

In order that the velocity field is real, it is necessary and sufficient that

$$\bar{\mathbf{a}}(k) = \mathbf{a}(-k), \quad k \in \mathbb{Z}^3. \quad (1.6.6)$$

The condition $\nabla \cdot \mathbf{u} = 0$ is equivalent to

$$k \cdot \mathbf{a}(k) = k_1 a_1(k) + k_2 a_2(k) + k_3 a_3(k) = 0. \quad (1.6.7)$$

The first equation in (1.6.4) implies that

$$2\pi i k \times \mathbf{a}(k) = \alpha \mathbf{a}(k). \quad (1.6.8)$$

We take the cross-product of the above equation with k , use (3.2.8) and the vector triple-product formula to obtain

$$2\pi i |k|^2 \mathbf{a}(k) = \alpha (k \times \mathbf{a}(k)) = \frac{\alpha^2}{2\pi i} \mathbf{a}(k). \quad (1.6.9)$$

Thus, the eigenvalue $\alpha = \pm 2\pi i |k|$. We may now check that for every $b \in \mathbb{C}^3$ such that $b \cdot k = 0$ the function

$$\mathbf{u}(x) = \left(b \pm i \frac{k}{|k|} \times b \right) e^{2\pi i k \cdot x} \quad (1.6.10)$$

is a divergence free eigenfunction of the curl operator with eigenvalue $\alpha = \pm 2\pi i |k|$. Finally, in order to construct real vector eigenfunctions we choose $b(k) \in \mathbb{C}^3$ such that

$$\bar{b}(k) = b(-k), \quad k \cdot b(k) = 0, \quad (1.6.11)$$

and define the Beltrami field $\mathbf{B}_k(x; b(k))$ as above.

Chapter 2

Onsager's theory: the 2D vortex gas

2.1 Introduction

During the early 1940's several scientists, including Chandrasekhar, Heisenberg, Kolmogorov, and Onsager, turned their attention to turbulence [31]. One of the main ideas formulated in this period is of turbulence as a process of energy transfer from long to small length scale, or intuitively, a picture of an incompressible flow that consists of eddies within eddies. It is an interesting irony of turbulence, that the simplest physical caricature, a scaling law for energy and dissipation was formulated by a mathematician, Kolmogorov, whereas a more careful mathematical examination of energy and dissipation in fluid flows was discovered by a physical chemist, Onsager. In this chapter, we introduce Onsager's approach to turbulence expanding on his brief, but seminal, article [27]. In addition, to this paper, we also draw on some of Onsager's correspondence during that period, and his unpublished notes following [12].

Onsager's treatment of two and three dimensional flows is completely different. He studied 2D flows consisting of many point vortices using the principles of equilibrium statistical mechanics. He discovered an intriguing notion of negative temperature, which he speculated explained the clustering of point vortices into giant vortices. In unpublished work, he also derived a mean-field equation to describe the form of the most-likely flow. This equation was rediscovered more than twenty years later by Joyce and Montgomery [22]. For 3D flows, he studied the notion of an energy cascade in Fourier space and carefully formulated the idea of energy transfer from one shell to another.¹ This led him to speculate that ideal flows, i.e. solutions to the Euler equations, which formally conserve energy, could in fact dissipate energy if the solutions were not smooth. This aspect of the problem has received extensive mathematical attention in the

¹He was also the first to use the word 'cascade' to describe this process.

past ten years [10]. In this lecture, we focus on the 2D theory. The 3D theory is treated in the next lecture.

2.2 The vortex gas and negative temperature

The starting point for Onsager's theory in 2D are Kirchhoff's equations for a system of point vortices in a *bounded* domain, Ω . In order to present the essentials of the theory in a concrete manner, let us first consider a situation where we have N identical point vortices, each with strength κ , located at points $\{z_1, \dots, z_N\} \subset \Omega$ ². Kirchhoff's equation (1.4.13) then specializes to the Hamiltonian system

$$\kappa \dot{x}_k = \partial_{y_k} H(x_1, y_1, \dots, x_n, y_n), \quad (2.2.1)$$

$$\kappa \dot{y}_k = -\partial_{x_k} H(x_1, y_1, \dots, x_n, y_n), \quad k = 1, \dots, N. \quad (2.2.2)$$

It is immediate from the form of the equation that the coordinates (x_k, y_k) of $z_k \in \Omega$ are canonically conjugate. In more mathematical terms, we say that equation (2.2.1) is a Hamiltonian system with the standard symplectic form $\sum_{k=1}^N dy_k \wedge dx_k$.

A fundamental feature of Hamiltonian systems is that the symplectic form is preserved under the flow. In particular, the volume form

$$dV_{2N} = \frac{1}{N!} \left(\sum_{k=1}^N dx_k \wedge dy_k \right)^N = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \dots \wedge dx_N \wedge dy_N, \quad (2.2.3)$$

is left invariant under the flow (Liouville's theorem). Further, a solution $\mathbf{z}(t)$ to (2.2.1) is restricted to the energy hypersurface

$$\mathcal{E} = \{\mathbf{z} \in \Omega^N \mid H(z_1, \dots, z_n) = E\}, \quad (2.2.4)$$

where $E = H(\mathbf{z}(0))$ denotes the initial value of the Hamiltonian. Observe also that the Hamiltonian diverges if two distinct vortices approach one another. Thus, there are no collisions between vortices, and for each initial configuration \mathbf{z}_0 with $z_j \neq z_k$ for all distinct j, k , (2.2.1) has a unique solution $\mathbf{z}(t)$ on the time interval $(-\infty, \infty)$ with $\mathbf{z}(0) = \mathbf{z}_0$.

A fundamental hypothesis of statistical mechanics is that in the large N limit, typical Hamiltonian systems, i.e. those without hidden symmetries, will have ergodic behavior. In the context of point vortices, this assumption means that the proportion of time a typical trajectory $\mathbf{z}(t)$ spends in a relatively open subset $U \subset \mathcal{E}$ of phase space is proportional to the $2N-1$ dimensional volume of the subset U obtained by restricting the volume form dV_N to the hypersurface \mathcal{E} . That is,

$$\lim_{T \rightarrow \infty} \int_0^T \mathbf{1}_U(\mathbf{z}(t)) dt = \frac{|U|_{2N-1}}{|\mathcal{E}|_{2N-1}}, \quad (2.2.5)$$

²In this section, we denote the coordinates of a point $z \in \mathbb{R}^2$ by (x, y) . This is at variance with the notation in (1.4.13), but fits better with standard notation for Hamiltonian systems. We also denote $\mathbf{z} = (z_1, \dots, z_N) \in \Omega^N$.

where $|\cdot|_{2N-1}$ is the $2N - 1$ dimensional volume form obtained by restricting the volume form dV_{2N} to the hypersurface \mathcal{E} as follows. If $dl^2 = \sum_{j=1}^n dx_j^2 + dy_j^2$ is the length element on Ω^{2N} , and dV_{2N-1} is the associated $2N - 1$ dimensional volume element, then the $2N - 1$ dimensional volume element on \mathcal{E} is [24, §1.3]

$$dA_{2N-1} = \frac{dV_{2N-1}}{|\nabla H|}, \quad |U|_{2N-1} = \int_U dA_{2N-1}. \quad (2.2.6)$$

The probability measure on \mathcal{E} obtained by normalizing the restricted volume form dS_{2N-1} as above is called the *microcanonical measure* or the *microcanonical ensemble*.

Despite the fact that our starting point is the Euler equations, the assumption of ergodicity removes time dependence from the problem by equating two distinct probabilistic notions – the left-hand-side of (2.2.5) is a frequentist measurement of likelihood of an event, whereas the right-hand-side associates a probability to sets that is independent of time. We stress that ergodicity is an assumption that underlies Onsager’s theory, not a theorem! In fact, very few models in statistical mechanics are rigorously justified from the standpoint of ergodic theory. In most instances, ergodicity should be viewed as an assumption, that has useful consequences (see[11]).

The computation of restricted volumes can be cumbersome, and it is often more convenient to work with the unrestricted volume form in Ω^N , sampling states in proportion to the probability density

$$p_{\beta,N}(\mathbf{z}) dV_{2N} = \frac{1}{Z_{\beta,N}} e^{-\beta H_N(\mathbf{z})} dV_{2N}, \quad Z_{\beta,N} = \int_{\Omega^N} e^{-\beta H_N(\mathbf{z})} dV_{2N}. \quad (2.2.7)$$

Here $\beta \in \mathbb{R}$ is (at first) a phenomenological parameter. The probability density (2.2.7) is called the *canonical measure*, and the related law on Ω^N is called the *canonical ensemble*. The underlying dynamical system is no longer (2.2.1), but instead a system that augments (2.2.1) with both dissipation and random forcing. That is, the system of point vortices is assumed to be interacting with a thermal bath.

The large N -limit is termed the thermodynamic limit. In this limit the parameter β has an interpretation as the inverse temperature. For the ideal gas, the temperature has an intuitive interpretation as the mean energy per particle, and it is always positive. Further, an interesting calculation reveals that both the canonical and microcanonical measure have exactly the same large- N limit³. More generally, the assumption that both these ensembles have the same limit, is termed the *equivalence of ensembles*. Again, this should be viewed as an assumption, not a theorem, though it is considerably easier to check in practice than the assumption (2.2.5).

In general, the temperature has a thermodynamic definition in terms of the number-of-states function. For the N -vortex model, the number-of-states

³Outlined in lecture. To be added as an exercise to notes with correct constants.

function is defined as follows [11, Ch.2]. Let

$$\Phi_N(E) = \int_{\Omega^N} \mathbf{1}_{H(\mathbf{z}) < E} dV_{2N}. \quad (2.2.8)$$

By definition, the function $\Phi_N(E)$ is a monotonically increasing function of E on the domain $(-\infty, \infty)$. Further, since the Ω is bounded, the $2N$ -dimensional volume of Ω^N is simply A^N where A is the area of the domain Ω , and we have

$$\Phi'_N(E) \geq 0, \quad \Phi_N(-\infty) = 0, \quad \Phi_N(+\infty) = A^N. \quad (2.2.9)$$

The inverse temperature of the system is defined to be [11, Ch.2] (assuming the thermodynamic limit exists)⁴

$$\beta(E) = \lim_{N \rightarrow \infty} \frac{d}{dE} \log \Phi'_N(E) = \lim_{N \rightarrow \infty} \frac{\Phi''_N(E)}{\Phi'_N(E)}. \quad (2.2.10)$$

Onsager applied this calculation to the vortex gas and noted the following interesting feature. Since the domain is bounded, $\Phi_N(E)$ is an increasing function with a finite limit as $E \rightarrow +\infty$. Thus, it is natural to expect that the thermodynamic limit $\Phi'(E) \rightarrow 0$ as $E \rightarrow \pm\infty$, so that it has a maximum at some critical value of the energy, E_c where $\Phi''(E_c) = 0$. But then (2.2.10) shows that the inverse temperature $\beta(E)$ is *negative* for $E > E_c$!⁵

Onsager interpreted negative temperature states as a sign of clustering of vortices in a rough analogy with crystallization. A basic model of phase transitions in thermodynamics goes as follows: assuming the temperature (or inverse temperature β is held fixed), the equilibrium phase of the system is determined by minimizing the Helmholtz free energy

$$F = U + \frac{1}{\beta} S. \quad (2.2.11)$$

Here U is the internal energy and S is the entropy of the system (see footnote below (2.4.3)). At low temperatures ($\beta \rightarrow \infty$), the free energy is minimized when the entropy is low, thus the system settles into an ordered state (e.g a crystal). At high temperatures $\beta \rightarrow 0$, disorder dominates since there is only a low cost for high entropy. This heuristic order-disorder argument can be applied to the vortex gas. But now, if the temperature is negative, the free energy is minimized by an ordered state.

2.3 The combinatorial origin of the entropy

Onsager's paper [27] is rather cryptic on what the ordered negative-temperature states should be, but in unpublished work he had derived a mean-field equation that described these states. The mean-field equation was rediscovered by

⁴In revision, explain for hard-sphere gas, as done in lecture.

⁵We are implicitly assuming that there is a single inflection point E_c and that $\Phi''_N(E) > 0$ for $E < E_c$ and $\Phi''_N(E) < 0$ for $E > E_c$. Figure to be added.

Joyce and Montgomery [22]. The mean-field equations are derived on the basis of a second postulate: in the thermodynamic limit, the most likely state (i.e. distribution of vortices in Ω) minimizes the Boltzmann entropy subject to the constraint that the total energy and total vorticity are prescribed. In this section, we explain the combinatorial origin of the entropy in a simplified setting. The minimization principle is applied in the next section.

In the simplest setting, the point vortex model consists of N vortices with equal strength κ/N located at points $\{z_1, \dots, z_N\} \subset \Omega^N$, with Hamiltonian

$$H_N(\mathbf{z}) = -\frac{\kappa^2}{2N^2} \sum_{j \neq k} G(z_j, z_k), \quad (2.3.1)$$

In this section, we explain the combinatorial origin of the entropy by discretizing the set of states for the point vortex model as follows. Instead of the open set $\Omega \subset \mathbb{R}^2$ we consider a finite lattice $\Omega_M \subset \Omega$ with M points

$$\Omega_M = \{a_1, \dots, a_M\}, \quad a_i \in \Omega, \quad i = 1, \dots, M, \quad (2.3.2)$$

and we assume that the vortices are constrained to lie at these points. Thus, each state of the vortex lattice model is a vector $\mathbf{z} = (z_1, \dots, z_N) \in \Omega_M^N$. In order to formulate a thermodynamic limit for the lattice model which extends naturally to the point vortex model, we define the empirical measure for both models,

$$\rho_N(dz) = \frac{1}{N} \sum_{j=1}^N \delta(z - z_k) dz, \quad (2.3.3)$$

where dz denotes the area element in Ω . We rewrite the energy of the point-vortex model in the form

$$H_N(\rho_N) = -\frac{\kappa^2}{2} \int_{(\Omega \times \Omega) \setminus \{(z, z): z \in \Omega\}} G(z, z') \rho_N(dz) \rho_N(dz'), \quad (2.3.4)$$

to stress that it is only a function of the empirical density. The diagonal is excised from the domain of integration to remove the infinite self-energy of a point vortex.

Formula (2.3.4) also extends to the lattice model. Let n_1, \dots, n_M the total number of vortices at each site $a_i \in \Omega_M$ for the lattice vortex model, i.e.,

$$n_i = \#\{k | z_k = a_i\}, \quad \sum_{i=1}^M n_i = N. \quad (2.3.5)$$

The empirical measure for the lattice model is completely determined by the fraction of vortices at each site:

$$\rho_N(z) = p_1 \delta(z - a_1) + p_2 \delta(z - a_2) + \dots + p_M \delta(z - a_M), \quad p_i = \frac{n_i}{N}. \quad (2.3.6)$$

Here p implicitly depends on N , but we will take limits $N \rightarrow \infty$ such that p is fixed. Thus, the energy of the lattice model is the quadratic form

$$H(p) = -N^2 \frac{\kappa^2}{2} p^T G p, \quad (2.3.7)$$

where G denotes the symmetric $M \times M$ Green's matrix,

$$G_{ij} = \begin{cases} G(a_i, a_j), & i \neq j \\ 0, & i = j. \end{cases} \quad (2.3.8)$$

In order to have a meaningful energy in the limit, we must assume that

$$\lim_{N \rightarrow \infty} \frac{H(p)}{N^2} = E, \quad (2.3.9)$$

for some constant $E \in \mathbb{R}$.

We do not associate discrete time dynamics in the lattice vortex model. Instead, we use H to define a microcanonical measure on Ω_M^N and focus on a simpler question: How do we describe the limit of the microcanonical ensemble?

Given an energy level E , the microcanonical ensemble for the lattice model is the uniform distribution on states $\mathbf{z} \subset \Omega_M^N$ with energy $H(\rho_N) = N^2 E$. Since we have assumed that all vortices are identical, the energy of a state depends only on the number of vortices, n_1, \dots, n_M , at each lattice site. The total number of such states is given by the multinomial coefficient

$$\binom{N}{n_1, n_2, \dots, n_M} = \frac{N!}{n_1! n_2! \cdots n_M!}. \quad (2.3.10)$$

The entropy appears as the dominant term when the multinomial coefficient is approximated by Stirling's formula in the regime $N \rightarrow \infty$, $p(N) \rightarrow p$. Recall that Stirling's formula may be written in the form

$$\log n! \sim n(\log n - 1) + \frac{1}{2} \log n + \frac{1}{2} \log 2\pi. \quad (2.3.11)$$

Therefore, retaining only the dominant terms, we obtain

$$\begin{aligned} \log \binom{N}{n_1, n_2, \dots, n_M} &= \log N! - \sum_{i=1}^M \log n_i! \\ &\sim N(\log N - 1) - \sum_{i=1}^M n_i(\log n_i - 1) = -N \sum_{i=1}^M \frac{n_i}{N} \log \frac{n_i}{N} := -NS_M(p), \end{aligned} \quad (2.3.12)$$

where we have defined the Boltzmann entropy ⁶

$$S_M(p) = \sum_{i=1}^M p_i \log p_i. \quad (2.3.13)$$

⁶The definition of entropy in information theory and mathematics is the negative of the usual convention in physics. Thus, mathematicians minimize entropy, rather than maximizing it. Similarly, the Helmholtz free energy and entropy are related through $F = E + TS$ (math), not $F = E - TS$ (physics); here E is the internal energy, T is the temperature, and S is the entropy.

Thus, the microcanonical ensemble has the following approximate description: given an energy level E , we first determine all p such that

$$p_i \geq 0, \quad \sum_{i=1}^M p_i = 1, \quad -\frac{1}{2}p^T G p = E. \quad (2.3.14)$$

Geometrically, this set is the intersection of the $M - 1$ dimensional simplex with the quadric energy surface. For each p such that (2.3.14) holds, there are approximately $e^{-NS_M(p)}$ equally likely arrangements of the vortices. Since N is very large, and $S_M < 0$, the most likely arrangement of vortices corresponds to the value of p such that S_M is minimized. In summary, the most likely value of p is determined by minimizing $S_M(p)$ subject to the constraints (2.3.14).

2.4 The mean-field equation for identical vortices

We now return to the point vortex model. In the thermodynamic limit $N \rightarrow \infty$ we assume that the empirical densities, $\rho_N(z)$ converge to a limiting density, $\rho(z)$. More precisely, we assume

$$\int_U \rho(z) dz = \lim_{N \rightarrow \infty} \int_U \rho_N(dz) = \lim_{N \rightarrow \infty} \frac{\#\{z_k \in U\}}{N}, \quad (2.4.1)$$

for every open set $U \subset \Omega$.

The minimization principle for the point vortex model is obtained by taking the continuum ($M \rightarrow \infty$) limit of the vortex lattice model. In this limit, the vector $p = (p_1, \dots, p_M)$ is replaced by the density $\rho(z)$, and the discrete sums in (2.3.13)–(2.3.14) are replaced by integrals over Ω . The limiting vorticity and energy are given by

$$\omega(z) = \kappa \rho(z), \quad H(\rho) = -\frac{\kappa^2}{2} \int_{\Omega \times \Omega} G(z, z') \rho(dz) \rho(dz') dz dz'. \quad (2.4.2)$$

Similarly, the Boltzmann entropy relative to the uniform measure on Ω is

$$S(\rho) = \int_{\Omega} \rho(z) \log \rho(z) dz. \quad (2.4.3)$$

We minimize the Boltzmann entropy subject to the constraint that the energy and the total number of particles is fixed. Without loss of generality, we may rescale so that the constraints are

$$\int_{\Omega} \rho(z) dz = 1, \quad H(\rho) = E, \quad (2.4.4)$$

for some fixed constant $E \in (-\infty, \infty)$. The constraints may be included in the maximization problem by adding Lagrange multipliers, say α and β . Thus, our

problem is to minimize the functional

$$I(\rho) := S(\rho) + \alpha \int_{\Omega} \rho + \beta H(\rho). \quad (2.4.5)$$

The Euler-Lagrange equations for an extremum of the functional $I(\rho)$ are

$$\log \rho(z) = \beta \kappa^2 \int G(z, z') \rho(z') dz' - \alpha. \quad (2.4.6)$$

The calculation so far could have applied to the thermodynamic limit of a system of point charges, each of magnitude κ/N . We now reconnect with fluid mechanics. The vorticity $\omega(z) = \kappa\rho$ is related to the stream function ψ by (1.4.5). Thus, (2.4.6) is equivalent to

$$\omega(z) = \kappa\rho(z) = e^{-\alpha} e^{\beta\kappa\psi}. \quad (2.4.7)$$

The Lagrange multiplier α must be determined by the condition that (2.4.4) is satisfied. We now use (1.4.2), and change our notation for the constant $e^{-\alpha}$ to notation more common in statistical mechanics, to obtain the *mean-field equation*⁷

$$\Delta\psi(z) = \frac{1}{Z_{\beta}} e^{\beta\psi}, \quad z \in \Omega, \quad Z_{\beta} = \int_{\Omega} e^{\beta\psi}. \quad (2.4.8)$$

$$\psi(z) = 0, \quad z \in \partial\Omega. \quad (2.4.9)$$

It is a surprising fact, of some depth, that the mean-field equation arises in conformal geometry, where it is called the Liouville equation. As explained below, the equation is solvable only for $\beta \in (\beta_c, \infty)$ where $\beta_c < 0$ is a critical *negative* inverse temperature. Thus, the mean-field equation (2.4.8) has nontrivial negative temperature states for a range $\beta_c < \beta < 0$. The existence of these solutions uses very interesting mathematics, including the Moser-Trudinger inequality and complete integrability. We first discuss these ideas. We then introduce a family of infinitely many mean-field equations and turn to an important issue regarding the infinite self-energy of the vortex gas and the conservation laws for a 2D fluid.

2.5 Exact solutions to the mean-field equation

The mean-field equation is exactly solvable on a disk. To be concrete, let $\Omega = B_1$ be the unit ball in \mathbb{R}^2 . We consider the equation

$$\Delta\psi = \frac{1}{Z_{\beta}} e^{\beta\psi}, \quad z \in B_1, \quad (2.5.1)$$

$$\psi = 0, \quad z \in \partial B_1. \quad (2.5.2)$$

⁷In the literature (see especially [20]), this equation is often written

$$-\Delta\psi(z) = \frac{1}{Z_{\beta}} e^{-\beta\psi}.$$

One equation may be obtained from the other by changing $\psi \mapsto -\psi$. Regrettably, Lions defines the vorticity with the opposite sign of the choice in [21], causing some inconsistency.

It is clear that $\Delta\psi > 0$; thus, by the maximum principle, any solution to (2.5.1) must be negative. A celebrated result of Gidas, Ni and Nirenberg [15, Thm.1], then implies that ψ must be radially symmetric. We may therefore write $\psi = \psi(r)$, where $r = |z|$, and reduce equation (2.5.1) to the ordinary differential equation

$$\psi'' + \frac{1}{r}\psi' - \frac{1}{Z_\beta}e^{\beta\psi} = 0, \quad \psi' = \frac{\partial\psi}{\partial r}. \quad (2.5.3)$$

The boundary conditions and elliptic regularity imply

$$\psi(1) = 0, \quad \psi'(0) = 0. \quad (2.5.4)$$

Further, Theorem 1 [15] also implies that ψ is strictly increasing at all other points

$$\frac{\partial\psi}{\partial r} > 0, \quad 0 < r < 1. \quad (2.5.5)$$

(Note that $\psi < 0$, so $|\psi|$ is strictly decreasing.) Solutions to (2.5.3) of this form are obtained by phase plane analysis and some explicit calculations. We make the change of variables

$$r = \log t, \quad \varphi(t) = \psi(r) + \frac{2t}{\beta}, \quad (2.5.6)$$

to transform equation (2.5.3) into the one-dimensional Hamiltonian system

$$\ddot{\varphi} - \frac{1}{Z_\beta}e^{\beta\varphi} = 0. \quad (2.5.7)$$

Thus, we have the conservation law

$$\frac{1}{2}\dot{\varphi}^2 - \frac{1}{\beta Z_\beta}e^{\beta\varphi} = E, \quad (2.5.8)$$

where E is a constant of motion. We may separate variables in the equation above to obtain

$$\frac{d\varphi}{\sqrt{2\left(E + \frac{1}{\beta Z_\beta}e^{\beta\varphi}\right)}} = dt. \quad (2.5.9)$$

The change of variables $s = e^{\beta\varphi/2}$ followed by a rescaling, reduces the left hand side to the standard form

$$\frac{du}{\sqrt{u^2 - 1}} \quad (2.5.10)$$

which may be integrated to yield $\cosh^{-1}u$. In all these calculations, we assume that Z_β is a positive constant. Finally, Z_β must be determined by self-consistency, i.e.

$$Z_\beta = 2\pi \int_0^1 e^{\beta\psi(r)} r \, dr. \quad (2.5.11)$$

After some manipulations ⁸, we find that the mean-field vortex density is

$$\rho(r) = \frac{1-a}{\pi} \frac{1}{(1-ar^2)^2}, \quad a = \frac{\beta}{8\pi + \beta}, \quad -8\pi < \beta < \infty. \quad (2.5.12)$$

2.6 Existence of solutions to the mean-field equation

For arbitrary domains Ω , there are no general symmetry principles, and the existence of solutions to the mean-field equation (2.4.8)–(2.4.9) must be established using other methods. In the positive temperature region, this follows from routine energy methods. Roughly, when $\beta \geq 0$ is held fixed, the problem of minimizing the functional $I(\rho)$ defined in (2.4.5) may be seen as the minimization problem for a convex functional on a convex set. On general grounds, such a problem has a unique minimizer, which further solves the partial differential equation (2.4.8) with boundary condition (2.4.9). However, for $\beta < 0$, the existence of minimizers and the existence of $\beta_c < 0$ relies on a critical inequality. Here is an outline of these ideas.

2.6.1 The positive temperature regime

Fix $\beta \in [0, \infty)$, consider the set

$$\mathcal{S} = \left\{ \rho \in L^1(\Omega) \mid \rho \geq 0, \int_{\Omega} \rho = 1, \int_{\Omega} \rho \log(1 + \rho) < \infty \right\}. \quad (2.6.1)$$

and the functional

$$I_{\beta} : \mathcal{S} \rightarrow \mathbb{R} \cup \{\infty\}, \quad I_{\beta}(\rho) = \beta S(\rho) + H(\rho). \quad (2.6.2)$$

It is easy to check that \mathcal{S} is a convex subset of $L^1(\Omega)$: that is, if ρ_1 and ρ_2 are elements of \mathcal{S} , then so is

$$(1 - \theta)\rho_1 + \theta\rho_2, \quad \theta \in [0, 1]. \quad (2.6.3)$$

The functional I_{β} is strictly convex. Indeed, the function $x \mapsto x \log x$ is convex on $[0, \infty]$, using the conventions $0 \log 0 = 0$ and $\infty \log \infty = \infty$ to extend the value of the function from $(0, \infty)$ to $[0, \infty]$. It then follows that $\rho \mapsto S(\rho)$ is convex as a map from $\mathcal{S} \rightarrow \mathbb{R} \cup \{\infty\}$. The strict convexity of the kinetic energy is a consequence of the identity

$$H(\rho) = \frac{1}{2} \int_{\Omega} |\nabla \psi|^2. \quad (2.6.4)$$

We apply this identity to $\rho(\theta)$ and use the Cauchy-Schwarz inequality to find

$$H(\rho(\theta)) \leq (1 - \theta)H(\rho_1) + \theta H(\rho_2), \quad 0 < \theta < 1, \quad (2.6.5)$$

⁸More details to be added, or outlined as exercise.

with equality only if $\rho_1 = \rho_2$. These observations show that I_β is a strictly convex function on a convex subset of a complete metric space. Thus, I_β has a unique minimizer, denoted ρ_β . In order to show that ρ_β solves (2.4.8), this abstract existence theorem must be combined with (standard) elliptic regularity theory. The interested reader may find the details in [20].

These conclusions may be summarized in the following

Theorem 1. *Fix $\beta \in [0, \infty)$. Assume $\Omega \subset \mathbb{R}^2$ is bounded, simply connected and $\partial\Omega$ is $C^{2,\alpha}$, for some $0 < \alpha \leq 1$. Then (2.4.8)–(2.4.9) has a unique solution, denoted ρ_β . Further, ρ_β is the unique minimizer of the functional I_β on the set $\mathcal{S} \subset L^1(\Omega)$.*

The assumptions on $\partial\Omega$ are included for completeness in the statement of the theorem. These assumptions are more restrictive than necessary – all that is required is that Ω be regular enough that there there is a well-defined solution to (1.4.2)–(1.4.3).

2.6.2 The negative temperature regime

The more interesting regime is $\beta < 0$. In this regime, the properties of (2.4.8) are more transparent in a dual variation problem in terms of ψ . We note that equation (2.4.8) is the Euler-Lagrange equation for the functional

$$F_\beta(\psi) = \frac{1}{\beta} \log Z_\beta + \frac{1}{2} \int_\Omega |\nabla\psi|^2. \quad (2.6.6)$$

Again, the functional $F(\psi)$ is the Helmholtz free energy of the system: the first term is β times the entropy, and the second term is the internal energy of vortices (the kinetic energy). Thus, (2.6.6) is equivalent to (2.4.5), if we restrict attention to ρ such that $\int_\Omega \rho = 1$. However, the entropic term is no longer convex. In order to establish the existence of a unique minimizer, it is necessary to control the entropy by the energy. Remarkably, a fundamental geometric inequality – Moser’s sharp form of Trudinger’s inequality [23] – yields exactly what is needed.

We first state the inequality in its standard form. The weaker form needed to study F_β is an easy corollary. Given a domain $\Omega \subset \mathbb{R}^2$, the homogeneous Sobolev space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm

$$\|f\|_{H_0^1} := \left(\int_\Omega |\nabla f|^2 \right)^{1/2}. \quad (2.6.7)$$

Theorem 2 (Moser-Trudinger). *Assume $\Omega \subset \mathbb{R}^2$ is bounded. There exists a universal constant $C_{\text{MT}} > 0$ such that*

$$\sup_{u \in H_0^1(\Omega), \|u\|_{H_0^1} \leq 1} \int_\Omega e^{4\pi u^2} \leq C_{\text{MT}} |\Omega|. \quad (2.6.8)$$

Corollary 1. For every $g \in H_0^1(\Omega)$ and $\beta \in \mathbb{R}$,

$$\log \left(\int_{\Omega} e^{\beta g} \right) \leq \frac{\beta^2}{16\pi} \int_{\Omega} |\nabla g|^2 + \log(C_{\text{MT}} |\Omega|). \quad (2.6.9)$$

Proof of Corollary. For brevity, we use $\|\cdot\|$ to denote $\|\cdot\|_{H_0^1(\Omega)}$. Given $g \in H_0^1$ and $\beta \in \mathbb{R}$ we note that

$$\beta g \leq |\beta g| \leq \frac{\beta^2}{16\pi} \|g\|^2 + 4\pi \frac{g^2}{\|g\|^2}. \quad (2.6.10)$$

We apply (2.6.8) with $f = g/\|g\|$ to obtain

$$\int_{\Omega} e^{\beta g} \leq e^{\frac{\beta^2}{16\pi} \|g\|^2} \int_{\Omega} e^{4\pi f^2} \leq e^{\frac{\beta^2}{16\pi} \|g\|^2} (C_{\text{MT}} |\Omega|). \quad (2.6.11)$$

We now take logarithms to obtain (2.6.9). \square

Theorem 3 (Negative temperature states). Assume $\beta \in (-8\pi, 0)$, and assume Ω is bounded and has a smooth boundary. The functional F_{β} has a minimizer in $H_0^1(\Omega)$ that solves the mean-field equation (2.4.8).

Further, if Ω is simply connected, this solution is unique.

Proof of existence. Assume $\beta \in (-8\pi, 0)$. We apply Corollary 1 with $g = \beta\psi$ to obtain the inequality

$$\log Z_{\beta} \leq \frac{\beta^2}{8\pi} H(\psi) + \log(C_{\text{MT}} |\Omega|). \quad (2.6.12)$$

Since $\beta < 0$ it follows that

$$F_{\beta}(\psi) = \frac{1}{\beta} \log Z_{\beta} + H(\psi) \quad (2.6.13)$$

$$\geq \left(1 + \frac{\beta}{8\pi}\right) H(\psi) + \frac{1}{\beta} \log(C_{\text{MT}} |\Omega|). \quad (2.6.14)$$

When $\beta > -8\pi$, it follows that $f_{\beta} := \inf_{\psi \in H_0^1(\Omega)} F_{\beta}(\psi) > -\infty$. Thus, we may choose a minimizing sequence $\{\psi_n\}_{n=1}^{\infty}$ such that $F_{\beta}(\psi_n) \rightarrow f_{\beta}$. Without loss of generality, we may suppose that $F_{\beta}(\psi_n) < f_{\beta} + 1$. It then follows from (2.6.14) that the sequence $\{\psi_n\}_{n=1}^{\infty}$ is uniformly bounded in H_0^1 . Thus, we may extract a subsequence, also denoted $\{\psi_n\}_{n=1}^{\infty}$ that converges weakly in H_0^1 and a.e to a limiting function $\psi \in H_0^1$. The energy $H(\psi_n)$ is weakly lower semicontinuous in H_0^1 . The a.e. convergence of ψ_n , when combined with (2.6.9), shows that $\lim_{n \rightarrow \infty} Z_{\beta}(\psi_n) = Z_{\beta}(\psi)$. Thus, ψ must minimize F_{β} since

$$f_{\beta} \leq F_{\beta}(\psi) \leq \liminf_{n \rightarrow \infty} F_{\beta}(\psi_n) = f_{\beta}. \quad (2.6.15)$$

\square

The existence proof presented above is a classic application of the direct method in the calculus of variations. However, *uniqueness* is more delicate. For example, the assumption that Ω is simply connected is necessary. The uniqueness proofs rely on interpreting (2.4.8) in terms of conformal geometry [25].

2.7 The mean-field equation for a mixture

2.7.1 The limiting variational problem

We now extend the mean-field equation to point vortices with distinct vorticity. Let us consider a system with s distinct species, such that:

1. A point vortex of species j has strength κ_j/N , for some fixed constants $\kappa_1, \kappa_2, \dots, \kappa_s$.
2. There are N_1, N_2, \dots, N_s vortices of species $j = 1, 2, \dots, s$ located at positions $\mathbf{z}_j = (z_{1,j}, \dots, z_{N_j,j})$. The associated empirical measure is

$$\rho_{j,N} = \frac{1}{N} \sum_{k=1}^{N_j} \delta(z - z_{k,j}), \quad N = N_1 + N_2 + \dots + N_s. \quad (2.7.1)$$

3. As $N \rightarrow \infty$, we assume that the relative proportion of species remains fixed, that is

$$\lim_{N \rightarrow \infty} \frac{N_j}{N} = p_j, \quad j = 1, \dots, s \quad (2.7.2)$$

for fixed positive numbers $p_j > 0$, $j = 1, \dots, s$ such that $\sum_{j=1}^s p_j = 1$.

In the thermodynamic limit, each empirical measure $\rho_{j,N}$ converges to a limiting density ρ_j . The vorticity and Hamiltonian of the limiting field is

$$\omega(z) = \sum_{j=1}^s \kappa_j \rho_j(z), \quad H(\rho) = -\frac{1}{2} \int_{\Omega \times \Omega} G(z, z') \omega(z) \omega(z') dz dz'. \quad (2.7.3)$$

The Boltzmann entropy is the functional

$$S(\rho) = \sum_{j=1}^s \int_{\Omega} \rho_j \log \rho_j dz, \quad \rho = (\rho_1, \dots, \rho_s), \quad (2.7.4)$$

and the mean-field equation for the stream function ψ is obtained by minimizing the Boltzmann entropy subject to the constraints

$$H(\rho) = E, \quad \int_{\Omega} \rho_j(z) dz = p_j, \quad j = 1, \dots, s. \quad (2.7.5)$$

2.7.2 Variational principle for a lattice model

Let us briefly explain why the functional of Section 2.4 has to be modified as above. The combinatorial derivation of Section 2.3 may be extended to s species as follows. Again we discretize the domain Ω to the lattice Ω_M with M lattice sites defined in equation (2.3.2). As in equation (2.3.5) Let $n_{1,j}, \dots, n_{M,j}$ denote the total number of vortices of species j at each lattice site

$$n_{ij} = \#\{k | z_{k,j} = a_i\}, \quad \sum_{i=1}^M n_{ij} = N_j. \quad (2.7.6)$$

The empirical measure for each species may now be expressed as

$$\rho_{j,N}(z) = \sum_{i=1}^M p_{ij} \delta(z - a_i), \quad p_{ij} = \frac{n_{ij}}{N}. \quad (2.7.7)$$

The Hamiltonian for this lattice model is given by summing over the interactions between all species. As in (2.3.7) we obtain

$$H(p) = -\frac{1}{2} \sum_{j=1}^s \sum_{k=1}^s \kappa_j \kappa_k p_j^T G p_k, \quad p_j^T = (p_{1j}, p_{2j}, \dots, p_{Mj}). \quad (2.7.8)$$

(The factor of N^2 in equation (2.3.7) is absent from equation (2.7.8) because we have scaled the vortex strengths in step 1 above). If we define the empirical vorticity

$$\omega_N(z) = \sum_{j=1}^s \kappa_j \rho_{j,N}(z). \quad (2.7.9)$$

then the Hamiltonian for the lattice model is also obtained by substituting (2.7.8) in (2.7.4), and ignoring the infinite self-energies in the system. This explains why (2.7.4) is the limiting energy.

The entropy is additive because the total number of states computed in (2.3.10) is now given by a product of multinomials. That is,

$$e^{-NS_M(p)} \approx \prod_{j=1}^s \binom{N_j}{n_{1j}, \dots, n_{Mj}}. \quad (2.7.10)$$

We take logarithms and use Stirling's formula to obtain the Boltzmann entropy for the lattice model

$$S_M(p) = \sum_{j=1}^s \sum_{i=1}^M p_{ij} \log p_{ij}. \quad (2.7.11)$$

In the $M \rightarrow \infty$ limit, we obtain the entropy functional $S(\rho)$ in (2.7.4).

2.7.3 The mean-field equation

We introduce Lagrange multipliers $\{\alpha_i\}_{i=1}^s$ and β to deal with the constraints, and minimize the functional

$$I(\rho) = S(\rho) + \beta H(\rho) + \sum_{j=1}^s \alpha_j \int_{\Omega} \rho_j. \quad (2.7.12)$$

The functional I has a unique minimizer $\rho = (\rho, \dots, \rho_s)$ for $\beta \in (\beta_c, \infty)$ that satisfies the Euler-Lagrange equations

$$\log \rho_j = -\alpha_j + \beta \kappa_j \int_{\Omega} G(z, z') \omega(z') dz', \quad j = 1, \dots, s. \quad (2.7.13)$$

We again observe that the integral above is simply the stream function. Thus,

$$\rho_j = e^{-\alpha_j} e^{\beta \kappa_j \psi}, \quad j = 1, \dots, s. \quad (2.7.14)$$

The constants α_j are determined by the constraint (2.7.4), and we may write

$$e^{-\alpha_j} = \frac{p_j}{Z_j}, \quad Z_{j,\beta} = \int_{\Omega} e^{\beta \kappa_j \psi} \quad j = 1, \dots, s. \quad (2.7.15)$$

Finally, we relate the stream function to the vorticity through (1.4.2) and (2.7.3) to obtain the mean-field equation

$$\Delta \psi = \sum_{j=1}^s \frac{p_j}{Z_{j,\beta}} e^{\beta \kappa_j \psi}. \quad (2.7.16)$$

The mean-field equation (2.7.16) admits the following dual variational expression. Let us define the partition function

$$Z_{\beta} = \prod_{j=1}^s Z_{j,\beta}^{p_j} \quad (2.7.17)$$

Then equation (2.7.16) is the Euler-Lagrange equation for the functional

$$F_{\beta}(\psi) = \frac{1}{\beta} \log Z_{\beta} + \frac{1}{2} \int_{\Omega} |\nabla \psi|^2. \quad (2.7.18)$$

The simplest example is the behavior of a neutral vortex gas consisting of an equal number of plus and minus vortices of unit magnitude. That is, $s = 2$, $\kappa_1 = 1$, $\kappa_2 = -1$ and $p_1 = p_2 = 1/2$ in the notation above. We further expect that the symmetry of the problem ensures

$$Z_{\beta} := \int_{\Omega} e^{-\beta \psi} = \int_{\Omega} e^{\beta \psi}, \quad (2.7.19)$$

(this requires a proof). In this situation, the mean-field equation becomes the sinh-Poisson equation

$$\Delta \psi = \frac{1}{Z_{\beta}} \sinh \beta \psi. \quad (2.7.20)$$

This equation was known to Onsager and was rediscovered by Joyce and Montgomery [22].

More generally, equation (2.7.16) has a natural continuum limit as the number of species $s \rightarrow \infty$ in such a manner that the measures

$$\sum_{j=1}^s p_j \delta_{\kappa_j}(ds) \rightarrow \mu(ds), \quad (2.7.21)$$

where $\mu(ds)$ is a probability measure on the line, and the measures converge in the weak topology. The partition function (2.7.17) converges to

$$Z_{\beta} = \exp \int_{\mathbb{R}} \left(\log \int_{\Omega} e^{\beta s \psi(z)} dz \right) \mu(ds), \quad (2.7.22)$$

and the variational principle remains unchanged. In particular, we may express the mean-field equation in the form

$$\Delta\psi = -\frac{1}{\beta} \frac{\delta}{\delta\psi} \log Z_\beta. \quad (2.7.23)$$

Here $\delta/\delta\psi$ denotes the first variation of the functional Z_β with respect to the stream function ψ .

2.8 The Miller-Robert theory

2.8.1 Integrals of motion for 2D flows

In summary, we may say that the use of Kirchhoff's model has allowed us to derive an interesting mean-field equation for 2D fluids by a direct application of standard ideas in statistical mechanics. However, this model has some flaws.

The approximation of 2D flows by point vortices is unphysical, since all point vortices have infinite self-energy. A truly consistent mean-field theory must rely on the use of flows with finite total energy. The fundamental well-posedness theorem for 2D flows, due to Yudovich [17], asserts that if $\omega_0 \in L^\infty(\Omega)$, then the Euler equations define a global continuous dynamical systems in $L^\infty(\Omega)$ ⁹ More precisely, equipping L^∞ with the weak-* topology, there is a continuous mapping

$$\mathbb{R} \times L^\infty(\Omega) \rightarrow L^\infty(\Omega), \quad (t, \omega_0) \mapsto \omega_t, \quad (2.8.1)$$

such that $\omega_t \in L^\infty(\Omega)$ is the vorticity field at time t for the unique solution to the Euler equations with initial data $\mathbf{u}_0 = \text{curl}^{-1}\omega_0$. (It is convenient here to use the notation ω_t to denote the vorticity field $\omega(z, t)$).

The Hamiltonian for the vorticity field, i.e. the kinetic energy,

$$H(\omega) = -\frac{1}{2} \int_\Omega \int_\Omega G_\Omega(z, z') \omega(z) \omega(z') dz dz' = \frac{1}{2} \int_\Omega |\mathbf{u}|^2 < \infty, \quad (2.8.2)$$

because the Green's function is locally integrable and $\omega \in L^\infty(\Omega)$. Further, the Hamiltonian remains constant along the orbit,

$$H(\omega_t) = H(\omega_0), \quad t \in \mathbb{R}. \quad (2.8.3)$$

Yudovich's proof also establishes that the Lagrangian flow map

$$\Phi : \mathbb{R} \times \Omega \rightarrow \Omega, \quad (t, z) \mapsto \Phi_t(z), \quad \frac{d\Phi_t(z)}{dt} = \mathbf{u}(\Phi_t(z), t), \quad \Phi_0(z) = z, \quad (2.8.4)$$

is an area-preserving diffeomorphism for each $t \in \mathbb{R}$. The vorticity is transported by the flow map, and we have

$$\omega(z, t) = \omega_0(\Phi_t^{-1}(z)). \quad (2.8.5)$$

⁹We assume throughout this chapter that Ω is bounded. This ensures that $\omega \in L^1 \cap L^\infty$, in accordance with the hypotheses in [17].

As a consequence, the flow has infinitely many integrals of motion, in addition to the Hamiltonian. Let us define the vorticity distribution function

$$\mu(s) = \frac{1}{|\Omega|} \int_{\Omega} \mathbf{1}_{\{\omega_0(z) \leq s\}} \quad s \in (-\infty, \infty). \quad (2.8.6)$$

We find using (2.8.5) and $\det(D\Phi_t) = 1$ that

$$\int_{\Omega} h(\omega(z, t)) dz = \int_{\Omega} h(\omega_0(z)) dz, \quad h \in C_0(\mathbb{R}). \quad (2.8.7)$$

Thus, the flow preserves the vorticity distribution function ¹⁰.

2.8.2 Young measures and weak convergence in L^∞

Onsager's theory provides a consistent method to derive a steady state fluid flow from an assumption about point vortices. Further, as explained in Section 2.7, we may incorporate more and more 'species' of point vortices, so that we obtain a mean-field equation that is in accordance with any distribution $\omega \in L^\infty(\Omega)$. On first sight, it appears that the mean-field equation (2.7.16) could describe the mean-field limit for a given initial vorticity field $\omega_0 \in L^\infty$ that takes a finite number of values $\kappa_1, \dots, \kappa_2$ on subsets of Ω of measure p_j , and β is determined by the constraint that $\int |\nabla\psi|^2 = 2H(\omega_0)$. But this is false! The right-hand-side of (2.7.16) may take values outside $[-\|\omega_0\|_\infty, \|\omega_0\|_\infty]$ violating the constraints. Thus, it is necessary to redevelop Onsager's theory based solely on vorticity fields in $L^\infty(\Omega)$, without invoking Kirchhoff's point-vortex approximation, ensuring compatibility with the integrals of motion described above.

A mean-field model in accordance with (2.8.7) was introduced independently by Miller [?] and Robert [?]. The main physical idea in the theory is as follows – we assume that over a long time scale the flow mixes an initial vorticity distribution in space on all scales in a way that the likelihood of observing a particular value of the vorticity at any point in the domain is given by the conserved distribution function. More precisely, we assume that for each $z \in \Omega$

$$\lim_{T \rightarrow \infty} \int_0^T h(\omega(z, t)) dt = \int_{-\infty}^{\infty} h(s) \mu(ds), \quad h \in C(\Omega). \quad (2.8.8)$$

While both theories lead to the same mean-field equation, Robert's approach is mathematically rigorous and combines weak convergence methods for partial differential equations with large deviation theory. For these reasons, we adopt Robert's approach. The use of weak convergence methods is natural here since the Euler equations in 2D define a continuous dynamical system in $L^\infty(\Omega)$ as described in (2.8.1). Thus, each initial condition ω_0 has a well-defined ω -limit set, and it is of basic interest to classify these limit sets. ¹¹ For each sequence of

¹⁰In fact, it preserves other combinatorial invariants when $\omega_0 \in C^\infty(\Omega)$. See [2, Ch.1]

¹¹There is an unfortunate clash in notation at this point: the same symbol ω is used to denote the concept of ω -limit sets in dynamical systems, as well as the vorticity in a fluid flow.

times $\{t_n\}_{n=1}^\infty$ such that $t_n \rightarrow \infty$, we may extract a subsequence (also denoted by $\{t_n\}_{n=1}^\infty$), such that $\omega(\cdot, t_n)$ converges weakly to a point in the ω -limit set of ω_0 . However, we expect the vorticity to oscillate increasingly rapidly in space as $t \rightarrow \infty$, and in order to understand the convergence of nonlinear functionals of $\omega(\cdot, t)$, we focus on their distributions. More formally, for each solution we introduce the family of Young measures, $\nu_{z,t} = \delta_{\omega(z,t)}$. Thus,

$$\int_{-\infty}^{\infty} h(s) \nu_{z,t}(ds) = h(\omega(z,t)). \quad (2.8.9)$$

The limiting Young measures for each weakly convergent sequence captures the distribution of the weak-* limits. More precisely, for each weakly convergent sequence $\omega(\cdot, t)$ there exists a map $\nu : \Omega \rightarrow \mathcal{P}$ Young measure such that

$$\lim_{n \rightarrow \infty} \int_U h(\omega(z, t_n)) dz = \int_U \int_{\mathbb{R}} h(s) \nu_z dz, \quad (2.8.10)$$

for each open set $U \subset \Omega$ and $h \in C_0(\mathbb{R})$. The mixing assumption (2.8.8) may be restated as an assumption on the convergence of the Young measures:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \nu_{z,t} dt = \mu, \quad z \in \Omega, \quad (2.8.11)$$

with the topology of weak convergence on $\mathcal{P}(\mathbb{R})$ ¹².

Assumption (2.8.8) bears a formal similarity to the ergodicity assumption (2.2.5) underlying the point-vortex model, since both equations relate time averages to space averages. However, the right hand side of equation (2.2.5) is the microcanonical measure for a Hamiltonian dynamical system (Kirchhoff's equations), whereas the right hand side of (2.8.8) is an average over the distribution function of the vorticity, not a microcanonical measure. In fact, the main subtlety in the Robert-Miller theory lies in the definition of the microcanonical and canonical measures, since these are measure on the infinite-dimensional phase space $L^\infty(\Omega)$. In order to explain these issues, we first introduce another natural discretization of the velocity field.

2.8.3 Permutations as discrete rearrangements

For simplicity we assume $\Omega = \mathbb{T}^2$ and consider an initial vorticity field that takes only two values $\omega_0(z) = \pm 1$ on sets of equal measure, denoted $A_\pm \subset \Omega$ respectively. Thus, the vorticity distribution function is

$$\mu(s) = \frac{1}{2} (\mathbf{1}_{\{s \geq -1\}} + \mathbf{1}_{\{s \geq 1\}}), \quad s \in (-\infty, \infty), \quad (2.8.12)$$

and the underlying assumption about long time behavior of the flow is

$$\lim_{T \rightarrow \infty} \int_0^T h(\omega(z,t)) dt = \frac{1}{2} (h(-1) + h(1)). \quad (2.8.13)$$

¹²Add appendix on weak convergence

We discretize the problem as follows. For any even positive integer N , let

$$z_{jk} = \left(\frac{j}{N}, \frac{k}{N} \right), \quad 1 \leq j, k \leq N, \quad (2.8.14)$$

denote points on the uniform grid with side $1/N$. This set of points is denoted Ω_N . The initial vorticity distribution, denoted $\omega_N \in L^\infty(\Omega)$ is an assignment of values $\omega_{jk} = \pm 1$ to each square with lower-right corner z_{jk} . It is assumed that there are an equal number of plus and minus one's, so that the distribution function of $\omega_{0,N}$ is (2.8.12).

The natural discretization of a volume-preserving diffeomorphism in this setting is provided by permutations of the squares in the lattice. Since the grid consists of N^2 squares, each permutation $\sigma \in S_{N^2}$, defines an area-preserving map $\Phi_\sigma : \Omega \rightarrow \Omega$. The analogues of (2.8.5) and (2.8.7) for the discrete model are immediate

$$\omega_{N,\sigma}(z) = \omega_N(\Phi_\sigma^{-1}(z)), \quad \int_\Omega h(\omega_{N,\sigma}(z)) dz = \int_\Omega h(\omega_N(z)) dz, \quad h \in C_0(\mathbb{R}). \quad (2.8.15)$$

Let $P_N \subset S_{N^2}$ consist of the permutations $\sigma \in S_{N^2}$ such that

$$H(\omega_{N,\sigma}) = H(\omega_N) \quad (2.8.16)$$

In contrast with the point-vortex models, we no longer need to renormalize the Hamiltonian by subtracting the infinite self-energy of point vortices, nor do we need to rescale the energy by N when we set $\omega = \omega_{0,N}$.

The microcanonical measure is the uniform distribution on the finite set P_N . The canonical measure at inverse temperature $\beta \in \mathbb{R}$ is the probability distribution on all of S_{N^2} , defined by

$$p_{N,\sigma} = \frac{1}{Z_{N,\beta}} e^{-\beta H(\omega_{N,\sigma})}, \quad Z_{N,\beta} = \sum_{\sigma \in S_{N^2}} e^{-\beta H(\omega_{N,\sigma})}. \quad (2.8.17)$$

The definitions above account scrupulously for the rearrangements of a given vorticity field, but they are unwieldy. For example, the set P_N may consist only of the identity permutation because of the 'hard' constraint (2.8.16). Similarly, the asymptotics of both the microcanonical and canonical ensembles are difficult to analyze based on the above definitions. Finally, the limiting oscillations of the vorticity field are captured by Young measures. For these reasons, we adopt a simpler discrete model.

2.8.4 The canonical measure

Let us now consider an arbitrary bounded domain Ω , and an arbitrary initial vorticity field ω_0 with vorticity distribution function μ . We discretize the problem by considering a grid $\Omega_N \subset \Omega$, for example as in the previous section. The discretized initial vorticity distribution is denoted $\omega_{N,0}$. Rather than use a probability distribution on permutations to define the analogue of a phase space

volume, we introduce a probability distribution ν_z at each lattice site $z \in \Omega_N$, that accounts for the possible values that the vorticity takes at z . For example, if the initial vorticity field takes a discrete set of values $\kappa_1, \dots, \kappa_s$, then ν_z is assumed to take values $\kappa_1, \dots, \kappa_s$. The key physical assumption here is that the fluid flow mixes the vorticity field so that all initial values of the vorticity are seen in the neighborhood of any lattice site in a manner that is consistent with the conservation law (2.8.7). Further, we assume that the vorticity fluctuations at the different lattice sites are independent.

These two assumptions imply that the law on the discretized vorticity field, written $\omega_N \in L^\infty(\Omega_N)$, is the product measure¹³

$$\Pi(d\omega_N) = \prod_{z \in \Omega_N} \nu_z(d\omega_N(z)). \quad (2.8.18)$$

Each function $h \in C(\Omega_N)$ may be 'lifted' into a functional $\tilde{h} : L^\infty(\Omega_N) \rightarrow \mathbb{R}$,

$$\tilde{h}(\omega_N) = \frac{1}{|\Omega_N|} \sum_{z \in \Omega_N} h(\omega_N(z)). \quad (2.8.19)$$

When $\nu_z = \mu$ for each $z \in \Omega_N$, the product measure Π is invariant under permutations of lattice sites, and for any $h \in C(\Omega_N)$ we find

$$\int_{L^\infty(\Omega_N)} \tilde{h}(\omega_N) \Pi(d\omega_N) = \int_{\mathbb{R}} h(s) \mu(ds). \quad (2.8.20)$$

The canonical measure on $L^\infty(\Omega_N)$ is defined by including the Hamiltonian and vorticity distribution as exponential weights. We set

$$\Pi_{\text{can}}(d\omega_N) = \frac{1}{Z_{N,\beta,h}} \exp\left(-\tilde{h}(\omega_N) - \beta H(\omega_N)\right) \Pi(d\omega_N), \quad (2.8.21)$$

with the partition function

$$Z_{N,\beta,h} = \int_{L^\infty(\Omega_N)} \exp\left(-\tilde{h}(\omega_N) - \beta H(\omega_N)\right) \Pi(d\omega_N), \quad (2.8.22)$$

The parameters β and h are Lagrange multipliers, so that the vorticity distribution and energy constraints are met in the form

$$\mathbb{E}_{\Pi_{\text{can}}}(\tilde{h}) = \int_{\mathbb{R}} h(s) \mu(ds), \quad \mathbb{E}_{\Pi_{\text{can}}}(H) = H(\omega_{0,N}). \quad (2.8.23)$$

Computing the variation of $\log Z_{N,\beta,h}$ with respect to β and h , allows us to express the constraints in the form

$$-\frac{\delta}{\delta h} \log Z_{N,\beta,h} = \mu(ds), \quad -\frac{\partial}{\partial \beta} \log Z_{N,\beta,h} = H(\omega_{0,N}). \quad (2.8.24)$$

In the second equality here, it is simplest to compute the variation by choosing $h \in C(\Omega_N)$ so that it is non-zero only at a fixed lattice site z .

¹³Clearly, the space $L^\infty(\Omega_N)$ is isomorphic to $\mathbb{R}^{|\Omega_N|}$; we adopt this notation to make clear that our goal is to introduce a probability measure on $L^\infty(\Omega)$.

2.8.5 The mean-field equation

The Miller-Robert theory is the continuum limit of this model. In the continuum limit, the fundamental object is a map $\nu : \Omega \rightarrow \mathcal{P}$ that is determined by an entropic variational principle with constraints. The appropriate entropy here is the Kullback-Leibler distance, defined by

$$S(\nu) = \int_{\Omega} \int_{\mathbb{R}} \log \frac{d\nu_z}{d\mu} \nu_z(ds) dz, \quad (2.8.25)$$

when ν_z is absolutely continuous with respect to μ a.e in z . We set $S(\nu) = -\infty$ when ν_z is not absolutely continuous with respect to μ on a set of positive measure. The most likely Young measure is determined by minimizing $S(\nu)$ with respect to the continuum limit of the constraints (2.8.23)

$$H(\bar{\omega}) = H(\omega_0), \quad \int_{\Omega} \nu_z(ds) dz = \mu(ds), \quad (2.8.26)$$

where $\bar{\omega}$ is the mean vorticity field

$$\bar{\omega}(z) = \int_{\mathbb{R}} s \nu_z(ds), \quad z \in \Omega. \quad (2.8.27)$$

Let us illustrate the consequences of this minimization problem in a concrete instance. Assume the vorticity field ω_0 takes s distinct values $\kappa_1, \kappa_2, \dots, \kappa_s$ on sets of measure $|\Omega|p_1, \dots, |\Omega|p_s$, with $p_j > 0$, $1 \leq j \leq s$, and $\sum_{j=1}^s p_j = 1$. Thus, the vorticity distribution is

$$\mu(ds) = \sum_{j=1}^s p_j \delta_{\kappa_j}(ds). \quad (2.8.28)$$

A Young measure ν_z is absolutely continuous with respect to μ if and only if it is of the form

$$\nu_z(ds) = \sum_{j=1}^s \rho_j(z) \delta_{\kappa_j}(ds), \quad 0 \leq \rho_j(z) \leq 1, \quad \sum_{j=1}^s \rho_j(z) = 1, \quad z \in \Omega. \quad (2.8.29)$$

Such a Young measure has mean vorticity and Kullback-Leibler entropy

$$\bar{\omega}(z) = \sum_{j=1}^s \kappa_j \rho_j(z), \quad S(\nu) = \sum_{j=1}^s \int_{\Omega} \rho_j(z) \log \left(\frac{\rho_j(z)}{p_j} \right) dz. \quad (2.8.30)$$

In order to minimize the entropy subject to the constraints (2.8.23) we introduce Lagrange multipliers $\{\alpha_j\}_{j=1}^s$ and β to impose the vorticity distribution and energy constraints

$$\int_{\Omega} \rho_j(z) = p_j, \quad j = 1, \dots, s, \quad H(\bar{\omega}) = H(\omega_0). \quad (2.8.31)$$

Finally, we also need a Lagrange multiplier $\gamma(z)$ to impose the 'hard' constraint $\sum_{j=1}^s \rho_j(z) = 1$. We may think of $\gamma(z)$ as a 'pressure', since the constraint $\sum_{j=1}^s \rho_j(z) = 1$ is analogous to incompressibility. In summary, we must minimize the functional

$$S(\nu) + \sum_{j=1}^s \alpha_j \int_{\Omega} \rho_j + \beta H(\bar{\omega}) + \int_{\Omega} Z(z) \left(\sum_{j=1}^s \rho_j(z) \right). \quad (2.8.32)$$

Elementary calculations similar to equations (2.7.13)–(2.7.15) yield

$$\rho_j(z) = e^{-\gamma(z)} e^{-\alpha_j} p_j e^{\beta \kappa_j \psi}, \quad e^{\alpha_j} = \int_{\Omega} e^{-\gamma(z)} e^{\beta \kappa_j \psi}, \quad e^{\gamma(z)} = \sum_{i=1}^s p_i e^{-\alpha_i} e^{\beta \kappa_i \psi}. \quad (2.8.33)$$

Since the vorticity and stream function are related through $\Delta \psi = \omega$, and the mean vorticity is given by (2.8.30), we find the mean-field equation

$$\Delta \psi = e^{-\gamma(z)} \sum_{j=1}^s \frac{p_j \kappa_j e^{\beta \kappa_j \psi}}{Z_{j,\beta}}. \quad (2.8.34)$$

This mean-field equation admits the following unified formulation for an arbitrary vorticity distribution function μ that is analogous to equation (2.7.23). Robert expresses (2.8.34) in the following manner. Given a function $\alpha(s)$ and a real number β , let us define the 'pointwise' partition function

$$Z_{\alpha,\beta}(z) = \int_{\mathbb{R}} s \exp(-\alpha(s) + \beta s \psi(z)) \mu(ds). \quad (2.8.35)$$

Then the mean-field equation (2.8.36) takes the form

$$\Delta \psi = -\frac{1}{\beta} \frac{d}{d\psi} \log Z_{\alpha,\beta}. \quad (2.8.36)$$

Of course, α and β are not independent variables, but must be determined by the constraints (2.8.36). In contrast with (2.7.23), it is immediate that the right hand side of (2.8.36) always lies in the range $[-\|\omega_0\|_{\infty}, \|\omega_0\|_{\infty}]$.

2.9 Turkington's model

Chapter 3

Komogorov's law and Onsager's criterion for dissipation in ideal flows

3.1 Introduction

In this chapter, we describe both Kolmogorov and Onsager's approach to fully developed turbulence. Kolmogorov derived a scaling law for the power spectrum of turbulent flows that is remarkable for its simplicity and predictive power. Once one assumes the fundamental role of the energy dissipation rate, ε , the structure of the power spectrum of turbulent flows follows from dimensional analysis. However, the theory is phenomenological – that is, at present we do not know how to connect the universal energy dissipation rate to the Navier-Stokes or Euler equations; nor do we understand how to develop a suitable theory of turbulent ensembles that provides a firm foundation for the implicit probabilistic assumptions in Kolmogorov's theory. The issue is not just one of mathematical rigor and completeness – while Kolmogorov's analysis makes successful predictions about the power spectrum of the velocity field, analogous predictions about other structure functions are incorrect.

Onsager's work complements Kolmogorov's theory, but separates the issue of a statistical description of the fluid flow, from that of energy dissipation in ideal flows. In order to get to the heart of the matter in a simpler and direct manner, we first present the essence of Kolmogorov's ideas, relying on an informal, visual understanding of 'flow statistics'. This is followed by a description of Onsager's ideas, including some rigorous statements.

3.2 Normalization

Since dimensional analysis plays an important role in this chapter, we will revert to the equations in dimensional form. The Navier-Stokes equations are

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}, \quad (3.2.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (3.2.2)$$

where ν is the kinematic viscosity.

Our interest lies in describing the statistical properties of flows such as those shown in Figure 1.1.2. By flow statistics we typically mean quantities such as the correlation of the velocity field that may be extracted from time series data. While these statistics are unambiguous in a laboratory, if we view the Navier-Stokes equations or Euler equations as our starting point, then it is necessary to describe precisely what we mean by 'random fluid flows'. Kolmogorov implicitly assumes the existence of isotropic, homogeneous turbulence in the following sense: there exists an ensemble of solutions to the Navier-Stokes equations in \mathbb{R}^3 such that:

- (i) the law of the velocity field is invariant under translations and rotations in space;
- (ii) the expected value of the kinetic energy at any point in space-time is finite:

$$\mathcal{K}(t) = \frac{1}{2} \overline{|\mathbf{u}(x, t)|^2} < \infty. \quad (3.2.3)$$

- (iii) The energy dissipation at any point in space-time is defined by

$$\varepsilon_\nu(t) = \nu \overline{|\nabla \mathbf{u}(x, t)|^2}, \quad (3.2.4)$$

The fundamental phenomenological assumption of Kolmogorov's theory is that the energy dissipation remains *strictly positive* in the limit $\nu \rightarrow 0$,

$$\liminf_{\nu \rightarrow 0} \varepsilon_\nu(t) = \varepsilon(t) > 0. \quad (3.2.5)$$

Since the statistics are invariant under translation, the parameters \mathcal{K} and ε do not depend on x . However, we do allow them to depend on t in order to describe freely decaying turbulence.

The mean energy in any bounded domain, $\Omega \subset \mathbb{R}^d$ is $\mathcal{K}(t)|\Omega|$. Since the flow is defined on \mathbb{R}^d the total energy of any such solution is infinite. In order to formulate the main ideas, without the additional complexity of infinite energies at each step, we will consider flows on a periodic box with length L , and then take the limit as the box size $L \rightarrow \infty$. At first sight, the assumption of periodicity is at variance with flows such as those in Figure 1.1.2. However, an essential aspect of Kolmogorov's theory is *universality*. The universal power spectrum for turbulent flows on \mathbb{T}^3 depends only on a single positive number, the mean

dissipation rate ε , and the universal flow fields on \mathbb{T}^3 describe the *local* behavior of fully developed turbulent flows in many domains. For instance, what is shown in Figure 1.1.2(a) is a form of decaying turbulence – the size of eddies grows as we move downstream along the x -axis. In this example, universality means that if we blow-up a small region of the flow at two fixed locations x_1 and x_2 , the statistical properties of these flows should be identical to that of isotropic, homogeneous turbulence, except for a scaling factor that is determined by the ratio of the eddy sizes at x_1 and x_2 .

In the rest of this chapter, we assume that Ω is a periodic box with sides of length L . The Fourier coefficients of the velocity field are defined as follows:

$$\mathbf{a}(\mathbf{k}, t) = \frac{1}{L^d} \int_{L\mathbb{T}^d} \mathbf{u}(x, t) e^{-2\pi i x \cdot \mathbf{k}} dx, \quad \mathbf{k} \in \frac{1}{L} \mathbb{Z}^d, \quad (3.2.6)$$

so that the velocity field may be expressed as the infinite sum

$$\mathbf{u}(x, t) = \sum_{\mathbf{k} \in L^{-1} \mathbb{Z}^d} \mathbf{a}(\mathbf{k}, t) e^{-2\pi i x \cdot \mathbf{k}}, \quad x \in L\mathbb{T}^d. \quad (3.2.7)$$

The incompressibility condition (3.2.2) is equivalent to

$$\mathbf{k} \cdot \mathbf{a}(\mathbf{k}, t) = 0, \quad \mathbf{k} \in L^{-1} \mathbb{Z}^d. \quad (3.2.8)$$

The above normalization of the Fourier coefficients is chosen so that we may ascribe a meaning to the energy density in the limit $L \rightarrow \infty$. More precisely, when the Fourier coefficients are defined by (3.2.6), Plancherel's theorem takes the form

$$\frac{1}{L^d} \int_{L\mathbb{T}^d} |\mathbf{u}(x, t)|^2 dx = \sum_{\mathbf{k} \in L^{-1} \mathbb{Z}^d} |\mathbf{a}(\mathbf{k}, t)|^2. \quad (3.2.9)$$

The quantity on the left is the kinetic energy per unit volume, which we expect to have a finite limit as $L \rightarrow \infty$.

Our interest lies in velocity fields $\mathbf{u}(x, t)$ that are random. For such random fields the mean kinetic energy density and the mean energy dissipation are defined by

$$\mathcal{K}(t) = \frac{1}{2L^d} \int_{L\mathbb{T}^d} \overline{|\mathbf{u}(x, t)|^2}, \quad \varepsilon_\nu(t) = \frac{\nu}{L^d} \int_{L\mathbb{T}^d} \overline{|\nabla \mathbf{u}(x, t)|^2} dx. \quad (3.2.10)$$

When the law of $\mathbf{u}(x, t)$ is invariant under translations, we may also write

$$\mathcal{K}(t) = \frac{1}{2} \overline{|\mathbf{u}(x, t)|^2}, \quad \varepsilon_\nu(t) = \nu \overline{|\nabla \mathbf{u}(x, t)|^2}, \quad x \in L\mathbb{T}^d. \quad (3.2.11)$$

Averaging over the randomness, Plancherel's theorem now implies

$$\mathcal{K}(t) = \frac{1}{2} \sum_{\mathbf{k} \in L^{-1} \mathbb{Z}^d} \overline{|\mathbf{a}(\mathbf{k}, t)|^2}. \quad (3.2.12)$$

The *power spectrum*, $\mathcal{E}_L(d\mathbf{l}, t)$, of the random field $\mathbf{u}(x, t)$ is the measure on \mathbb{R}^d with atoms of magnitude $|\mathbf{a}(\mathbf{k}, t)|^2$ at each wave number $\mathbf{k} \in L^{-1}\mathbb{Z}^d$.

The normalization of Fourier coefficients in (3.2.6) is chosen so that the kinetic energy per unit volume and the power spectrum have a limit as $L \rightarrow \infty$. In the particular case of homogeneous, isotropic turbulence, we assume the power spectrum has a radially symmetric limiting density $\mathcal{E}(|\mathbf{k}|, t)$ such that for any $0 < r < R < \infty$,

$$\int_r^R \mathcal{E}(s, t) ds = \lim_{L \rightarrow \infty} \sum_{\mathbf{k} \in L^{-1}\mathbb{Z}^d} \mathbf{1}_{\{r \leq |\mathbf{k}| \leq R\}} \overline{|\mathbf{a}(\mathbf{k}, t)|^2}. \quad (3.2.13)$$

The power spectrum provides a natural notion of the energy contained within a shell with inner radius r and outer radius R in \mathbf{k} -space. Of particular importance is a decomposition of the power spectrum into shells that are logarithmically spaced – we consider the energy in shells $\{(r_k, r_{k+1})\}_{k=-\infty}^{\infty}$ with $r_k = \lambda^k$ for some $\lambda > 1$. The power spectrum then provides a notion of the mean energy at wave-number λ^k . The total energy in the system is obtained by summing over all wave-numbers

$$\mathcal{K}(t) = \int_0^\infty \mathcal{E}(r, t) dr. \quad (3.2.14)$$

Kolmogorov's main result is as follows: in the limit $\nu \rightarrow 0$, the power spectrum of isotropic, homogeneous turbulence satisfies the scaling law

$$\mathcal{E}(k, t) = \varepsilon(t)^{2/3} k^{-5/3} F(l_\nu(t)k), \quad k = |\mathbf{k}| > 0. \quad (3.2.15)$$

where the length scale l_ν , called the Kolmogorov length, is given by

$$l_\nu(t) = \nu^{3/4} \varepsilon(t)^{-1/4}, \quad (3.2.16)$$

and F is a universal function that is positive and satisfies $F(0) > 0$.

In what follows, we first review some basic distinctions between two and three dimensional flows, so that the phenomenological basis for Kolmogorov's theory is clear. We then establish (3.2.15) by dimensional analysis, including cut-offs for $\nu > 0$, and interpret it as an *energy cascade* from long to short length scales. Finally, we extend Kolmogorov's approach to 2D flows, deriving an analogous power-law due to Kraichnan, which describes an *inverse cascade* in two dimensions.

3.3 Energy and energy dissipation

The fundamental phenomenological difference between flows in two and three dimensions is the possibility of enhanced dissipation in three dimensional flows. In order to formulate this idea precisely, we review the relation between energy and energy-dissipation for weak and strong solutions. Classical solutions to

the Navier-Stokes equations on a time interval $[0, T]$ satisfy the fundamental energy-energy dissipation identity

$$\frac{1}{2} \int_{L\mathbb{T}^d} |\mathbf{u}(x, t)|^2 dx + \nu \int_0^t \int_{L\mathbb{T}^d} |\nabla \mathbf{u}(x, t)|^2 dx dt = \frac{1}{2} \int_{L\mathbb{T}^d} |\mathbf{u}(x, 0)|^2 dx, \quad 0 \leq t \leq T. \quad (3.3.1)$$

Further, since the flow is incompressible,

$$\int_{L\mathbb{T}^d} |\nabla \mathbf{u}(x, t)|^2 dx = \int_{L\mathbb{T}^d} |\boldsymbol{\omega}(x, t)|^2 dx, \quad (3.3.2)$$

so that it is the vorticity that determines the rate of energy dissipation.

As we show below, classical solutions to the Euler equations are insufficient to describe the energy dissipation in turbulence, and it is necessary to consider weak solutions to the equations. The energy identities above have to be interpreted with some care for weak solutions. The incompressibility condition (1.2.6) is linear and may be interpreted in the sense of distributions if \mathbf{u} is a distribution. As a consequence, the identity (3.3.2) is very robust (we defer technical statements to later sections). However, the Leray-Hopf weak solutions to the Navier-Stokes equations are constructed as a limit of Galerkin approximations and are only known to satisfy the energy *inequality* for a.e. $t \in (0, T)$

$$\frac{1}{2} \int_{L\mathbb{T}^d} |\mathbf{u}(x, t)|^2 dx + \nu \int_0^t \int_{L\mathbb{T}^d} |\nabla \mathbf{u}(x, t)|^2 dx dt \leq \frac{1}{2} \int_{L\mathbb{T}^d} |\mathbf{u}(x, 0)|^2 dx. \quad (3.3.3)$$

In order to formulate the idea of *anomalous dissipation*, or *dissipation in ideal flows* precisely, we fix an initial datum $\omega_0 \in L^2$, and consider a sequence of flows, labeled $\{\omega^{(n)}\}_{n=1}^\infty$ with viscosities $\{\nu_n\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} \nu_n = 0$ ¹. We then ask whether

$$\liminf_{\nu \rightarrow 0} \int_0^t \int_{L\mathbb{T}^d} |\nabla \mathbf{u}^\nu(x, t)|^2 dx dt > 0? \quad (3.3.4)$$

3.3.1 No anomalous energy dissipation in two-dimensions

As we have seen, the evolution of the vorticity is completely different in two and three dimensions. In two dimensions, the vorticity field is a scalar, and it satisfies the transport equation

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega. \quad (3.3.5)$$

The effect of the convective nonlinearity is simply to rearrange the distribution of vorticity in space. Further, the diffusive term has the effect of smoothing the

¹This is equivalent to considering a sequence of flows on a fixed (non-dimensional) domain \mathbb{T}^2 with Reynolds number, $\{Re_n\}_{n=1}^\infty$, such that $\lim_{n \rightarrow \infty} Re_n = \infty$. In practice, the increase in Reynolds number is obtained by increasing the flow velocity or length scale L , *not* the viscosity. We have chosen to vary ν_n for consistency with the dimensional analysis in this chapter.

vorticity field. As a consequence, every smooth solution to (3.3.1) satisfies a maximum (and minimum) principle

$$\min_{x \in \mathbb{T}^2} \omega(x, 0) \leq \min_{x \in \mathbb{T}^2} \omega(x, t) \leq \max_{x \in \mathbb{T}^2} \omega(x, t) \leq \max_{x \in \mathbb{T}^2} \omega(x, 0), \quad 0 \leq t \leq T. \quad (3.3.6)$$

Similarly, all L^p norms of the vorticity are uniformly bounded. This may be seen as follows. Assume $h : \mathbb{R} \rightarrow \mathbb{R}$ is a positive convex function. We multiply equation (3.3.1) by $h'(\omega)$ and integrate by parts, to obtain the a priori estimate

$$\frac{d}{dt} \int_{L\mathbb{T}^2} h(\omega(x, t)) dx = -\nu \int_{\mathbb{T}^2} h''(\omega) |\nabla \omega|^2 dx \leq 0, \quad (3.3.7)$$

since $h''(s) \geq 0$ for a convex function. In particular, choosing $h(s) = s^2$, we find

$$\int_{L\mathbb{T}^2} |\omega(x, t)|^2 dx \leq \int_{L\mathbb{T}^2} |\omega(x, 0)|^2 dx. \quad (3.3.8)$$

Let us now assume that the initial datum $\omega_0 \in L^1 \cap L^\infty$ (this is the assumption underlying Yudovich's theorem for weak solutions to the Euler equations on the bounded domain $L\mathbb{T}^2$). It follows immediately that the mean dissipation in the flow satisfies the estimate

$$\frac{\nu}{TL^2} \int_0^T \int_{L\mathbb{T}^2} |\nabla \mathbf{u}^\nu(x, t)|^2 dx dt \leq \frac{\nu}{L^2} \int_{L\mathbb{T}^2} |\omega_0(x)|^2 dx. \quad (3.3.9)$$

Thus, the mean dissipation in the flow converges to zero at rate $O(\nu)$.²

3.3.2 Vortex stretching and energy dissipation in three dissipation

How could the mean energy dissipation, ε , in three dimensional flows remain strictly positive? A heuristic mechanism for enhanced energy dissipation in three dimensions, termed vortex stretching, was first proposed by G.I. Taylor. Recall that the circulation is conserved in sufficiently smooth, ideal flows (see Section 1.3 and (1.3.11)). Let us consider a vortex tube as shown in Figure 3.3.1, and let us assume that the flow stretches the tube as shown. The conditions of incompressibility and Kelvin's theorem imply

$$l_0 a_0 = l a, \quad \omega_0 a_0 = \omega a, \quad (3.3.10)$$

so that the stretching of the vortex tube leads to growth of the vorticity

$$\omega = \omega_0 \frac{l}{l_0}. \quad (3.3.11)$$

Since the above flows are smooth, the effect of adding viscosity is a regular perturbation on a time scale of order $O(1/\nu)$. Thus, on this time scale the

²Check on convergence of NSE solutions to Yudovich solutions under this assumption. Ladyzhenskaya?

vorticity field for a viscous flow $\omega^\nu(x, t)$ behaves as shown in Figure 3.3.1. But then we see that the dissipation in the three dimensional flow is significantly greater than that of a two dimensional flow, since the L^2 -norm of the vorticity is amplified as in (??).

This heuristic argument, while plausible, contains several flaws, and is hard to pin down (though see [21]). For example, we are implicitly assuming that the flow is unstable to small perturbations which cause vortex tubes to stretch. However, the Euler equations are reversible in time, so that vortex stretching is reversible! At present, there are no rigorous construction of solutions (weak or classical) to the Navier-Stokes equations that satisfy the lower bound (3.3.4). This is one of the central obstructions to understanding turbulence.

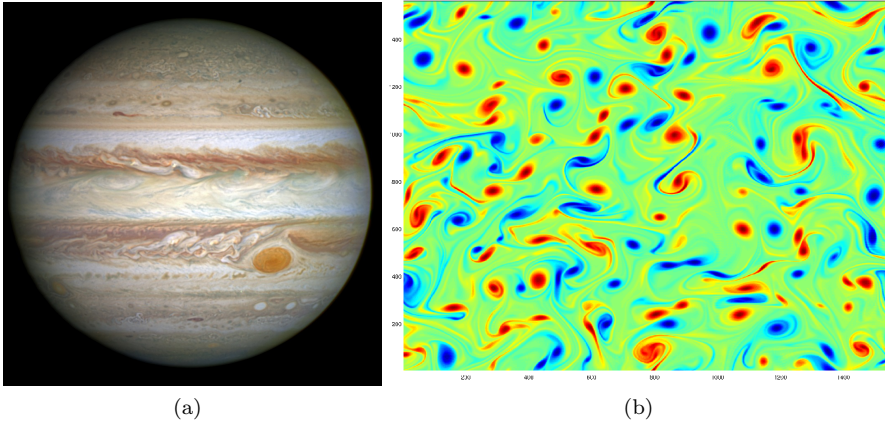


Figure 3.3.1: **Vortex stretching.** (a) An initial vortex tube – we assume that ω has constant magnitude ω_0 on the ring with perimeter l_0 and cross-sectional area a_0 ; (b) the length of the tube increases to l and its area decreases to a because of incompressibility. The vorticity must increase because of Kelvin's theorem.

3.4 Kolmogorov's scaling law

We now turn to a derivation of Kolmogorov's law (3.2.15) based on dimensional analysis. We denote the dimensions of length and time as L and T respectively. The velocity, pressure gradient and kinematic viscosity have dimensions ³

$$[\mathbf{u}] = \frac{L}{T}, \quad [\nabla p] = \frac{L}{T^2}, \quad [\nu] = \frac{L^2}{T}. \quad (3.4.1)$$

³Fix notation for L , or is this OK?

The dimensions of the mean energy and energy dissipation rate are

$$[\mathcal{K}(t)] = \frac{L^2}{T^2}, \quad [\varepsilon(t)] = \frac{L^2}{T^3}. \quad (3.4.2)$$

The normalization of the Fourier coefficients in (3.2.6) and the Plancherel theorem ensures that the dimensions of $\mathbf{a}(\mathbf{k})$ and the power spectrum are given by

$$[\mathbf{a}] = \frac{L}{T}, \quad [\mathcal{E}(k, t) dk] = [|\mathbf{a}|^2] = \frac{L^2}{T^2}, \quad [\mathcal{E}(k, t)] = \frac{L^3}{T^2}. \quad (3.4.3)$$

Kolmogorov's analysis proceeds as follows. The basic cartoon of the fluid flow is that it is assumed to consist of coherent structures, or eddies, on many length scales. Any form of coherent fluid motion, for example a large eddy is very unstable and shorter length scales are rapidly excited (the evolution equations in the Fourier modes are introduced below). This causes a transfer of energy from long to short scales. Since ν is small, the action of viscosity on the long wavelengths is very weak, and most of the energy is transferred to smaller wavelengths. We expect this process to occur in a self-similar manner, until the length scales are so small that viscosity becomes the dominant factor.

We ignore the role of the mean energy, \mathcal{K} in the system, and treat the mean-energy dissipation, ε as the primary parameter in the problem then there is only one intrinsic length scale in the system: the Kolmogorov length scale $l_\nu = \nu^{3/4} \varepsilon^{-1/4}$ introduced in (3.2.16). The exponents of ν and ε are fixed by their dimensions noted above. At length scales smaller than l_ν^{-1} , equivalently at wave numbers $k \gg l_\nu^{-1}$, the flow is dominated by viscosity and the nonlinear effects of convection are negligible. The wavenumber $k \ll l_\nu^{-1}$ is termed the *inertial range* and $k \gg l_\nu^{-1}$ is called the *dissipation range*. We expect self-similar transfer of energy between eddies to dominate in the inertial range. The specific form of this *energy cascade* is again determined by dimensional analysis.

The power-spectrum of the flow is assumed to depend on only the parameters ε , k , and ν . Since k has dimensions L^{-1} , and ε and \mathcal{E} have dimensions given in (3.4.2) and (3.4.3), we find that

$$\mathcal{E}_\nu(k, t) = \varepsilon^{2/3} k^{-5/3} F(kl_\nu) \quad (3.4.4)$$

as noted in (3.2.15). In the limit $\nu \rightarrow 0$, there is only one dimensional parameter, ε , and (3.4.4) reduces to

$$\mathcal{E}(k, t) = F(0) \varepsilon^{2/3} k^{-5/3}. \quad (3.4.5)$$

While it is conventional to focus on a 'turbulent equilibrium' when the dissipation rate $\varepsilon(t) =$ is independent of time, this requires us to assume that energy dissipation is balanced by the input of energy at long length scales. Unfortunately, this means that the mean-energy in the system is infinite! Indeed,

$$\mathcal{K}(t) = \int_0^\infty \mathcal{E}(k, t) dk = F(0) \varepsilon^{2/3} \int_0^\infty k^{-5/3} dk = +\infty. \quad (3.4.6)$$

The divergence is at low wave-numbers and arises in part from the fact that Kolmogorov ignored a second dimensional parameter, the mean energy $\mathcal{K}(t)$, in his analysis. In our view, it is more natural to include this parameter, both because it models a situation of freely decaying turbulence, and also because the inclusion of the energy resolves the above divergence.

More precisely, let us postulate the existence of random fields that solve the Euler equations whose law is invariant under translations and rotations in space, and have finite mean energy and positive dissipation. Then a new length scale,

$$l_*(t) = \frac{\mathcal{K}^{3/2}(t)}{\varepsilon(t)}, \quad (3.4.7)$$

is determined by the balance between the energy and the dissipation. Dimensional analysis as above, yields the universal law ⁴

$$\mathcal{E}(k, t) = \varepsilon^{2/3} k^{-5/3} G(kl_*(t)), \quad (3.4.8)$$

where $G(s)$ is a positive, universal function such that

$$\int_0^\infty s^{-5/3} G(s) ds = 1. \quad (3.4.9)$$

In particular, we expect freely decaying turbulence to be described by a statistically self-similar velocity fields that solve the Euler equations, with parameters $\mathcal{K}(t)$ and $\varepsilon(t)$ such that equations (3.4.8)–(3.4.9) hold and the mean energy and dissipation depend on time through

$$\dot{\mathcal{K}}(t) = -\varepsilon(t). \quad (3.4.10)$$

While the existence of such random fields is unknown for the Euler equations, later in these notes, we will establish the existence of such random fields in a simpler model problem.

3.5 Onsager's criterion and dissipation in ideal flows

1. Equations in Fourier space.
2. Criterion for smoothness related to Kolmogorov law.
3. Rigorous results: (Duchon-Robert).

Suppose $\varphi \in C_c^\infty(\mathbb{R}^3)$ is even and non-negative; set $\varphi_\varepsilon(x) = \varepsilon^{-3}\varphi(x/\varepsilon)$ and

$$D_\varepsilon(u)(x, t) = \frac{1}{4} \int_{\mathbb{T}^3} \nabla \varphi_\varepsilon(y) \cdot \delta \mathbf{u}(x, y) |\delta \mathbf{u}(x, y)|^2 dy, \quad \delta \mathbf{u}(x, y) = \mathbf{u}(x+y) - \mathbf{u}(y). \quad (3.5.1)$$

⁴This should be somewhere in the literature, but where?

Theorem 4 (Local energy balance for the Navier-Stokes equations). (a) Suppose $\mathbf{u} \in L^3(0, T; L^3)$ is a weak solution to the Navier-Stokes equations, and define $D_\varepsilon(\mathbf{u}) \in L^1((0, T) \times \mathbb{T}^3)$ as in (3.5.1). As $\varepsilon \rightarrow 0$, the functions $D_\varepsilon(\mathbf{u})$ converge to a distribution $D(\mathbf{u})$ that is independent of the choice of φ .
 (b) The following local form of energy conservation is satisfied:

$$\partial_t \left(\frac{1}{2} |\mathbf{u}|^2 \right) + \nabla \cdot \left(\mathbf{u} \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \right) + \frac{1}{Re} |\nabla \mathbf{u}|^2 - \frac{1}{2Re} \Delta |\mathbf{u}|^2 = -D(\mathbf{u}). \quad (3.5.2)$$

(c) If $\mathbf{u} \in L^\infty((0, T); L^2) \cap L^2((0, T); \dot{H}^1)$ is a Leray-Hopf weak solution, then $D(\mathbf{u})$ is a non-negative measure.

Theorem 5 (Local energy balance for the Euler equations). (a) Assume $\mathbf{u} \in L^3(0, T; L^3(\mathbb{T}^3))$ is a weak solution to the Euler equations. The following local form of energy conservation is satisfied:

$$\partial_t \left(\frac{1}{2} |\mathbf{u}|^2 \right) + \nabla \cdot \left(\mathbf{u} \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \right) = -D(\mathbf{u}), \quad (3.5.3)$$

where $D(\mathbf{u})$ is the limit of the distributions $D_\varepsilon(\mathbf{u})$ defined in (3.5.1) as $\varepsilon \rightarrow 0$.
 (b) If $\mathbf{u} \in L^\infty((0, T); L^2) \cap L^2((0, T); \dot{H}^1)$ is a limit of Leray-Hopf weak solutions, then $D(\mathbf{u})$ is a non-negative measure.

3.6 Rigorous results: the Euler equations as a differential inclusion

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