Lectures on Partial Differential Equations

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Abstract

These are my incomplete lecture notes for the graduate introduction to PDE at Brown University in Fall 2005. The lectures on Laplace's equation and the heat equation are included here. Typing took too much work after that. I hope what is here is still useful. Andreas Klöckner's transcript of the remaining lectures is also posted on my website. Those however have not been proofread. Comments are welcome by email.

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1.1 Introduction

Geometry and physics are the two main sources for problems in partial differential equations. Laplace's equation is fundamental, and arises in both contexts. The main sources for this chapter are John [7, Ch. 6] and Gilbarg and Trudinger [5, Ch. 2].

1.1.1 Minimal surfaces and Laplace's equation

First the geometric context. Take a closed wire frame, dip it in soap solution and pull it out. What is the shape of the soap film? One mathematical formulation of this problem is the following. Let us suppose the surface can be described as a graph over an open subset U of $\mathbb{R}^2$. Any smooth function $u : U \to \mathbb{R}$, yields a surface $x_3 = u(x_1, x_2)$. The area of this surface is

$$I(u) = \int_U \sqrt{|Du|^2 + 1}\, dx. \tag{1.1}$$

Our notation here is $Du = (\partial_{x_1} u, \partial_{x_2} u)$ and $dx = dx_1 dx_2$. If the shape of the wire frame is known (say given by a function $f : \partial U \to \mathbb{R}^3$), we require that u satisfy the *boundary condition* $u = f$ for $x \in \partial U$. Thus, the problem is to find a function u that minimizes $I(u)$ subject to the boundary condition. In calculus you learn that the gradient of a function must vanish at an extremum. Our problem is similar: we wish to minimize a *functional*, thus we must find the analogue of its gradient. The way to do this is to assume that one has a solution to the problem, and then say that first order changes must be zero. Suppose we have found a solution u, then if we consider a variation φ that is smooth and $\varphi = 0$ on ∂U, we expect that

$$\left.\frac{dI(u + \varepsilon\varphi)}{d\varepsilon}\right|_{\varepsilon=0} = 0. \tag{1.2}$$

We substitute in (1.1) and integrate by parts to find

$$0 = \int_U \frac{D\varphi \cdot Du}{\sqrt{|Du|^2 + 1}}\, dx = -\int \varphi \,\mathrm{div}\left(\frac{Du}{\sqrt{|Du|^2 + 1}}\right) dx. \tag{1.3}$$

There is a wonderful trick at this point. Since φ is arbitrary, we can in fact deduce that u must satisfy the *minimal surface equation*

$$\mathrm{div}\left(\frac{Du}{\sqrt{|Du|^2 + 1}}\right) = 0, \quad x \in U. \tag{1.4}$$

Observe that there is no real need to suppose that the surface is two-dimensional, and the same equation would result for higher-dimensional minimal 'surfaces'. The geometric quantity on the left hand side is n times the mean curvature; thus minimal surfaces must have *zero mean curvature.* We proceed from (1.3) to (1.4) as follows. If the mean curvature is sufficiently regular (say, continuous) then the set where it is (strictly) positive is open. If we choose a bump function φ that is positive in any open ball contained within this set, then we contradict (1.3). Thus, the set where the mean curvature is strictly positive is empty. Similarly for the set where it is strictly negative.

The minimal surface equation is nonlinear, and unfortunately rather hard to analyze. A simpler version of the equation is obtained by linearization: we assume that $|Du|^2 \ll 1$ and neglect it in the denominator. Thus, we are led to *Laplace's equation*

$$\mathrm{div} Du = 0. \tag{1.5}$$

The combination of derivatives $\mathrm{div} D = \sum_{i=1}^n \partial_{x_i}^2$ arises so often that it is denoted $\triangle$. The combination of the PDE and boundary condition on u is called the Dirichlet problem

$$\triangle u = 0, \quad x \in U, \qquad u = f, \qquad x \in \partial U. \tag{1.6}$$

One may continue in this vein. Another classical geometric problem is to determine surfaces with prescribed curvature (zero mean curvature being just one example). To build intuition, consider a two dimensional graph embedded in $\mathbb{R}^3$ with principal curvatures κ_1 and κ_2. As stated, κ_i depend on the embedding of the surface in $\mathbb{R}^3$. Remarkably, the product $K = \kappa_1 \kappa_2$ does not. This is the intrinsic or Gaussian curvature. The problem of determining surfaces with prescribed Gaussian curvature K in n-dimensions leads to the Monge-Ampére equation

$$\det(D^2 u) = K(x)(1 + |Du|^2)^{(n+2)/2}, \quad x \in U. \tag{1.7}$$

As before, this equation is nonlinear, and the Laplacian appears if one were to cheat and linearize. (This has to be done more carefully than what I said in class). The convex function $|x|^2/2$ solves (1.7) with $K = (1+|x|^2)^{-(n+2)/2}$. If we consider a small perturbation of this surface, that is we set $u = |x|^2/2 + \varepsilon v$, and retain only first order terms in ε we obtain,

$$\triangle v = \frac{(n+2)x \cdot Dv}{(1 + |x|^2)}, \quad x \in U. \tag{1.8}$$

where we use the expansion

$$\det\left(I+\varepsilon D^2 v\right)=1+\varepsilon\triangle v+O(\varepsilon^2).$$

1.1.2 Fields and Laplace's equation

Laplace's equation and Poisson's equation are also central equations in classical (ie. 19th century) mathematical physics. For example, distributions of mass or charge ρ in space induce gravitational or electrostatic potentials determined by Poisson's equation

$$\triangle u=\rho. \tag{1.9}$$

Of course, if $\rho\equiv 0$ this reduces to Laplace's equation. Since these equations are linear, the solvability of Poisson's equation is closely tied to solvability of Laplace's equation. Laplace's equation also arises in the description of the flow of incomressible fluids.

1.1.3 Motivation v. Results

An understanding of the context of the PDE is of great value. Different viewpoints suggest different lines of attack and Laplace's equation provides a perfect example of this. For example, in the physical context it is natural as a first step to consider special solutions for a point mass or point charge. This leads to the powerful tool of Green's function and Poisson's integral formula. In the geometric context, it is natural to expect that minimal surfaces should not have local peaks and should be smooth (for dimples and crinkles would cause the curvature to increase). This viewpoint leads to the maximum principle and elliptic regularity theorems. Both viewpoints also suggest that solutions with rotational symmetry should be important. Notice however that while this intituition serves as a guide, our task is to deduce this behavior from the equation. Theorems once proved hold independent of any particular application.

1.2 Notation

To be more careful let us fix notation. U will denote an open set in $\mathbb{R}^n$. The set is usually bounded and connected. ∂U denotes the boundary of U. We work with open sets because they are good for calculus. A point $x\in\mathbb{R}^n$ has coordinates $(x_1,\ldots,x_n)$. The derivative of a function $f:U\to\mathbb{R}^m$ is written Df and has components $(Df)_{ij}=\partial_{x_j}f_i,\quad i=1,\ldots,m, j=1,\ldots,n$. The convention is that the derivative Df_i of each component f_i is a row vector. Sometimes for brevity, we will write $f_{i,j}$ to denote $\partial_{x_j} f_i$. As a rule of thumb write out an equation in components to build some intuition about it. Sometimes it is also more convenient to write $\nabla\cdot$ instead of div.

While we will use little functional analysis this semester, it is helpful to use some of the notation of function spaces for brevity and clarity. If $f : U \to \mathbb{R}$ is continuous, we write $f \in C(U)$. Moreover, if $k \geq 1$ is an integer, we say that $f \in C^k(U)$ if f has k derivatives and these are continuous.

A delicate point: It is crucial to distinguish between the space $C(U)$ and $C(\bar{U})$ – the space of functions continuous on the closure of U. Suppose U is bounded. An example of a function in $C(\bar{U})$ is the distance from the boundary $f(x) = \mathrm{dist}(x, \partial U) = \inf_{y\in\partial U} |x-y|$. Observe that $C(U)$ is strictly larger than $C(\bar{U})$ because $1/\mathrm{dist}(x, \partial U)$ is in $C(U)$ but not $C(\bar{U})$.

1.3 The mean value inequality

The question of what is meant by a solution to a PDE is not as straightfoward as it may first seem. By a classical solution to Laplace's equation we mean a solution in the most direct sense: u is a C^2 function such that $\triangle u = 0$. It turns out that it is useful also to have notions of sub and super-solutions to an equation.

Definition 1.1. A function $u \in C^2(U)$ is *harmonic* if $\triangle u = 0, x \in U$. $u \in C^2(U)$ is *subharmonic* if $\triangle u \geq 0$, and u is *superharmonic* if $-u$ is subharmonic.

Later we will weaken this definition of subharmonicity to continuous functions.

Example 1.2. If A is a positive semi-definite matrix, $u = x^t A x$ is subharmonic.

A fundamental property of subharmonic functions is the mean value inequality. In all that follows ω_n denotes the $n-1$ dimensional measure of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n | |x| = 1\}$. It is an interesting calculus exercise to show that

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{1.10}$$

where the Gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt. \tag{1.11}$$

Warning: some authors use the notation ω_n to mean the volume (ie. n dimensional Lebesgue measure) of the unit ball $B(0,1) = \{x \in \mathbb{R}^n | |x| \leq 1\}$. Once one has calculated ω_n, it is easy to show that the volume of the ball $B(x,r) = \{y \in \mathbb{R}^n | |x-y| \leq r\}$ is

$$|B(x,r)| = |B(0,r)| = |B(0,1)|r^n = \frac{\omega_n}{n} r^n. \tag{1.12}$$

Theorem 1.3. *Let $u \in C^2(U)$ be subharmonic. Suppose $x \in U$ and $r <$ $\mathrm{dist}(x, \partial U)$. Then*

$$u(x) \leq \frac{1}{\omega_n r^{n-1}} \int_{S(x,r)} u(y) dS_y := \fint_{S(x,r)} u(y) dS_y, \tag{1.13}$$

and

$$u(x) \leq \frac{n}{\omega_n r^n} \int_{B(x,r)} u(y) dy := \fint_{B(x,r)} u(y) dy. \tag{1.14}$$

Corollary 1.4. *If $u \in C^2(U)$ is harmonic, then it is both subharmonic and superharmonic, and we have the* mean-value property

$$u(x) = \fint_{S(x,r)} u(y) dS_y = \fint_{B(x,r)} u(y) dy. \tag{1.15}$$

Proof. The proof follows from a simple integration by parts. Observe that

$$\int_{B(x,r)} \triangle u(y) dy = \int_{S(x,r)} \frac{\partial u}{\partial n} dS_y.$$

Since the surface is a sphere, the outward normal at $y \in S(x,r)$ is $\omega = (y-x)/|y-x|$. Therefore,

$$\frac{\partial u}{\partial n}(y) = \frac{d}{dr} u(x + \rho\omega),$$

and we may change variables in the integral from y to ω. The standard (and inconsistent) notation for the $n-1$ dimensional volume element on a sphere $|\omega| = 1$ with respect to the solid angle is $d\omega$. Thus, $dS_y = r^{n-1} d\omega$ and we obtain

$$\begin{aligned} \int_{S(x,r)} \frac{\partial u}{\partial n}(y) dS_y &= r^{n-1} \frac{d}{dr} \int_{|\omega|=1} u(x + r\omega) d\omega \\ &= r^{n-1} \frac{d}{dr} \left(r^{1-n} \int_{S(x,r)} u(y) dS_y \right). \end{aligned}$$

Hence, we have

$$0 \leq \int_{B(x,r)} \triangle u(y)dy = \omega_n r^{n-1} \frac{d}{dr} \fint_{S(x,r)} u(y)dS_y.$$

Thus, the mean value $\fint_{S(x,r)} u(y)dS_y$ is increasing. Since u is continuous $\lim_{r\to 0} \fint_{S(x,r)} u(y)dS_y = u(x)$. This proves (1.13). The mean value inequality for balls is obtained by integrating the mean value inequality on spheres,

$$\begin{aligned} \int_{B(x,r)} u(y)dy &= \int_0^r \int_{|\omega|=1} u(x+\rho\omega)d\omega\rho^{n-1}d\rho \\ &= \int_0^r \omega_n \fint_{S(x,\rho)} u(y)dS_y\rho^{n-1}d\rho \geq u(x)\omega_n \int_0^r \rho^{n-1}d\rho = u(x)|B(x,r)|. \end{aligned}$$

□

1.4 Maximum principles

The mean value inequality has unexpectedly strong consequences. In this section U is assumed bounded and connected.

Theorem 1.5 (Strong maximum principle). *Suppose $u \in C^2(U)$ is subharmonic. Suppose there exists a point $x \in U$ where $u(x) = \sup_U u$. Then u is constant. Consequently, if u is harmonic and attains its minimum or its maximum in U it is constant.*

Proof. Let $M = \sup_U u$. We may decompose U into the sets $U_1 = \{y|u(y) < M\}$ and $U_2 = \{y|u(y) = M\}$. Since u is continuous, U_1 is open and U_2 is closed. By assumption, U_2 is not empty. Let x be any point in U_2. We apply the mean value ineqality to deduce that

$$M = u(x) \leq \fint_{B(x,r)} u(y)dy \leq M.$$

Therefore, $u(y) = M$ on any ball $B(x,r) \subset U$. In particular, this implies that U_2 is open. Since a connected set cannot be decomposed into two nonempty, disjoint open sets, it must be that $U = U_2$. □

The adjective strong in this maximum principle refers to the conclusion that u is a constant if it attains its maximum. The following estimate is also of great value. Its strength is that it usually holds for scalar elliptic problems where one does not have a mean value inequality.

Theorem 1.6 (Weak maximum principle). *Suppose $u \in C^2(U) \bigcap C(\bar{U})$ is subharmonic. Then $\max_{x\in\overline{U}} u = \max_{x\in\partial U} u$. For harmonic u*

$$\min_{\partial U} \leq u(x) \leq \max_{\partial U} u, \quad x \in U.$$

The weak maximum principle follows immediately from the strong maximum principle. Here is an independent proof, that provides additional intuition.

Proof. First, let us suppose that u is strictly subharmonic, that is $\triangle u > 0$. Since $u \in C(\bar{U})$ it attains its maximum at some point $x \in \bar{U}$. If $x \in U$ we derive a contradiction as follows. At a maximum it is necessary that $Du(x) = 0$ and $D^2 u$ be negative semi-definite. But then $\triangle u = \mathrm{Tr}(D^2 u) \leq 0$ contradicting strict subharmonicity. Thus, $\max_{\overline{U}} u = \max_{\partial U} u$.

If u is not strictly harmonic, let $\varepsilon > 0$ and consider the strictly subharmonic function $u^\varepsilon = u + \varepsilon |x|^2$. We then have

$$\max_{\overline{U}} u \leq \max_{\overline{U}} u^\varepsilon = \max_{\partial U} u^\varepsilon \leq \max_{\partial_U} u + \varepsilon \max_{x\in\partial U} |x|^2.$$

The left hand side is independent of ε, thus one may let $\varepsilon \to 0$ to deduce the weak maximum principle. □

Taken as a geometric statement, this means that the graph of u is always saddlelike. A probabilistic proof of the maximum principle will be presented after we consider the heat equation in a few weeks.

We now obtain our first uniqueness theorem for the Dirichlet problem to Laplace's and Poisson's equation. Notice however that we have not shown existence.

Theorem 1.7. *Suppose $u, v \in C^2(U) \bigcap C(\bar{U})$ satisfy $\triangle u = \triangle v$ in U, and $u = v$ on ∂U. Then $u = v$ in U.*

Proof. Let $w = u - v$. Then $\triangle w = 0$ in U and $w = 0$ on ∂U. Therefore, $0 \leq w \leq 0$ in U by the weak maximum principle. □

The following inequality shows that a non-negative harmonic function cannot oscillate very much.

Theorem 1.8 (Harnack's inequality). *Let $u \in C^2(U)$ be a non-negative harmonic function in U. Then for any connected $U' \subset\subset U$ there exists a constant $C(U, U')$ such that*

$$\sup_{U'} u \leq C \inf_{U'} u. \tag{1.16}$$

Proof. Let $x \in U$ and choose $r < 1/4\text{dist}(x, \partial U)$. For any $x_1, x_2 \in B(x,r)$ we apply the mean value inequality to find

$$u(x_1) = \frac{n}{\omega_n r^n} \int_{B(x_1,r)} u(y)dy \leq \frac{n}{\omega_n r^n} \int_{B(x,2r)} u(y)dy.$$

Similarly,

$$u(x_2) = \frac{n}{\omega_n (3r)^n} \int_{B(x_2,3r)} u(y)dy \geq \frac{n}{\omega_n (3r)^n} \int_{B(x,2r)} u(y)dy.$$

We combine the inequalities, and take the sup over x_1 and inf over x_2 to obtain

$$\sup_{B(x,r)} u \leq 3^n \inf_{B(x,r)} u. \tag{1.17}$$

Now if $U' \subset\subset U$ let $x_1, x_2 \in \bar{U}'$ be such that $u(x_1) = \sup_{U'} u$ and $u(x_2) = \inf_{U'} u$. We connect x_1 and x_2 by a closed arc Γ contained in $\bar{U}'$. Let $r < 1/4\text{dist}(\Gamma, U')$. Since $\bar{U}'$ is compact, it may be covered by a finite number of balls, say N, of size r. Then combining the inequality (1.17) over the balls that cover Γ we obtain $u(x_1) \leq 3^{nN} u(x_2)$. □

Remark 1.9. The mean value inequality, weak maximum principle and Harnack's inequality are a priori estimates. The adjective a priori refers to the fact that they must hold for all harmonic functions in U, even if we do not know yet that these functions exist. The a priori estimates are usually used to then prove existence. A general theme, and typically the hardest step in PDE, is to prove suitable a priori estimates. The best a priori inequalities usually encode geometric or physical meaning.

1.5 The fundamental solution

Laplace's equation is invariant under rotations. This is not unexpected, physically, we expect the field induced by a point charge to be rotationally symmetric. To this end, let us look for solutions to Laplace's equation of the form $u(x) = \psi(r)$ where r denotes the radial coordinate $r = |x|$. We subsitute in Laplace's equation to find

$$\triangle u = \psi'' + \frac{(n-1)}{r}\psi' = 0.$$

Integrate once to find

$$\psi' = C_1 r^{1-n}, \qquad n \geq 2, \quad r > 0,$$

and integrate again to find

$$\begin{aligned}\psi &= C_1 \log r + C_2, \qquad n = 2, \quad r > 0 \\ \psi &= \frac{C_1}{2-n} r^{2-n} + C_2, \qquad n \geq 3, \quad r > 0.\end{aligned}$$

The constant C_2 is of little interest, and is usually taken to be zero. (For example, in the physical context all that matters is a potential difference, not a potential itself, thus the value of u can be taken to be zero at ∞ for convenience). As will become clear, there is a good reason to choose $C_1 = 1/\omega_n$. We thus have obtained radially symmetric solutions to Laplace's equation on the punctured space $\mathbb{R}^n \backslash \{0\}$. These are the fundamental solutions,

$$\psi(r) = \frac{1}{2\pi} \log r, \qquad n = 2, \tag{1.18}$$

$$\psi(r) = \frac{r^{2-n}}{(2-n)\omega_n}, \qquad n \geq 3. \tag{1.19}$$

Remark 1.10. The special role of two-dimensions is a recurrent theme in the study of Laplace's equation. In two-dimensions, harmonic functions on a domain U are in one-one correspondence with analytic (or holomorphic) functions, and to each harmonic function is assigned its harmonic conjugate.

Remark 1.11. At this point, you should note that if $n \geq 3$ then $\psi(r) < 0$ for $r < 0$, and $\psi(r) \to -\infty$ as $r \to 0$. In two-dimensions, ψ is neither bounded above or below. This has the consequence that in two-dimensions *subharmonic functions* also satisfy Liouville's theorem: a subharmonic function bounded above is constant. This is false for $n \geq 3$.

There is no need to restrict oneself to symmetry about the origin. We use the translation invariance of Laplace's equation to define the fundamental solution with pole at ξ

$$K(x, \xi) = \psi(|x - \xi|). \tag{1.20}$$

The singularity at ξ is very interesting. It turns out that $K(x, \xi)$ does not solve Laplace's equation (in all of $\mathbb{R}^n$), but instead solves Poisson's equation with a Dirac measure δ_ξ on the right hand side.

$$\triangle_x K = \delta_\xi. \tag{1.21}$$

Intuitively, $K(x, \xi)$ measures the influence a point source exerts at a point x. Notice from the formula that the influence is symmetric, $K(x, \xi) = K(\xi, x)$. More rigorously, we should use some care to interpret (1.21). One way is to simply multiply by a smooth test function φ with compact support in $\mathbb{R}^n$, integrate by parts and replace (1.21) by the apparently more legitimate statement

$$\varphi(\xi) = \int_{\mathbb{R}^n} K(x,\xi) \triangle_x \varphi(x)\, dx. \tag{1.22}$$

This works because even though the integrand is singular, it is still locally integrable. However, let us derive a general identity using calculus alone. To this end, we will use the following identities obtained by integration by parts (Green's identities). Suppose U is an open set with a C^1 boundary ∂U, and let $u, v \in C^2(\overline{U})$. Then

$$\int_U v \triangle u\, dx = -\int_U Dv \cdot Du\, dx + \int_{\partial U} v \frac{\partial u}{\partial n} dS, \tag{1.23}$$

$$\int_U (v\triangle u - u \triangle v)\; dx = \int_{\partial U} \left(v\frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) dS. \tag{1.24}$$

We will use this identity when U is a domain containing ξ, and $v(x) = K(x,\xi)$. Consider the domain $U \backslash B(\xi,\varepsilon)$, that is U with a small hole punched out. The boundary of this domain is $\partial U \cup S(\xi,\varepsilon)$. Apply Green's second identity (1.24) with $v = K(x,\xi)$. Then,

$$\int_U v\triangle u\, dx = \int_{\partial U} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS + \int_{S(\xi,\varepsilon)} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS.$$

Of course, we would like to obtain the limit of the second term as $\varepsilon \to 0$. If $x \in S(\xi,\varepsilon)$ we have

$$\frac{\partial v}{\partial n}(x) = -\psi'(\varepsilon) = -\frac{1}{\omega_n \varepsilon^{n-1}},$$

(the negative sign because the outward normal from $U\backslash B(\xi,\varepsilon)$ points towards ξ). Thus,

$$-\int_{S(\xi,\varepsilon)} u \frac{\partial v}{\partial n} dS = \fint_{S(\xi,\varepsilon)} u dS \to u(\xi) \qquad \text{as} \quad \varepsilon \to 0.$$

If $n \geq 3$, $v = \varepsilon^{2-n}/((2-n)\omega_n)$ on $S(\xi,\varepsilon)$ and

$$\begin{aligned}\int_{S(\xi,\varepsilon)} v \frac{\partial u}{\partial n} dS &= \frac{\varepsilon^{2-n}}{(2-n)\omega_n} \int_{S(\xi,\varepsilon)} \frac{\partial u}{\partial n} dS \\ &= \frac{\varepsilon^{2-n}}{(2-n)\omega_n} \int_{B(\xi,\varepsilon)} \triangle u\, dx = \frac{\varepsilon^2}{n(2-n)} \triangle u(y(\varepsilon)),\end{aligned}$$

where y_ε is some point in $B(\xi,\varepsilon)$ guaranteed by the mean value theorem. Since $\triangle u$ is continuous, this term vanishes as $\varepsilon \to 0$. A similar calculation for $n=2$ yields

$$\int_{S(\xi,\varepsilon)} v\frac{\partial u}{\partial n} dS = \left(\varepsilon^2 \log \varepsilon\right) \triangle u(y(\varepsilon)) \to 0.$$

To summarize, only the normal flux of $K(x,\xi)$ contributes and we are left with the following important identity. For every $u \in C^2(\overline{U})$

$$u(\xi) = \int_U K(x,\xi)\triangle u(x)\,dx + \int_{\partial U}\left(\frac{\partial K(x,\xi)}{\partial n_x}u - K(x,\xi)\frac{\partial u}{\partial n_x}\right) dS_x. \quad (1.25)$$

Of particular, importance is the case when u is harmonic. In this case, only the boundary term remains, and we obtain a representation for u in the interior in terms of its boundary values

$$u(\xi) = \int_{\partial U}\left(\frac{\partial K(x,\xi)}{\partial n_x}u - K(x,\xi)\frac{\partial u}{\partial n_x}\right) dS_x. \quad (1.26)$$

This doesn't yet constitute a solution to the Dirichlet or Neumann problem. The identity (1.26) allows us to obtain u in U if we know *both* u and $\partial u/\partial n$ on the boundary. In either the Dirichlet or Neumann problem, only one of these is prescribed and the other is unknown. However, this identity can be used as a basis for a solution. For example, in the Neumann problem, one could take the limit of ξ to the boundary and obtain a system of equations on the boundary alone. This is the basis for numerical methods known as boundary integral methods. We will not follow this approach. Instead, we first establish existence of solutions for simple domains.

1.6 Green's function and Poisson's integral formula

An important feature of identities (1.25) and (1.26) is that they remain valid when K is replaced by $K+w$ for any harmonic w. By a judicious choice of w one may obtain a solution to the Dirichlet problem. We say that $G(x,\xi)$ is a Green's function of the Dirichlet problem if

$$\triangle_x G = \delta_\xi, \quad x,\xi \in U, \qquad G(x,\xi) = 0, \quad x \in \partial U. \quad (1.27)$$

Observe that G yields an integral representation for a solution to the Dirichlet problem. We replace K by G in (1.26) to obtain

$$u(\xi) = \int_{\partial U} \frac{\partial G(x,\xi)}{\partial n_x} u(x) dS_x. \quad (1.28)$$

The catch is to actually find w such that $w = -K$ on the boundary. One can only do this explicitly for domains with some symmetry. The time-honoured way of achieving this is through the method of images. As a heuristic, imagine you had a unit positive charge at the point ξ. We would like to add charges at different points outside U such that their field cancels that of ξ on the boundary ∂U.

Example 1.12 (Poisson's integral formula for the half-space). Let U be the half-space $\{x \in \mathbb{R}^n | x_n > 0\}$. Fix $\xi \in U$. Every point ξ has an image ξ^* under reflection in the plane $x_n = 0$. In coordinates, $\xi^* = (\xi_1, \ldots, \xi_{n-1}, -\xi_n)$. Clearly, ξ^* does not lie in U. Therefore, $K(x, \xi^*)$ is harmonic in U. Moreover, if $x_n = 0$, then $|\xi - x| = |\xi^* - x|$. In order to construct a Green's function we choose

$$G(x, \xi) = K(x, \xi) - K(x, \xi^*).$$

An explicit computation of $\partial G / \partial n_x$ on the boundary $x_n = 0$ yields *Poisson's integral formula* for the half-space.

$$u(\xi) = \frac{2\xi_n}{\omega_n} \int_{\mathbb{R}^{n-1}} |x - \xi|^{-n} u(x)\, dx. \tag{1.29}$$

Note that here $dx = dx_1 \ldots dx_{n-1}$ denotes $n-1$ dimensional Lebesgue measure on $\mathbb{R}^{n-1}$.

A little bit more work is needed to find the Green's function for the unit ball $B(0,1)$. The natural notion of symmetry in this case, is inversion in the unit sphere $S(0,1)$. Given $\xi \in B(0,1)$ we define its image under inversion,

$$\xi^* = \frac{\xi}{|\xi|^2}. \tag{1.30}$$

As shown in the homework, inversion is a conformal transformation (and a fundamental one at that). The reason inversion is appropriate to the circle is the following calculation. If $|x| = 1$, then

$$|x - \xi| = |\xi||x - \xi^*|.$$

Let us verify this.

$$|x - \xi|^2 = |x|^2 + |\xi|^2 - 2x \cdot \xi = 1 + \xi^2 - 2x \cdot \xi,$$

and we use $|x| = 1$ and (1.30) to obtain

$$|x - \xi^*|^2 = 1 + |\xi|^{-2} - 2|\xi|^{-2} x \cdot \xi = |\xi|^{-2} |x - \xi|^2.$$

Therefore, for $n \geq 3$ an appropriate choice for the Green's function is

$$G(x,\xi) = K(x,\xi) - |\xi|^{2-n} K(x,\xi^*),$$

and for $n = 2$,

$$G(x,\xi) = \frac{1}{2\pi} \log \left(\frac{|x-\xi|}{|\xi||x-\xi^*|} \right).$$

You should show that in both cases,

$$G(x,\xi) = G(\xi,x) \quad x,\xi \in \overline{B(0,1)}.$$

The derivative of the Green's function in both cases is

$$D_x G = \frac{1}{\omega_n} \left(\frac{x-\xi}{|x-\xi|^n} - |\xi|^{n-2} \frac{x-\xi^*}{|x-\xi^*|^n} \right) = \frac{(1-|\xi|^2)x}{\omega_n |x-\xi|^n}.$$

Finally, we observe that on the unit sphere $\partial G/\partial n_x = DG \cdot x$ and substitute in (1.28) to obtain

$$u(\xi) = \frac{1}{\omega_n} \int_{S(0,1)} \frac{1-|\xi|^2}{|x-\xi|^n} u(x) dS_x.$$

For a ball of radius r we may rescale to obtain *Poisson's integral formula* for a ball

$$u(\xi) = \frac{r^2-|\xi|^2}{\omega_n r} \int_{S(0,r)} |x-\xi|^{-n} u(x) dS_x, \quad \xi \in B(x,r). \tag{1.31}$$

Note that $\xi = 0$ yields the mean-value property. We call the following expression the *Poisson kernel*

$$H(x,\xi) = \frac{r^2-|\xi|^2}{\omega_n r |x-\xi|^n}. \tag{1.32}$$

The following properties of the Poisson kernel are fundamental.

Theorem 1.13. *(a)* $H(x,\xi) > 0$, $\xi \in B(0,r)$.

(b) $\triangle_\xi H(x,\xi) = 0$, $\xi \in B(0,r), x \in S(0,r)$.

(c) $\int_{S(0,r)} H(x,\xi) dS_x = 1$.

(d) Let $0 < \delta < 2$ *and* $A_\delta = \{|y| = r, |y-x| \geq \delta\}$. *Then*

$$\lim_{\xi \to y,\, |\xi|<r} H(x,\xi) = 0$$

uniformly for $y \in A_\delta$.

Proof. (a) is immediate. (b) is proven by observing that for fixed x, $K(x,\xi)$ is harmonic in ξ as long as $|x-\xi|>0$. Moreover, $|\xi|^{2-n}K(x,\xi^*)$ is also harmonic as it is obtained by Kelvin's transformation from $K(x,\xi)$. Thus, G is harmonic in ξ, and so is its derivative $H=\partial G/\partial n_x$. A clever proof of (c) is to substitute $u\equiv 1$ in the identity (1.31). (d) is immediate. $\square$

We have proved uniqueness for the Dirichlet problem via the maximum principle. Poisson's integral formula yields existence for $B(0,r)$ immediately, and is the basis for Perron's method for general domains.

Theorem 1.14. *Let $U=B(0,r)$ and $f:\partial U\to\mathbb{R}$ be continuous. Then the function u defined by*

$$u(\xi)=\begin{cases}\int_{S(0,r)}H(x,\xi)f(x)dS_x, & \xi\in U\\ f(\xi), \quad \xi\in\partial U,\end{cases} \tag{1.33}$$

is harmonic in U and continuous on $\overline{U}$.

Proof. Since H is harmonic in ξ and f is continuous we may differentiate under the integral sign to deduce that u is harmonic. (If you have not done this sort of thing before, you should justify it by taking finite differences and passing to the limit using dominated convergence).

It only remains to show that u is continuous onto the boundary. To this end, fix $y\in\partial U$ and use Theorem 1.13 (c) to write

$$u(\xi)-f(y)=\int_{S(x,r)}H(x,\xi)\left(f(x)-f(y)\right)dS_x.$$

Fix $\varepsilon>0$ and choose $\delta>0$ so that $|f(x)-f(y)|<\varepsilon$ for $|x-y|<\delta$. Separate the integral into two pieces, one where $|x-y|<\delta$ and the other where $|x-y|\geq\delta$. On the first piece, we have

$$\left|\int_{S(x,r),|x-y|<\delta}H(x,\xi))\left(f(x)-f(y)\right)dS_x\right|\leq\varepsilon\int_{S(x,r)}H(x,\xi)dS_x=\varepsilon.$$

We have used $H>0$ here. On the second piece, we use Theorem 1.13 (d) to interchange limits, and obtain

$$\left|\int_{S(x,r),|x-y|\geq\delta}H(x,\xi)(f(x)-f(y))dS_x\right| \\ \leq 2\max|f|\int_{S(x,r),|x-y|\geq\delta}H(x,\xi)dS_x<\varepsilon,$$

for $|\xi-y|$ sufficiently small. $\square$

Remark 1.15. The method of proof here is as important as the theorem itself and crops up under the name 'approximate identities' in various places in analysis. One example that you may have seen is the proof of Fejér's theorem on the convergence of Fourier series (see for example, [10, Ch.2]).

Remark 1.16. A nice property of the Poisson integral formula is its independence of dimension. This is *not* true for the wave equation. A hint why this is true is provided by the homework problem on the reduced wave operator $\triangle u + cu = 0$.

1.7 The mean value property revisited

Observe that when $\xi = 0$ Poisson's integral formula reduces to the mean value property. This leads to the following characterization of harmonic functions.

Theorem 1.17. *A function $u \in C(U)$ is harmonic if and only if for every ball $B(x, r) \subset U$ it satisfies the mean value property.*

Proof. We only need prove that the mean value property implies u is harmonic. Suppose then that u has the mean value property. Let $B(x, r) \subset U$. Define a harmonic function v in $B(x, r)$ by $v = u$ on $S(x, r)$, and extend it to the interior by Poisson's integral formula (1.31). Then $v - u$ satisfies the mean value property on any $B(y, \rho) \subset B(x, r)$. Thus, it satisfies the maximum principle, and $v \equiv u$ in $B(x, r)$. Thus, u is harmonic. $\square$

1.8 Harmonic functions are analytic

Our goal is to study the regularity of harmonic functions. We will show that harmonic functions are analytic (denoted C^ω). To show that u is C^∞ is nice, to show that it is C^ω is better. Heuristically, this is like saying its nice to be rich, but its nicer to be a billionaire. The gap between C^∞ and C^ω functions will be of interest for the heat equation too.

We will adopt L. Schwartz's slick notation for multi-variable calculus. A multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{Z}_+$. The height of a multi-index is $|\alpha| = |\alpha_1| + \ldots + |\alpha_n|$. Monomials are denoted $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, and we have the multinomial expansion

$$(x_1 + \ldots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha,$$

where $\alpha! = \alpha_1! \dots \alpha_n!$. The operator ∂^α is defined to be $\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$. Taylor's expansion about a point $y \in \mathbb{R}^n$ now takes the simple (and cryptic) form

$$f(y) \sim \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial^\alpha f(x)(x-y)^\alpha := \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha f(x)(x-y)^\alpha$$

The expansion above is only formal, and even if it converges, it may not converge to $f(x)$. By convergence of an infinite series $\sum_\alpha a_\alpha$ is meant absolute convergence $\sum_\alpha |a_\alpha| < \infty$. Functions that are always locally represented by convergent power series expansions are *analytic.*

Definition 1.18. A function $f : U \to \mathbb{R}$ is *real analytic* in U if at every $x \in U$, there is $B(x,r) \subset U, r > 0$ such that f is represented by the convergent power series expansion

$$f(y) = \sum_{\alpha} c_\alpha (y-x)^\alpha, \quad y \in B(x,r). \tag{1.34}$$

Necessarily, $c_\alpha = \alpha!^{-1} \partial^\alpha f(x)$. More precisely, (see [7, Ch 3.3])

Theorem 1.19. *If $f \in C^\omega(U)$ then $f \in C^\infty(U)$ and for any $x \in U$ there exists $B(x,r) \subset U$ such that $c_\alpha = (\alpha!)^{-1} \partial^\alpha f(x)$. Moreover there exists $M > 0$, such that for every $y \in B(x,r)$ and $\alpha \in \mathbb{Z}_+^n$ we have*

$$|\partial^\alpha f(y)| \le \frac{M|\alpha|!}{r^{|\alpha|}}. \tag{1.35}$$

The growth estimate on the derivatives is necessary and sufficient. If f is C^∞ and satisfies (1.35), it is equal to its Taylor series. Analytic functions satisfy an *identity theorem* that general C^∞ functions don't. In words, analytic functions are determined completely by all their derivatives at a single point. The proof below should remind you of the proof of the strong maximum principle.

Theorem 1.20. *Let U be an open, connected set in $\mathbb{R}^n$. Suppose $f \in C^\omega(U)$. Let $x \in U$. Then f is determined uniquely by $\partial^\alpha f(x), \alpha \in \mathbb{Z}_+^n$.*

Proof. Suppose $f, g \in C^\omega(U)$ such that $\partial^\alpha f(x) = \partial^\alpha g(x), \alpha \in \mathbb{Z}_+^n$. Let $h = f - g$. Decompose U into

$$U_1 = \{y \in U | \partial^\alpha h(y) = 0, \quad \text{all } \alpha \in \mathbb{Z}_+^n\},$$

$$U_2 = \{y \in U | \partial^\alpha h(y) \neq 0, \quad \text{some } \alpha \in \mathbb{Z}_+^n\}.$$

U_2 is open by the continuity of f. U_1 is also open, since if $y \in U_2$, then $h = 0$ in a neighborhood of y by the power series representation. Since U is connected, and U_2 is nonempty by assumption, $h \equiv 0$. ∎

We often need C^∞ functions that are not analytic; for example, to construct C^∞ "bump" functions. The identity theorem rules out analytic functions for this purpose. An example that is usually chosen is the following. Let $p > 0$ and consider the function

$$g(x) = \begin{cases} e^{-x^{-p}}, & x > 0 \\ 0, & x \le 0. \end{cases} \tag{1.36}$$

These functions are C^∞ but not analytic, because all derivatives at zero exist and vanish. You may have seen a brute force proof of this in a class on advanced calculus. However, the following quantitative estimate [7, p.73] is more interesting, and highlights the role of (1.35). We will need it to study Tychonoff's counterexample for the heat equation.

Example 1.21. (a)Show that we can find $\theta = \theta(p) > 0$ with $0 < \theta < 1$ such that

$$|g^{(k)}y| \le \frac{k!}{(\theta y)^k} \exp\left(-\frac{1}{2y^p}\right), \qquad y > 0. \tag{1.37}$$

(b) Show that there exist M, r depending only on p such that

$$|g^{(k)}y| \le M(k!)^{1+1/p} r^{-k} \quad k \in \mathbb{N}, \quad y \in \mathbb{R}. \tag{1.38}$$

Estimate (1.38) reveals that while we have good control over the derivatives, it isn't good enough to yield analyticity (as in (1.35)). One way to "fill the gap" between C^∞ and C^ω is to grade C^∞ functions through estimates such as (1.38). There is a fair bit of classical analysis that revolves around the question of characterizing functions for which the identity theorem holds [11]. Such C^∞ functions are called *quasianalytic.*

The definition of analyticity stated has the obvious flaw that it would seem extremely hard to verify if a given function is analytic. One has to show it is C^∞, and then one has to show that the growth estimates (1.35) hold. The power of the theory of *complex* analytic functions is that we only need to check if a function is (complex) differentiable: it is then automatically analytic. For a precise statement, see [7, p.70]. With all of this background, here is the result for harmonic functions.

Theorem 1.22. *Let u be harmonic in U. Then $u \in C^\omega(U)$.*

Proof. Without loss of generality, we may suppose that $0 \in U$. Let $r = \operatorname{dist}(0, \partial U)$ in (1.31). We show that u is analytic in $B(0,r)$. The trick is to write the Poisson kernel in the form

$$H(x,\zeta) = \frac{r^2 - \sum_{j=1}^n \zeta_j^2}{\omega_n r \left(\sum_{j=1}^n (x_j - \zeta_j)^2\right)^{n/2}}, \qquad \zeta = \xi + i\sigma \in \mathbb{C}^n, \quad \xi \in B(0,r).$$

This expression reduces to (1.32) when $\sigma = 0$. Let us show that $H(x,\zeta)$ is analytic in ζ. The numerator is a polynomial, thus analytic. The denominator is differentiable everywhere it is not zero. Since

$$\sum_{j=1}^{n} (x_j - \zeta_j)^2 = (|x-\xi|^2 - |\sigma|^2) + 2i(x-\xi)\cdot\sigma,$$

if $|\sigma| < |x-\xi|/2$ we see that the denominator is bounded away from zero. Since $x \in S(0,r)$, this is uniformly true for $|\sigma| < (r - |\xi|)/2$. Thus, $H(x,\zeta)$ is complex differentiable in the region $\{\xi \in B(0,r), |\sigma| < r - |\xi|/2\}$. □

We are now guaranteed estimates on derivatives as in (1.35). However, it is of interest to obtain such estimates from scratch. These can be derived either through the mean value theorem or Poisson's integral formula.

Theorem 1.23. *Let u be harmonic in U and let $K \subset U$ be compact. Then for any multi-index α we have*

$$\sup_K |\partial^\alpha u| \le \left(\frac{n|\alpha|}{d}\right)^{|\alpha|} \sup_U |u|, \qquad d = \operatorname{dist}(K, \partial U). \tag{1.39}$$

Proof. Suppose $B(x,r) \subset U$. We first prove estimates on all derivatives of u depending only on u on $S(x,r)$. These are of independent interest. Since u is harmonic, so is $\partial_{x_j} u$. We apply the mean value theorem to $\partial_{x_j} u$ to obtain

$$\partial_{x_j} u(x) = \frac{n}{\omega_n r^n}\int \partial_{y_j} u(y)\,dy = \frac{n}{\omega_n r^n}\int_{S(x,r)} u(y)\nu_j(y)\,dS_y,$$

where $\nu(y)$ denotes the outer normal at y. Take absolute values to find

$$|\partial_{x_j} u(x)| \le \frac{n}{r}\max_{S(x,r)} |u| \le \frac{n}{r}\sup_U |u|.$$

We can now iterate this estimate. If α is a multi-index with height 2 (say, $\partial^2_{x_j x_k}$), apply the estimate to concentric balls of radius $r/2$ and r to obtain,

$$|\partial^2_{x_j x_k} u(x)| \le \frac{n}{r/2}\max_{S(x,r/2)} |\partial_{x_k} u| \le \left(\frac{n}{r/2}\right)^2 \max_{S(x,r)} |u|.$$

(why is the second inequality true?). One may continue this procedure for any multi-index α. We choose $|\alpha|$ equally spaced concentric balls, and find

$$|\partial^\alpha u(x)| \le \left(\frac{n|\alpha|}{r}\right)^{|\alpha|} \max_{S(x,r)} |u|. \tag{1.40}$$

A uniform bound over K is obtained by choosing r as large as possible, and replacing $\max_{S(x,r)} |u|$ by $\sup_U |u|$. □

Remark 1.24. The best constant in this estimate and the radius of convergence of the power series about the center of the ball is considered in the homework.

1.9 Compactness and convergence

The characterization of harmonic functions by their mean value property leads to simple compactness criterion. In all that follows we will consider a sequence of harmonic functions $u_k : U \to \mathbb{R}$, $k \in \mathbb{Z}_+$.

Theorem 1.25. *Suppose u_k converges uniformly on compact subsets of U to a function $u : U \to \mathbb{R}$. Then u is harmonic.*

Proof. Fix $B(x,r) \subset\subset U$. Since $u_k \to u$ uniformly on $\overline{B(x,r)}$, the limit u is continuous. Moreover, since $u_k(x) = \fint_{B(x,r)} u_k(y)dy$ we may interchange limits (why?) to find $u(x) = \fint_{B(x,r)} u(y)dy$. This holds for every $B(x,r) \subset\subset U$. Thus, u is harmonic. $\square$

As an application of Harnack's inequality, one obtains the following strong convergence theorem.

Theorem 1.26 (Harnack's convergence theorem). *Suppose u_k is a monotone increasing sequence, and for some $\xi \in U$ the sequence $\{u_k(\xi)\}$ is bounded. Then u_k converges uniformly on compact subsets of U to a harmonic function.*

Proof. $\lim_{k\to\infty} u_k(\xi)$ exists. Thus, for any $\varepsilon > 0$ there exists N such that $0 \le u_l(\xi) - u_k(\xi) < \varepsilon$ for $l \ge k \ge N$. Let $U' \subset\subset U$ with $\xi \in U'$. By Harnack's inequality

$$\sup_{U'}(u_l - u_k) \le C\left(\inf_{U'}(u_l - u_k)\right) \le C\varepsilon.$$

Thus, $u_l - u_k$ is a uniformly Cauchy sequence in $\overline{U'}$. Now apply the previous theorem. $\square$

It is quite remarkable that convergence of a sequence of functions at one point implies convergence everywhere. Finally, we have a simple and powerful compactness theorem. For analytic functions, this is usually called Montel's theorem. Bounded families of harmonic functions are called *normal families*.

Theorem 1.27. *If u_k is a bounded sequence, there is a subsequence converging uniformly on compact subsets to a harmonic function.*

Proof. The fundamental compactness criterion for continuous functions is the Arzela-Ascoli theorem. Since we have assumed a uniform bound, say $\sup_k \sup_U |u_k| \leq M$, we only need check equicontinuity. But Theorem 1.23 yields a uniform (in k) estimate on all derivatives on any $U' \subset\subset U$. □

1.10 Perron's method for the Dirichlet problem

We have solved the Dirichlet problem for the ball through the Poisson integral formula. Heuristically, one may think of solving the Dirichlet problem on a domain U by piecing together the loacl solution on balls. This is a vague idea made precise in Perron's construction.

The setup is as follows. U is open, bounded and connected; $f : \partial U \to \mathbb{R}$ is continuous. We wish to solve the problem

$$\triangle u = 0, \quad x \in U, \tag{1.41}$$

$$u = f, \quad x \in \partial U \tag{1.42}$$

We know that if a solution exists, it is unique. Thus, we need only come up with a solution. Many elliptic equations in geometry and physics arise through problems of determining extrema (eg. *minimal* surfaces, *least* energy, ...). Perron's method is based on characterizing the solution through a pointwise extremal property suggested by the mean value inequality. We begin with the weaker notion of subharmonicity considered in the HW.

Definition 1.28. A function $u \in C(U)$ is subharmonic if for every $x \in U$ there exists $\delta(x) > 0$ such that $\overline{B(x,\delta)} \subset U$ and

$$u(x) \leq \fint_{S(x,r)} u(y)dS_y, \quad 0 < r \leq \delta.$$

It is not assumed that v is C^2, and it is not assumed that $v \in C(\overline{U})$. The boundary condition is included in our class of functions through an inequality.

Definition 1.29. Let S_f denote the class of subharmonic functions on U such that $v \in C(\overline{U})$ and $v \leq f$ on ∂U.

We use an inequality (instead of equality) to ensure that S_f is non-empty. Let $m = \min_{\partial U} f$ and $M = \max_{\partial U} f$. Then the constant function $v \equiv m$ is in S_f. The basic result of Perron's method is the following.

Theorem 1.30. *The function $u(x) = \sup_{S_f} v(x)$ is harmonic in U.*

The key insight is to separate the existence of a candidate harmonic function (that is (1.41)) from study of the boundary condition (1.42). It is remarkable that with essentially no geometric hypothesis on the boundary ∂U we still have a suitable candidate u. To prove the theorem, we begin with some basic properties of subharmonic functions.
1. Suppose $\{v_1, \ldots, v_m\} \in S_f$. Then $\max_j v_j := v \in S_f$. This is immediate from the mean value inequality. The maximum $v(x) = v_k(x)$ for some $k = k(x)$. Let $\delta(x) = \min_j \delta_j(x)$. Then for any $0 < r < \delta(x)$ we have

$$v(x) = v_k(x) \leq \fint_{S(x,r)} v_k(y)\, dS_y \leq \fint_{S(x,r)} v(y)\, dS_y.$$

We piece together harmonic functions on balls through *harmonic lifting.*.

Definition 1.31. Suppose v is subharmonic and $B(x,r) \subset U$. The harmonic lifting of v on the ball $B(\xi, r)$ is defined by

$$V(x) = \begin{cases} w(x), & x \in B(\xi, r), \\ v(x), & x \in U \backslash B(\xi, r), \end{cases}$$

where $w : B(\xi, r) \to \mathbb{R}$ is the harmonic function such that $v = w$ on $S(x,r)$. The following property of V was problem 1 on HW1.

2. If $v \in S_f$ then $V \in S_f$ and $v \leq V$.

Proof of Theorem 1.30. We first note that u is well-defined since $\sup_U v \leq M$ by the weak maximum principle. Fix $\overline{B(\xi, r)} \subset U$. By definition, there exists a sequence of functions $v \in S_f$ such that $v_k(\xi) \to u(\xi)$. We replace the sequence $\{v_k\}$ by the increasing sequence

$$\bar{v}_k = \max_{1 \leq j \leq k} (v_1, \ldots, v_k, m).$$

Then $\bar{v_k} \in S_f$ and $v_k(\xi) \leq \bar{v}_k(\xi) \leq u(\xi)$ with convergence as $k \to \infty$. Now replace $\bar{v}_k$ with its harmonic lifting over $B(\xi, r)$, denoted V_k. Since V_k are bounded below by m and converge at the point ξ, we may apply Harnack's convergence theorem (Theorem 1.26) to deduce that V_k converges uniformly on compacts subsets $\overline{B(x, \rho)} \subset B(\xi, r)$ to a harmonic function V. It is clear that $V \leq u$. We claim that $V = u$ in $B(\xi, r)$. Suppose not. Then there exists $\zeta \in B(\xi, r)$ such that $V(\zeta) < u(\zeta)$. Thus, there must be $w \in S_f$ such that $V(\zeta) < w(\zeta) \leq u(\zeta)$. Now consider the sequence $w_k = \max(V_k, w)$ and their harmonic liftings W_k. As before we find a harmonic limit W with

$$V(x) \leq W(x) \leq u(x), \quad x \in B(\xi, r), \qquad \text{and} \quad V(\xi) = W(\xi) = u(\xi).$$

By the strong maximum prinicple, $V = W$. Thus, $V = u$ and u is harmonic. $\square$

The attainment of boundary values is tied to the regularity of the boundary. In the Perron method, the regularity of the boundary is used to construct suitable *barrier functions*. Since we all have an intuitive notion of the boundary of a set, let us recall that the precise definition here is $\partial U = \overline{U} \backslash U$. Since our only requirement on U is that it is open, bounded and connected, you can guess that ∂U may be very rough in general. To check your understanding, show that ∂U is compact.

Definition 1.32. A $C(\overline{U})$ function w is a barrier at $y \in \partial U$ relative to U if (a) w is subharmonic, and (b) $w(y) = 0$ and $w(x) < 0, x \in \partial U, x \neq y$.

Here are some examples of barriers. First, suppose that U is smooth and convex. At every point $y \in \partial U$ there is a tangent plane such that U is on one side of the tangent plane. Without loss of generality, we may suppose after a translation and rotation that $y = 0$ and the tangent plane is $x_n = 0$ and U is contained in the lower-half plane $\{x_n < 0\}$. The function x_n now serves as a barrier. In the HW you will show that the existence of a barrier is a local property. Thus, if there is a tangent plane at y such that $B(y, r) \cap U$ is on one side of the tangent plane, we may use the barrier above. This yields a useful heuristic: the boundary point is bad only when we have 'inward corners'.

Another barrier for this problem may be constructed as follows. Fix y and let $z \in \mathbb{R}^n \backslash \overline{U}$ lie on the outward normal through y and let $r = |y - z|$. Then $\overline{B(z, r)} \cap \overline{U} = \{y\}$, and

$$w = K(y, z) - K(x, z)$$

is a barrier at y. As before, we may weaken the conditions on the domain. We say that U satisfies an *exterior sphere condition* if at any point $y \in \partial U$ there exists a ball $B(z, r) \subset \mathbb{R}^n \backslash \overline{U}$ such that $\overline{B(z, r)} \cap \overline{U} = \{y\}$. Then w may be used as a barrier.

A more careful analysis can be used to weaken this further. We say that u satisfies an *exterior cone condition* if at any point $y \in \partial U$ there exists a finite right circular cone C with vertex y such that $\overline{C} \cap \overline{U} = \{y\}$. One may then choose a negative harmonic function w in the exterior of the cone as a barrier. The existence of such a function is part of the homework.

Definition 1.33. A boundary point $y \in \partial U$ is *regular* if there exists a barrier at y. Boundary points that are not regular are called *exceptional*. A domain U is regular if all its boundary points are regular.

Theorem 1.34. *Let* $u = \sup_{S_f} v$*. Then* $\lim_{x\to y,\, x\in U} u(x) = f(y)$ *at every regular boundary point* y*.*

Proof. Fix $\varepsilon > 0$ and $N = \max(m, M) = \max_{\partial U} |f|$. Let w be a barrier function at y. Since f and w are continuous on ∂U, we may choose $\delta > 0$ and $A > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$ and $Aw(x) \leq -2M$ if $|x - y| \geq \delta$. We use the barrier to construct a suitable subfunction $v \in S_f$. Define $v(x) = f(y) + Aw(x) - \varepsilon$, $x \in \overline{U}$. Definition 1.32 implies that v is subharmonic and in $C(\overline{U})$. In order to confirm that $v \in S_f$ we only need check the boundary values. If $x \in \partial U$ and $|x - y| < \delta$ we have

$$v(x) = f(y) + Aw(x) - \varepsilon < f(y) - \varepsilon < f(x),$$

and if $|x - y| \geq \delta$, we have

$$v(x) = f(y) + Aw(x) - \varepsilon \leq -N - \varepsilon < f(x).$$

Thus, $v \in S_f$, consequently $v(x) \leq u(x), x \in U$, and

$$f(y) - \varepsilon = \lim_{x\to y,\, x\in U} v(x) \leq \liminf_{x\to y,\, x\in U} u(x). \tag{1.43}$$

In order to prove the opposite inequality, we use a trick. Suppose the boundary data is $-f$, and consider the Perron function $\bar{u} = \sup_{S_{-f}} v$. Observe that $u(x) \leq f(x)$ (why?) and $-\bar{u}(x) \leq -f(x)$, $x \in \partial U$. Since u and $\bar{u}$ are harmonic, this implies $\bar{u} \leq -u$, $x \in \overline{U}$ by the maximum principle. We apply (1.43) to $\bar{u}$ to obtain

$$-f(y) - \varepsilon \leq \liminf_{x\to y,\, x\in U} \bar{u}(x) \leq \liminf_{x\to y,\, x\in U} -u(x) = -\limsup_{x\to y,\, x\in U} u(x).$$

Therefore,

$$f(y) - \varepsilon \leq \liminf_{x\to y,\, x\in U} u(x) \leq \limsup_{x\to y,\, x\in U} u(x) \leq f(y) + \varepsilon.$$

□

We are now able to characterize domains such that the Dirichlet problem is solvable.

Theorem 1.35. *The Dirichlet problem on a bounded domain* U *is solvable for arbitrary continuous boundary values if and only if every boundary point is regular.*

Proof. We have just demonstrated existence of a solution if every boundary point is regular. Conversely, if the problem is solvable for every continuous $f : \partial U \to \mathbb{R}$, it is certainly solvable when $f(x) = -|x-y|, x, y \in \partial U$. Such a solution constitutes a barrier, thus y is regular. $\square$

This is a cheap result. What is desired is a more meaningful characterization of regularity of boundary points. The exterior sphere and cone conditions are sufficient conditions, but also rely on exact solutions of positive harmonic functions – a limited approach. A complete answer to the problem is provided by Wiener's beautiful *characterization* of regular boundary points. But in order to appreciate it, one must first consider an entirely different approach to the existence problem and some counterexamples.

1.11 Energy methods and Dirichlet's principle

The characterization of a harmonic function as a pointwise maximum is an essentially one-dimensional feature: our unkown u is a scalar. Many (most?) interesting problems in PDE require the study of systems of equation (the unkown $u : U \to \mathbb{R}^m, m > 1$), and do not have maximum principles. Here is an analogy that may be useful. Laplace's equation is a useful approximation to the physical problem of determining the equilibrium displacement of an elastic membrane. (A membrane is a surface that resists stretching, but does not resist bending). Similarly, Poisson's equation describes the equilibrium of a membrane subjected to a load. Thus, they are the simplest prototypes of problems of static equilibrium in the theory of elasticity. On physical grounds, one expects such equlibria to be characterized by least energy. We formulate a simple version of this, which represents a powerful general idea. We shall studiously avoid the maximum principle.

The treatment here is straight from Evans [3, p.42] with some changes of notation for consistency with these notes. Let U be an open bounded set with C^1 boundary. Consider the boundary value problem

$$\begin{aligned} \triangle u &= g, \quad x \in U && (1.44) \\ u &= f, \quad x \in \partial U. && (1.45) \end{aligned}$$

Theorem 1.36. *There is at most one solution $u \in C^2(\overline{U})$.*

Proof. As always, let u and v be two solutions in $C^2(\overline{U})$ and consider their difference $w = u - v$. Then $\triangle w = 0$ in U, and $w = 0$ on ∂U. Integrate by parts to find

$$0 = \int_U w \triangle w \, dx = \int_U |Dw|^2 \, dx.$$

Since Dw is continuous, it must vanish identically. Therefore, w is a constant, and must be zero because of the boundary condition. □

The method of this uniqueness proof is of great value. Physically, the integral on the right is the elastic energy of the membrane. The quantity on the left is the work done by the applied force g in displacing the membrane. The equality between these two reflects conservation of energy. *Dirichlet's principle* characterizes the solution to (1.44) and (1.45) through least energy. We introduce the *energy functional*

$$I[w] = \int_U \left(\frac{1}{2}|Dw|^2 + wg \right) \, dx, \tag{1.46}$$

for w in the class of admissible functions

$$\mathcal{A} = \{ w \in C^2(\overline{U}) \mid w = g \;\text{ on }\; \partial U \}. \tag{1.47}$$

Theorem 1.37 (Dirichlet's principle). *Assume $u \in C^2(\overline{U})$ solves (1.44) and (1.45). Then*

$$I[u] = \min_{w \in \mathcal{A}} I[w]. \tag{1.48}$$

Conversely, if $u \in \mathcal{A}$ satisfies (1.48), then u solves (1.44) and (1.45).

Proof. Let $u \in C^2(\overline{U})$ solve (1.44) and (1.45) and $w \in \mathcal{A}$. Then (1.44) implies

$$0 = \int_U (-\triangle u + g)(u - w) \, dx = \int_U \left(Du \cdot D(u - w) + g(u - w) \right) \, dx,$$

Rearrange terms in this identity as follows

$$\int_U \left(|Du|^2 + gu \right) \, dx = \int_U \left(Du \cdot Dw + gw \right) \, dx.$$

We almost have $I[u]$ upto a factor of $1/2$. The trick is to observe that

$$\begin{aligned} \int_U Du \cdot Dw dx \le \int_U |Du||Dw| dx \quad &\le \quad \left(\int_U |Du|^2 dx \right)^{1/2} \left(\int_U |Dw|^2 dx \right)^{1/2} \\ &\le \quad \frac{1}{2} \left(\int_U |Du|^2 \, dx + \int_U |Dw|^2 \, dx \right). \end{aligned}$$

Rearrange terms again to find $I[u] \le I[w]$.

The converse is more interesting, since we must deduce from the variational principle that the minimizer solves the PDE (1.44). We have to choose a rich enough class of test functions. Let v be any C_c^∞ function (that is C^∞ and with compact support in U). Let $w = u + tv$, $t \in \mathbb{R}$. Observe that $w \in \mathcal{A}$. Then $i(t) := I[u + tv]$ is a differentiable function of t because it is simply the quadratic expression

$$\begin{aligned} i(t) &= \int_U \left(\frac{1}{2}|Du + tDv|^2 + g(u + tv) \right) dx \\ &= I[u] + t \left(\int_U (Du \cdot Dv + vg)\, dx \right) + \frac{t^2}{2} \int_U |Dv|^2\, dx. \end{aligned}$$

$I[u]$ cannot be the minimum unless the linear term vanishes, that is

$$0 = i'(0) = \int_U (Du \cdot Dv + vg)\ dx = \int_U (-\triangle u + g)\, v\, dx.$$

Since this holds for each $v \in C_c^\infty(U)$, we must have (1.44). □

The basic method of this proof is important, and reappears in many guises throughout the calculus of variations. However, the smoothness assumptions we adopt are far from optimal, as is the choice of function class. A more sophisticated analysis requires the introduction of Sobolev spaces, ie. we will do this again. To indicate some of the subtleties, let me point out a deft dodge of a delicate issue. As we have learnt through Perron's theorem, the attainment of boundary values is tricky. Now look again at the statements of the theorems and observe that there are *no assumptions* on f or g. All assumptions have been subsumed into the existence of solutions. In fact, it is not even clear that the class of admissible solutions is nonempty!

1.12 Potentials of measures

The fundamental solution is physically interpreted as the potential induced by a point charge. We now consider potentials induced by general charge distributions. Mathematically, a charge distribution is a signed measure μ. It gives rise to a potential by superposition. The following normalization will be convenient: it seems easier (for me) to work with positive measures.

Definition 1.38. The potential of a signed measure μ is defined by

$$u_\mu(x) = \int_{\mathbb{R}^n} |x - y|^{n-2} \mu(dy) = \frac{2-n}{\omega_n} \int_{\mathbb{R}^n} K(x, y) \mu(dy), \tag{1.49}$$

for all x such that the integral is well-defined (this is always so, if μ is a positive measure, though $u_\mu(x)$ could equal $+\infty$).

Observe that if μ has support in a compact set F then $u_\mu(x)$ is harmonic in $\mathbb{R}^n\backslash F$. Explicit (and surprising) examples of such potentials may be found in [9]. Here are some classical examples in $\mathbb{R}^3$.

Example 1.39. Consider a sphere $S(0,R)$ with uniform charge density λ. The potential can be found directly by integration. A highlight of seventeenth century mathematics was the surprising result that outside the sphere this potential is *exactly* that of a point mass (charge) of magnitude $4\pi R^2\lambda$ at the center of the sphere (this is on HW 3). More generally, in $\mathbb{R}^n$ we have

$$u_\mu(\xi) = \lambda\omega_n R^{n-1}|\xi|^{2-n}. \tag{1.50}$$

Example 1.40. Consider a uniform distribution of charge with (linear) density λ on the interval $(-a,a)$. That is,

$$\mu(dx_1\, dx_2\, dx_3) = \lambda \mathbf{1}_{-a<x_1<a} dx_1 \delta_0(dx_2)\delta_0(dx_3).$$

The potential at any $\xi \in \mathbb{R}^3$ with $b^2 = \xi_2^2 + \xi_3^2 > 0$ is given by,

$$u_\mu(\xi) = \lambda \int_{-a}^{a} \frac{dx_1}{\sqrt{(x_1-\xi_1)^2+b^2}},$$

which may be integrated (use the substitution $x_1 - \xi_1 = b\tan\theta$) to yield

$$u_\mu(\xi) = \lambda \log\left(\frac{\sqrt{(a-\xi_1)^2+b^2}+a-\xi_1}{\sqrt{(a+\xi_1)^2+b^2}-(a+\xi_1)} \right). \tag{1.51}$$

Geometrically, the surfaces of equal potential are ellipsoids of revolution with foci at the endpoints of the wire. Observe that the potential is divergent on the wire itself (its natural to say $u_\mu = +\infty$).

Example 1.41. If we consider a circular disk in the (x_1,x_2) plane with radius a and uniform (area) density λ we may use radial symmetry to reduce to the case $\xi_2 = 0$ and we find as above

$$u_\mu(\xi) = \lambda \int_0^a \int_0^{2\pi} \frac{1}{\sqrt{r^2 - 2r\xi_1\cos\theta + |\xi|^2}} d\theta\, r dr.$$

The inner integral cannot be done without elliptic integrals. If we restrict attention to points along the axis we have $\xi_1 = \xi_2 = 0$ and

$$u_\mu(\xi) = 2\pi\lambda\left(\sqrt{a^2+|\xi|^2} - |\xi|\right).$$

Observe that as $\xi \to \infty$ we have $u_\mu(\xi) = (\pi a^2\lambda)|\xi|^{-1} = q|\xi|^{-1}$, where $q = \pi a^2\lambda$ is the total charge. This simply expresses the fact that from a distance the disk looks like a point charge. (Work this out for Example 1.58 too).

It is natural to construct potentials from measures. It is also important to be able to obtain measures that represent potentials. To this end, we will need the following important uniqueness theorem.

Theorem 1.42. *If $u_\mu = 0$ a.e then $\mu \equiv 0$.*

Proof. The theorem is far from obvious. Since we allow signed measures it is not clear that we could not have positive and negative charges whose influences cancelled out. By the Riesz representation theorem, we must show

$$\int_{\mathbb{R}^n} f(x)\mu(dx) = 0, \tag{1.52}$$

for every $f \in C_0(\mathbb{R}^n)$ (continuous functions such that $\lim_{x\to\infty} f = 0$). It suffices to show (1.52) for functions dense in $C_0(\mathbb{R}^n)$. What we know is

$$\int_{\mathbb{R}^n} |x-\xi|^{2-n}\mu(dx) = 0, \quad \text{or} \quad \int_{\mathbb{R}^n} K(x,\xi)\, dx = 0 \tag{1.53}$$

for almost every $\xi \in \mathbb{R}^n$. These functions are *not* dense in $C_0(\mathbb{R}^n)$ but the theorem is still true!

There is a clever calculation that saves the day. It will suffice to show (1.52) for $f \in C_c^\infty$ (C^∞ functions with compact support). If f is any such function, we consider $\breve{f} = f(-x)$. We consider the convolution,

$$\breve{f} \star \mu(y) = \int_{\mathbb{R}^n} \breve{f}(y-x)\mu(dx),$$

and notice that (1.53) is identical to $\breve{f} \star \mu(0) = 0$. Now every $\breve{f}$ of this form, admits a representation (see (1.22)) $\breve{f} = \triangle \breve{f} \star K$. Thus,

$$\mu \star \breve{f} = \mu \star (K \star \triangle \breve{f}) = (\mu \star K) \star \triangle \breve{f} = 0,$$

by (1.53). The interchange of limits has to be justified, and I am sweeping some things under the rug, but this sketch is enough at this stage. $\square$

1.13 Lebesgue's thorn

We now return to the question of regularity of boundary points. The two-dimensional situation is very special. Without loss of generality suppose $0 = y \in \partial U$ and suppose $z = re^{i\theta}$ defines complex numbers in polar coordinates. Suppose there is a neighborhood $B(0,r)$ such that a single valued branch of θ is defined in $U \cap B(0,r)$. Then the function

$$w = \mathrm{Re}(\frac{1}{\log z}) = \frac{\log r}{\log^2 r + \theta^2}$$

is a barrier at 0. All we require of U in order to obtain a single branch of θ is a simple (ie. non-self-intersecting curve) with endpoint 0 that lies completely in $\mathbb{R}^2\backslash U$. For example, if U is the unit ball slit along a simple curve (however rough), it is a regular domain. Moreover, the Riemann mapping theorem states that every simply-connected domain can be mapped analytically to the unit ball. Thus, purely topological, and no geometric, information on U can determine that a domain is regular.

For $n \geq 3$ this is not true. The following example due to Lebesgue provides some intuition for exceptional points and shows the havoc created by a little asymmetry in Example 1.40. The exposition here is based on Courant and Hilbert [1, p.303], except that we construct a family of thorns indexed by $\beta > 0$. For the following calculations we revert to notation (x, y, z) for points in $\mathbb{R}^3$ (with all the flaws this implies).

Fix $\beta > 0$. Let μ be the measure concentrated on the x-axis on $(0, 1)$ with non-uniform density $x^\beta dx, x \in (0, 1)$. The potential of μ is

$$u(x, y, z) = \int_0^1 \frac{\xi^\beta}{\sqrt{(x-\xi)^2 + b^2}}\, dx, \quad b^2 = y^2 + z^2.$$

The integral may be computed explicitly when $\beta = 1$ (this is Lebesgue's counterexample), but it is not any harder to analyze for any $\beta > 0$. We study the asymptotics of u as $x, b \downarrow 0$ with $b/x \to 0$. To leading order

$$u(x, y, z) \sim \frac{1}{\beta} - 2x^\beta \log b, \tag{1.54}$$

and therefore if we consider the limit as $x, b \to 0$ along the surface

$$b = e^{-c/2x^\beta}, \quad c > 0, \tag{1.55}$$

we find $\lim u = \beta^{-1} + c$. One may now cheat and construct for any $c > 0$ a solution to Laplace's equation (not Poisson's equation) in the domain exterior to the level set $u^{-1}\{\beta^{-1} + c\}$ with constant boundary values $\beta^{-1} + c$. Simply use the function above. The point 0 is an exceptional point for this exterior problem. To find an exceptional point for an interior problem, we use inversion in a circle, and convert the domain above, to one with an exponentially thin spike or thorn.

That is the outline. Let us nail down(1.54). We separate the integral into two pieces, one on $(0, 2x)$ and the other on $[2x, 1)$.

$$\int_{2x}^1 \frac{\xi^\beta}{\sqrt{(x-\xi)^2 + b^2}}\, d\xi = \int_x^{1-x} \frac{(t+x)^\beta}{\sqrt{t^2 + b^2}}\, dt,$$

after the translation $t = \xi - x$. Observe that

$$|\xi| \leq |x - \xi| + |\xi| \leq 2|x - \xi|, \quad \xi \geq 2x.$$

Therefore, the integrand above is bounded by $2|t|^{\beta-1}$ which is integrable for $\beta > 0$. One may now use the dominated convergence theorem to deduce

$$\lim_{x\to 0, b\to 0} \int_{2x}^{1} \frac{\xi^\beta}{\sqrt{(x-\xi)^2 + b^2}}\, d\xi = \int_0^1 t^{\beta-1}\, dt = \frac{1}{\beta}. \tag{1.56}$$

On the range $\xi \in (0, 2x)$ we translate and rescale setting $\eta = b/x$ to obtain

$$\begin{aligned}\int_0^{2x} \frac{\xi^\beta}{\sqrt{(\xi - x)^2 + b^2}}\, dt &= x^\beta \int_{-1}^{1} \frac{(t+1)^\beta}{\sqrt{t^2+\eta^2}}\, dt \qquad (1.57)\\ &= x^\beta \left(\int_{-1}^{1} \frac{(t+1)^\beta - 1}{\sqrt{t^2+\eta^2}}\, dt + \int_{-1}^{1} \frac{1}{\sqrt{t^2+\eta^2}}\, dt. \right)\end{aligned}$$

The first integral is now regular (check!) and as $\eta \to 0$ we have (use dominated convergence for example)

$$\int_{-1}^{1} \frac{(t+1)^\beta - 1}{\sqrt{t^2+\eta^2}}\, dt \to \int_{-1}^{1} \frac{(t+1)^\beta - 1}{t} dt := C_\beta.$$

The second integral may be computed exactly by the change of variable $t = \eta \tan\theta$. We then have

$$\int_{-1}^{1} \frac{1}{\sqrt{t^2+\eta^2}} = \log\left(\frac{\sqrt{1+\eta^2}+1}{\sqrt{1+\eta^2}-1} \right).$$

Finally, observe that

$$\lim_{\eta\to 0} \log\left(\frac{\sqrt{1+\eta^2}+1}{\sqrt{1+\eta^2}-1} \right) + 2\log\eta = \log 4.$$

To summarize, the integral in equation (1.57) is asymptotic to

$$x^\beta(C_\beta + \log 4 - 2\log\eta) = -2x^\beta \log b + x^\beta(C_\beta + \log 4 + 2\log x).$$

As $x \to 0$, the second term converges to 0 independent of b. To be precise, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $0 < x < \delta, \eta = b/x < \delta$ we have

$$\left| u(x,y,z) - \beta^{-1} + 2x^\beta \log b \right| < \varepsilon.$$

If we choose $b = e^{-c/2x}$ we find that the limit of u is $1 + c$. The surface $u^{-1}\{1+c\}$ is a surface of rotation with an exponentially thin spike near the origin.

The measure μ was only needed to generate this potential. Let U_c denote the domain exterior to $u^{-(1+c)}$. If we solve the exterior problem for Laplace's equation in U with boundary values $u = 1 + c$ with the decay condition $u \to 0$ at ∞ we recover our function u. The origin must be an exceptional point of U_c since we can obtain as limit any value between 1 and $1 + c$ by approaching along a suitable curve. To connect this exterior problem to an interior problem we use Kelvin's transformation. Inverting in the sphere of radius 1/2 with center $(1/2, 0, 0)$ we obtain a domain with an internal thorn. The tip of the thorn is an exceptional point. For detailed calculations of the inversion, see [1].

1.14 The potential of a compact set

We will formalize a useful physical cartoon. A conductor is a body U with charges that are free to move without resistance. In electrostatics, the potential in a conductor must be constant; if not, currents would flow to equilibrate the potential difference. Most materials have some resistance and if we sit around and wait long enough, the potential would become constant. Even though the conductor is at constant potential, the charge on it need not be constant. It turns out, that the charge of a conductor must reside on its surface, and concentrates at corners. A rather spectacular example of this principle is lightning: the surface of the earth is at constant potential, most of the charge accumulates on the extremities of rough surfaces (eg. trees), and they get hit by lightning first.

In what follows F will denote a compact set in $\mathbb{R}^n, n \geq 3$. We use F^o to denote its interior (which may be empty), and ∂F to denote the boundary. For the next few sections, $U = \mathbb{R}^n \backslash F$ will be unbounded. Verify that $\partial U = \partial F$. Let $U_k = U \cap B(0,k)$ denote a sequence of increasing, bounded, open sets. We consider the Perron function with boundary values $u_k = 1$ on $\partial U = \partial F$ and $u_k = 0$ on $|x| = r$. u_k is well-defined once k is large enought that $F \subset B(0,k)$. It follows from Harnack's convergence theorem that the sequence u_k has a harmonic limit $u : U \to \mathbb{R}$.

Definition 1.43. The electrostatic potential (or simply potential) of F is the function $p_F : \mathbb{R}^n \to \mathbb{R}$ defined by

$$p_F(x) = \begin{cases} 1, & x \in F, \\ u(x), & x \in U. \end{cases} \tag{1.58}$$

Remark 1.44. It is a homework problem to show that p_F depends only on F and not on the approximating domains U_k.

Example 1.45. The potential of $F = \overline{B(0,R)}$ is given by

$$p_F(x) = \begin{cases} 1, & |x| \le R \\ R^{n-2}|x|^{2-n}, & |x| > R. \end{cases}$$

The potential of the sphere $S(0,R)$ is the same. In both cases, the potential is generated by a uniform measure of density $\lambda_R := (\omega_n R)^{-1}$ on $S(0,R)$ (see Example 1.39 and equation (1.50)). Observe that the total charge on the ball or sphere is R^{n-2}. This example reflects the principle that the charge of a conductor is concentrated on its surface.

Theorem 1.46. *Suppose ∂F is C^2. Then p_F is the potential of a positive measure concentrated on ∂F. This measure is unique, and we call it the charge on F.*

Proof. Since ∂F is smooth, p_F solves the Dirichlet problem with boundary value 1 on ∂F. We use Poisson's representation formula (1.26) to obtain for every $\xi \in U$

$$\begin{aligned} u(\xi) &= \int_{\partial U} \left(\frac{\partial K(x,\xi)}{\partial n_x} - K(x,\xi)\frac{\partial u}{\partial n_x} \right) dS_x \\ &= -\int_{F^o} \triangle_x K(x,\xi)\, dx + \int_{\partial F} K(x,\xi) \frac{\partial u}{\partial n_x} dS_x \\ &= \frac{(2-n)}{\omega_n} \int_{\partial F} |x-\xi|^{2-n} \frac{\partial u}{\partial n_x} dS_x. \end{aligned}$$

We switched signs when we switched from integration over ∂U to ∂F because the outward normal switches sign. The first term has vanished because $K(x,\xi)$ is harmonic in F^o. Finally, $\partial u/\partial n \le 0$ by the maximum principle. This gives the desired non-negative surface measure (see (1.49)). Uniqueness follows from Theorem 1.42. $\square$

We would like to extend this principle to arbitrary compact sets. This requires an approximation argument.

Theorem 1.47. *Let $F \subset \mathbb{R}^n$ be compact. Then there is a sequence of approximating compact sets F_k such that*

1. $F = \cap_{k=0}^{\infty} F_k$.
2. $F \subset F_k^o$.

3. $\mathrm{dist}(F, F_k) \leq k^{-1}$.

4. ∂F_k *is* C^∞.

Proof. It is easy to construct compact sets with nonempty interior that approximate F. For any $\varepsilon > 0$ the ε-thickening of F is the set

$$F^\varepsilon = \{x \in \mathbb{R}^n \,| \mathrm{dist}(x, F) \leq \varepsilon.\}$$

This set is compact, and F is contained within the interior of F^ε. However, the boundary of F^ε need not be smooth, and this is fixed through the important technique of *mollification*.

Let φ be a smooth function with support in $B(0,1)$ such that $\varphi \geq 0$ and $\int \varphi dx = 1$. Observe that for any $\beta > 0$, the scaled copy $\varphi_\beta := \beta^{-n}\varphi(y\beta^{-1})$ has support in the ball $B(0,\beta)$ and $\int_{\mathbb{R}^n} \varphi_\beta(x)\,dx = 1$. Let $\mathbf{1}_A$ denote the indicator function for a set A. For every k we consider the $1/k$-thickening $F^{1/k}$, and construct the smooth function

$$\psi(x) = \int_{\mathbb{R}^n} \varphi_{1/k^2}(y)\mathbf{1}_{F^{1/k}}(x-y)\,dx.$$

Heuristically, we are smoothing the boundary of $F^{1/k}$ over a scale $1/k^2$. More precisely, observe that $\psi(x) = 1$ for all $x \in F^{1/(2k)}$, and $0 < \psi(x) < 1$ for $x \in \partial F^{1/k}$, therefore

$$F^{1/2k} \subset \psi^{-1}\{1\} \subset F^{1/k}.$$

It would seem that $\psi^{-1}\{1\}$ is the desired compact set with smooth boundary, but the smoothness of ψ isn't enough to deduce this. There is a fix for this problem too, and again the method is very important.

First, we refine the inclusion above. We use compactness to say that there is a $\delta > 0$ with $\psi(x) \geq 1-\delta$ for $x \in \partial F^{1/k}$. Thus, for any $c \in [1-\delta, 1]$ we have

$$F^{1/2k} \subset \psi^{-1}[c,1] \subset F^{1/k}.$$

The factor of δ is needed for the following reason. Ideally, one would like to say that since ψ is C^∞, its inverse image $\psi^{-1}\{c\}$ is a C^∞ surface ($n-1$ dimensional manifold to be precise). But this need not be true for any fixed value c (its certainly *not* true for $c = 1$). However, a powerful result called Sard's theorem (see [12, Ch. 3] for a proof), allows us to say that this is true for *almost every* value $c \in \mathbb{R}$! Thus, there is a c such that $F_k = \psi^{-1}[c,1]$ and $\psi^{-1}\{c\}$ is C^∞. $\square$

Observe that $p_{F_k} \to p_F$ and the convergence is uniform on compact subsets of U (use Harnack's convergence theorem). We can now strengthen Theorem 1.46.

Theorem 1.48. *Consider a compact set $F \subset \mathbb{R}^n, n \geq 3$. There is a unique positive measure μ_F supported on ∂F such that p_F is the potential of μ_F.*

Proof. Consider a sequence of approximations as in (1.47). Let μ_k denote the charge (measure) associated to F_k. We may choose $R > 0$ so that all the μ_k are supported within a fixed ball $B(0,R)$. These measures satisfy a uniform bound. To see this, we follow Gauss. Consider the potential p_R of the ball $B(0,R)$. Recall from Example 1.45 that this potential is generated by the measure $\lambda_R dS$ on $S(0,R)$. Since $p_R = 1$ in $B(0,R)$ we have

$$
\begin{aligned}
\mu_k(\mathbb{R}^n) &:= \int_{\mathbb{R}^n} \mu_k(dx) = \int_{B(0,R)} \mu_k(dx) \\
&= \int_{B(0,R)} p_R(x)\mu_k(dx) = \int_{B(0,R)} \int_{S(0,R)} \frac{\lambda_R}{|x-y|^{n-2}} dS_y \mu_k(dx) \\
&= \int_{S(0,R)} \int_{B(0,R)} \frac{\mu_k(dx)}{|x-y|^{n-2}} \lambda_R dS_y = \int_{S(0,R)} p_{F_k}(y)\lambda_R dS_y \leq R^{n-2}.
\end{aligned}
$$

Here we have used Definitions 1.38 and 1.43 and Fubini's theorem.

Therefore, the measures μ_k are precompact in the weak-* topology and we may extract a subseqence (also denoted μ_k) that converges to a weak-* limit μ. That is, $\int_{\mathbb{R}^n} f(x)\mu_k(dx) \to \int_{\mathbb{R}^n} f(x)\mu(dx)$ for every continuous f that vanishes at infinity. We need to show that μ is suported on ∂F. Choose a continuous function f with compact support that does not intersect ∂F. Then $\operatorname{dist}(\operatorname{supp}(f), \partial F) > 0$ and for large enough k, $\int_{\mathbb{R}^n} f(x)\mu_k(dx) = 0$. Thus, we find that

$$
\int_{\mathbb{R}^n} f\mu(dx) = \lim_{l\to\infty} \int_{\mathbb{R}^n} f\mu_{k_l}(dx) = 0.
$$

This is equivalent to $\operatorname{supp}(\mu) \subset \partial F$.

Since the support of $\mu \subset \partial F$, for every ξ that is not in ∂F we find that

$$
\int_{\mathbb{R}^n} |x-\xi|^{n-2}\mu_{k_l}(dx) \to \int_{\mathbb{R}^n} |x-\xi|^{n-2}\mu(dx).
$$

(The function $|x-\xi|^{2-n}$ is not continuous in $\mathbb{R}^n$, but we only need continuity in the vicinity of ∂F). The term on the left is simply $p_{F_{k_l}}(\xi)$ and that on the right is $p_F(\xi)$. Thus, p_F is the potential of μ, hence μ is unique by Theorem 1.42. $\square$

Remark 1.49. The uniform bound on the measures obtained above is of independent interest.

1.15 Capacity of compact sets

The capacity of a compact set is the total charge $\mu_F(\mathbb{R}^n)$ of the potential p_F. It is called the capacity, because this turns out to be the maximal charge that can be placed on F when it is held at constant potential. Observe from Theorem 1.46 that if ∂F is smooth, the charge μ_F has density $\partial p_F/\partial n_x dS_x$ and in this case we find,

$$\operatorname{cap}(F) = \mu_F(\mathbb{R}^n) = \frac{2-n}{\omega_n}\int_{\partial F}\frac{\partial p_F}{\partial n_x}dS_x.$$

An observation at this point is that p_F is harmonic in $\mathbb{R}^n\backslash F$, therefore if we take *any* smooth surface Σ enclosing F we have

$$\operatorname{cap}(F) = \frac{2-n}{\omega_n}\int_{\Sigma}\frac{\partial p_F}{\partial n_x}\,dS_x. \tag{1.59}$$

Observe also that if ∂F is C^2, then $p_F = 1$ on ∂F and we have

$$\operatorname{cap}(F) = \frac{2-n}{\omega_n}\int_{\partial F} p_F\frac{\partial p_F}{\partial n_x}dS_x = \frac{2-n}{\omega_n}\int_U |Dp_F|^2\,dx.$$

There is another physical interpretation of capacity that avoids electrostatics, due to Polya [13]. Think of a body held at constant temperature in a uniform medium with zero temperature at infinity. For example, if you are standing outside on a cold day, to a good approximation your surface is at constant temperature and the temperature at infinity is constant (which may as well be zero). The capacity is the heat lost by your body in unit time in steady state. This heat simply propagates through space, and we could evaluate it on any surface Σ.

We are now ready to state Wiener's criterion for the regularity of a boundary point. Let U be an open, connected subset of $\mathbb{R}^n$.

Theorem 1.50. *Suppose $y \in \partial U$ and $\lambda \in (0,1)$. Define the compact sets*

$$F_k = \{x \in \mathbb{R}^n\backslash U \,\Big|\, \lambda^{k+1} \le |x-y| \le \lambda^k\}.$$

Then $y \in \partial U$ is regular if and only if

$$\sum_{k=0}^{\infty}\lambda^{k(2-n)}\operatorname{cap}(F_k) = \infty. \tag{1.60}$$

Of course, (1.60) is equivalent to the statement that a boundary point is exceptional if and only if the series converges. If (1.60) holds for one $\lambda \in (0,1)$ then it holds for every $\lambda \in (0,1)$. Heuristically, this criterion says that the complement of U must occupy enough space near y, ruling out Lebesgue's thorns.

Let us prove some basic properties of capacity.

1. If $F_1 \subset F_2$ then $\mathrm{cap}(F_1) \le \mathrm{cap}(F_2)$.
2. If F_k is a nested sequence of approximating compact sets with $\cap_{k=}^{\infty} F_k = F$ then $\mathrm{cap}(F) = \lim_{k\to\infty} \mathrm{cap}(F_k)$.
3. $\mathrm{cap}(F_1 \cup F_2) \le \mathrm{cap}(F_1) + \mathrm{cap}(F_2)$.

The proofs make liberal use of Gauss' trick. First (1). For brevity, let p_i, μ_i denote the potentials and charges of $F_i, i = 1, 2$. Since $p_2(x) = 1$ for all $x \in F_2$ we have

$$\begin{aligned}\mathrm{cap}(F_1) = \int_{\mathbb{R}^n} p_2(x)\mu_1(dx) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{2-n}\mu_2(dy)\mu_1(dx) \\ = \int_{\mathbb{R}^n} p_1(y)\mu_2(dy) &\le \int_{\mathbb{R}^n} \mu_2(dy) = \mathrm{cap}(F_2).\end{aligned}$$

As for (2), we know that μ_{F_k} converge weakly to μ_F. The issue here is that weak-* convergence of measures only allows us to deduce $\mathrm{cap}(F) \le \liminf_{k\to\infty} \mathrm{cap}(F_k)$ in general. However, we may use the positivity of the measures and our knowledge of the support of F_k. Pick a smooth function φ with compact support that is equal to 1 in a neighborhood of $F_1 \supset F_k \supset F$. Then by the definition of weak convergence

$$\mathrm{cap}(F) = \int_{\mathbb{R}^n} \varphi(x)\mu_F(dx) = \lim_{k\to\infty}\int_{\mathbb{R}^n} \varphi(x)\mu_{F_k}(dx) = \lim_{k\to\infty} \mathrm{cap}(F_k).$$

Finally, (3). Observe that $p_{F_1\cup F_2} \le p_{F_1} + p_{F_2}$ by the maximum principle. Therefore, applying Gauss' trick again

$$\begin{aligned}\mathrm{cap}(F_1 \cup F_2) = \int_{\mathbb{R}^n} p_{F_1\cup F_2}\mu_{F_1\cup F_2}(dx) &\le \int_{\mathbb{R}^n} (p_{F_1} + p_{F_2})\mu_{F_1\cup F_2}(dx) \\ = \int_{\mathbb{R}^n} p_{F_1\cup F_2}(\mu_{F_1}(dy) + \mu_{F_2}(dy)) &= \mathrm{cap}(F_1) + \mathrm{cap}(F_2).\end{aligned}$$

It turns out that a stronger inequality is true:

$$\mathrm{cap}(F_1 \cup F_2) + \mathrm{cap}(F_1 \cup F_2)$$

As you can see, the analysis of capacity has the flavour of measure theory. Here are some examples to get a better feel for capacity.

Example 1.51. The capacity of a ball, is the same as the capacity of a sphere, and we have

$$\text{cap}(\overline{B(0,R)}) = \text{cap}(S(0,R)) = \int_{S(0,R)} \lambda_R dS = R^{n-2}.$$

Example 1.52. Note that the inequality in (3) can be strict. Suppose $F_1 \subset F_2$ are nested spheres. Then the potential $p_{F_1 \cup F_2} = p_{F_2}$, and $\text{cap}(F_1 \cup F_2) = \text{cap}(F_2)$. This is called the *screening effect.*

Example 1.53. The capacity of a line segment in $\mathbb{R}^3$ is zero. Suppose the line segment is held at unit potential. Consider example 1.40. We showed that $u_\mu = \infty$ on the segment for any $\lambda > 0$. Thus, each of these potentials dominates the potential of the line segment. Hence, the capacity of the line segment cannot exceed λ for any $\lambda > 0$.

1.16 Variational principles for capacity

The study of capacity allows us to encounter several variational principles. The following principle explains the choice of the term ‘capacity’: the charge is maximal over all measures that keep F at constant potential.

Theorem 1.54. *Let $F \subset \mathbb{R}^n$ be compact. Then*

$$\text{cap}(F) = \sup\{\mu(F) |\, \text{supp}(\mu) \subset F, \quad u_\mu(x) \leq 1 \quad \text{for} \quad x \in F\}. \tag{1.61}$$

Proof. The proof is a direct consequence of Gauss’ trick. Let μ be a measure with support in F, and F_k a smooth approximation to F. We have

$$\mu(F) = \int_{\mathbb{R}^n} p_{F_k} \mu(dx) = \int_{F_k} u_\mu(y) \mu_{F_k} dy \leq \int_{F_k} \mu_{F_k}(dy) = \text{cap}(F_k).$$

Therefore, $\sup_\mu \mu(F) \leq \text{cap}(F_k)$. Let $k \to \infty$ to find $\sup_\mu(F) \leq \text{cap}(F)$. □

Capacity may also be characterized through minimum energy. The appropriate notion of energy here is the Coulomb energy. In the sequel, a signed measure μ has positive and negative parts $\mu^\pm$ with $\mu = \mu^+ - \mu^-$. The positive measure $|\mu| = \mu^+ + \mu^-$. The total variation of μ is $\|\mu\| = |\mu|(\mathbb{R}^n)$. We will only consider finite measures, that is $\|\mu\| < \infty$.

Definition 1.55. The Coulomb energy of a finite measure μ is

$$E[\mu] = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{2-n} \, \mu(dx) \, \mu(dy). \tag{1.62}$$

The mutual energy of two finite measures μ_1, μ_2 is

$$E[\mu_1, \mu_2] = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{2-n} \, \mu_1(dx) \, \mu_2(dy). \tag{1.63}$$

The most important property of the Coulomb energy is that it is positive definite.

Theorem 1.56. *The Coulomb energy has the following properties:*

1. *If $E(|\mu|) < \infty$, then $E[\mu] \geq 0$ with equality only if $\mu \equiv 0$.*
2. *If ν is a finite measure with $E[|\nu|] < \infty$ then*

$$|E[\mu, \nu]|^2 \leq E[\mu]E[\nu], \tag{1.64}$$

with equality for nonzero μ, ν if and only if $\nu = c\mu$ for some $c \in \mathbb{R}$.

3. *The map $\mu \mapsto E[\mu]$ is strictly convex: if $0 < t < 1$, and $\mu \neq \nu$ then*

$$E[t\mu + (1-t)\nu] < tE[\mu] + (1-t)E[\nu]. \tag{1.65}$$

We will assume this theorem for now. Its proof is best understood after we study the heat equation. The hard part is (1); (2) and (3) are direct consequences. An important variational principle is the following.

Theorem 1.57 (Gauss' principle). *Let μ be a non-negative, finite measure supported on F. The quadratic form*

$$G[\mu] = E[\mu] - \mu(F) \geq -\frac{1}{2}\mathrm{cap}(F), \tag{1.66}$$

with equality if and only if $\mu = \mu_F$, the charge associated to F.

Proof. The proof is a classic demonstration of what is called the *direct method in the calculus of variations*. It consists of the following steps.

1. $G[\mu]$ is bounded below: Let $\beta = \inf_{x,y \in F} |x-y|^{2-n}$. Observe that $\beta > 0$ since F is compact. We use the definition of the Coulomb energy (1.62) and $\mu \geq 0$ to obtain

$$E[\mu] = \frac{1}{2} \int_F \int_F |x-y|^{2-n} \mu(dx)\mu(dy) \geq \frac{\beta}{2}(\mu(F))^2. \tag{1.67}$$

This is called a *coercivity* bound. Now use this estimate in (1.66)

$$G[\mu] \geq \frac{\beta}{2}(\mu(F))^2 - \mu(F) \geq -\frac{1}{2\beta}. \tag{1.68}$$

2. Infimizing sequences are precompact: Let μ_k be a sequence such that $\lim_{k\to\infty} G[\mu_k] = \inf G$. We use (1.68) again to find for large enough k

$$1 + \inf G \geq \frac{\beta}{2}(\mu_k(F))^2 - \mu_k(F)$$

Therefore, the sequence $\mu_k(F)$ is bounded. Thus, the sequence μ_k is precompact in the weak-* topology, and we may extract a convergent subsequence (also denoted μ_k) with limit μ.

3. G is weakly lower semicontinuous: that is

$$G[\mu] \leq \liminf_{k\to\infty} G[\mu_k] = \inf_{\mu} G[\mu]. \tag{1.69}$$

Therefore, μ is a minimizer. It is only necessary to study the Coulomb energy, for by choosing $\varphi \in C_c^\infty$ with $\varphi = 1$ on a neighborhood of F we have

$$\mu(F) = \int_{\mathbb{R}^n} \varphi(x)\mu(dx) = \lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi(x)\mu_k(dx) = \mu_k(F).$$

The proof uses the quadratic nature of E. For any M, observe that

$$u_M(x) = \int_{\mathbb{R}^n} \min(M, |x-y|^{2-n})\mu(dy)$$

is a continuous functions. Therefore,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} u_M(x)\mu_k(dx) = \int_{\mathbb{R}^n} u_M(x)\mu(dx).$$

Let $M \to \infty$ and use the monotone convergence theorem to obtain

$$E[\mu] = \lim_{k\to\infty} E[\mu, \mu_k].$$

We now pass to the limit in the inequality

$$0 \leq E[\mu - \mu_k] = E[\mu_k] + E[\mu] - 2E[\mu, \mu_k]$$

to deduce that $E[\mu] \leq \liminf_{k\to\infty} E[\mu_k]$.

4. The minimizer is unique: Suppose we had two minimizers, μ and ν. We consider a linear combination $t\mu + (1-t)\nu$ and find

$$G(t\mu + (1-t)\nu) < tG(\mu) + (1-t)G(\nu) = \min G.$$

5. The minimizer is μ_F. Let u_μ denote the potential of the measure μ and let $\nu \geq 0$ be a finite measure. Since $G(\mu+\varepsilon\nu) \geq G(\mu)$ we take a limit to find

$$0 \leq \liminf_{\varepsilon \downarrow 0} \frac{G(\mu+\varepsilon\nu)-G(\mu)}{\varepsilon} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{2-n}\mu(dy)\nu(dx) - \nu(F)$$
$$= \int_{\mathbb{R}^n} (u_\mu(x)-1)\nu(dx). \tag{1.70}$$

In particular, we have the equality

$$0 = \int_{\mathbb{R}^n} (u_\mu(x)-1)\mu(dx). \tag{1.71}$$

This is because $\mu - \varepsilon\mu$ is also non-negative for small ε and we may set $\nu = -\mu$ in (1.70). Now consider the difference $E[\mu-\mu_F]$. A direct computation yields

$$0 \leq E[\mu-\mu_F] = \int_F (u_\mu - p_F)(\mu-\mu_F)(dx)$$
$$= \int_F (u_\mu - 1)\mu(dx) - \int_F (u_\mu-1)\mu_F(dx) \leq 0,$$

since $p_F = 1$ on F and equality (1.71) holds. Therefore, $\mu = \mu_F$.

6. We may now evaluate the minimum by setting $\mu = \mu_F$. We use (1.71) to find $2E[\mu] = -\mu_F(F) = -\mathrm{cap}(F)$. □

Remark 1.58. Steps (1),(2) and (4) are very general. There is another proof of (3) which uses only convexity, not the quadratic nature of E. The equation in step (5) is called the Euler-Lagrange equation.

Theorem 1.59 (Kelvin's principle).

$$\frac{1}{2\mathrm{cap}(F)} = \inf\{E[\mu]\,|\mu \geq 0, \mathrm{supp}(\mu) \subset F, \quad \mu(F) = 1\}, \tag{1.72}$$

with the understanding that the left hand side is $+\infty$ *when* $\mathrm{cap}(F) = 0$.

Proof. It is instructive to repeat the direct method, noting that $E \geq 0$ gives the lower bound, and $\mu(F) = 1$ gives pre-compactness. However, we may also use Gauss' principle. Given any $\mu \geq 0$ with $\mu(F) = 1$ and any $t > 0$ we use (1.66) with $t\mu$ to obatin the estimate

$$t^2 E[\mu] - t\mu(F) \geq -\frac{\mathrm{cap}(F)}{2},$$

with equality if and only if $\mu = t\mu_F$. We set $t = \mathrm{cap}(F)$ when this is positive. If $\mathrm{cap}(F) = 0$, then we have $E[\mu] \geq t^{-1}$ for every $t > 0$. □

2 The heat equation

2.1 Motivation

Let us begin with some motivation. The heat equation is the simplest physical model for the spread of mass or temperature by diffusion. Think for example of the transport of smoke in still air. We think of smoke particles as being much larger than the air molecules, yet sufficiently small that they feel the random kicks of collisions with many atoms. The average displacement of any particle is zero, yet there are fluctuations about this mean that increase with time. The macroscopic manifestation of these fluctuations is diffusion. This physical picture was proposed in 1905 by Einstein [2] at a time when the existence of atoms was in doubt, and his theoretical predictions were confirmed by Perrin.

This microscopic picture of diffusion underlies a classical theory of diffusion derived by Euler (though the name usually attach to the heat equation is that of Fourier). There are two steps in the modeling process. Let $u(x,t)$ denote the density of smoke at a position in space. If the smoke only gets transported (and not created or destroyed) then we may use conservation of mass to write

$$\partial_t u + \operatorname{div}(\mathbf{J}) = 0,$$

where $\mathbf{J}$ denotes the flux of particles at any point in space. Since there are two unknowns, we need another equation. This is a *constitutive relation* usually called *Fick's law*. The flux is related to the density (or temperature) u through $\mathbf{J} = -\alpha^2 Du$. The experimental basis for Fick's law is that heat flows in the direction of steepest descent. Here α^2 is a material constant that we may choose to be 1 (or 1/2 for probabilists). We are thus led to the heat (or diffusion) equation

$$\partial_t u = \triangle u, \quad x \in U, \quad t > 0. \tag{2.1}$$

We shall almost always be concerned with the Cauchy problem: here we are given an initial field

$$u(x,0) = f(x), \quad x \in U, \tag{2.2}$$

and the task is to propagate it in space. Let us note that Laplace's equation corresponds to the special case when the field $u(x,t)$ is independent of t.

2.2 The fundamental solution

John derives the fundamental solution using Fourier analysis. Here is a different derivation based on scaling. We have already noted that the Laplacian is invariant under rotations. Now observe that the heat equation is invariant under the *parabolic scaling* $x \to \lambda x, t \to \lambda^2 t$. More precisely, if u solves (2.1) on the domain $x \in \mathbb{R}^n, t \in (0,\infty)$, then so does the rescaled function $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$. Moreover, the physical context above suggests that the heat equation conserves mass: explicitly, if $U = \mathbb{R}^n$ then upon integration (and assuming sufficiently rapid decay at infinity) we have

$$\partial_t \left(\int_{\mathbb{R}^n} u \, dx \right) = 0.$$

These observations suggest we seek a fundamental solution of the form

$$u(x,t) = t^{-n/2} g(|x|t^{-1/2}) := t^{-n/2} g(\xi), \tag{2.3}$$

where $\xi = |x|t^{-1/2}$ is called the *similarity variable*. Substitute this ansatz in (2.1) to obtain

$$\partial_t u = -\frac{1}{2t^{1+n/2}} \left(\xi g' + ng \right) = -\frac{\xi^{1-n}}{2t^{1+n/2}} \left(\xi^n g \right)',$$

and

$$\triangle u = \frac{1}{t^{1+n/2}} \left(g'' + \frac{n-1}{\xi} g' \right) = \frac{\xi^{1-n}}{t^{1+n/2}} \left(\xi^{n-1} g' \right)'.$$

We equate these terms to obtain the differential equation

$$\left(\xi^{n-1} g' \right)' + \frac{1}{2} \left(\xi^n g \right)' = 0.$$

In order that g is smooth, we require $g'(0) = 0$ (g is symmetric under rotations in x) . Therefore, integrating in ξ we have

$$g' + \frac{\xi}{2} g = 0,$$

with solution $g = Ce^{-\xi^2/4}$. The constant is determined by the normalization $\int_{\mathbb{R}^n} u(x,t)\, dx = \int_{\mathbb{R}^n} g(|x|)\, dx = 1$ (this is similar to the normalization for the fundamental solution to Laplace's equation). This yields the fundamental solution or *heat kernel*

$$k(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad x \in \mathbb{R}^n, t > 0. \tag{2.4}$$

We must justify the term fundamental solution. This is based on the following properties of k which should remind you of the Poisson kernel.

1. $k(x,t)$ solves $\partial_t k = \triangle_x k$, $x \in \mathbb{R}^n, t > 0$.
2. $k(x,t) > 0, \quad x \in \mathbb{R}^n,\ t > 0$.
3. $\int_{\mathbb{R}^n} k(x,t)dx = 1$ for every $t > 0$.
4. For any $\delta > 0$, $\lim_{t\downarrow 0} k(x,t) = 0$ uniformly in $|x| \geq \delta$.
5. For any $\delta > 0$,
$$\lim_{t\downarrow 0} \int_{|x|\geq\delta} k(x,t)\,dx = 0.$$
6. $k(x,t)$ is C^∞ in $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

Properties (1), (2) and (3) follow immediately from our construction of k. Let us check (4). For any $|x| \geq \delta, t > 0$ we have $e^{-|x|^2/4t} \leq e^{-\delta^2/4t}$ and $t^{-n/2}e^{-\delta^2/4t} \to 0$ as $t \to 0$. Here is a quantitative version of (5) which will give you some practice with Gaussian integrals. Observe that for $y \in \mathbb{R}$ and any $a > 0$

$$\int_{|y|>a} e^{-y^2}dy = 2\int_{y>a} e^{-y^2}dy < 2\int_{y>a} \frac{y}{a}e^{-y^2}dy = \frac{e^{-a^2}}{a}.$$

The second inequality is called Chebyshev's inequality. In $\mathbb{R}^n$, the following crude bound will suffice. Since the set $|y| \geq a$ is contained within $\{|y_i| \geq a/\sqrt{n}, 1 \leq i \leq n\}$, we have

$$\int_{|y|>a} e^{-y^2}dy \leq \left(\int_{|y_1|\leq a/\sqrt{n}} e^{-y_1^2}dy_1\right)^n \leq \frac{n^{n/2}e^{-a^2}}{a^n}.$$

Now let us apply this to the heat kernel. Let $y = x/2\sqrt{t}$ so that

$$\int_{|x|>\delta} k(x,t)dx = \frac{1}{\pi^{n/2}}\int_{|y|>2\delta/\sqrt{t}} e^{-|y|^2}dy \leq \left(\frac{tn}{4\delta^2\pi}\right)^{n/2} e^{-4\delta^2/t}.$$

Finally, it is clear that the kernel is C^∞ as long as $t > 0$.

To summarize, the heat kernel is an approximate identity that solves $\partial_t k = \triangle_x k$. This allows us to construct solutions to the Cauchy problem.

Theorem 2.1. *Let f be a bounded continuous function on $\mathbb{R}^n$. Then*

$$u(x,t) = \int_{\mathbb{R}^n} k(x-y,t)f(y)\,dy = \int_{\mathbb{R}^n} k(y,t)f(x-y)\,dy, \quad t > 0, \qquad (2.5)$$

defines a C^∞ solution to the heat equation $\partial_t u = \triangle u$ on the domain $\mathbb{R}^n \times (0,\infty)$. u attains the initial values f in the following sense

$$\lim_{t\downarrow 0} u(x,t) = f(x)$$

for every $x \in \mathbb{R}^n$. Thus, defining $u(x,0) = f(x)$ we obtain a continuous function $u(x,t)$ on $\mathbb{R}^n \times [0,\infty)$.

Proof. The proof is similar to the proof of Theorem 1.14. In order to show that u solves the heat equation, we differentiate under the integral sign, and use $\triangle_x k(x-y,t) = \triangle_{x-y} k(x-y,t) = \partial_t k(x-y,t)$ to deduce that u solves the heat equation.

In order to give a careful justification for differentiation under the integral sign you may use finite differences. For example, to show that $\partial_t u$ exists, proceed as follows. Fix $x \in \mathbb{R}^n$, $t > 0$ and let $h \neq 0$. Let M denote $\sup |f|$. By definition

$$\begin{aligned} \frac{u(x,t+h)-u(x,t)}{h} &= \int_{\mathbb{R}^n} \frac{k(y,t+h)-k(y,t)}{h} f(x-y)dy \\ &= \int_{\mathbb{R}^n} \partial_t k(y,t+\theta h) f(x-y)dy, \end{aligned}$$

for some $0 < \theta < 1$ by the mean value theorem. Therefore,

$$\begin{aligned} &\left| \frac{u(x,t+h)-u(x,t)}{h} - \int_{\mathbb{R}^n} \partial_t k(y,t) f(x-y)dy \right| \\ &\qquad \leq M \int_{\mathbb{R}^n} |\partial_t k(y,t+\theta h) - \partial_t k(y,t)|\, dy \leq Mh \int_{\mathbb{R}^n} |\partial_t^2 k(y,t+\theta' h)| dy, \end{aligned}$$

for some $0 < \theta' < 1$. Here we used the mean value theorem again, and the assumption that f is bounded. Observe that $\partial_t^2 k(y,t)$ has Gaussian decay as $y \to \infty$. In particular, it is integrable. Now take the limit $h \to 0$ to conclude that u is differentiable in t.

In order to show that the initial values are attained, we must use the properties of an appromixate identity. By property (3), for any $t > 0$ we have

$$u(x,t) - f(x) = \int_{\mathbb{R}^n} k(y,t) \left(f(x-y) - f(x)\right) dy.$$

Since f is continuous, we may choose $\delta > 0$ such that $|f(x) - f(x-y)| < \varepsilon$ for $|y| < \delta$. Estimate the integral separately on $|y| < \delta$ and $|y| \geq \delta$. We use properties (2) and (3) to obtain

$$\left|\int_{|y|<\delta} k(y,t)\left(f(x-y)-f(x)\right)\,dy\right|$$
$$\leq \int_{|y|<\delta} k(y,t)\,|f(x-y)-f(x)|\,dx \leq \varepsilon \int_{\mathbb{R}^n} k(y,t)\,dy = \varepsilon.$$

On the other hand, when $|y|>\delta$ we use property (5) and the uniform bound $\sup|f| = M < \infty$ to obtain

$$\int_{|y|>\delta} k(y,t)|f(x-y)-f(x)|\,dy$$
$$\leq 2M\int_{|y|>\delta} k(y,t)\,dy \leq 2M\left(\frac{tn}{4\delta^2\pi}\right)^{n/2} e^{-4\delta^2/t}.$$

$\square$

Remark 2.2. Observe that there is no need to assume that f is bounded. To justify differentiation under the integral sign we only need a growth assumption on f that is beaten by the decay of $e^{-|y|^2/4t}$. For example, the proof would work even if $|f(x)| \leq Me^{|y|^\alpha}, \alpha < 2$.

2.3 Uniqueness of solutions

Theorem 2.1 makes no mention of uniqueness: with good reason, as the following counterexample of Tychonoff shows. We construct a series solution

$$u(x,t) = \sum_{k=0}^{\infty} g_k(t)x^{2k}. \tag{2.6}$$

If we formally differentiate, we have

$$\sum_{k=0}^{\infty} \dot{g}_k(t)x^{2k} = u_t = u_{xx} = \sum_{k=0}^{\infty}(2k+2)(2k+1)g_{k+1}x^{2k}.$$

Equate coefficients to obtain the necessary conditions

$$\dot{g_k} = (2k+2)(2k+1)g_{k+1}, \quad \text{with solution} \quad g_k(t) = \frac{g_0^{(k)}(t)}{(2k)!}. \tag{2.7}$$

We choose $g_0(t)$ to ensure convergence of (2.6). For any $p > 1$, let $g_0(t) = e^{-t^{-p}}, t > 0$. As shown in the HW, there exists $\theta = \theta(p) > 0$ such that

$$|g_0^{(k)}(t)| \leq \frac{k!}{(\theta t)^k} e^{-1/2t^p}, \quad t > 0. \tag{2.8}$$

Therefore, we may susbtitute (2.8) in (2.7) to obtain for any $t > 0$

$$\sum_{k=0}^{\infty} |g_k(t)||x|^{2k} \leq e^{-1/2t^p} \sum_{k=0}^{\infty} \frac{k!}{2k!} \left(\frac{|x|^2}{\theta t}\right)^k \leq \exp\left(-\frac{1}{2t^p} + \frac{|x|^2}{\theta t}\right).$$

Thus, $u(x,t)$ is well defined for every $t > 0$. Moreover, the assumption $p > 1$ ensures $\lim_{t\to 0} u(x,t) = 0$ uniformly on compact sets. A similar bound shows that the series for u_t and u_{xx} converge. We now have infinitely many (for every $p > 1$) solutions to the heat equation with initial data $u(x,0) = 0$.

This places us in an interesting quandary. The heat equation is based on simple modeling assumptions (see § 2.1) and should not be so complicated. The essential flaw here is that $k(x,t) > 0$ for every $x \in \mathbb{R}^n, t > 0$. In short, heat propagates infinitely fast from any one point to another, and there is nothing to protect against a great blast of heat from infinity. To quote verbatim, from [10, Ch. 67]:

> To the applied mathematician [the counterexample] is simply an embarrassment reminding her of the defects of a model which allows an unbounded speed of propagation. To the numerical analyst it is just a mild warning that the heat equation may present problems which the wave equation does not. But the pure mathematician looks at it with the same simple pleasure with which a child looks at a rose which has just been produced from the mouth of a respectable uncle by a passing magician.

One requires other considerations to obtain uniqueness. The optimal result in this direction is Widder's theorem, which asserts that solutions to the heat equation that are bounded on one sided (say $u \geq 0$) are unique. Indirectly, we can conclude that all of Tychonoff's solutions change sign.

2.4 The weak maximum principle

The lectures follow John with no change. You should also look at the discussion of analyticity of solutions.

2.5 The mean value property

Harmonic functions are characterized by the mean value property. A similar, but more subtle characterization holds for the heat equation. Spheres are level sets of the fundamental solution for Laplace's equation. This motivates the appropriate geometric notion for the heat equation.

Definition 2.3. For $x \in \mathbb{R}^n, t \in \mathbb{R}$ we define the heat ball

$$E(x,t;r) = \{(y,s) \in \mathbb{R}^{n+1} \Big| s \leq t, \quad k(x-y,t-s) \geq \frac{1}{r^n}\} \tag{2.9}$$

The following geometric properties of the heat ball are basic.
1. Translation: $E(x,t;r)$ is the ball $E(0,0,;r)$ translated to (x,t).
2. Scaling: $E(0,0;r)$ is obtained from $E(0,0;1)$ by the parabolic scaling $y \to ry, s \to r^2 s$.
3. $E(0,0;r)$ is a closed, convex set with boundary given by

$$|y| = R_r(s) := \left(-2ns\log\left(\frac{r^2}{-4\pi s}\right)\right)^{1/2}, \quad -\frac{r^2}{4\pi} \leq s \leq 0. \tag{2.10}$$

(It is natural to include the limit $y(0) = 0$.)

The following theorem is surprisingly recent [14]. It works for more general domains in $\mathbb{R}^{n+1}$. To be concrete, we will work with strips $V_T = U \times (0,T)$ for fixed $T > 0$. Let $C_1^2(V_T)$ denote functions on V_T such that $\partial_t u$ and $D^2 u$ exist and are continuous.

Theorem 2.4. *Let $u \in C_1^2(V_T)$ be a subtemperature (a subsolution of the heat equation). Then for every $E(x,t;r) \subset V_T$*

$$u(x,t) \leq \frac{1}{4r^n}\int\int_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\,dy\,ds. \tag{2.11}$$

In particular, temperatures (solutions to the heat equation) satisfy

$$u(x,t) = \frac{1}{4r^n}\int\int_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\,dy\,ds. \tag{2.12}$$

Conversely, if $u \in C_1^2(V_T)$ and (2.12) holds for every $E(x,t;r) \subset V_T$ then $\partial_t u = \triangle u$ in V_T.

Remark 2.5. Of course, one would like a stronger converse; namely, if u is continuous and (2.12) holds for every $E(x,t;r)$ then u solves the heat equation. This is true but we will need a strong maximum principle first. Once one has characterized temperatures through a mean value property, Harnack's inequality, Harnack's convergence theorem and compactness theorems follow as for harmonic functions

Remark 2.6. Set $u = 1$ and use the scaling property of $E(x, t; r)$ to find

$$4 = \int_{E(0,0,1)} \frac{|y|^2}{s^2}\, dy\, ds.$$

This is the volume of the heat ball with the weighted measure $|y|^2 s^{-2} dyds$. What is surprising is that this volume is independent of n (compare with the behavior of ω_n). It is interesting to compute this volume directly (see Remark 2.8).

Proof. 1. We may suppose $x = 0$ and $t = 0$ after translating the heat ball. For brevity, we denote $E(0, 0; r)$ by $E(r)$ and define

$$\psi(y, s, r) = \log\left(\frac{r^n e^{|y|^2/4s}}{(-4\pi s)^{n/2}}\right), \quad \varphi(r) = \frac{1}{4r^n} \int\int_{E(r)} u(y, s) \frac{|y|^2}{s^2}\, dy\, ds.$$

The heat ball is the set $\{\psi \geq 0\}$. The proof relies on the following identity (called a *monotonicity formula*) which holds for every $u \in C^3(V_T)$:

$$\varphi'(r) = \frac{-n}{r^{n+1}} \int\int_{E(r)} \psi\left(\partial_s u - \triangle u\right)\, dy\, ds. \tag{2.13}$$

The theorem follows immediately from this identity. If u is a subsolution, then $\varphi'(r) \geq 0$ which is stronger than (2.12). For solutions we have equality. Conversely, if we have equality for every heat ball and $u_t - \triangle u$ is continuous, it must vanish in V_T since ψ is positive in $E(r)$.

2. The proof of (2.13) is a clever calculation. We rescale variables $y = ry'$, $s = r^2 s'$ to obtain

$$\varphi(r) = \frac{1}{4} \int\int_{E(1)} u(ry', r^2 s') \frac{|y'|^2}{s'^2}\, dy'\, ds'.$$

Now differentiate to obtain

$$\begin{aligned}
\varphi'(r) &= \frac{1}{4} \int\int_{E(1)} \left(y' \cdot D_y u + 2rs'\partial_s u\right) \frac{|y'|^2}{s'^2}\, dy' ds' \\
&= \frac{1}{4r^{n+1}} \int\int_{E(r)} \left(y \cdot D_y u + 2s\partial_s u\right) \frac{|y|^2}{s^2}\, dyds := A + B.
\end{aligned} \tag{2.14}$$

To simplify B we have to play with the derivatives of ψ. We have

$$y \cdot D_y \psi = \frac{|y|^2}{2s}, \quad \text{and} \quad s\partial_s \psi = -\frac{n}{2} - \frac{|y|^2}{4s}. \tag{2.15}$$

Substitute the first of these relations in B to obtain

$$B = \frac{1}{r^{n+1}} \int\int_{E(r)} \partial_s u \, y \cdot D_y \psi \, dy \, ds.$$

The integrand can be rewritten as

$$\partial_s u \, y \cdot D_y \psi = D_y \cdot (y\psi\partial_s u) - n\psi\partial_s u - \psi y \cdot \partial_s D_y u.$$

The last term can be further expanded as

$$\psi y \cdot \partial_s D_y u = \partial_s(\psi y \cdot D_y u) - (y \cdot D_y u)\partial_s \psi.$$

Since $\psi = 0$ on the boundary of $E(r)$ we integrate by parts to obtain

$$\begin{aligned} B &= \frac{1}{r^{n+1}} \int\int_{E(r)} (-n\psi\partial_s u + y \cdot D_y u \, \partial_s \psi) \, dy \, ds \\ &= -\frac{1}{r^{n+1}} \int\int_{E(r)} (n\psi\partial_s u + y \cdot D_y u \left[\frac{n}{2s} + \frac{|y|^2}{4s^2}\right] \\ &= -\frac{n}{r^{n+1}} \int\int_{E(r)} (\psi\partial_s u + D_y \psi \cdot D_y u) - A. \end{aligned} \tag{2.16}$$

We combine (2.14) and (2.16) and integrate by parts to obtain the monotonicity formula (2.13). □

It is also of interest to get a mean value theorem on heat spheres (that is, on the surface $\partial E(x,t;r)$. For completeness, this is stated below as a consequence of Theorem 2.4.

Theorem 2.7. *Suppose $u \in C_1^2(V_T)$ solves the heat equation. Then for any $E(x,t;r) \subset V_T$ we have the mean value property on heat spheres*

$$u(x,t) = \frac{1}{2r^n} \int_{-r^2/4\pi}^{0} \left(\int_{|\omega|=1} u(x + R\omega, t+s) d\omega \right) \frac{R_r(s)^n}{-s} \, ds. \tag{2.17}$$

Conversely, if $u \in C_1^2(V_T)$, and (2.17) holds for every $E(x,t;r) \subset V_T$, then $u_t = \triangle u$.

Proof. Suppose $x = 0$ and $t = 0$. We rewrite $\varphi(r)$ using polar coordinates

$$\begin{aligned}\varphi(r) &= \frac{1}{4r^n}\int\int_{E(r)} u(y,s)\frac{|y|^2}{s^2}\,dyds \\ &= \frac{1}{4r^n}\int_{-r^2/4\pi}^{0}\int_0^R\int_{|\omega|=1} u(\rho\omega,s)\,d\omega\frac{\rho^{n+1}}{s^2}d\rho\,ds.\end{aligned}$$

Here R denotes $R_r(s)$ for brevity. We differentiate with respect to r to obtain

$$\varphi'(r) = \frac{-n\varphi(r)}{r} + \frac{1}{4r^n}\int_{-r^2/4\pi}^{0}\int_{|\omega|=1} u(\rho\omega,s)\,d\omega\frac{\rho^{n+1}}{s^2}\bigg|_{\rho=R}\frac{dR}{dr}\,ds. \quad (2.18)$$

If u solves the heat equation, $\varphi'(r) = 0$ and $\varphi(r) = u(x,t)$. From the definition of $R_r(s)$ (see (2.10)) we have

$$R\frac{dR}{dr} = -\frac{2ns}{r}.$$

We substitute in (2.18) and rearrange terms to obtain (2.17). To prove the converse, we simply observe that (2.17) is equivalent to $\varphi'(r) = 0$. $\square$

Remark 2.8. If $u \equiv 1$, Theorem 2.7 implies

$$1 = \frac{\omega_n}{2r^n}\int_{-r^2/4\pi}^{0}\frac{R_r(s)^n}{-s}\,ds.$$

Let us check this directly. Let I denote the integral. First rescale $s = -r^2 t$, and use (2.10) to find

$$I = \frac{\omega_n}{2}\int_0^{1/4\pi}(-2nt\log(4\pi t))^{n/2}\frac{dt}{t}.$$

Set $p = -n^{-1}\log(4\pi t)$ in I and cancel many factors of 2 and n to find

$$\begin{aligned}I &= \frac{\omega_n}{n\pi^{n/2}}\int_0^{\infty} e^{-n/2}p^{n/2}dp \\ &= \frac{\omega_n}{n\pi^{n/2}}\Gamma\left(\frac{n}{2}+1\right) = \frac{\omega_n}{2\pi^{n/2}}\Gamma\left(\frac{n}{2}\right) = 1.\end{aligned}$$

More generally, if we consider weights $|y|^\alpha s^\beta$ and consider

$$\int_{E(r)} |y|^\alpha |s|^\beta\,dy\,ds = C_{\alpha,\beta}r^{n+2+\alpha+2\beta},$$

a similar calculation yields

$$C_{\alpha,\beta} = \frac{\omega_n}{(n+\alpha)\theta(4\pi)^\theta}\left(\frac{(2n)}{\theta}\right)^{n+\alpha}\Gamma(\frac{n+\alpha}{2}+1),$$

where $\theta = \beta + 1 + (n+\alpha)/2$. Everything cancels and this reduces to 1/4 when $\theta = n/2$. Strangely, the combination $\alpha = 2$, $\beta = -2$ is the only one that leads to $C_{\alpha,\beta}$ independent of n.

2.6 The strong maximum principle

We split the boundary of V_T into two pieces: $\partial_i V_T, i = 1, 2$. The parabolic boundary $\partial_1 V_T$ consists of $U \times \{0\}$ and $\partial U \times [0, T]$.

Theorem 2.9. *Suppose U is open, bounded and connected. Suppose $u \in C(\bar{V}_T)$ satisfies the mean value inequality (2.11). Then*

$$\max_{\bar{V}_T} u \leq \max_{\partial_1 V_T} u. \tag{2.19}$$

If the maximum is attained at $(x,t) \in V_T$ then u is constant in $\bar{V}_t$.

Remark 2.10. The strong maximum principle applies to subtemperatures by the mean value property. Here we isolate the property of subtemperatures (the inequality (2.11) that is needed for a strong maximum principle.

Remark 2.11. The theorem is more delicate than the strong maximum principle for subharmonic functions. We cannot deduce that u is constant on all of V_T, only that it was constant in the past $0 \leq s \leq t$. This comes down to the geometry of heat balls, especially that (x,t) sits at the top of $E(x,t,r)$, not in its interior.

Proof. 1. Let $M = \max_{\bar{V}_T} u$. Suppose $u(x,t) = M$. For some $r > 0$, we have $E(x,t) \subset \bar{V}_t$. Apply the mean value property (2.11) to obtain

$$0 \leq \frac{1}{4r^n}\int\int_{E(x_0,t_0;r)} (u(x_0,t_0) - M)\,\frac{|y|^2}{s^2}\,dy\,ds \leq 0.$$

Thus, $u \equiv M$ in $E(x,t)$. At this point, the simple topological argument that we used for Laplace's equation does not work. We can no longer say that $u^{-1}\{M\}$ is open.

2. Let us consider the set of all $(y,s), s < t$ that can be linked to (x,t) by a line segment $\gamma(\tau) = (1-\tau)(x,t) + \tau(y,s), \tau \in [0,1]$. On any such segment we have $u \equiv M$. Indeed, if $f : [0,1] \to \mathbb{R}$ is defined by $f(\tau) = u(\gamma(\tau))$ we see that $f^{-1}\{M\}$ is nomempty ($f(0) = M$), closed (by continuity) and open (by step 1). It is important here that $s < t$.

3. Finally, since V_t is connected to any point $(y, s) \in V_t$ can be connected to (x, t) by a polygonal path (the set of points that can be so connected to (x, t) is nonempty, open and relatively closed in V_t). Thus, $u \equiv M$ in V_t and by continuity, in $\bar{V}_t$.

4. If $v \equiv M$, (2.19) is trivial. If v is not identically constant, since $\max_{\bar{V}_T} u$ is attained, it can be attained only on $\partial_1 \bar{V}_T$. □

Theorem 2.12. *If $u \in C(V_T)$ satisfies the mean value property (2.12) for every $E(x, t; r) \subset V_T$, then u solves the heat equation in V_T.*

Proof. Fix $(x, t) \in V_T$ and consider a parabolic cylinder $C = \{(y, s) | |y - x| \leq \delta, t - \delta^2 \leq s \leq t\}$ contained within V_T. We may solve the heat equation on such a cylinder with boundary values prescribed on the parabolic boundary $\partial_1 C$ by the method of Green's function (see [7]). Then the difference $u - v = 0$ on the parabolic boundary of C, and satisfies the mean value property in C. By the strong maximum principle, $u \equiv v$. □

2.7 Difference schemes

Discretizations of PDEs are used to prove existence theorems and for the practical matter of computing solutions by numerical methods. They are also of intrinsic interest. We only consider Laplace's equation on a uniform grid, but you will see that there is a natural notion of harmonic functions on graphs, with many familiar properties. This is of interest in combinatorics and in engineering (for example, electrical circuits are discretizations of Maxwell's equation).

Let $\mathbb{Z}^n$ denote the integer lattice. We will work with subsets of the spatial grid $h\mathbb{Z}^n$. To make the analogy with Laplace's equation transparent, similar notation will be adopted. For example, $\{|\omega| = 1\}$ will denote the 'unit sphere' in $\mathbb{Z}^n$, that is the set of $2n$-points $\pm e_i$ where e_i are the unit vectors in $\mathbb{R}^n$. Points in a domain $U_h \subset h\mathbb{Z}^n$ will be denoted by x. The boundary of U_h, denoted ∂U_h, is the set of points $y \in \mathbb{Z}^n \backslash U_h$ such that $y = x + h\omega$ for some $x \in U_h, |\omega| = 1$. A set U_h is connected (!) if for every $x, y \in U_h$ there is a finite 'walk' $x_i, i = 0, \ldots, N$ such that $x_0 = x$, $x_N = y$ and $x_{i+1} = x_i + h\omega_i$ for some $|\omega_i| = 1$. The discrete Laplacian is the difference operator

$$\triangle_h u(x) = \frac{1}{2nh^2} \sum_{|\omega|=1} \left(u(x + h\omega) - u(x) \right).$$

A function $u : U_h \cup \partial U_h \to \mathbb{R}$ is harmonic if $\triangle_h u = 0$ in U_h. Similarly, u is subharmonic if $\triangle_h u \geq 0$. Subharmonic functions satisfy a strong maximum principle.

Theorem 2.13. *If U_h is finite and connected, and $u : U_h \cup \partial U_h \to \mathbb{R}$ is subharmonic. Then*

$$\max_{U_h \cup \partial U_h} u = \max_{\partial U_h} u.$$

The maximum is attained in U_h if and only if u is constant.

Proof. 1. Since $U_h \cup \partial U_h$ is finite the maximum is attained. If the maximum is attained on ∂U_h there is nothing to prove. So let us suppose u attains its maximum M at $x \in U_h$. As in the continuous setting we have

$$M = u(x) \leq \frac{1}{2n} \sum_{|\omega|=1} u(x + h\omega) \leq M.$$

Thus, $u(y) = M$ for every neighbor y of x.

2. Since U_h is connected, every point $z \in U_h$ is connected to x by a walk in U_h. By step (1), the value at every point on this walk is M. Thus, u is a constant. □

The discrete Dirichlet problem is to solve

$$\triangle_h u = 0, \quad x \in U_h, \tag{2.20}$$

$$u = f, \quad x \in \partial U_h. \tag{2.21}$$

The use of the maximum principle as an existence tool is transparent in the discrete setting.

Theorem 2.14. *Suppose U_h is bounded and connected. The Dirichlet problem has a unique solution.*

Proof. 1. Let N denote the number of points in U_h. Equations (2.20) and (2.21) form a linear system of N equations (one for each $x \in U_h$). Schematically, we may write this as $Au = B$.

2. If $f \equiv 0$, then $B = 0 \in \mathbb{R}^N$ and we have $u \equiv 0$ by the maximum principle. Therefore, the nullspace of A consists of only $\{0\} \subset \mathbb{R}^N$. Thus, A is of full rank, and there is a unique solution for every f. □

The discrete analog of Poisson's integral formula is instructive. We construct a basis for the solution space as follows. Fix $y \in \partial U_h$, and solve the

Dirichlet problem with boundary data $f(z) = 0, z \neq y,\ f(y) = 1$. Call this solution $H(x,y)$. Then the solution to (2.20) and (2.20) for arbitrary f is

$$u(x) = \sum_{y \in \partial U_h} H(x,y) f(y). \tag{2.22}$$

2.8 Random walks

The probabilistic interpretation of the discrete equations adds depth to our understanding of harmonic functions. What follows is a heuristic description to give you a flavour of the subject. We imagine a drunken walker on the grid taking steps with equal probability in each coordinate direction. More precisely, consider a sequence of independent random variables ω_i uniformly distributed on the sphere $|\omega| = 1$. The walk starting at 0 is denoted

$$W_m = h \sum_{k=1}^{m} \omega_k.$$

The walk starting at x is $x + W_m$. These walks have the *Markov property*

$$P(W_{m+1} \in A \,|W_1, \ldots, W_m) = P(W_{m+1} \in A \,|W_m), \quad A \subset \mathbb{Z}^n.$$

That is knowledge of the entire trajectory $W_1, \ldots, W_m$ tells us no more about about W_{m+1} than knowledge of only W_m. A deeper *strong Markov property* is also true: we may replace m by certain admissible random times, M, called stopping times. Loosely speaking, a stopping time is a random time that does not look into the future. Here is a useful example: Fix a set $G \subset \mathbb{Z}^n$ and let us define the first hitting time $M = \inf\{k \,|W_k \in G\}$ and the last hitting time $N = \sup\{k \,|W_k \in G \text{ and } W_l \text{ is not in } G \text{ for } l > k\}$. Observe that M relies only on $W_1, \ldots, W_M$, but N requires knowledge of $W_N, W_{N+1}, \ldots$. M is a stopping time, N is not. In fact, M is the most important example of a stopping time, termed the first passage time to G.

The *(super)martingale property* of (sub)harmonic functions is fundamental.

Theorem 2.15. *Suppose $u : U_h \cup \partial U_h \to \mathbb{R}$ is subharmonic. Let $X_m = x + W_m$ be a random walk started at $x \in U_h$. If $X_m \in U_h$*

$$\mathbb{E}(u(X_{m+1} \,|X_m) \geq u(X_m). \tag{2.23}$$

Proof.

$$\mathbb{E}(u(X_{m+1} \,|X_m) = \frac{1}{2n} \sum_{|\omega|=1} u(X_m + h\omega) \geq u(X_m),$$

since u is subharmonic. □

Supermartingales are the probabilistic analog of increasing functions. If we have an upper bound, we can conclude that a limit exists. The deeper property, discovered by Doob and Hunt, is that Theorem 2.15 holds when m is replaced by a stopping time M. Let us call this the strong martingale property. The probabilistic solution to the Dirichlet problem is immediate: Fix $x \in U_h$, and consider a random walk $x+W_k$. Let M be the first passage time to ∂U_h. Suppose u is harmonic in $U_h \cup \partial U_h$. By the martingale property, $\mathbb{E}(u(X_M)) = u(x)$. Since $X_M \in \partial U_h$, this implies

$$u(x) = \mathbb{E}(f(x+W_M)). \tag{2.24}$$

The weak and strong maximum principles are easy consequences of this formula. The weak maximum principle is clear from the inequality $\min f \leq \mathbb{E}(f(x+W_M)) \leq \max f$. This ensures uniqueness of solutions. We compare (2.22) and (2.24) to obtain

$$\mathbb{E}(f(x+W_m)) = \sum_{y \in \partial U_h} H(x,y) f(y).$$

Therefore, $H(x,y)$ is the probability that a walk begun at x exits the domain at $y \in \partial U_h$. If U_h is connected, $H(x,y) > 0$ for every $x \in U_h$, $y \in \partial U_h$ (since there is a path connecting x and y) and the strong maximum principle follows. Yet another way of thinking about (2.24) is to treat the solution formula as a map $x \mapsto \nu_x$ where

$$\nu_x = \sum_{y \in \partial U_h} H(x,y) \delta_y,$$

is a probability measure concentrated on ∂U_h with $\nu_x(\{y\}) = H(x,y)$. This is called the exit measure for x, and the solution of the Dirichlet problem is $u(x) =< f, \nu_x >$.

Let me conclude this discussion of discrete equations, with some comments on numerical solutions. One may use any numerical method (for example, Gaussian elimination) to solve the linear equations (2.20) and (2.21). To make contact with the theory for Laplace's equation here are two other methods. The first is to run $N \gg 1$ random walks, $W_\cdot^{(l)}$ and average

$$u(x) = \mathbb{E}(f(x+W_M)) \approx \frac{1}{N} \sum_{l=1}^{N} f(x + W_{M_x}^{(l)}).$$

The following algorithm is called the *method of relaxation.* It is Perron's method in disguise. For any initial guess $u^{(0)}$ we constructs a sequences of iterates $u^{(l)} : U_h \cap \partial U_h \to \mathbb{R}$ defined by

$$u^{(l+1)}(x) = \frac{1}{2n} \sum_{|\omega|=1} u^{(l)}(x + h\omega).$$

It is a good exercise to check that $u^{(l)}$ converges to u.

2.9 Brownian motion

The passage from random walks on discrete grids $h\mathbb{Z}^n$ to the Brownian motion W_t on $\mathbb{R}^n$ involves measure theoretic subtleties, which are ignored here. What follows is a brief description of what is involved, and a sample of interesting results such as the following celebrated theorem of Kakutani: The solution to the Dirichlet problem on bounded regular domains is

$$u(x) = \mathbb{E}(f(x + W_{T_x})), \tag{2.25}$$

where $T_x = \inf\{t|x + W_t \in \partial U\}$. This is the simplest version of what is known as the Feynman-Kac formula. It allows us to view the solution to a PDE as an average over a sum of paths. Feynman's formula holds for the Schrödinger equation $iu_t = \triangle u$ and is more subtle than Kac's version for the heat equation.

The passage to the limit $h \to 0$ relies on two fundamental results in probability theory for random walks. It will suffice to consider $n = 1$, so that ω_k is a sequence of coin tosses $\{-1, +1\}$ (this assumption aids intuition, but is not necessary).

Theorem 2.16 (Central limit theorem).

$$\lim_{N\to\infty} P\left(a \le \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \omega_k \le b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\,dx, \quad a \le b \in \mathbb{R}. \tag{2.26}$$

There is a deeper 'path space' version of the same theorem. To a random walk on the grid $h\mathbb{Z}$ we associate a function $W^{(h)} : [0,1] \to \mathbb{R}$ defined by interpolating in the natural way. Let $h = N^{-1/2}$ for $m \in \mathbb{Z}_+$, and set

$$W^{(h)}(t) = h \sum_{k=1}^{[Nt]} \omega_k, \quad [Nt] = \sup\{l \in \mathbb{Z}\,|\,l \le Nt\}.$$

The central limit theorem quantifies $\lim_{h\to 0} P(a \le W^{(h)}(1) \le b)$. This is the probability that $W^{(h)}(1)$ passes through the 'gate' $[a, b]$ as $h \to 0$. A finer version of this is not hard to obtain. Think of a walker passing through a finite number of slalom gates $[a_j, b_j], 1 \leq j \leq M$ at times $0 = t_0 < t_1 < t_2 < \ldots < t_M \leq 1$. The central limit theorem also implies

$$\lim_{h\to 0} P(a_j \leq W^{(h)}(t_j) \leq b_j, 1 \leq j \leq M)$$
$$= \int_{a_N}^{b_N} \cdots \int_{a_0}^{b_0} \prod_{j=1}^{M} g(x_j - x_{j-1}, t_j - t_{j-1}) dx_i, \tag{2.27}$$

where $g(t,x)$ is related to the kernel $k(x,t)$ of the heat equation by a nuisance factor of 2

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} = k(x, t/2).$$

The limit (2.27) is called convergence of finite-dimensional distributions, but it is not enough. What we would really like is (i) a limiting path W_t with the slalom-gate property

$$P(a_i \leq W(t_i) \leq b_i, i = 1, \ldots, N) = \int_{a_N}^{b_N} \cdots \int_{a_0}^{b_0} \prod_{i=1}^{N} g(t_i - t_{i-1}, x_i - x_{i-1}) dx_i. \tag{2.28}$$

and (ii) convergence of the paths $W^{(h)} \to W$ in a sufficiently strong topology.

For example, (ii) is needed in the Dirichlet problem for the following reason. If we consider *functionals of the path* such as the first-passage time to a level $c \in \mathbb{R}$ (that is $T_c^{(h)} = \inf\{t \,\big|\, W^{(h)}(t) \geq a\}$), we would like $P(T_c^{(h)} \in [\alpha, \beta]) \to P(T_c \in [\alpha, \beta])$.

Both (i) and (ii) are true. A limiting path, the *Brownian motion* $W(t)$ (usually written W_t and called a 'process' by probabilists) was constructed by Norbert Wiener in 1923. More precisely, equation (2.28) is a prescription of a measure on a 'cylinder set' in the space $C([0,1])$. One uses these sets to obtain a Borel σ-algebra, $\mathcal{B}$, on $C([0,1])$. Wiener constructed a probability measure on the measure space $(C([0,1]), \mathcal{B})$ such that (2.28) holds. This measure has the amazing property that it is concentrated on continuous functions which are (a) nowhere differentiable, (b) Hölder continuous for any exponent $\alpha < 1/2$, and (c) oscillate wildly in the sense that

$$-1 = \liminf_{t\to 0} \frac{W(t)}{\sqrt{2t \log\log t}} \leq \limsup_{t\to 0} \frac{W(t)}{\sqrt{2t \log\log t}} = 1.$$

Property (ii) was obtained by Donsker in 1951.

Theorem 2.17 (Donsker's invariance principle). *The measures* $P^{(h)}$ *on* $(C([0,1]),\mathcal{B})$ *induced by the paths* $W^{(h)}$ *converge in the weak-* topology to the Wiener measure.*

Wiener's construction of Brownian motion is a watershed in probability theory. It gives firm meaning to our physical understanding that the random motion of particles gives rise to diffusion. The combination of a limiting measure (Wiener) and a convergence theorem (Donsker) is a central theme in modern probability theory.

With all this propaganda out of the way, I have to confess that it would takes use too far afield to prove Theorem 2.17. A good, even if occasionally pedantic, source for this material is [8]. However, here is a short proof of Theorem 2.16 which may explain why Donsker's theorem involves the weak-* topology (Note: probabilists use weak to mean weak-*).

Proof of Theorem 2.16. 1. Let $F(dx)$ be the measure that has a jump of size $1/2$ at ± 1. The assumption that ω_k are identically distributed coin-tosses is the assumption, that for all k, $P(a \leq \omega_k \leq b) = \int_{[a,b]} F(dx)$. The assumption that ω_k are independent is the statement $P(a \leq \sum_{k=1}^{N} \omega_k \leq b) = \int_{[a,b]} (F\star \ldots \star F)(dx)$ (N-fold convolution). Therefore, (2.26) is the statement that suitably rescaled measures $F_N(dx)$ converge weakly to $g(1,x)dx$.

2. A sequence of probability measures $F_N(dx)$ on $\mathbb{R}$ converges weak-* to a measure $F_*(dx)$ if and only if $\int_{\mathbb{R}} e^{-i\xi x} F_N(dx) \to \int_{\mathbb{R}} e^{-i\xi x} F_*(dx)$. (A proof is outlined in the HW). The Fourier transform of $g(1,x)dx$ is $e^{-\xi^2/2}$.

3. The convergence of Fourier transforms is more transparent (to me) in the following notation

$$\begin{aligned}
\mathbb{E}\left(\exp\left(-i\xi\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\omega_k\right)\right) &= \mathbb{E}\left(\prod_{k=1}^{N}\exp\left(-i\frac{\xi}{\sqrt{N}}\omega_k\right)\right) \\
&= \prod_{k=1}^{N}\mathbb{E}\left(\exp\left(-i\frac{\xi}{\sqrt{N}}\omega_k\right)\right) = \left(\cos\frac{\xi}{\sqrt{N}}\right)^N \to e^{-\xi^2/2},
\end{aligned}$$

as $N \to \infty$. The second equality may be taken as a definition of independence. The third equality is the assumption that all ω_k are identically distributed. □

2.10 The Feynman-Kac formula

Once one has a measure on $C([0,1])$ it can be extended to a measure on $C([0,\infty))$ by σ-additivity. Brownian motion in $\mathbb{R}^n$ is obtained by taking n

independent Brownian motions in $\mathbb{R}$. The cylinder sets are generated by cubes $[a_j, b_j]^n$ and (2.28) has the obvious extension to $\mathbb{R}^n$. Henceforth, we write $W_t : [0,\infty) \to \mathbb{R}^n$ to denote a Brownian motion with independent components $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$.

We first use Brownian motion to interpret solutions to the Cauchy problem for the heat equation

$$u_t = \frac{1}{2}\triangle u, \quad x \in \mathbb{R}^n, t > 0; \quad u(x,0) = f(x), \quad x \in \mathbb{R}^n,$$

for bounded, continuous initial data f. We then have $u(x,t) = \int_{\mathbb{R}^n} f(x-y)g(y,t)\,dy$. But (2.28) implies $g(y,t)\,dx = P(W_t \in [y, y+dy])$. Therefore,

$$\begin{aligned}\mathbb{E}(f(x+W_t)) &= \int_{\mathbb{R}^n} f(x+y)P(W_t \in [y, y+dy]) \\ &= \int_{\mathbb{R}^n} f(x+y)g(y,t)\,dy = \int_{\mathbb{R}^n} f(x-y)g(y,t)\,dy.\end{aligned}$$

Thus, we obtain

$$u(x,t) = \mathbb{E}\left(f(x+W_t)\right). \tag{2.29}$$

In short, we sum over all Brownian paths starting at x and running backwards in time till they hit the boundary $t = 0$.

Example 2.18. This formulation makes some problems very simple. For example [7][p.213], if $u_1, \ldots, u_n$ are solutions to $v_t = v_{ss}/2$, $s \in \mathbb{R}$, then the product $u(x,t) = \prod_{k=1}^n u_k(x_k, t)$ solves $u_t = \triangle u/2$, $x \in \mathbb{R}$. Let f_k denote $u_k(s,0)$, and $f(x) = \prod_{k=1}^n f_k(x_k)$. Since $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ we have

$$\prod_{k=1}^n u_k(x_k,t) = \prod_{k=1}^n \mathbb{E}\left(f_k(x_k + W_t^{(k)})\right) = \mathbb{E}\left(\prod_{k=1}^n f_k(x_k + W_t^{(k)})\right) = u(x,t).$$

The similarity between (2.25) and (2.29) is striking. As far as Brownian paths go, there is little difference between the heat equation and Laplace's equation. What is essential is only the right notion of a stopping time. To unify the two problems, let us suppose we are solving the initial boundary value problem for the heat equation on a cylinder $V_a = U \times (0,a)$ for fixed $a > 0$ and bounded, connected $U \subset \mathbb{R}^n$. Suppose continuous boundary data f is prescribed on $\partial_1 V$. Let $T_x = \inf\{t > 0 | x + W_t \in \partial_1 V\}$. Then

$$u(x,t) = \mathbb{E}\left(f(x + W_{T_x})\right).$$

The strong maximum principle is now obvious. However, this approach trades one difficulty for another. While we have a unified solution formula, explicit computation are the same as before. Moreover, it takes a lot of measure theory to nail down the notion of a stopping time (for example, the fact that T_x is well-defined under the minimal hypothesis that U is open and bounded).

But we gain a lot. For example, the following intuitive picture of boundary regularity emerges. To be concrete let us consider the Dirichlet problem for Laplace's equation and Kakutani's formula (2.25). Thus, for every continuous $f : \partial U \to \mathbb{R}$ we obtain a harmonic function given by (2.25). However, we know that the Dirichlet problem is not solvable unless every boundary point is regular. How should we understand regularity of boundary points? For example, an intuitive picture of Lebesgue's thorn emerges. Let U be a domain with a thorn at the origin (see Figure 2.1). Brownian paths that start within U cannot squeeze through the thorn and always exit before hitting 0. Therefore, if we prescribe boundary values f, the value $f(0)$ is never felt. Another such example is Littlewood's crocodile in Figure 2.2. Here the values of f on the left boundary cannot be felt by the Brownian motion. In complete generality, one has the following identifications [6]:

1. u defined by (2.25) is the Perron function.
2. A point $y \in \partial U$ is regular if and only if $P(T_y = 0) = 1$ (Brownian motion immediately exits U). This is *not* true for a point at a tip of a thorn.
3. For $y \in \partial U$, $P(T_y = 0) = 1$ if and only if Wiener's criterion (Theorem 1.50) holds.

(3) is an example of a $0-1$ law for a tail event. Such results typically arise in the following context. If $\{a_k\}_{k=1}^{\infty}$ is a sequence of independent random numbers, consider the sum $\sum_{k=1}^{\infty} a_k$. Convergence of this sum does not depend on any finite number of terms: $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=N}^{\infty} a_k$ converges for every $N \in \mathbb{Z}_+$. If A is the event that this series converges then A is a 'tail-event'. A very clever trick of Kolmogorov shows that every tail event A is independent of itself(!) so that $P(A)^2 = P(A)$, thus $P(A) = 0$ or 1. This may suggest some vague explanation for Wiener's criterion, (amazingly, Wiener's original proof is purely analytic, and simpler probabilistic proofs appeared much later).

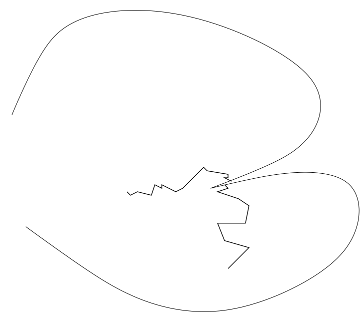

Figure 2.1: Lebesgue's thorn: Brownian motion started at the interior can never exit at the tip of the thorn..

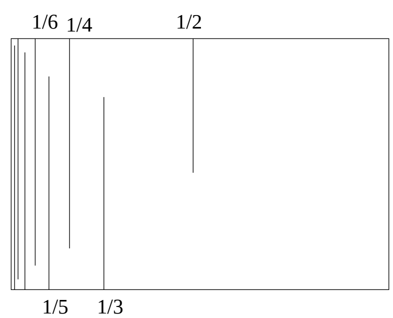

Figure 2.2: Littlewood's crocodile: a barrier of lenth $1 - 1/n$ is placed at $x = 1/n$. Brownian motion started in the interior can never reach the boundary $\{x = 0\}$.

2.11 Can one hear the shape of a drum?

The initial boundary value problem for the heat equation on a bounded domain can be solved by the method of separation of variables. A nonzero function $u : U \to \mathbb{R}$ is an eigenfunction for the Dirichlet problem with eigenvalues $\lambda \in \mathbb{R}$ if $-\triangle u = \lambda u$ in U and $u = 0$ on ∂U. The following theorem on eigenvalues of the Laplacian will be assumed here.

Theorem 2.19. *Suppose $U \subset \mathbb{R}^n$ is open, bounded and connected, and has C^1 boundary ∂U. There is a sequence of increasing eigenvalues*

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \dots \tag{2.30}$$

with $\lambda_k \to \infty$, and associated eigenfunctions u_k such that

$$\triangle u_k + \lambda_k u_k = 0, x \in U, \quad u_k(x) = 0, x \in \partial U.$$

Moreover, (a) $0 < u_1(x), x \in U$, (b) $\{u_k\}_{k=1}^{\infty}$ form a complete orthonormal basis for $L^2(U)$.

The uniqueness of λ_1 and positivity of λ_1 and u_1 is known as the Perron-Frobenius theorem. Clearly the eigenvalues depend on the domain. An interesting converse theorem, is the extent to which the domain depends on the eigenvalues. Physically, in $\mathbb{R}^2$, the eigenvalues are the frequencies of the modes of vibration of a membrane with boundary ∂U. Therefore, the question may be stated as follows: can one hear the shape of a drum? More precisely, does knowledge of the sequence $\{\lambda_k\}_{k=1}^{\infty}$ determine U?

I will outline Kac's elegant proof of a beautiful result of Weyl. To state Weyl's theorem, we need the *spectral measure* $A(d\lambda)$ defined by the increasing function

$$A(\lambda) = \sum_{k=1}^{\infty} \mathbf{1}_{\lambda_k \leq \lambda}. \tag{2.31}$$

This is the measure obtained by placing a unit mass at every λ_k.

Theorem 2.20 (Weyl).

$$\lim_{\lambda \to \infty} \frac{A(\lambda)}{\lambda^{n/2}} = \frac{|U|}{(2\pi)^{n/2}\Gamma(n/2)}. \tag{2.32}$$

Here $|U|$ is the n-dimensional volume of U. Thus, one can hear the volume of U! What we will actually prove is the equivalent assertion

Theorem 2.21 (Kac).

$$\lim_{t\to 0_+}(2\pi t)^{n/2}\sum_{k=1}^{\infty} e^{-\lambda_k t} = |U|. \tag{2.33}$$

The sum $h(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} = \int_0^\infty e^{-t\lambda}A(d\lambda)$, $t > 0$ is known as the Dirichlet series or Laplace transform of the spectral measure. The Dirichlet series is useful because it is tractable, invertible (ie. knowledge of $h(t)$ for $t \in (0,\infty)$ determines A), and the asymptotics of $h(t)$ as $t \to 0$ and ∞ determine the asymptotics of A as $\lambda \to \infty$ and 0 respectively.

It is easy to show that (2.20) implies (2.21). This is sometimes called an Abelian theorem. Assume (2.32). We integrate by parts to obtain

$$\int_0^\infty e^{-\lambda t}A(d\lambda) = t\int_0^\infty e^{-\lambda t}A(\lambda)d\lambda.$$

Rescale by $p = t\lambda$ to obtain

$$(2\pi t)^{n/2}\sum_{k=1}^{n} e^{-\lambda_k t} = (2\pi)^{n/2}\int_0^\infty e^{-p}t^{n/2}A(\frac{p}{t})\,dp.$$

The integrand converges poinwtise to a multiple of $p^{n/2}$ by (2.32), and the interchange of limits may be justified by the dominated convergence theorem ((2.32) also implies $A(\lambda) \le C_\varepsilon \lambda^{n/2+\varepsilon}$ for any $\varepsilon > 0$). The proof of Weyl's theorem from (2.33) is more interesting. This is called a Tauberian theorem (more precisely, the Hardy-Littlewood-Karamata Tauberian theorem or Karamata's theorem) . A half page proof can be found in [4, XIII.5], but this is by no means a trivial result.

Proof of Kac's theorem. 1. Let g_U denote the fundamental solution for the heat equation on U with Dirichlet boundary conditions.

$$\partial_t g_U = \triangle g_u, \quad g_U(x,y;0) = \delta_y(x), \quad g_U(x,y,t) = 0, \quad x \in \partial U.$$

This is also called the absorbing boundary condition: think of particles diffusing in U that are eaten at the boundary. Observe that we always have the monotonicity formula:

$$U \subset V \quad \Rightarrow g_U(x,t) \le g_V(x,y,t), \quad x,y \in U, t > 0$$

with strict inequality if $U \neq V$.

2. Kac's proof relies on a representation for the Green's function g_U in terms of the eigenfunctions u_k. For every $t>0$, $g_U \in L^2(U)$ and admits the representation

$$g_U(x,y,t) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} u_k(x).$$

The coefficients are found by orthonormality. For every $t>0$

$$\int_U g_U(x,y,t) u_l(x)\,dx = e^{-\lambda_k t} c_l.$$

and we may let $t \to 0$ to find that $u_l(y) = c_k$. To rigorously justify this we need to show that u_l is continuous in the interior of U. This is believable, and will be proved in Sem 2. Thus, we obtain the symmetric formula

$$g_U(x,y,t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} u_k(x) u_k(y).$$

3. We now choose $x=y$, integrate both sides over U, and use $\int_U u_k^2(x)\,dx = 1$ to obtain

$$\int_U g_U(x,x,t)\,dx = \sum_{k=1}^{\infty} e^{-\lambda_k t} = h(t).$$

Finally, Kac's insight is that for short time the particle cannot 'feel the boundary' and there should be little difference between $g_U(x,x,t)$ and $g_{\mathbb{R}^n}(x,x,t)$. We certainly have the upper bound

$$g_U(x,x,t) \leq g_{\mathbb{R}^n}(x,x,t) = \frac{1}{(2\pi t)^{n/2}}.$$

Therefore,

$$\limsup_{t\to 0} (2\pi t)^{n/2} h(t) \leq \int_U 1\,dx = |U|.$$

To obtain the lower bound, we fix $\varepsilon > 0$, and consider all points $x \in U$ such that $\operatorname{dist}(x, \partial U) > \varepsilon$. Call this set $U_{-\varepsilon}$. We then have the uniform estimate

$$g_U(x,x,t) \geq g_{B(x,\varepsilon)}(x,x,t) = g_{B(0,\varepsilon)}(0,0,t), \quad x \in U_{-\varepsilon}.$$

We then have

$$\liminf_{t\to 0} (2\pi t)^{n/2} h(t) \geq |U_{-\varepsilon}| \left(\lim_{t\to 0} (2\pi t)^{n/2} g_{B(0,0,\varepsilon)} \right) = |U_{-\varepsilon}|.$$

I leave the last step as an exercise (you do *not* need to solve explicitly, a scaling argument will do the trick). $\square$

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