On the Initial Value Problem for the Basic Equations of Hydrodynamics

Eberhard Hopf

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1 Introduction.

Let $x = (x_1, \ldots, x_n)$ denote a point in $\mathbb{R}^n$, and $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_n$ denote the volume element in $\mathbb{R}^n$. Let $u(x,t) : G \to \mathbb{R}^n$ be a time-dependent vector field defined on an open subset $\hat{G}$ of x-t-space with components $u_i, i = 1, \ldots, n$. G will be called a domain for brevity, though we do not assume that $\hat{G}$ is connected. Domains in x-space will be denoted G, in x-t-space they will be denoted $\hat{G}$.

A vector field $u(x,t)$ which is C^1 in x on an x-t-domain $\hat{G}$ is *divergence free* if it satisfies the differential equation

$$\operatorname{div} u = \frac{\partial u_i}{\partial x_i} = 0. \tag{1.1}$$

Here and throughout this paper we use the summation convention. There is also another characterization of divergence free vector fields that does not involve derivatives. We say that a scalar or vector-valued function $v(x,t)$ on $\hat{G}$ belongs to *class N on* $\hat{G}$ if and only if $v \equiv 0$ outside a suitable compact subset of this region. The functions of this class, thus vanish on a boundary strip of $\hat{G}$. The alternative characterization is as follows. A field $u(x,t)$ which is C^1 in x on $\hat{G}$ is called *divergence free* on $\hat{G}$ if

$$\iint_{\hat{G}} u_i \frac{\partial h}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{1.2}$$

for any function $h(x,t)$ of class N in $\hat{G}$ that is C^1 in x on $\hat{G}$. The equivalence of (1.1) and (1.2) is a consquence of Gauss' Theorem (which is applicable because $h \in N$ in $\hat{G}$) and because of the fundamental lemma of the calculus of variations. If we introduce the scalar product of two vector fields $v(x,t)$ and $w(x,t)$ on $\hat{G}$ as

$$\iint_{\hat{G}} v_i w_i\,\mathrm{d}x\,\mathrm{d}t,$$

we can say that "a field u which is C^1 in x is divergence-free in $\hat{G}$" means that u is orthogonal in $\hat{G}$ to the gradient of a function of class N that is C^1 in x. [1].

[1] The formulation of these terms in x-t-space rather than just in x-space is advantageous for our problem. Applications of Hilbert space theory can be found in the following works: O. Nikodym, Sur un théorème de M.S. Zaremba

The following counterpart of this fact is of interest here. A vector field $h' : \hat{G} \to \mathbb{R}^n$ which is continuous in x is a gradient field (that is, $h'_i = \partial h/\partial x_i$ for some $h : \hat{G} \to \mathbb{R}$ that is C^1 in x), if and only if h' is orthogonal to every divergence-free vector field that is C^1 in x in $\hat{G}$.

Necessity is once more a consequence of the Integral Theorem. Sufficiency is obtained as follows. By considering $w(x,t) = \varphi(t)\omega(x)$ with scalar φ we reduce to the corresponding claim for x-regions G. Thus, assume

$$\int_G w_i h'_i \,\mathrm{d}x = 0 \tag{1.3}$$

for any smooth divergence-free field $w(x)$ of class N in G. The claim follows if we can show that the circulation of the field h'

$$\int_{\mathfrak{C}} h'_i \,\mathrm{d}x_i = \int_{\mathfrak{C}} h'_s \,\mathrm{d}s, \tag{1.4}$$

vanishes along every closed path $\mathfrak{C}$ in G. It is easy to see that this needs to be shown only for continuously curved paths without self-intersections. We will obtain this vanishing through a suitable choice of fields w. For any given small $\varepsilon > 0$, there is a vector field $w(x)$ which is smooth and divergence-free in G and which has the following properties: w is non-zero only in a closed tube around $\mathfrak{C}$ of thickness $< \varepsilon$. On any plane tube section that cuts $\mathfrak{C}$ orthogonally, the vector w forms an angle $< \varepsilon$ with the normal direction (i.e. the direction of $\mathfrak{C}$ in the section). The sectional flow of w, which is independent of the exact shape of the section because w is divergence-free, is equal to 1. This fact suffices to prove that the circulation along $\mathfrak{C}$ vanishes. We consider such a field $w(x)$ for fixed (but sufficiently small) ε. If we let $\mathrm{d}F$ denote the hypersurface element on these tube sections and if we choose the arc length s along $\mathfrak{C}$ as the parameter transverse to the sections, we can write the volume element $\mathrm{d}x$ in the tube as $\rho(x)\,\mathrm{d}F\,\mathrm{d}s$, where we assume ρ to be continuous in a neighborhood of $\mathfrak{C}$ and equal to 1 on $\mathfrak{C}$. Then

$$\int h'_i w_i \,\mathrm{d}x = \int h'_w |w| \rho \,\mathrm{d}F\,\mathrm{d}s.$$

If we replace the component h'_w by the component h'_s taken at the intersection of $\mathfrak{C}$ with the section, $|w(x)|$ by the component $w_s(x)$ taken in a direction normal to $\mathrm{d}F$ and ρ by 1, then the right-hand side integral becomes

$$\int h'_s \left[\int w_s \,\mathrm{d}F\right] \mathrm{d}s = \int h'_s \,\mathrm{d}s,$$

i.e. the circulation. Based upon the properties of the field w noted above, we can meanwhile easily prove that that the error introduced by these approximations goes to zero with ε. The claim is proven.

The basic equations of Navier-Stokes equations for the movement of a homogeneous, incompressible liquid are

$$\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_\beta \partial x_\beta}, \tag{1.5}$$

where μ is a positive constant, namely the *kinematic viscosity coefficient* and

$$\operatorname{div} u = 0.$$

concernant les fonctions harmoniques. J. Math. pur appl., Paris, Sér. IX, **12** (1933), 95–109; J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta math., Uppsala **63** (1934), 193–248; H. Weyl, The method of orthogonal projection in potential theory. Duke math J. **7** (1940), 411–444.

Let $u(x,t), p(x,t)$ be a solution in an x-t-region $\hat{G}$ which we assume to be continuous along with all the occurring derivatives $u_t,\ u_x,\ u_{xx}$. We now introduce a new time-dependent vector field $a = a(x,t)$ which is divergence-free in $\hat{G}$. The only requirements on a are that is of class N in $\hat{G}$ and sufficiently smooth: a and the derivatives a_t, a_x, a_{xx} should be continuous in $\hat{G}$. Since $a \in N$ in G and

$$u_\alpha \frac{\partial u_i}{\partial x_\alpha} = \frac{\partial u_i u_\alpha}{\partial x_\alpha},$$

we have the identities

$$\begin{aligned}
\iint_{\hat{G}} a_i \frac{\partial u_i}{\partial t}\,\mathrm{d}x\,\mathrm{d}t &= -\iint_{\hat{G}} \frac{\partial a_i}{\partial t} u_i\,\mathrm{d}x\,\mathrm{d}t,\\
\iint_{\hat{G}} a_i u_\alpha \frac{\partial u_i}{\partial x_\alpha}\,\mathrm{d}x\,\mathrm{d}t &= -\iint_{\hat{G}} \frac{\partial a_i}{\partial x_\alpha} u_\alpha u_i\,\mathrm{d}x\,\mathrm{d}t,\\
\iint_{\hat{G}} a_i \frac{\partial^2 u_i}{\partial x_\beta \partial x_\beta}\,\mathrm{d}x\,\mathrm{d}t &= -\iint_{\hat{G}} \frac{\partial a_i}{\partial x_\beta}\frac{\partial u_i}{\partial x_\beta}\,\mathrm{d}x\,\mathrm{d}t = \iint_{\hat{G}} \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since $\operatorname{div} a = 0$ and $a \in N$, we also have

$$\iint_{\hat{G}} a_i \frac{\partial p}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Therefore, we find that the field $u(x,t)$ satisfies the following condition

$$\iint_{\hat{G}} \frac{\partial a_i}{\partial t} u_i\,\mathrm{d}x\,\mathrm{d}t + \iint_{\hat{G}} \frac{\partial a_i}{\partial x_\alpha} u_\alpha u_i\,\mathrm{d}x\,\mathrm{d}t + \mu \iint_{\hat{G}} \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta} u_i\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{1.6}$$

for any sufficiently smooth field $a(x,t)$ on $\hat{G}$ with the properties

$$\operatorname{div} a = 0 \ \text{ in } \hat{G}, \quad a \in N \ \text{ in } \hat{G}. \tag{1.7}$$

In addition, we need to take into account that u is divergence-free, that is,

$$\iint_{\hat{G}} \frac{\partial h}{\partial x_i} u_i\,\mathrm{d}x\,\mathrm{d}t = 0, \quad h \in N \ \text{ in } \ \hat{G}. \tag{1.8}$$

These identities hold for sufficiently smooth functions in the respective classes. We have thus reduced the basic equations to the form of equations between linear functional operators of arbitrary fields and functions a and h. The essential part of this is that the unknown field u on which these operators depend occurs without any derivatives.

We still need to convince ourselves that we may revert from equations (1.6) and (1.8) to the differential form (1.5) of the equations if we restrict ourselves to sufficiently smooth solutions u. We already know that under this assumption (1.8) implies $\operatorname{div} u = 0$ in $\hat{G}$. For a sufficiently smooth u, we may undo all the integrations-by-parts. It then follows that

$$\iint_{\hat{G}} a_i \left\{ \frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} - \mu \frac{\partial^2 u_i}{\partial x_\beta \partial x_\beta} \right\} \mathrm{d}x\,\mathrm{d}t$$

holds for every sufficiently smooth field $a(x,t)$ of the form (1.7). Using the theorem proved above, we may conclude that the term in curly braces is the gradient of a uniquely determined function

$p(x,t)$. Thus, the differential equations of motion (1.5) must hold in $\hat{G}$. We see that this integral form of the equations exactly expresses the physical demand that the pressure be unique.

It is natural to build the general mathematical theory on the integral form of the equations. But then it is appropriate to rid ourselves of the artificial restriction to smooth solution fields u. The occurrence of the quadratic forms

$$\int u_i u_i \, \mathrm{d}x, \quad \int \frac{\partial u_i}{\partial x_\beta} \frac{\partial u_i}{\partial x_\beta} \, \mathrm{d}x$$

in the energy equation leads us to base the problem on a Hilbert space of vector fields. It is a methodical advantage that in this broader framework the problem of regularity of solutions u can be entirely separated from the problem of existence.[2]

The common initial value problem of the basic equations of hydrodynamics is the following. We need to find the solution $u(x,t)$ in a prescribed, moving region $G(t)$ $(t \geqslant 0)$ of x-space, while $u(0)$ in $G(0)$ is prescribed (together with a suitably formulated condition of continuous continuation for $t \to 0$) and the boundary values at the boundary of $G(t)$, $t > 0$ are also given (with a suitably formulated sense of continuation). J. Leray dedicated three sizable works to this problem in the early 1930s[3]. These inquiries had already forced Leray to use the methods of Hilbert space and the integral interpretation of the equations in three dimensions[4]. In his works, Leray solved the question of existence for all $t > 0$ in the following cases, a) $G = \mathbb{R}^2$ under the added condition of finite kinetic energy, b) G is a fixed oval with zero boundary values, c) $G = \mathbb{R}^3$ under the added condition of finite kinetic energy. The remarkable analysis that Leray dedicates to the question of regularity point to a strange difference between the dimensions $n = 2$ and $n > 2$. If G is the entire plane, the proof of infinite differentiability is successful, but the methods that one should view as natural fail for $n \geqslant 3$. Even for arbitrary smoothness of all prescribed data, the proof of smoothness of the solution did not work out. The other strange thing is the failure of the uniqueness proof in three dimensions. These questions are still not answered satisfactorily. It is hard to believe that the initial value problem of viscous liquids for $n = 3$ should have more than one solution, and more attention should be paid to the settling of the uniqueness question. However, recent research indicates that for nonlinear partial differential problems the number of independent variables has significant influence on the local properties of solutions.

The present work is dedicated to the initial value problem with the integral form of the equations viewed as their primary form. We leave aside the questions of regularity and uniqueness. We hope to come back to these things as well as to the proof of the energy equation (which is easy in our context) in later memoranda. The main point of this work is that the construction of approximate solutions (that takes such broad space in Leray's work) may be replaced by a simpler process, applicable to a much broader classes of partial differential problems. We also hope to come back to this issue later. This method enables the solution of the initial value problem for all $t > 0$ in substantial generality. However, in this article what matters more to us is the exposition of the basic method, rather than the generality of the results. We restrict ourselves to the case that

[2] Compare the treatment of quadratic variation and linear differential problems by methods of Hilbert spaces in R. Courant and D. Hilbert, Methoden der mathematischen Physik, Volume 2, Berlin 1937, Chapter VII.

[3] J. Leray, a) Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'Hydrodyamique. J.Math.pur.appl. Paris, Sér. IX **12** (1933) 1–82; b) Essay sur les mouvements plans d'un liquide visqueux que limitent des parois. c) loc. cit. in footnote 1.

[4] A long while before this, C.W. Oseen had based his well-known hydrodynamic inquiries on a form of the basic equations that is free of second derivatives. However, he only succeeded in proving existence for sufficiently small times. Cf. his work Hydrodynamik (Leipzig 1927)

the x-region G is fixed in time, but otherwise completely arbitrary, and where u has vanishing boundary values. The boundary condition will be defined in terms of Hilbert space–broad enough to guarantee solvability, and narrow enough to guarantee the uniqueness of the solution, at least in two dimensions[5]. In pure existence theory, the space dimension n will not play any role.

2 The function space H'. Solutions of class H'.

We define the class H with respect to an x-t-region $\hat{G}$ to mean the space of all measurable functions $f : \hat{G} \to \mathbb{R}$ with finite norm

$$\iint_{\hat{G}} f^2 \, \mathrm{d}x \, \mathrm{d}t.$$

H is a real Hilbert space. We will mean weak and strong convergence in with respect to this norm in what follows. Recall that a sequence of functions $f \in H$ converges weakly if first, the norms of all f remain below a fixed value and second, if

$$\iint_{\hat{G}} f g \, \mathrm{d}x \, \mathrm{d}t \to \iint_{\hat{G}} f^* g \, \mathrm{d}x \, \mathrm{d}t$$

holds for any fixed function $g \in H$. While maintaining the first condition, the second one may be weakened to the effect that the sequence of numbers

$$\iint_{\hat{G}} f g \, \mathrm{d}x \, \mathrm{d}t$$

converges for any fixed g in a set that is strongly dense in H. Then there exists one, and essentially only one weak limit function f^* in $\hat{G}$. Here we have used an x-t-region. We will also use the same terms for a purely spatial x-region G. In this case, we will base our considerations on the norm

$$\int_G f^2 \, \mathrm{d}x.$$

We remind the reader of that a sequence of functions with uniformly bounded norms is weakly compact (F. Riesz's Theorem). The following criterion for strong convergence, used extensively by Leray, will also be necessary here. For a sequence of functions that converges weakly in $\hat{G}$ to a limit function f^*, we have

$$\overline{\lim} \iint_{\hat{G}} f^2 \, \mathrm{d}x \, \mathrm{d}t \geqslant \iint_{\hat{G}} (f^*)^2 \, \mathrm{d}x \, \mathrm{d}t,$$

where equality holds if and only if $f \to f^*$ in the strong sense. All these notions transfer to vector fields u, v on $\hat{G}$ if we use the scalar product

$$\iint_{\hat{G}} u_i v_i \, \mathrm{d}x \, \mathrm{d}t$$

and the corresponding norm.

[5]If G is the entire x-space, the boundary condition thus phrased becomes the condition of finite kinetic energy and finite dissipation integral.

The phrasing of the boundary condition is suggested by the work of R. Courant and D. Hilbert, Methoden der mathematischen Physik, Vol. 2, Berlin 1937, Chap. VII, 1, 3rd section.

Lemma 2.1 *If the vector fields $u(x,t)$ converge weakly in $\hat{G}$ to a limit field $u^*(x,t)$, then*

$$\overline{\lim} \iint_{\hat{G}} u_i u_i \,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_{\hat{G}} u_i^* u_i^* \,\mathrm{d}x\,\mathrm{d}t.$$

Equality holds if and only if the convergence in $\hat{G}$ is strong.

Like Leray, we need generalized (purely spatial) derivatives of functions $f(x,t)$ and fields $u(x,t)$.

Definition 2.1 *A function $f : \hat{G} \to \mathbb{R}$ belongs to the space H' if and only if it has the following properties: f belongs to H in $\hat{G}$, and there exist n functions denoted $f_{\prime,i}$ in H such that*

$$\iint_{\hat{G}} h f_{\prime,i} \,\mathrm{d}x\,\mathrm{d}t = -\iint_{\hat{G}} \frac{\partial h}{\partial x_i} f \,\mathrm{d}x\,\mathrm{d}t, \quad i = 1, 2, \ldots, n, \tag{2.1}$$

for every function $h(x,t)$ which is continuous in $\hat{G}$ along with its derivatives and which belongs to class N.

The class H' obviously contains any f that is C^1 in x such that f and all $\partial f/\partial x_i$ belong to H in $\hat{G}$. For such an f, we have $\partial f/\partial x_i = f_{\prime,i}$. This follows from the integral theorem and the demand that h must belong to N, that is that h vanishes outside a certain compact subset of $\hat{G}$. Obviously, generalized derivatives $f_{\prime i}$ in G are uniquely determined except for the values on an x-t set of measure zero in the case of $f \in H'$.

Lemma 2.2 *If a sequence of functions in H' converge weakly to f^* and for all f in the sequence the expressions*

$$\iint_{\hat{G}} f^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_{\hat{G}} f_{\prime i} f_{\prime i} \,\mathrm{d}x\,\mathrm{d}t$$

are uniformly bounded, then f^* also belongs to H' in $\hat{G}$ and every x-derivative $f_{\prime i}$ converges weakly to the corresponding x-derivative $f^*_{\prime i}$.

Proof Every f satisfies (2.1), where h is an arbitrary admissible function. The right hand sides converge to

$$-\iint_{\hat{G}} \frac{\partial h}{\partial x_i} f^* \,\mathrm{d}x\,\mathrm{d}t.$$

For a fixed h and i, the left hand sides converge along the sequence of the f's. The set of admissible functions h is strongly dense in the Hilbert space H. Thus, for any fixed i the sequence of the $f_{\prime i}$ is weakly convergent. If we let f_i^* denote the limit function, then from (2.1), we conclude that

$$\iint_{\hat{G}} h f_i^* \,\mathrm{d}x\,\mathrm{d}t = -\iint_{\hat{G}} \frac{\partial h}{\partial x_i} f^* \,\mathrm{d}x\,\mathrm{d}t$$

holds for any admissible h and i. By Definition 2.1, f^* belongs to H' in $\hat{G}$, and because of uniquness of the x-derivative, we have $f_i^* = f^*_{\prime i}$. □

We say a vector field is of class H' in $\hat{G}$ if this is the case for all components. There are no derivatives on u in the integral form of the basic equations (1.6)–(1.8). However, it is practical to make a weak differentiability assumption like membership in the class H' on the solutions u. We may then write for the friction term in (1.6)

$$\mu \iint_{\hat{G}} \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta} u_i \ \mathrm{d}x \ \mathrm{d}t = -\mu \iint_{\hat{G}} \frac{\partial a_i}{\partial x_\beta} u_{i,\beta} \ \mathrm{d}x \ \mathrm{d}t. \tag{2.2}$$

Definition 2.2 *A field $u : \hat{G} \to \mathbb{R}^n$ is a solution of class H' of the basic equations of hydrodynamics if it satisfies the following conditions:*

a) *$u \in H'$ in $\hat{G}$.*

b) Vanishing divergence: *any function h which is of class N in $\hat{G}$ and C^1 in x satisfies the relation (1.8).*

c) Equations of motion: *any field $a(x,t)$ that is of class N in $\hat{G}$, divergence-free and continuous along with its derivatives a_t, a_x, a_{xx} satisfies the relation (1.6).*

Observe that under the condition a) the term in the basic equations (1.6) which is nonlinear in u is a valid Lebesgue integral for any admissible field a. This is already the case if $u \in H$ in $\hat{G}$. The condition of incompressibility (b) is equivalent with

$$\operatorname{div} u \equiv u_{i,i} = 0$$

for a.e. $(x,t) \in \hat{G}$ because of assumption (a).

We will define all integrands in the basic equations (1.6) outside of $\hat{G}$ to be zero. The integrals can then be extended over all x-t-space. With this convention, the following theorem, which we would like to prove here even though it is not needed in this paper, holds:

Theorem 2.1 *A solution of class H' satisfies the equation*

$$\int_{t=\tau} a_i u_i \,\mathrm{d}x = \int\int_{t<\tau} \frac{\partial a_i}{\partial t} u_i \,\mathrm{d}x \,\mathrm{d}t + \int\int_{t<\tau} \frac{\partial a_i}{\partial x_\alpha} u_\alpha u_i \,\mathrm{d}x \,\mathrm{d}t - \mu \int\int_{t<\tau} \frac{\partial a_i}{\partial x_\beta} u_{i,\beta} \ \mathrm{d}x \ \mathrm{d}t \tag{2.3}$$

for a.e. value of τ.

Proof Observe that $h(t)a(x,t)$ is also an admissible field if $h(t)$ is an arbitrary C^1 function of t. If we replace a by ha in equation (1.6), which we abbreviate as,

$$\int\int K[a,u] \,\mathrm{d}x \,\mathrm{d}t = \int_{-\infty}^{\infty} \left\{ \int_{t=\tau} K[a,u] \right\} \mathrm{d}\tau = 0,$$

it follows that the equation

$$\int_{-\infty}^{\infty} h(\tau) \left\{ \int_{t=\tau} K \,\mathrm{d}x \right\} \mathrm{d}\tau + \int_{-\infty}^{\infty} h'(\tau) \left\{ \int_{t=\tau} a_i u_i \,\mathrm{d}x \right\} \mathrm{d}\tau = 0 \tag{2.4}$$

is also satisfied. The terms in curly braces are Lebesgue-integrable functions of τ on $-\infty < \tau < \infty$ that vanish for all large $|\tau|$. The validity of (2.4) for abritray $h(\tau)$ with continuous $h'(\tau)$ is equivalent to the fact that

$$\int_{t=\tau} a_i u_i \,\mathrm{d}x = \int_{-\infty}^{t} \left\{ \int_{t \text{ fixed}} K \,\mathrm{d}x \right\} \mathrm{d}t = \int\int_{t<\tau} K \ \mathrm{d}x \ \mathrm{d}t$$

for a.e. τ. □

In (2.3), the left hand side is defined for just a.e. τ, while the right hand side is an absolutely continuous function of τ. In fact, one can prove that a solution of class H' in $\hat{G}$ can be changed on a set of x-t measure zero such that the new u satisfies (2.3) without exception, i.e. for any admissible a and any τ. But we will not elaborate further on this here.

3 The no-slip boundary condition. The initial value problem

The cross sections $t = \text{const}$ of the x-t-region $\hat{G}$ are x-regions $G(t)$. By using only terms of the Hilbert space, we need to get as close as possible to the boundary condition that a function $g(x,t)$ and a field $u(x,t)$ vanish on the boundary of $G(t)$ for all t. This can be achieved by obtaining g from functions of class N in $\hat{G}$ through a limit process. In doing so, it is necessary to use sufficiently effective bounds on the spatial x-derivatives (but not on the t-derivatives) of the approximating functions, so that the "vanishing" remains intact along the boundaries of the x-regions $G(t)$. We express the boundary condition by membership in the following function class $H'(N)$.

Definition 3.1 *A function $g(x,t)$ is said to be of class $H'(N)$ in $\hat{G}$ if it is the weak limit of a sequence of functions $\gamma(x,t)$, which are C^1 in x, belong to N in $\hat{G}$, and for which the expressions*

$$\iint_{\hat{G}} \gamma^2 \, \mathrm{d}x \, \mathrm{d}t + \iint_{\hat{G}} \gamma_{'i}\gamma_{'i} \, \mathrm{d}x \, \mathrm{d}t \tag{3.1}$$

are uniformly bounded.[6]

It follows from Lemma 2.2 that for a given x-t-region G the class $H'(N)$ is contained in the class H'.

Lemma 3.1 *Let $\hat{G}$ by a cylinder set $x \subset G$, $0 < t < T$. Let $g(x,t)$ be the weak limit in $\hat{G}$ of a sequence of functions $\gamma(x,t)$, C^1 in x, that are of the following kind. For each γ there is a compact subset of the x-region G such that γ vanishes for x outside that set. The integrals* (3.1) *are uniformly bounded for γ in the sequence. Then g belongs to $H'(N)$.* [7]

Proof Observe the difference between the class of γ admissible in this lemma and the narrower class of γ of Definition 3.1. Membership of γ in N in the x-t-region $\hat{G}$ in the present case requires that γ vanishes sufficiently close to $t = 0$ and $t = T$. But since only x-derivatives occur in (3.1), this difference is inconsequential. If we replace the present γ by functions $\varphi(t)\gamma(x,t)$, where φ is continuous in $(0,T)$ and

$$\varphi = \begin{cases} 0 & \text{for } 0 < t < \varepsilon,\ T - \varepsilon < t < T, \\ 1 & \text{for } 2\varepsilon < t < T - 2\varepsilon, \end{cases}$$

[6]Cf. Courant-Hilbert, l.c. footnote 5, p. 218. The definition of the boundary condition of vanishing given there is only seemingly stronger than ours. By S. Saks' Theorem the sequence of arithmetic means of a weakly convergent sequence has a strongly convergent subsequence. It follows from this theorem and from Lemma 2.2 that for any g in $H'(N)$, there exists a sequence of functions γ of the above-mentioned kind such that

$$\gamma \to g, \quad \gamma_{'i} \to g_{'i}$$

holds in the strong sense.

[7]If G is $\mathbb{R}^n$, the class $H'(N)$ coincides with the class H'. In this case the admissible γ are strongly dense in the function space H' in the sense of the norm (3.1).

and otherwise $0 < \varphi < 1$ ($\varepsilon \to 0$), then Definition 3.1 applies to the new $\tilde{\gamma} = \varphi\gamma$. Thus g belongs to $H'(N)$. □

Lemma 3.2 *The relations*

$$\iint_{\hat{G}} g_{'i} f \ \mathrm{d}x \ \mathrm{d}t = -\iint_{\hat{G}} g f_{'i} \ \mathrm{d}x \ \mathrm{d}t \quad (i = 1, 2, \ldots, n)$$

are satisfied by any f of class H' in $\hat{G}$ and any g of class $H'(N)$ in $\hat{G}$.

Proof By Definition 2.1, the relations hold for any specified f and for any γ that is C^1 in x and of class N in $\hat{G}$. By Definition 3.1, g is a weak limit of a sequence of such γ with uniformly bounded integrals (3.1). In addition to $\gamma \to g$, we also have $\gamma_{'i} \to g_{'i}$ weakly in $\hat{G}$ by Lemma 2.2. The relations that hold for f, γ thus also hold for f, g. □

To facilitate a more convenient form of the initial condition, we also introduce the class $H(N)$. In doing so, we restrict ourselves to x-space and vector fields $u(x)$ that are defined in an x-region G. If we only consider functions $f(x)$ that belong to both the classes H and N, then it is clear that the strong closure of these sets of functions is identical to H. The same is true of vector fields in G. However, a difference arises if we restrict ourselves to divergence-free fields in G.

Definition 3.2 *A divergence-free vector field in G of class H is said to be of class $H(N)$ if it is a weak limit of C^2, divergence-free vector fields that belong to N.*[8]

One easily proves the following: If the field $u(x)$ is divergence-free and of class $H(N)$ and if the function $\varphi(x)$ is of class H', then

$$\int_G u_i \varphi_{'i} \, \mathrm{d}x = 0.$$

Membership of a divergence-free field in $H(N)$ obviously replaces the boundary condition of vanishing on the normal component.

We may now state the existence theorem for the hydrodynamic initial value problem.

Theorem 3.1 (Existence theorem) *Let G be an open subset of $\mathbb{R}^n$. Let the field $U(x)$ be divergence-free in G and of class $H(N)$, but otherwise arbitrary. Then there is a field $u(x,t)$ defined for all $t > 0$ in G with the following properties:*

A. In any x-t-cylinder region $x \subset G$, $0 < t < T$, u is a solution of class H' of the basic equations of hydrodynamics (cf. Definition 2.2).

B. "Vanishing of the boundary values" for $t > 0$: u belongs to $H'(N)$ in every cylinder regions.

C. Initial condition: As $t \to 0$, $u(\cdot, t) \to U$ strongly.

[8]By Saks' Theorem, it is then also the strong limit of just these vector fields.

4 Simplification of the problem. The approximation procedure.

To construct solutions of the initial value problem for an x-region G constant in time, we start with the equation

$$\int_G a_i u_i \,\mathrm{d}x\Big|_{t=\tau'} - \int_G a_i u_i\Big|_{t=\tau} = \tag{4.1}$$
$$\int_\tau^{\tau'}\int_G \frac{\partial a_i}{\partial t} u_i \ \mathrm{d}x\ \mathrm{d}t + \int_\tau^{\tau'}\int_G \frac{\partial a_i}{\partial x_\alpha} u_\alpha u_i\ \mathrm{d}x\ \mathrm{d}t + \mu\int_\tau^{\tau'}\int_G \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta} u\ \mathrm{d}x\ \mathrm{d}t.$$

Lemma 4.1 *Assume $u : G\times(0,\infty)\to\mathbb{R}^n$ belongs to class H in every cylinder set $x\subset G$, $0<t<T$. Let it satisfy equation (4.1) for all $\tau'>\tau>0$ and for any C^2 vector field a such that*

$$a = a(x), \quad \operatorname{div} a = 0 \ \text{ in } G, \quad a\in N \ \text{ in } G. \tag{4.2}$$

Then u satisfies the basic equation (1.6) for the semi-infinite cylinder $\hat{G}$: $x\subset G$, $t>0$ and for any field admissible there (c.f condition (c) in the definition 2.2 of a weak solution).

Proof If we write (4.1) in the abbreviated form

$$f(\tau') - f(\tau) = \int_\tau^{\tau'} g(t),$$

we see that the equation

$$\int_0^\infty \varphi'(t) f(t)\,\mathrm{d}t + \int_0^\infty \varphi(t) g(t)\,\mathrm{d}t = 0$$

must be satisfied for any φ that is continuously differentiable in $(0,\infty)$ and which vanishes for all sufficiently small and large t. If we once more write the equation out in full, we recognize that equation (1.6) is satisfied in the semi-infinite cylinder by any field $a=\varphi(t)a(x)$, where $a(x)$ is a C^2 vector field satisfying (4.2), and $\varphi(t)$ is one of the arbitrary functions permitted above. But now any $a(x,t)$ permitted by condition (c) in the definition 2.2 of a solution may be approximated in the semi-infinite cylinder $\hat{G}$ by sums of fields of such special shape that in the basic equation (1.6) integration and limit may be interchanged. For example, one could always arrange that the convergence of the fields and their derivatives up to a prescribed order in $\hat{G}$ is uniform and that the approximating fields all vanish outside a fixed compact subset of $\hat{G}$.

It is thereby clear that a field $u(x,t)$ which satisfies (4.1) to the extent specified in the lemma, and which is further divergence-free and which belongs to class H' in any cylinder section satisfies the full scope of the definition 2.2 of a solution on any cylinder section. □

The following lemma yields an even better basic equation.

Lemma 4.2 *There is a sequence a^ν of C^2, linearly independent vector fields in G satisfying (4.2),*

$$a = a^\nu(x), \quad \operatorname{div} a^\nu = 0 \ \text{in } G, \quad a^\nu \in N \ \text{ in } G, \tag{4.3}$$

with the following property. An arbitrary C^2 vector field field in G of the form (4.2) is the uniform limit in G of a sequence of finite linear combinations of the field $a^\nu(x)$, with uniform convergence of the derivatives up to second order in G. For a given $a(x)$, only such linear combinations occur in this approximation that have the value zero outside a certain compact subset of G which only depends on a.

Based on this Lemma, it is clear that a vector field $u(x,t)$ which is of class H in each cylinder section and which satisfies the basic equation (4.1) for all $\tau' > \tau > 0$ and for any field a^ν of the dense sequence automatically satisfies (4.1) for all vector fields a admitted above. In summary, the basic equations (1.6) can be replaced by the equations (4.1) with (4.3).

Equations (4.1) and (4.3) yield an affine coordinate representation of the basic equations of hydrodynamics in the space of divergence-free vector fields. The affine system of coordinate vectors (4.3) can be transformed into an orthogonal system in the sense of the bilinear form

$$\int_G v_i w_i \,\mathrm{d}x$$

through a simple linear transformation. We may also assume that the sequence (4.3) verifies the condition

$$\int_G a_i^\lambda a_i^\nu \,\mathrm{d}x = \delta_{\lambda,\nu}. \tag{4.4}$$

Lemma 4.3 *The orthonormal system $a^\nu(x)$ is complete in the space of divergence-free fields $U(x)$ of class $H(N)$ in G.*

The proof results from Definition 3.2 and Lemma 4.2.

The Approximation Procedure. The kth approximation step consists simply of considering only the first k equations of the infinitely many basic equations (4.1), (4.3),

$$a = a^\nu(x) \quad (\nu = 1, 2, \ldots, k) \tag{4.5}$$

and trying to solve those through the ansatz

$$u = u^k(x,t) = \sum_{\nu=1}^{k} \lambda_\nu(t) a^\nu(x), \tag{4.6}$$

with scalar factors $\lambda_\nu = \lambda_\nu^k$ to be determined. This ansatz is automatically divergence-free and satisfies the non-slip boundary condition because of (4.3). That is,

$$\operatorname{div} u^k = 0 \text{ in } G, \quad u^k \in N \text{ in } G. \tag{4.7}$$

Since only differentiable $\lambda(t)$ need to be considered and since the admissible fields a do not depend on t, the first k equations (4.1) may be written in the form

$$\int_G a_i \frac{\partial u_i}{\partial t} \,\mathrm{d}x = \int_G \frac{\partial a_i}{\partial x_\alpha} u_\alpha u_i \,\mathrm{d}x + \mu \int_G \frac{\partial^2 a_i}{\partial x_\beta \partial x_\beta} u_i \,\mathrm{d}x. \tag{4.8}$$

By orthonormality (4.4), the k equations (4.8), (4.5) together with (4.6) yield a system of ordinary differential equations

$$\frac{\mathrm{d}\lambda_\nu}{\mathrm{d}t} = F_\nu(\lambda_1, \ldots, \lambda_k) \quad (\nu = 1, 2, \ldots, k) \tag{4.9}$$

for the λ, in which the right hand sides $F_\nu = F_\nu^k$ are polynomials in λ with constant coefficients. The equations (4.5), (4.6), and (4.8), or the equivalent equations (4.9), share with the strict hydrodynamic equations the important property that for their solutions, the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_G u_i u_i \,\mathrm{d}x = -\mu \int_G \frac{\partial u_i}{\partial x_\beta} \frac{\partial u_i}{\partial x_\beta} \,\mathrm{d}x \tag{4.10}$$

holds. Namely, since the equations (4.8) hold for all fields (4.5), they also hold for their linear combinations (4.6) $u = u^k$. The energy equation follows in the usual way (and without difficulties at the boundary) since because of (4.7)

$$\int_G \frac{\partial u_i}{\partial x_\alpha} u_\alpha u_i \,\mathrm{d}x = \int_G \frac{\partial K}{\partial x_\alpha} u_\alpha \,\mathrm{d}x = 0, \quad \left(K = \frac{1}{2} u_i u_i\right)$$

and

$$\int_G \frac{\partial^2 u_i}{\partial x_\beta \partial x_\beta} u_i \,\mathrm{d}x = -\int_G \frac{\partial u_i}{\partial x_\beta} \frac{\partial u_i}{\partial x_\beta} \,\mathrm{d}x \quad (u = u^k).$$

It follows from (4.10) that

$$\int_G u_i u_i \,\mathrm{d}x = \lambda_1^2 + \cdots + \lambda_k^2 \quad (u = u^k)$$

never increases. From this we conclude that any solution of the differential system (4.9) begun at $t = 0$ exists for all $t > 0$.

The approximation procedure may very easily be interpreted formally in the following manner. We think of both sides of the Navie-Stokes differential equations and the solution u formally as if they were expanded in the orthonormal system of the fields a^ν: $u = \lambda_\nu a^\nu$. We then obtain purely formally a system of infinitely many differential equations of first order for the infinitely many scalar Fourier coefficients λ. Our kth step then simply consists of only considering the first k of these equations and setting all unknowns with indices $\nu > k$ to zero. The way in which we subsequently prove our existence theorem simultaneously yields a statement regarding the convergence properties of this simplest and most natural approximation method.

We choose the initial values of the $\lambda_\nu(t)$ at $t = 0$ to be the Fourier coefficients of the expansion of the given field $U(x)$ in the a^ν. While the solutions $\lambda(t)$ in the kth step generally depend on k, these initial values are independent of them. By the assumption that $U \in H(N)$ and by the completeness lemma 4.3, we have

$$u_k(\cdot, 0) \to U(\cdot) \quad \text{strongly} \quad (k \to \infty). \tag{4.11}$$

5 Proof of the Existence Theorem.

We summarize the properties of the sequence of vector fields which we will need in the following:

a) Each u^k is C^2 in $\hat{G}$ and divergence-free for $x \subset G$, $t > 0$.

b) $u^k(x, t)$ vanishes if x lies outside a compact subset of the x-region G that only depends on k.

c) $u^k(x, t)$ satisfies the equation (4.8) ($t \geqslant 0$) and the equation (4.1) ($\tau' > \tau \geqslant 0$) in the k cases (4.3) ($\nu = 1, 2, \ldots, k$).

d) The integrals

$$\int_G u_i u_i \,\mathrm{d}x, \quad \int_0 \int_G \frac{\partial u_i}{\partial x_\beta} \frac{\partial u_i}{\partial x_\beta} \,\mathrm{d}x \,\mathrm{d}t, \quad (u = u^k(x, t))$$

are bounded independent of k, t, and T.

e) The initial values $u^k(x,0)$ satisfy the limit relationship (4.11).

Observe that the uniform bound, 5(d) above, follows immediately from the time-integrated energy equation (4.10) and (4.11).

First step. Each field $a^\nu(x)$ is continuous in G and different from zero only in a compact subset of G. If we apply the first half of (d) to the right hand side of (4.8) ($a = a^\nu$) by estimating the term linear in $u = u^k$ by means of the Schwarz Inequality and the term quadratic in u by means of an absolute bound for the derivatives of a, we obtain the following: The right hand side of (4.8) ($a = a^\nu$, $u = u^k$, $k \geqslant \nu$) is uniformly bounded for fixed ν for all k and t. The same is true of the left hand side

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_G a_i u_i \, \mathrm{d}x.$$

For fixed ν, the functions of time

$$\int_G a_i^\nu(x) u_i^k(x,t) \, \mathrm{d}x$$

satisfy a Lipschitz condition for all $t \geqslant 0$ that is independent of k. Furthermore, they remain uniformly bounded for all t and k. So by a well-known selection (compactness) theorem there exists for an arbitary, fixed ν a sequence of integers k' such that

$$\lim_{k' \to \infty} \int_G a_i^\nu(x) u_i^k(x,t) \, \mathrm{d}x \tag{5.1}$$

exists for any $t \geqslant 0$, in fact uniformly so in any finite t-interval. The sequence of k' depends on the index ν, but we may pick the sequence belonging to the index $\nu + 1$ as a subsequence of the previous one. By means of a diagonal argument we may then form a fixed sequence of integers (which we will once again label as k') for which the limit statement above holds properly for any fixed $\nu = 1, 2, \ldots$. In the sequel, we will operate on this sequence of k'.

Second step. We will now prove that the sequence of fields $u^{k'}(x,t)$ converges weakly in the x-region G for each fixed $t \geqslant 0$. For the purposes of our proof, we now fix an arbitrary value t_0 of t and observe that by the first half of 5(d) the sequence of these fields ($t = t_0$) is weakly compact in G. The claim will be proven when we show the sequence has a unique weak limit. Suppose $u^*(x,t_0)$ is a weak limit and k'' a subsequence of the k' (this subsequence will depend on t_0) such that

$$\lim_{k'' \to \infty} \int_G w_i(x) u_i^{k''}(x,t_0) \, \mathrm{d}x = \int_G w_i(x) u_i^*(x,t_0) \, \mathrm{d}x$$

for each vector field $w(x)$ of class H. In the case $w = a^\nu$, the value of the right hand side is already fixed by the limit (5.1). If u^* and u^{**} are two weak limits and if v is their difference, then

$$\int_G a_i^\nu v_i \, \mathrm{d}x = 0$$

for each ν. By Definition 3.2 the fields u^*, u^{**} and thus also v belong to class $H(N)$. However, by Lemma 4.3 the fields a^ν span $H(N)$. From this we conclude

$$\int_G v_i v_i \, \mathrm{d}x = 0$$

and thus the claim.

Consequently, there is a field u^* which is well-defined in G for all $t > 0$ such that

$$\lim_{k' \to \infty} \int_G w_i(x) u_i^{k'}(x,t)\,\mathrm{d}x = \int_G w_i(x) u_i^*(x,t)\,\mathrm{d}x \tag{5.2}$$

for each field $w(x)$ $(w \in H)$ and for each $t > 0$. The field u^* satisfies condition (B) of the existence theorem 3.1 at the end of Section 3. This follows from (b) and the second half of 5(d) by applying Lemma 3.1. One easily proves that $u^{k'} \to u^*$ also holds weakly in x and t $(0 < t < T)$.

Third step. Here we prove that $u^*(x,t)$ satisfies condition (A) of the existence theorem. In each cylinder region $x \subset$G, $0 < t < T$, u^* belongs to class H', which is, as we remarked, a superclass of $H'(N)$ (and because of (B)) it also belongs to the latter class). By the arguments in the first half of Section 4 we only need to show that u^* satisfies the equations (4.1) for every $a = a^\nu$ and for all $\tau' > \tau > 0$. By c), $u = u^*$ satifies these equations for the same τ, τ' and for the first k' fields a^ν. We now fix τ, τ' and the index ν and pass to the limit $k' \to \infty$. It is clear that on the left hand side of (4.1) u may be replaced by u^*. The same is true of the third integral on the right hand side (the first one is zero). Consider that in

$$\int_\tau^{\tau'} \left[\int_G w_i(x) u_i^{k'}(x,t)\,\mathrm{d}x \right] \mathrm{d}t$$

the inner integral is a uniformly bounded function with respect to k' because of the first half of d) and that we may apply Lebesgue's bounded convergence theorem to the outer t-integral. It requires some deeper thoughts that make use of the second half of d) to see that we may also interchange the limit $k' \to \infty$ and the integration in the second integral on the right hand side of (4.1). For this, we need the following lemma which we will prove later.

Lemma 5.1 *Consider a sequence of functions $f^k(x,t)$ which are C^1 in x for $x \subset G$, $0 < t < T$ and have the following properties. For each fixed t, f^k belongs to class N. For each fixed t, the $f^k(x,t)$ converge weakly in G to a function $f^*(x,t)$. The integrals*

$$\int_G f^2(x,t)\,\mathrm{d}x, \quad \int_0^T \int_G f_{'i} f_{'i}\ \mathrm{d}x\ \mathrm{d}t \quad (f = f^k)$$

remain uniformly bounded with respect to t and k. Then the f^k converge strongly to f^* on the x-t-region $x \subset QG$, $0 < t < T$, where Q is an arbitrary finite cuboid in x-space. In particular, the assertion holds for G itself if G is bounded.

We see that because of (a), (b), (d) and the result of the second step, the assumptions of the lemma are satisfied for the components of the sequence of fields $u^{k'}(x,t)$ for an arbitrary fixed T. Thus, it follows that

$$\int_0^T \int_{QG} (u_i - u_i^*)(u_i - u_i^*)\ \mathrm{d}x\ \mathrm{d}t \quad (u = u^{k'})$$

goes to zero for $k' \to \infty$ if Q is an arbitry finite cuboid of x-space. We can thus justify passing to the limit in the second integral on the right hand side of (4.1) $(a = a^\nu$, ν fixed). Recall that the factor a of the integrand vanishes outside a fixed compact subset C of G. If we choose $Q \supset C$ and $T > \tau'$, then for the integral

$$\int_\tau^{\tau'} \int_{QG} (a_{i,\alpha})(u_\alpha)\ \mathrm{d}x\ \mathrm{d}t \quad (a = a^\nu, u = u^{k'})$$

we have the following stuation. The first factor converges weakly in the area of integration to $a_{i,\alpha}u^*_\alpha$, while the second one converges strongly to u^*_i. As is well-known, this suffices for passage to the limit under the integral sign. We have thus shown that the field u^* satisfies the equations (4.1) for any field $a^\nu(x)$ and for all positive τ, τ'. The condition A) of the existence theorem is thus verified except for the freedom from divergence. This latter property, however, is trivially true, even for any fixed $t > 0$.

To complete the proof of the existence theorem, we only need to show that the initial condition C) is also satisfied. From the energy equation (4.10) follows

$$\frac{1}{2}\int_G u_i u_i \,\mathrm{d}x|_0 = \frac{1}{2}\int_G u_i u_i \,\mathrm{d}x|_T + \int_0^T \int_G \frac{\partial u_i}{\partial x_\beta}\frac{\partial u_i}{\partial x_\beta}\ \mathrm{d}x\ \mathrm{d}t \tag{5.3}$$

for each field u of our sequence. The left hand side tends to

$$\frac{1}{2}\int_G U_i U_i \,\mathrm{d}x$$

for $k' \to \infty$ because of (4.11). For $t = T$, the fields converge weakly to u^* in G. In an x-t-cylinder set, we have

$$u^{k'}_{i,\beta} \to u^*_{i,\beta}$$

weakly because of Lemma 2.2 and (d). By applying Lemma 2.1, (5.3) implies the inequality

$$\frac{1}{2}\int_G U_i U_i \,\mathrm{d}x \geqslant \frac{1}{2}\int_G u^*_i u^*_i \,\mathrm{d}x|_T + \mu\int_0^T \int_G u^*_{i,\beta}u^*_{i,\beta}\ \mathrm{d}x\ \mathrm{d}t$$

for an arbitrary $T > 0$. In particular,

$$\overline{\lim_{t\to 0}}\int_G u^*_i u^*_i \,\mathrm{d}x \leqslant \int_G U_i U_i \,\mathrm{d}x.$$

If we once again apply Lemma 2.1 to this last inequality, we recognize that the initial condition (C) is satisfied, which is what we wanted to show.

We will not go into detail on the question of strong convergence for a fixed t.

6 Proof of Lemma 5.1

The lemma is closely related to Rellich's Compactness Theorem and has a similar proof.[9].

Let us note from the start that the lemma, just like Rellich's Theorem, need not hold for G itself if G is infinite. A counterexample is given by the case where G is $\mathbb{R}^n$ and

$$f^k(x,t) = f(x_1 + k, x_2, \ldots, x_n)$$

[9]Cf. Courant-Hilbert, l.c. footnote 5, p. 218. In Rellich's Theorem, the boundedness of the x-integrals of the squares of the derivatives is assumed. Our boundedness assumption merely concerns the x-t-integral and is thus better adapted to the state of affairs in our problem.

Leray proves and uses a lemma even closer to Rellich's Theorem l.c. Footnote 1, p. 214, Lemma 2, which, like this theorem, only works with the x-integral. Our convergence proof is more direct.

with f belonging to H' and N in G. In this case, $f^* = 0$, but there is no strong covnergence to zero[10].

The proof of Lemma 5.1 arises from Friedrichs' Inequality: Let Q be a finite cuboid in $\mathbb{R}^n$. For any given $\varepsilon > 0$, there exists a finite number of fixed functions $\omega_\nu(x)$ which belong to H in Q such that the inequality

$$\int_Q f^2 \,\mathrm{d}x \leqslant \sum_\nu \left[\int_Q f\omega_\nu \,\mathrm{d}x\right]^2 + \varepsilon \int_Q f_{'i} f_{'i} \,\mathrm{d}x$$

is satisfied by any function $f(x)$ belonging to H' in Q[11]. For the proof of Lemma 5.1, we first note that for fixed t the functions $f^k(x,t)$ of the lemma are C^1 in G and of class N. If we define the functions to be zero outside G, then this statement remains valid if we relate it to the entire x-space instead of to G. In particular, any of the functions on any finite cuboid Q of x-space belongs to the class H'. The extension of the functions and the last statement were made possible by the assumption of membership in class N. This is however the only place where this assumption is used. We now fix a cuboid Q and an arbitrary number $\varepsilon > 0$ and pick the finitely many auxiliary functions $\omega_\nu(x)$ such that Friedrichs' Inequality holds in Q. We apply it to the functions

$$f(x,t) = f^k(x,t) - f^l(x,t), \tag{6.1}$$

which surely belong to H' in Q, for fixed t. By integration in t, we conclude that all the functions (6.1) satisfy the inequality

$$\int_0^T \int_Q f^2 \ \mathrm{d}x \ \mathrm{d}t \leqslant \sum_\nu \int_0^T \left[\int_Q f\omega_\nu \,\mathrm{d}x\right]^2 \mathrm{d}t + \varepsilon \int_0^T \int_Q f_{'i} f_{'i} \ \mathrm{d}x \ \mathrm{d}t. \tag{6.2}$$

By assumption (weak convergence for fixed t), we have

$$\lim_{k\to\infty, l\to\infty} \int_Q f\omega_\nu \,\mathrm{d}x = 0$$

for each fixed t. Moreover, because of the boundedness assumption (first half), the function of t

$$\int_Q (f^k - f^l)\omega_\nu \,\mathrm{d}x$$

remains uniformly bounded w.r.t. k, l. Thus the first term on the right hand side in (6.2) tends to zero for $k \to \infty$, $l \to \infty$. By assumption, the factor of ε for the functions (6.1) remains below a fixed bound. But

$$\overline{\lim}_{k\to\infty, l\to\infty} \int_0^T \int_Q (f^k - f^l)^2 \ \mathrm{d}x \ \mathrm{d}t \leqslant c\varepsilon$$

implies strong convergence of our sequence in the x-t-region $x \subset Q$, $0 < t < T$, since ε was arbitrary. We easily obtain that the limit function is the function $f^*(x,t)$ mentioned in the statement of the lemma. Thus, Lemma 5.1 is proven.

[10] We may only conclude the strong convergence of the approximate fields $u(x,t)$ to $u^*(x,t)$ in the cylinder sections if G is bounded. However, strong convergence is clearly true for arbitrary G. Leray deduced it for his approximations in the case where G is the entire x-space using complicated estimates of the distribution of energy over G. We hope to come back to the stronger convergence properties of our approximations at some later date.

[11] The ω_ν may be assumed to be orthogonal in Q. The inequality then represents an estimate of the difference in Bessel's inequality. You may find the proof of the inequality in Courant-Hilbert, l.c. footnonte 5, p. 218, Chap. VII, §3, Section 1. We may easily convince ourselves that the proof that is given there in 2 dimensions also works in n dimensions. Friedrichs' Inequality does not hold for arbitrary bounded regions.