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The Scaling Attractor and Ultimate Dynamics for Smoluchowski's Coagulation Equations

Govind Menon · Robert L. Pego

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Abstract We describe a basic framework for studying dynamic scaling that has roots in dynamical systems and probability theory. Within this framework, we study Smoluchowski's coagulation equation for the three simplest rate kernels $K(x, y) = 2$, $x + y$ and xy. In another work, we classified all self-similar solutions and all universality classes (domains of attraction) for scaling limits under weak convergence (Menon and Pego in *Commun. Pure Appl. Math.* **57**, 1197–1232, 2004). Here we add to this a complete description of the set of all limit points of solutions modulo scaling (the *scaling attractor*) and the dynamics on this limit set (the *ultimate dynamics*). A key tool is Bertoin's Lévy-Khintchine representation formula for eternal solutions of Smoluchowski's equation (Bertoin in *Ann. Appl. Probab.* **12**, 547–564, 2002a). This representation *linearizes* the dynamics on the scaling attractor, revealing these dynamics to be conjugate to a continuous dilation, and chaotic in a classical sense. Furthermore, our study of scaling limits explains how Smoluchowski dynamics "compactifies" in a natural way that accounts for clusters of zero and infinite size (dust and gel).

Keywords Dynamic scaling · Agglomeration · Coagulation · Coalescence · Infinite divisibility · Lévy processes · Lévy-Khintchine formula · Stable laws · Universal laws · Semi-stable laws · Doeblin solution

G. Menon (✉)
Division of Applied Mathematics, Box F, Brown University, Providence, RI 02912, USA
e-mail: menon@dam.brown.edu

R.L. Pego
Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
e-mail: rpego@cmu.edu

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1 Introduction

Smoluchowski's coagulation equation is a fundamental mean-field model of clustering processes. The merging of clusters of mass x and mass y to produce clusters of mass $x+y$ occurs at a mass–action rate modulated by a symmetric rate kernel $K(x,y)$. Formally, the evolution equation for the density $n(t,x)$ of the size distribution reads

$$\partial_t n(t,x) = \frac{1}{2}\int_0^x K(x-y,y)n(t,x-y)n(t,y)\,\mathrm{d}y - \int_0^\infty K(x,y)n(t,x)n(t,y)\,\mathrm{d}y. \tag{1.1}$$

Many kernels arising in applications are homogeneous, that is, there is γ such that $K(\alpha x,\alpha y)=\alpha^\gamma K(x,y)$ for every $\alpha,x,y>0$. As time proceeds, coagulation transports mass from small to large scales, and the typical cluster size grows. In order to study the long-time dynamics of this process in detail, it is necessary to rescale relative to a typical cluster size. An issue of particular relevance for homogeneous kernels is whether and how the size distribution develops toward self-similar form. In the physics literature, this is called the *dynamic scaling* problem.

We will restrict attention to the "solvable" kernels $K(x,y)=2$, $x+y$ and xy ($\gamma=0$, 1 and 2, respectively) which may be studied via the Laplace transform. These kernels are amenable to a complete mathematical analysis, and provide valuable heuristic hints for general kernels. They are also of physical interest. Smoluchowski himself used the approximation $K(x,y)\propto \frac{1}{2}(x^{1/3}+y^{1/3})(x^{-1/3}+y^{-1/3})\approx 2$ in his study of coagulation of colloidal particles executing Brownian motion, to obtain an explicit solution for monodisperse initial data [23]. All three kernels appear in Flory's model of polymerization, with the geometry of the polymer determining the kernel (see [10, Chap. 9] and the references therein). The additive kernel also arises in droplet formation in clouds [13], a phase transition for parking [7], and the study of random graphs [1] ($K=xy$). Furthermore, it has a striking connection with the study of the inviscid Burgers equation with random initial data. The Cole-Hopf solution develops shocks that cluster as time proceeds. Remarkably, the statistics of shock size for a large class of random initial data can be described by Smoluchowski's equation with additive kernel, via an elegant closure theorem of Carraro and Duchon [6] and Bertoin [3]. Our results have several implications regarding "universality" (domains of attraction) for shock statistics, some of which are spelled out in [19].

In this article we continue our study of dynamic scaling for the solvable kernels. We lay out a general framework for the analysis of dynamic scaling that is inspired by elements from both dynamical systems and probability theory. The main issues we address may be set out as a list of basic and general questions:

- What scaling solutions exist? Here we seek *self-similar solutions*, or fixed points of the dynamics modulo scaling.

- What are the domains of attraction of these scaling solutions? These comprise the *universality classes* for dynamic scaling.
- What limit points are possible under scaling dynamics in general? We call the set of such points the *scaling attractor* of the system.
- How can we describe the dynamics on the scaling attractor? We call this the *ultimate dynamics* of the system.
- How complicated can the ultimate dynamics be?

While this is evidently stated in dynamical terms, there is a deep analogy with the classical limit theorems of probability theory associated with necessary and sufficient conditions for convergence in the central limit theorem. The general theory, developed by the pioneers of probability in the 1930s and laid out beautifully in Feller's book [9], concerns the description of general scaling limits of the distribution of sums $S_n = \sum_{j=1}^n X_j$ of independent and identically distributed random variables. The limiting distributions that emerge along subsequences $n_k \to \infty$ are the *infinitely divisible laws*, and are parameterized in terms of measures satisfying certain finiteness conditions through the fundamental Lévy-Khintchine representation formula.

The analogy with Smoluchowski's coagulation equations begins with the fact from [20] that the initial value problem for (1.1) (interpreted in a suitable weak sense) yields well posed dynamics on the entire family of probability measures on $(0, \infty)$, through the association of a cluster size distribution ν_t with the probability distribution function

$$F_t(x) = \int_0^x y^\gamma \nu_t(\mathrm{d}y) \Big/ \int_0^\infty y^\gamma \nu_t(\mathrm{d}y),$$

where $\gamma = 0, 1, 2$ for $K = 2, x + y, xy$, respectively. (The size distribution measure $\nu_t(\mathrm{d}x) = n(t, x)\,\mathrm{d}x$ if there is a size density $n(t, x)$.) For $K = x + y$ this is just the mass distribution. No smoothness or moment conditions on the initial distribution F_0 are necessary to generate a solution, just as no conditions are needed on the initial common distribution of the random variables X_j to generate the distribution of S_n through n-fold convolution.

In the earlier article [20], we comprehensively treated the first two issues from the above framework, characterizing the approach to self-similarity for Smoluchowski's equation as $t \to \infty$ for $K = 2$ and $x + y$, and the approach to self-similar blow-up as t approaches the *gelation time* for $K = xy$, as follows:

- *Scaling solutions.* Up to normalization, there is a one-parameter family of self-similar solutions, corresponding to distribution functions written $F_{\rho,\gamma}$ for $0 < \rho \le 1$, $\gamma \in \{0, 1, 2\}$. The endpoint $\rho = 1$ delivers the unique (and classically known) self-similar solution with finite $\gamma + 1$st moment, and this solution has exponential decay as $x \to \infty$. For $0 < \rho < 1$, the solutions have infinite $\gamma + 1$st moment and are directly related to important heavy-tailed distributions of probability theory—Mittag-Leffler distributions for $K = 2$, and Lévy stable laws of maximum skewness for $K = x + y$ and xy. For $K = x + y$ these solutions were first discovered by Bertoin by a different argument [4].
- *Domains of attraction.* The classical self-similar solution with $\rho = 1$ attracts all solutions with finite $\gamma + 1$st moment. In general, the domains of attraction of self-

similar solutions are characterized by the power-law behavior (more precisely, regular variation) of the $\gamma+1$st moment distribution: An initial size distribution measure ν_0 lies in the domain of attraction of the self-similar solution $F_{\rho,\gamma}$ if and only if

$$\int_0^x y^{\gamma+1}\nu_0(\mathrm{d}y) \sim x^{1-\rho}L(x), \quad x\to\infty, \tag{1.2}$$

for some function L slowly varying at ∞ (meaning $L(\lambda x)/L(\lambda)\to 1$ as $\lambda\to\infty$ for all $x>0$). There are no other self-similar solutions or domains of attraction.

These results strongly resemble central results in classical probability: First, the normal distribution is the unique distribution of finite variance which is scale-invariant (meaning the distribution of S_n is a rescaling of that of the X_j). But more generally, there is a two-parameter family of (heavy-tailed) scale-invariant distributions, classified by tail behavior and skewness. These are the *stable laws* first characterized completely by Lévy (see [16] for a historical account). Second, the central limit theorem states that the normal law attracts all initial distributions with finite variance. But finite variance is not necessary, only sufficient. The domains of attraction of all the stable laws are completely classified in terms of regular variation and skewness of the initial 2nd-moment distribution function.

For Smoluchowski's equations, our results above relate to a number of others in the literature that we will touch upon in the discussion at the end of this paper. The results on scaling solutions show that folklore regarding uniqueness of self-similar solutions is false. The results on domains of attraction show that the asymptotic self-similar profile (if any) is selected by the tails of the initial data, and this is particularly delicate for the heavy-tailed solutions ($0<\rho<1$). This is a form of sensitive dependence on initial conditions, and suggests the need for a deeper investigation of *all* scaling limit points.

In probability theory, this leads to the infinitely divisible laws. Thus, in probabilistic language, our present aim is to study infinite divisibility for Smoluchowski's coagulation equations. In this we are motivated by Bertoin's characterization of *eternal solutions* for Smoluchowski's equation with additive kernel $K=x+y$ [4]. Eternal solutions for this kernel are defined for all $t\in(-\infty,\infty)$ (i.e., they may be extended *backwards* in time globally), and thus they are "infinitely divisible" under clustering dynamics. Bertoin established a remarkable Lévy-Khintchine-type representation for these solutions. Here, we generalize this result to other solvable kernels, and show that it is a powerful tool for developing a comprehensive theory of scaling dynamics for these systems.

In the context of the framework above, our findings are as follows:

- *Scaling attractor.* The (proper) scaling attractor corresponds in one-to-one fashion with eternal solutions of Smoluchowski's equation, and these have a Lévy-Khintchine representation for each solvable kernel. This parameterizes the set of scaling limit points in terms of measures satisfying certain finiteness conditions. The measures can be described as backward-in-time limits of eternal solutions scaled to preserve the $\gamma+1$st moment (if finite). For example, for $K=x+y$ the measure H corresponding to an eternal solution ν_t satisfies

$$\int_{[0,x]} H(\mathrm{d}y) = \lim_{t\to-\infty} \int_{[0,x]} y^2 \nu_t\big(\mathrm{e}^t\,\mathrm{d}y\big)$$

at each point of continuity.

- *Ultimate dynamics.* The Lévy-Khintchine representation *linearizes the dynamics on the attractor*. As a consequence of basic scaling properties of Smoluchowski's equation, nonlinear evolution on the attractor is conjugate to a group of simple scaling transformations on the measures that generate the representation. For example, for $K = x + y$, if a distribution F_0 on the scaling attractor evolves to F_t, then the corresponding measure H_0 evolves to

$$H_t(\mathrm{d}x) = \mathrm{e}^{2t} H_0\big(\mathrm{e}^{-t}\,\mathrm{d}x\big).$$

  This representation makes precise the sensitive dependence of long-time dynamics on the tails of the initial size distribution—the ultimate dynamics on the scaling attractor is conjugate to a continuous dilation map.
- *Chaos.* We use the Lévy-Khintchine representation to construct orbits with complicated dynamics. The scaling attractor contains a dense family of scaling-periodic solutions. Furthermore, there are eternal solutions with trajectories *dense* in the scaling attractor—we call these *Doeblin solutions*. And, for any given scaling trajectory, there is a dense set of initial data whose forward trajectories shadow the given one.

In addition, this study of scaling limits reveals how Smoluchowski dynamics "compactifies" in a natural way that accounts for clusters of zero and infinite size (dust and gel). Considering defective limits on $(0,\infty)$ that concentrate probability at 0 and ∞ yields a well-posed dynamics of "extended solutions" on $[0,\infty]$. Proper solutions remain fundamental, but considering extended solutions with dust and gel helps to understand just how the tails of initial data determine long-time behavior.

We remark that scaling-periodic solutions are analogous to the *semi-stable* laws in probability theory [18]. Our "Doeblin solutions" are constructed by "packing the tails" of the Lévy measure in a fashion entirely analogous to the construction of *Doeblin's universal laws* in probability [9, XVII.9], for which the set of limit points of rescaled self-convolutions includes *all* the infinitely divisible laws. In this connection, it is interesting to note that the examples in [9, XVII.9] dismissed by Feller as "primarily of curiosity value," closely resemble modern treatments of chaos, and Doeblin's construction appears particularly prescient.

The solvable cases of Smoluchowski's equation correspond to sophisticated stochastic models with a rich theory (see [1, 5] for excellent reviews), so perhaps it is no accident that the classical probabilistic methods work so well. But let us stress that our work really relies only on the analytical methods for studying scaling limits that lie behind the classical limit theorems. These methods are simple and powerful and should be useful for understanding scaling phenomena in other applications that have no obvious probabilistic meaning. Thus, no knowledge of probability is presumed and (almost) all details are included (though there is no substitute for reading Feller!).

2 Statement of Results

In this section, we state our results precisely in a setting that unifies the treatment of dynamic scaling for all the solvable kernels.

Let E denote the open interval $(0,\infty)$, $\mathcal{M}$ the space of nonnegative Radon measures on E, and $\mathcal{P}$ the space of probability measures on E. We will always use the weak topology on $\mathcal{M}$ and $\mathcal{P}$. We also let $\bar{E}$ denote the closed half line with point at infinity, $\bar{E}=[0,\infty]=[0,\infty)\cup\{\infty\}$, and let $\bar{\mathcal{P}}$ be the space of probability measures on $\bar{E}$.

Rigorous theories for solutions where $\nu_t(\mathrm{d}x)=n(t,x)\,\mathrm{d}x$ is a general size-distribution measure on $E=(0,\infty)$, thus accounting for both continuous and discrete size distributions in one general setting, are based on the moment identity

$$\frac{d}{dt}\int_E f(x)\nu_t(\mathrm{d}x)=\frac{1}{2}\int_E\int_E \tilde{f}(x,y)K(x,y)\nu_t(\mathrm{d}x)\nu_t(\mathrm{d}y), \tag{2.1}$$

where f is a suitable test function and $\tilde{f}(x,y)=f(x+y)-f(x)-f(y)$ (see [12, 20, 22]). Let $m_\theta(t):=\int_E x^\theta\nu_t(\mathrm{d}x)$ denote the θth moment of ν_t. By the results of [20], for a solvable kernel of homogeneity γ, any initial measure ν_{t_0} with finite γth moment $m_\gamma(t_0)$ determines a unique continuous weak solution

$$\nu=\big(\nu_t(\mathrm{d}x),\ t\in[t_0,T_{\max})\big). \tag{2.2}$$

For convenience we can always scale the initial data so that

$$t_0:=\begin{cases}1 & (K=2),\\ 0 & (K=x+y),\\ -1 & (K=xy),\end{cases}\qquad\text{and}\quad m_\gamma(t_0)=\int_E x^\gamma\nu_{t_0}(\mathrm{d}x)=1. \tag{2.3}$$

Then $T_{\max}=\infty$ for $K=2$ and $x+y$, and $T_{\max}=0$ for $K=xy$. For each solvable kernel, $m_\gamma(t)=\int_E y^\gamma\nu_t(\mathrm{d}y)$ is an explicitly known function—from (2.1) with $f(x)=x^\gamma$ we find

$$m_\gamma(t)=\begin{cases}t^{-1} & (K=2),\\ 1 & (K=x+y),\\ |t|^{-1} & (K=xy).\end{cases} \tag{2.4}$$

The solution ν_t is typically not a probability measure because the total number of clusters decreases in time, but there is a naturally associated probability measure $F_t(\mathrm{d}x)$ with distribution function $F_t(x)$ defined by

$$F_t(x)=\int_{(0,x]}y^\gamma\nu_t(\mathrm{d}y)\Big/\int_E y^\gamma\nu_t(\mathrm{d}y). \tag{2.5}$$

In this way, we regard Smoluchowski's equation as defining a continuous dynamical system on the phase space $\mathcal{P}$.

2.1 Eternal Solutions

For exceptional initial data ν_{t_0} we may also solve backwards in time (meaning ν_{t_0} is *divisible* under clustering dynamics). The maximum possible interval of existence that can be obtained in this way is denoted $(T_{\min}, T_{\max})$, where $T_{\min}, T_{\max}$ depend only on the kernel and $\int_E x^\gamma \nu_{t_0}(\mathrm{d}x)$. With the normalization (2.3), the maximum possible interval of existence turns out to be that compatible with (2.4), namely

$$(T_{\min}, T_{\max}) = \begin{cases} (0, \infty) & (K = 2), \\ (-\infty, \infty) & (K = x + y), \\ (-\infty, 0) & (K = xy). \end{cases} \tag{2.6}$$

Solutions which are defined on this maximum interval of existence are the analog of infinitely divisible laws in probability.

Definition 2.1 A solution to Smoluchowski's equation that is defined for all $t \in (T_{\min}, T_{\max})$ is called an *eternal solution*.

2.2 The Scaling Attractor

A central idea in dynamical systems theory is to understand the long-time behavior of solutions through the notions of an attractor and ω-limit sets. Coagulation equations transport mass irreversibly from small to large scales, and to obtain a nontrivial description of asymptotic behavior, we must rescale solutions. We adopt the following definitions for such *scaling dynamical systems*. Below, $T_n \in [t_0, T_{\max})$, $\beta_n > 0$. We will often use the same letter to denote a measure and its distribution function, e.g., $F(x) = \int_{[0,x]} F(\mathrm{d}x)$.

Definition 2.2 The (proper) *scaling ω-limit set* of a solution ν to Smoluchowski's equation is the set of probability measures $\hat{F}$ on E for which there exist sequences $T_n \to T_{\max}$, $\beta_n \to \infty$, such that $F_{T_n}(\beta_n x) \to \hat{F}(x)$ at every point of continuity of $\hat{F}$.

Definition 2.3 The (proper) *scaling attractor*, $\mathcal{A}_{\mathrm{p}}$, is the set of probability measures $\hat{F}$ on E for which there exists a sequence of solutions $\nu^{(n)}$ defined for $t \in [t_0, T_{\max})$, and sequences $T_n \to T_{\max}$, $\beta_n \to \infty$, such that $F^{(n)}_{T_n}(\beta_n x) \to \hat{F}(x)$ at every point of continuity of $\hat{F}$.

As a consequence of continuous dependence of solutions on initial data (forward and backward in time), we will show that the scaling attractor is an invariant set, and that points on the proper scaling attractor and eternal solutions are in one-to-one correspondence.

Theorem 2.4

(a) *The proper scaling attractor $\mathcal{A}_{\mathrm{p}}$ is invariant*: *If ν is a solution of Smoluchowski's equation*, *and $F_t \in \mathcal{A}_{\mathrm{p}}$ for some t*, *then the solution is eternal and $F_t \in \mathcal{A}_{\mathrm{p}}$ for all $t \in (T_{\min}, T_{\max})$*.

(b) *A probability measure* $\hat{F}$ *belongs to* $\mathcal{A}_\mathrm{p}$ *if and only if* $\hat{F}(\mathrm{d}x) = x^\gamma \nu_{t_0}(\mathrm{d}x)$ *for some eternal solution* ν, *where* t_0 *is as in* (2.3).

The perfect definition of an attractor remains elusive (see, for example, the discussion in [14, Chap. 1.6]). Definition 2.3 is perhaps the simplest for dynamical systems. It also has the virtue of generalizing the probabilistic notion of domains of partial attraction [9, XVII]. However, some typical properties that hold in finite-dimensional dynamical systems do not hold here. For example, it need not be the case that every solution has a nonempty scaling ω-limit set. Nor is $\mathcal{A}_\mathrm{p}$ closed. Defective limits are possible, as shown in [20]. See [15] for a discussion of related issues in the probabilistic context. It turns out that we can cure these defects and account for limits that involve mass concentrating at zero or leaking to infinity, by the simple expedient of allowing limits to be probability measures on $\bar{E} = [0, \infty]$.

Definition 2.5 The full *scaling ω-limit set* of a solution ν to Smoluchowski's equation is the set of probability measures $\hat{F}$ on $\bar{E}$ with the property in Definition 2.2. The full *scaling attractor*, $\mathcal{A}$, is the set of probability measures $\hat{F}$ on $\bar{E}$ with the property in Definition 2.3.

The space $\bar{\mathcal{P}}$ of probability measures on $\bar{E}$, equipped with the weak topology, is compact—any sequence contains a converging subsequence. We will show that Smoluchowski dynamics naturally extends by continuity from $\mathcal{P}$ to $\bar{\mathcal{P}}$. Such "extended solutions" have probability distributions that may include atoms at 0 and ∞, allowing for the possibility that clusters have zero size ("dust") or infinite size ("gel") with positive probability. To interpret these physically, one should recognize that of course 0 and ∞ are idealizations relative to a given scale of measuring cluster size.

We defer detailed discussion of extended solutions to Sect. 9. There we extend Theorem 2.4 to relate the full scaling attractor $\mathcal{A}$ (now a compact set that is the closure of $\mathcal{A}_\mathrm{p}$) to the set of eternal extended solutions. Also, the Lévy-Khintchine representation and linearization theorems in the next two sections have elegant extensions involving extended solutions. First, however, we think it appropriate to focus on standard weak solutions, and develop the theory without dust in our eyes, so to speak.

2.3 Lévy-Khintchine Representations

In probability theory, infinitely divisible distributions are parameterized by the Lévy-Khintchine representation theorem, which expresses the log of the characteristic function (Fourier transform) in terms of a measure that satisfies certain finiteness conditions. In particular [9, XIII.7], a function $\omega(q)$ is the Laplace transform $\int_0^\infty \mathrm{e}^{-qx} F(\mathrm{d}x)$ of an infinitely divisible probability measure F supported on $[0, \infty)$ if and only if $\omega(q) = \exp(-\Phi(q))$ where the Laplace exponent Φ admits the representation

$$\Phi(q) = \int_{[0,\infty)} \frac{1 - \mathrm{e}^{-qx}}{x} G(\mathrm{d}x) \tag{2.7}$$

for some measure G on $[0,\infty)$ that satisfies

$$\int_{[0,x]} G(\mathrm{d}y) < \infty \quad \text{and} \quad \int_{[x,\infty)} y^{-1} G(\mathrm{d}y) < \infty \quad \text{for all } x > 0. \tag{2.8}$$

Equivalently,

$$\int_{[0,\infty)} (1 \wedge y^{-1}) G(\mathrm{d}y) < \infty. \tag{2.9}$$

We need a name for measures with this property, although none seems standard. The measure G characterizes the generator of a natural convolution semigroup associated with F—see the remark following Theorem 2.8—hence we call them *g-measures*. To handle defective limits, it is convenient to allow $y^{-1}G(\mathrm{d}y)$ to have an atom at ∞.

Definition 2.6 A measure G on $[0,\infty)$ is a *g-measure* if (2.9) holds. A pair (G, g_∞) is called a *$\overline{g}$-measure* on $\bar{E} = [0,\infty]$ if G is a g-measure and $g_\infty \geq 0$. g_∞ is called the charge of $y^{-1}G(\mathrm{d}y)$ at ∞, and we will abuse notation by letting G implicitly denote the pair (G, g_∞). In addition, we say that a g-measure (or $\overline{g}$-measure) G is *divergent* if

$$G(0) > 0 \quad \text{or} \quad \int_E y^{-1} G(\mathrm{d}y) = \infty. \tag{2.10}$$

Here recall we use the notation $G(x) = \int_{[0,x]} G(\mathrm{d}y)$. If $g_\infty = 0$ we identify G with $(G, 0)$. The space of $\overline{g}$-measures has a natural weak topology which will prove fundamental in our study of scaling dynamics.

Definition 2.7 A sequence of $\overline{g}$-measures $G^{(n)}$ *converges* to a $\overline{g}$-measure G as $n \to \infty$, if at every point $x \in (0,\infty)$ of continuity of G we have

$$\int_{[0,x]} G^{(n)}(\mathrm{d}y) \quad \to \quad \int_{[0,x]} G(\mathrm{d}y), \tag{2.11}$$

and

$$\int_{[x,\infty]} y^{-1} G^{(n)}(\mathrm{d}y) \quad \to \quad \int_{[x,\infty]} y^{-1} G(\mathrm{d}y). \tag{2.12}$$

The integrals in (2.12) include the charge at ∞, if any. We note that in view of the weak convergence implied by (2.11), convergence of $G^{(n)}$ to a g-measure G (having $g_\infty = 0$) is equivalent to (2.11) together with the tightness condition

$$\int_{[x,\infty]} y^{-1} G^{(n)}(\mathrm{d}y) \to 0 \quad \text{as } x \to \infty, \text{ uniformly in } n. \tag{2.13}$$

Bertoin's main theorem in [4] shows that eternal solutions for $K = x + y$ are in one-to-one correspondence with divergent g-measures. (More precisely, Bertoin formulated his result in terms of "Lévy pairs," separating the atom at the origin from a jump measure on $(0,\infty)$.) We extend this result as follows. Let ν be an arbitrary solution to Smoluchowski's equation for a solvable kernel of homogeneity γ. Since the γth moment of ν_t is finite, $x^{\gamma+1}\nu_t(\mathrm{d}x)$ is a g-measure. Rescaling, we associate with ν_t the g-measure G_t defined by

$$G_t(\mathrm{d}x) = x^{\gamma+1}\nu_t\big(\lambda(t)\,\mathrm{d}x\big), \quad \lambda(t) = \begin{cases} 1 & (K=2), \\ \mathrm{e}^t & (K=x+y), \\ |t|^{-1} & (K=xy). \end{cases} \tag{2.14}$$

This choice of rescaling ensures that if the total measure $G_t(E)$ is finite for some t, then it is constant: $G_t(E) = m_{\gamma+1}(t)/\lambda(t)^{\gamma+1} = \text{const}$.

One computes that

$$\int_0^\infty y^{-1}G_t(\mathrm{d}y) = \frac{m_\gamma(t)}{\lambda(t)^\gamma} = \begin{cases} t^{-1} & (K=2), \\ \mathrm{e}^{-t} & (K=x+y), \\ |t| & (K=xy). \end{cases} \tag{2.15}$$

The well-posedness theorem [20] implies that solutions of Smoluchowski's equation normalized according to (2.3) that exist on any time interval $[t, T_{\max})$ *are in one-to-one correspondence* with g-measures $\hat{G}$ that satisfy

$$\hat{G}(0) = 0 \quad \text{and} \quad \int_0^\infty y^{-1}\hat{G}(\mathrm{d}y) = m_\gamma(t)/\lambda(t)^\gamma,$$

via $\hat{G} = G_t$. Through studying the limit $t \downarrow T_{\min}$, we find that eternal solutions may be characterized as follows.

Theorem 2.8

(a) *Let ν be an eternal solution for Smoluchowski's equation with $K = 2, x+y$ or xy. Then there is a divergent g-measure H such that G_t converges to H as $t \downarrow T_{\min}$.*
(b) *Conversely, for every divergent g-measure H, there is a unique eternal solution ν such that G_t converges to H as $t \downarrow T_{\min}$.*
(c) *Let $\mathfrak{S}_p : \mathcal{A}_\mathrm{p} \to \mathcal{S}_d$ map the (proper) scaling attractor $\mathcal{A}_\mathrm{p}$ to the set $\mathcal{S}_d$ of divergent g-measures by $\mathfrak{S}_p(\hat{F}) = H$, where H is the divergent g-measure associated to the eternal solution ν such that $\hat{F}(\mathrm{d}x) = x^\gamma \nu_{t_0}(\mathrm{d}x)$ with t_0 as in (2.3). Then $\mathfrak{S}_p$ is a bicontinuous bijection.*

The procedure for obtaining the eternal solution ν from the divergent g-measure H is nonlinear and is different for each kernel (see Theorems 4.3, 5.4, and 6.1). It seems natural to call Theorem 2.8 a Lévy-Khintchine representation for the scaling attractor $\mathcal{A}_\mathrm{p}$—as we will see, eternal solutions are determined by the Laplace exponents associated with divergent g-measures.

In Sect. 9, the correspondence in Theorem 2.8 is expanded to one between eternal extended solutions and arbitrary $\overline{g}$-measures.

Remark 2.1 The characterization of the g-measure H via limits of rescaled measures from (2.14) is a nonlinear analog of a basic relation in probability theory, between an infinitely divisible distribution F on $[0,\infty)$ and the generator of the convolution semigroup naturally associated with the family of distributions $(F_t,\ t\ge 0)$ that satisfy $F_1 = F$ and the convolution property $F_t \star F_s = F_{t+s}$ $(t,s\ge 0)$. The Laplace transform of F_t is $\exp(-t\Phi(q))$ and, as discussed by Feller [9, XIII.9(a)], the generator of the semigroup is found by studying weak convergence of rescaled first-moment measures. One finds

$$\int_{[0,x]} t^{-1} y F_t(\mathrm{d}y) \quad \to \quad \int_{[0,x]} G(\mathrm{d}y), \quad t\downarrow 0,$$

where G is the measure in the Lévy-Khintchine representation (2.7).

2.4 Linearization of Ultimate Dynamics

There are two natural group actions on the class of eternal solutions that are related to scaling dynamics, arising from *time evolution* and *rescaling of size*. A straightforward but remarkable consequence of the scaling properties of Smoluchowski's equation is that nonlinear dynamics (time evolution) on the scaling attractor $\mathcal{A}_{\mathrm{p}}$ is conjugate to a simple linear scaling transformation of the divergent g-measures that correspond by Theorem 2.8.

Theorem 2.9 *Let ν be a solution of Smoluchowski's equation with $K=2$, $x+y$ or xy. Given scaling parameters $a>0$ and $b>0$, let*

$$\tilde\nu_t(\mathrm{d}x) = \begin{cases} a\nu_{at}(b\,\mathrm{d}x) & (K=2),\\ b\nu_{t+\log a}(b\,\mathrm{d}x) & (K=x+y),\\ ab^2\nu_{at}(b\,\mathrm{d}x) & (K=xy),\end{cases} \tag{2.16}$$

with associated probability distribution function

$$\tilde F_t(x) = \begin{cases} F_{at}(bx) & (K=2 \text{ or } xy),\\ F_{t+\log a}(bx) & (K=x+y).\end{cases} \tag{2.17}$$

Then $\tilde\nu$ is again a solution. If ν is eternal and H its associated divergent g-measure, then $\tilde\nu$ is eternal and its associated divergent g-measure is given by

$$\tilde H(x) = \begin{cases} ab^{-1}H(bx) & (K=2),\\ a^2b^{-1}H(a^{-1}bx) & (K=x+y),\\ a^{-2}b^{-1}H(abx) & (K=xy).\end{cases} \tag{2.18}$$

Proof The proof is simple, based on Theorem 2.8 and the scaling properties of Smoluchowski's equation. First, one checks that (2.16) determines a solution, by scaling the moment identity (2.1) in each case. Next, compute that if the g-measure G_t is associated with ν_t as in (2.14), then the corresponding g-measure associated with

$\tilde{\nu}_t$ is given by

$$\tilde{G}_t(\mathrm{d}x) = \begin{cases} x\tilde{\nu}_t(\mathrm{d}x) = ab^{-1}G_{at}(b\,\mathrm{d}x) & (K=2), \\ x^2\tilde{\nu}_t(\mathrm{e}^t\,\mathrm{d}x) = a^2b^{-1}G_{t+\log a}(a^{-1}b\,\mathrm{d}x) & (K=x+y), \\ x^3\tilde{\nu}_t(|t|^{-1}\,\mathrm{d}x) = a^{-2}b^{-1}G_{at}(ab\,\mathrm{d}x) & (K=xy). \end{cases} \tag{2.19}$$

Then take $t \downarrow T_{\min}$ and apply Theorem 2.8 to deduce (2.18). □

Theorem 2.10 *Let ν be an eternal solution with corresponding divergent g-measure H and let F_t be as in (2.5) for $K=2$, $x+y$ or xy. For each $t \in (T_{\min}, T_{\max})$, let $H_t = \mathfrak{S}_p(F_t)$ be the divergent g-measure associated to $F_t \in \mathcal{A}_\mathrm{p}$. Then*

$$H_t(x) = \begin{cases} tH(x) & (K=2), \\ \mathrm{e}^{2t}H(\mathrm{e}^{-t}x) & (K=x+y), \\ |t|^{-2}H(|t|x) & (K=xy). \end{cases} \tag{2.20}$$

Proof Take $b=1$ and put $t=t_0$ in (2.17), then substitute $a=t$, e^t, $|t|$ for $K=2$, $x+y$, xy, respectively, to obtain $\tilde{F}_{t_0} = F_t$. Then the corresponding divergent g-measure $H = \mathfrak{S}_p(\tilde{F}_{t_0})$ is found from (2.18). □

By this theorem, we see that in terms of the divergent g-measure that corresponds to the solution, the time evolution on the scaling attractor $\mathcal{A}_\mathrm{p}$ is governed by the linear equations

$$t\partial_t H_t = H_t \qquad (K=2), \tag{2.21}$$

$$(\partial_t + x\partial_x)H_t = 2H_t \qquad (K=x+y), \tag{2.22}$$

$$(t\partial_t - x\partial_x)H_t = -2H_t \qquad (K=xy). \tag{2.23}$$

2.5 How Initial Tails Encode Scaling Limits

The long-time scaling behavior is very sensitive to the initial distribution of the largest clusters in the system, as indicated by the characterization of domains of attraction via (1.2), and Theorem 2.10. In fact, the long-time scaling dynamics is encoded in the tails of initial data in a simple fashion related to the Lévy-Khintchine representation.

Theorem 2.11 *Let $\hat{F} \in \mathcal{A}_\mathrm{p}$ with associated divergent g-measure H. Let $\nu^{(n)}$ be any sequence of solutions defined for $t \ge t_0$, with associated initial g-measures given by $G^{(n)}(\mathrm{d}x) = x^{\gamma+1}\nu_{t_0}^{(n)}(\mathrm{d}x)$. Let $T_n \to T_{\max}$, $\beta_n \to \infty$. Then the following are equivalent*:

(i) *$F_{T_n}^{(n)}(\beta_n x) \to \hat{F}(x)$ as $n \to \infty$, at every point of continuity.*

(ii) *The rescaled initial g-measures* $\tilde{G}^{(n)}$ *defined by*

$$\tilde{G}^{(n)}(x) = \begin{cases} \beta_n^{-1} T_n \, G^{(n)}(\beta_n x) & (K = 2), \\ \beta_n^{-1} \mathrm{e}^{2T_n} \, G^{(n)}(\mathrm{e}^{-T_n} \beta_n x) & (K = x + y), \\ \beta_n^{-1} |T_n|^{-2} \, G^{(n)}(|T_n| \beta_n x) & (K = xy), \end{cases} \tag{2.24}$$

have the property that $\tilde{G}^{(n)}$ *converges to* H *as* $n \to \infty$.

This result generalizes to the full attractor $\mathcal{A}$, with H replaced by the corresponding $\overline{g}$-measure; see Sect. 9. We remark that in the proof it is shown that for the convergence in part (ii) to hold, it is necessary that $\mathrm{e}^{-T_n}\beta_n \to \infty$ for $K = x + y$, and $|T_n|\beta_n \to \infty$ for $K = xy$.

2.6 Signatures of Chaos

The dilational representation of dynamics in (2.20) in terms of the Lévy-Khintchine representation means that Smoluchowski dynamics on the scaling attractor is a continuous analog of a Bernoulli shift map, a classical paradigm for chaotic dynamics. We demonstrate the utility of this representation by constructing solutions with both chaotic and regular orbits, and by proving a shadowing theorem illustrating sensitive dependence on the tails.

2.6.1 Solutions with Dense Limit Sets

Theorem 2.12 *There exists an eternal solution* ν *whose scaling* ω*-limit set contains every element of the full scaling attractor* $\mathcal{A}$.

We call such solutions *Doeblin solutions* by analogy with Doeblin's universal laws. The construction follows Feller closely and relies only on general principles (separability of $\mathcal{P}$ and $\mathcal{S}_d$, and continuity of the bijection $\mathfrak{S}_p$). Theorem 2.12 tells us that $\mathcal{A}$ cannot be decomposed into invariant subsets.

2.6.2 Scaling Periodic Solutions

Another classical signature of chaos is the density of periodic solutions. The notion of periodicity generalizes as follows.

Definition 2.13 Let ν be a solution and define $F_t(x)$ by (2.5). We say ν is *scaling-periodic* if for some $t_1 > t_0$ and $\beta > 1$,

$$F_{t_1}(\beta x) = F_{t_0}(x) \quad \text{for all } x > 0. \tag{2.25}$$

These are analogous to semi-stable laws in probability theory [18]. The Lévy-Khintchine representation yields a simple characterization.

Theorem 2.14 *A scaling-periodic solution of Smoluchowski's equation with kernel* $K = 2$, $x + y$ *or* xy *is eternal, and its divergent* g*-measure* H *satisfies*

$$H(x) = aH(bx) \tag{2.26}$$

for some $a > 0$, $b > 1$ *such that either*

(i) $a = 1$ *and* H *is an atom at the origin.*
(ii) $a < 1$ *and* $ab > 1$.

Conversely, if H *is a measure on* $[0, \infty)$ *with* $H(x) = aH(bx)$, *where* $a > 0$, $b > 1$ *and* (i) *or* (ii) *hold, then* H *is a divergent* g*-measure and the corresponding eternal solution is scaling-periodic.*

Case (i) is simple but important. The corresponding scaling-periodic solutions are the self-similar solutions with exponential decay. More generally, all self-similar solutions are determined by divergent g-measures of the power-law form $H(x) = C_\rho x^{1-\rho}$, $0 < \rho \le 1$. Scaling-periodic solutions that are not self-similar solutions are generated by (ii). Thus, there are uncountably many scaling-periodic solutions. Moreover, the Lévy-Khintchine representation allows us to prove:

Theorem 2.15 *Scaling-periodic solutions are dense in* $\mathcal{A}$.

2.6.3 Shadowing and Sensitive Dependence on the Initial Tails

We show that asymptotically similar initial tails imply *shadowing* of scaled solution trajectories. To study shadowing, we note that the space $\bar{\mathcal{P}}$ of probability measures on $\bar{E}$ is metrizable and compact. We let $\operatorname{dist}(\cdot, \cdot)$ denote any metric on $\bar{\mathcal{P}}$ which induces the weak topology.

Theorem 2.16 *Let* ν *and* $\bar{\nu}$ *denote any two solutions of Smoluchowski's equation defined on* $[t_0, T_{\max})$, *and let the associated initial* g*-measures be*

$$G(\mathrm{d}x) = x^{\gamma+1}\nu_{t_0}(\mathrm{d}x), \qquad \bar{G}(\mathrm{d}x) = x^{\gamma+1}\bar{\nu}_{t_0}(\mathrm{d}x), \tag{2.27}$$

with Laplace exponents φ *and* $\bar{\varphi}$, *respectively, associated as in* (2.7). *Assume that*

$$\bar{\varphi}(q)/\varphi(q) \sim L(1/q) \quad \text{as } q \to 0, \tag{2.28}$$

where L *is slowly varying at* ∞. *Suppose that* $b(t) \uparrow \infty$ *as* $t \uparrow T_{\max}$, *and define*

$$(\bar{t}, \bar{b}) = \begin{cases} (t/L(b), b) & (K = 2), \\ (t - \log L(b\mathrm{e}^{-t}), b/L(b\mathrm{e}^{-t})) & (K = x + y), \\ (t\,L(|t|b), bL(|t|b)) & (K = xy), \end{cases} \tag{2.29}$$

so that $\bar{b}/\lambda(\bar{t}) = b/\lambda(t)$ *with* $\lambda(t)$ *as in* (2.14). *Then we have*

$$\operatorname{dist}\big(F_t\big(b(t)\,\mathrm{d}x\big), \bar{F}_{\bar{t}}\big(\bar{b}(t)\,\mathrm{d}x\big)\big) \to 0 \quad \text{as } t \to T_{\max}. \tag{2.30}$$

The simplest situation, requiring no readjustment of the scaling ($\bar{t} = t$, $\bar{b} = b$), is when $L = 1$. When condition (2.28) holds, the solutions ν and $\bar{\nu}$ have identical scaling ω-limit sets, for example. If one of the solutions in this theorem, say $\bar{\nu}$, is self-similar, then the sufficient condition (2.28) for shadowing in this theorem is equivalent to (1.2) (see [20], (5.3) and (5.7) for $K = 2$, and (7.2) and (7.4) for $K = x + y$). Hence (2.28) is also necessary, according to the classification theorem on domains of attraction. It appears that in general the sufficient condition for shadowing given in this theorem may not always be necessary. But we will not pursue this issue here.

The sensitivity of solutions to initial tails in the weak topology is revealed strikingly in Theorem 2.16. The topology of weak convergence is undoubtedly natural for limit theorems, for example, the approach to self-similarity. On the other hand, this topology cannot distinguish the tails, as the following "cut-and-paste" argument shows. Let $\hat{F} = x^\gamma \hat{\nu}_{t_0}$ and $\check{F} = x^\gamma \check{\nu}_{t_0}$ be given initial data for two arbitrary solutions, and define

$$\hat{F}^{(n)}(x) = \begin{cases} \hat{F}(x) \wedge \check{F}(n) & x < n, \\ \check{F}(x) & x \geq n. \end{cases} \tag{2.31}$$

Then $\hat{F}^{(n)} \to \hat{F}$ as $n \to \infty$, and $\hat{F}^{(n)}$ has Laplace exponent given by

$$\frac{\varphi^{(n)}(q)}{q} = \int_0^n \left(\frac{1 - \mathrm{e}^{-qy}}{qy}\right) y \hat{F}(\mathrm{d}y) + \int_n^\infty \left(\frac{1 - \mathrm{e}^{-qy}}{qy}\right) y \check{F}(\mathrm{d}y), \tag{2.32}$$

from which one sees easily that if $\check{G}(E) = \infty$, then $\varphi^{(n)}(q) \sim \check{\varphi}(q)$ as $q \to 0$. Thus, according to Theorem 2.16, the solution $\hat{\nu}^{(n)}$ generated by $\hat{F}^{(n)}$ will shadow $\check{\nu}$. This justifies the statement made in the introduction that for any given scaling trajectory, there is a dense set of initial data whose forward trajectories shadow the given one.

2.7 Plan of the Paper

In Sect. 3 we establish some basic facts regarding convergence of the measures that generate the Lévy-Khintchine representation. The analysis of eternal solutions is different in detail for the constant and additive kernels, so we treat these cases in turn, establishing Theorems 2.4, 2.8, and 2.11 for these kernels in Sects. 4 and 5. The multiplicative case reduces mathematically to the additive by a change of variables, and is treated briefly in Sect. 6. We emphasize that the results in this case concern the behavior of solutions approaching the gelation time, so perhaps this is the case of most interest physically.

With the Lévy-Khintchine representation in hand, we then construct Doeblin solutions in Sect. 7 and scaling-periodic solutions in Sect. 8. Extended solutions and the full scaling attractor $\mathcal{A}$ are studied in Sect. 9, and the shadowing Theorem 2.16 is proved in Sect. 10, where we also provide a streamlined treatment of the domains of attraction. The article concludes with a discussion of these results and related issues.

3 Laplace Exponents and Limits of $\overline{g}$-measures

The main analytic tool in the study of the solvable kernels is the Laplace transform. Recall that a sequence of probability measures $F^{(n)}$ is said to converge weakly to a probability measure F if the distribution functions $F^{(n)}(x)\to F(x)$ at every point of continuity of the limit. It is basic [9, XIII.1] that $F^{(n)}$ converges weakly to F if and only if the Laplace transforms converge pointwise:

$$\int_E \mathrm{e}^{-qx}F^{(n)}(\mathrm{d}x) \quad\to\quad \int_E \mathrm{e}^{-qx}F(\mathrm{d}x),\qquad \text{for all } q>0.$$

We will need the following refinements of this result for $\overline{g}$-measures. With any $\overline{g}$-measure G we associate "Laplace exponents" Φ and Ψ (with $\Phi=\partial_q\Psi$) defined for $q\in\mathbb{C}_+=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}$ by

$$\Phi(q)=\int_{\bar{E}}\frac{1-\mathrm{e}^{-qx}}{x}G(\mathrm{d}x),\qquad \Psi(q)=\int_{\bar{E}}\frac{\mathrm{e}^{-qx}-1+qx}{x^2}G(\mathrm{d}x). \tag{3.1}$$

If g_0 and g_∞ denote amplitudes of the atoms of the measure $(1\wedge y^{-1})G(\mathrm{d}y)$ at 0 and ∞, respectively, this means

$$\Phi(q)=qg_0+g_\infty+\int_{(0,\infty)}\frac{1-\mathrm{e}^{-qx}}{x}G(\mathrm{d}x), \tag{3.2}$$

$$\Psi(q)=\frac{1}{2}q^2g_0+qg_\infty+\int_{(0,\infty)}\frac{\mathrm{e}^{-qx}-1+qx}{x^2}G(\mathrm{d}x). \tag{3.3}$$

We use the terminology Laplace exponent in accordance with probabilists' usage. If we need to distinguish the two types of exponents, we will refer to Φ and Ψ as Laplace exponents of the first and second order, respectively. Observe that

$$\partial_q\Phi=\partial_q^2\Psi=\int_{[0,\infty)}\mathrm{e}^{-qx}G(\mathrm{d}x). \tag{3.4}$$

These functions are Laplace transforms of a positive measure, thus are completely monotone functions on $(0,\infty)$.

We note that the amplitude of the atom of $(1\wedge y^{-1})G(\mathrm{d}y)$ at ∞ is

$$g_\infty=\lim_{q\to0^+}\Phi(q)=\lim_{q\to0^+}q^{-1}\Psi(q), \tag{3.5}$$

thus the $\overline{g}$-measure G is a g-measure if and only if this vanishes. Furthermore, we claim that G is divergent if and only if

$$\lim_{q\to\infty}\Phi(q)=\infty,\quad \text{equivalently}\quad \lim_{q\to\infty}q^{-1}\Psi(q)=\infty. \tag{3.6}$$

To prove this, observe that

$$\Phi(q)\le qg_0+\int_{(0,\infty)}x^{-1}G(\mathrm{d}x).$$

Thus, if $\lim_{q\to\infty}\Phi(q)=\infty$, then G satisfies (2.10). Conversely, if G satisfies (2.10), then $\lim_{q\to\infty}\Phi(q)=\infty$ by the monotone convergence theorem. The proof for Ψ is similar. We integrate by parts and use Fubini's theorem to obtain

$$q^{-1}\Psi(q) = \frac{1}{2}qg_0 + g_\infty + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-qx}\right) \int_{(x,\infty)} y^{-2} G(\mathrm{d}y)\,\mathrm{d}x$$
$$\leq \frac{1}{2}qg_0 + g_\infty + \int_{(0,\infty)} y^{-1} G(\mathrm{d}y).$$

Thus, if $q^{-1}\Psi(q) \to \infty$, then G satisfies (2.10). The converse follows from the monotone convergence theorem.

Theorem 3.1 *Let $G^{(n)}$ be a sequence of $\overline{g}$-measures with Laplace exponents $\Phi^{(n)}$ and $\Psi^{(n)}$. Then, taking $n \to \infty$, the following are equivalent*:

(i) *$G^{(n)}$ converges to a $\overline{g}$-measure G with Laplace exponents Φ and Ψ.*
(ii) *$\Phi(q) := \lim_{n\to\infty} \Phi^{(n)}(q)$ exists for each $q > 0$.*
(iii) *$\Psi(q) := \lim_{n\to\infty} \Psi^{(n)}(q)$ exists for each $q > 0$.*

Proof (i) implies (ii): Fix $q > 0$ and let $\varepsilon > 0$. Equation (2.12) allows us to choose a such that a is a point of continuity for G and for every n

$$\int_{[a,\infty]} \mathrm{e}^{-qx} x^{-1} \left(G^{(n)}(\mathrm{d}x) + G(\mathrm{d}x)\right) \leq \mathrm{e}^{-qa} C < \varepsilon.$$

On the other hand, (2.11) guarantees

$$\int_0^a \left(1 - \mathrm{e}^{-qx}\right) x^{-1} G^{(n)}(\mathrm{d}x) \quad \to \quad \int_0^a \left(1 - \mathrm{e}^{-qx}\right) x^{-1} G(\mathrm{d}x).$$

Using again (2.12), we conclude that for large n, $|\Phi^{(n)}(q) - \Phi(q)| < \varepsilon$.

(ii) implies (i): 1. Claim: Φ is analytic in $\mathbb{C}_+$ and $\Phi^{(n)} \to \Phi$ uniformly on compact subsets of $\mathbb{C}_+$.

Proof Let $K \subset \mathbb{C}_+$ be compact. The claim follows from the estimate:

$$\sup_n \sup_{q\in K} \left|\Phi^{(n)}(q)\right| < \infty. \tag{3.7}$$

Indeed, by Montel's theorem, (3.7) implies $\{\Phi^{(n)}\}_{n=1}^\infty$ are a normal family of analytic functions (i.e., precompact in the uniform topology). Thus, every subsequence has a further subsequence converging uniformly to an analytic function. Since every subsequence converges (pointwise) to Φ, this implies $\Phi^{(n)} \to \Phi$ uniformly and Φ is analytic. It remains to prove (3.7). We integrate by parts to obtain

$$q^{-1}\Phi^{(n)}(q) = G^{(n)}(0) + \int_E \mathrm{e}^{-qx} \left(\int_{[x,\infty]} y^{-1} G^{(n)}(\mathrm{d}y)\right) \mathrm{d}x.$$

Thus, for any $a > 0$, $\sup_{\mathrm{Re}\, q > a} |q^{-1}\Phi^{(n)}(q)| \leq a^{-1}\Phi^{(n)}(a)$. Since $\Phi^{(n)}(q)$ converges for all $q > 0$, we have $\sup_n \sup_{\mathrm{Re}\, q > a} |q^{-1}\Phi^{(n)}(q)| < \infty$. This proves (3.7).

2. Cauchy's integral formula and the claim imply $\partial_q^k \Phi^{(n)} \to \partial_q^k \Phi$ for every $k \in \mathbb{N}$. Since $\partial_q \Phi^{(n)}$ are completely monotone, so is the limit $\partial_q \Phi$.

3. Thus, $\partial_q \Phi = \int_{[0,\infty)} \mathrm{e}^{-qx} G(\mathrm{d}x)$ is the Laplace transform of a measure G on $[0,\infty)$. We integrate with respect to q with $g_\infty := \Phi(0^+)$ defined to be the charge of $y^{-1}G(\mathrm{d}y)$ at ∞, and use Tonelli's theorem to obtain (3.1). Note that $\int_{[1,\infty]} x^{-1} G(\mathrm{d}x) < \infty$ because $\Phi(q) < \infty$ for each fixed q, so G is a $\overline{g}$-measure.

4. The convergence $\partial_q \Phi^{(n)} \to \partial_q \Phi$ is equivalent to weak convergence of $G^{(n)}$ to G on $[0,\infty)$, meaning (2.11) holds. This implies that for every point x of continuity of G, as $n \to \infty$ we have

$$\int_{[0,x]} \left(1 - \mathrm{e}^{-qy}\right) y^{-1} G^{(n)}(\mathrm{d}y) \quad \to \quad \int_{[0,x]} \left(1 - \mathrm{e}^{-qy}\right) y^{-1} G(\mathrm{d}y), \tag{3.8}$$

and together with (ii) this yields that $\int_{[x,\infty]} y^{-1} G^{(n)}(\mathrm{d}y)$ is bounded and

$$\int_{[x,\infty]} \mathrm{e}^{-qy} y^{-1} G^{(n)}(\mathrm{d}y) \quad \to \quad \int_{[x,\infty]} \mathrm{e}^{-qy} y^{-1} G(\mathrm{d}y).$$

From (ii) then follows (2.12). This proves $G^{(n)} \to G$.

(ii) implies (iii): This is due to $\Psi^{(n)}(q) = \int_0^q \Phi^{(n)}(s)\,\mathrm{d}s$ and monotonicity.

(iii) implies (ii): Since $\Psi^{(n)}(q) = \int_0^q \Phi^{(n)}(s)\,\mathrm{d}s \geq \frac{1}{2} q \Phi^{(n)}(\frac{1}{2}q)$, we find that (3.7) holds as in step 1 above. Then for every subsequence of $\Phi^{(n)}$ there is a further subsequence that converges on compact sets of $\mathbb{C}_+$ to an analytic limit Φ. This limit is unique due to (iii), and (ii) follows. (It follows also that $\Psi^{(n)} \to \Psi$ uniformly on compact sets.) □

4 The Constant Kernel

In this section we study eternal solutions and the Lévy-Khintchine representation in particular for the constant kernel $K = 2$. This kernel is technically easiest to deal with, and the general framework is most transparent. Theorems 4.2, 4.3, and 4.4 are the main technical results and serve to establish Theorems 2.4 and 2.8 for this kernel.

4.1 Preliminaries

Smoluchowski's equation with constant kernel $K = 2$ has a unique global solution in an appropriate weak sense given any initial size-distribution measure with finite zero-th moment [20, Sect. 2]. For convenience, we adopt the normalization in (2.3). The moment identity (2.1) is valid for all bounded continuous functions f on $\bar{E}$, and taking $f = 1$ we find that the total number density of clusters is $\nu_t(E) = t^{-1}$. Since $t\nu_t(E) = 1$, we associate to each solution a probability distribution function

$$F_t(x) = \int_{(0,x)} \nu_t(\mathrm{d}x) \Big/ \int_E \nu_t(\mathrm{d}x) = t\nu_t(x). \tag{4.1}$$

We also introduce the g-measures

$$G_t(\mathrm{d}x) = x\nu_t(\mathrm{d}x), \tag{4.2}$$

and associated Laplace exponents

$$\varphi(t,q) = \int_E \big(1-\mathrm{e}^{-qx}\big)\nu_t(\mathrm{d}x) = \int_{\bar{E}} \frac{1-\mathrm{e}^{-qx}}{x} G_t(\mathrm{d}x). \tag{4.3}$$

Notice that $q \mapsto \varphi(t,q)$ is strictly increasing with $\varphi(t,\infty) = \nu_t(E) = t^{-1}$, and $\partial_q\varphi(t,q)$ is the Laplace transform of the mass-distribution measure $x\nu_t(\mathrm{d}x)$, so is completely monotone. φ solves the simple equation

$$\partial_t\varphi = -\varphi^2, \tag{4.4}$$

for which the solution at any time $t > 0$ is determined from data at time $t_0 > 0$ according to

$$\varphi(t,q) = \frac{\varphi(t_0,q)}{1+(t-t_0)\varphi(t_0,q)}, \quad q \ge 0,\ t > 0. \tag{4.5}$$

Since $0 \le \varphi(t_0,q) < t_0^{-1}$, we see that given $F_{t_0} = t_0\nu_{t_0}$ an arbitrary probability measure, $\varphi(t,q)$ is well-defined on the time-interval $(0,\infty)$. But for $0 < t < t_0$, $\varphi(t,q)$ may not have completely monotone derivative, and thus may not define a (positive) measure. The map $q \mapsto \partial_q\varphi(t,q)$ is completely monotone for all $t \in (0,\infty)$ if and only if ν is an eternal solution.

Our study of convergence properties for solution sequences is based on pointwise convergence properties of φ, which are equivalent to convergence properties of the g-measures G_t according to the results of Sect. 3. We begin by proving the continuous dependence of solutions on initial data, based on the evident fact that $\varphi(t,q)$ is a continuous function of $\varphi(t_0,q)$.

Theorem 4.1 (Continuous dependence on data) *For Smoluchowski's equation with constant kernel $K = 2$, let $t_0 > 0$ and let $\nu^{(n)}$ be a sequence of solutions defined for $t \ge t_0$.*

(a) *If $\nu_{t_0}^{(n)}$ converges weakly to a measure $\hat{\nu}_0$ with $\hat{\nu}_0(E) = t_0^{-1}$, then for every $t \ge t_0$ we have that $\nu_t^{(n)}$ converges weakly to ν_t, the time-t solution with initial data $\nu_{t_0} = \hat{\nu}_0$.*

(b) *For any $t \ge t_0$, if $\nu_t^{(n)}$ converges weakly to a measure $\hat{\nu}$ with $\hat{\nu}(E) = t^{-1}$, then $\nu_{t_0}^{(n)}$ converges weakly to a measure $\hat{\nu}_0$ with $\hat{\nu}_0(E) = t_0^{-1}$, and $\hat{\nu} = \nu_t$, the time-t solution with initial data $\nu_{t_0} = \hat{\nu}_0$.*

Proof We prove part (b); part (a) is similar. Let $G_t^{(n)}(\mathrm{d}x) = x\nu_t^{(n)}(\mathrm{d}x)$ and $\hat{G}(\mathrm{d}x) = x\hat{\nu}(\mathrm{d}x)$, and let $\varphi^{(n)}(t,q)$ and $\hat{\varphi}(q)$ be the associated Laplace exponents as in (4.3). The hypothesis is equivalent to saying that the g-measures $G_t^{(n)}(\mathrm{d}x)$ converge to a nondivergent g-measure $\hat{G}(\mathrm{d}x) = x\hat{\nu}(\mathrm{d}x)$ with $\int_E x^{-1}\hat{G}(\mathrm{d}x) = t^{-1}$. This is equivalent to the statement that for all $q > 0$, $\varphi^{(n)}(t, q) \to \hat{\varphi}(q)$ as $n \to \infty$, where $\hat{\varphi}(\infty) = t^{-1}$ and $\hat{\varphi}(0^+) = 0$. Then it follows that

$$\varphi^{(n)}(t_0, q) = \frac{\varphi^{(n)}(t, q)}{1 - (t - t_0)\varphi^{(n)}(t, q)} \quad \to \quad \hat{\varphi}_0(q) := \frac{\hat{\varphi}(q)}{1 - (t - t_0)\hat{\varphi}(q)}. \tag{4.6}$$

Since $\varphi_0(0^+) = 0$ and $\hat{\varphi}_0(\infty) = t_0^{-1}$, we conclude that $\hat{\varphi}_0$ is the Laplace exponent for a measure $\hat{\nu}_0$ on E with $\hat{\nu}_0(E) = t_0^{-1}$, and that $\nu_{t_0}^{(n)}$ converges weakly to $\hat{\nu}_0$. We compare (4.6) with the explicit solution (4.5) to see that $\hat{\nu} = \nu_t$, where ν is the solution on $[t_0, \infty)$ with initial data $\hat{\nu}_0$. □

4.2 The Scaling Attractor and Eternal Solutions

We are ready to prove Theorem 2.4 for the kernel $K = 2$. First we consider part (b), the correspondence between the scaling attractor and eternal solutions.

Theorem 4.2 *A probability measure $\hat{F}$ is an element of the scaling attractor for Smoluchowski's equation with constant kernel $K = 2$ if and only if $\hat{F} = \nu_1$ for some eternal solution ν.*

Proof Let us first suppose that $\hat{F} = \nu_1$ for some eternal solution ν and show that $\hat{F} \in \mathcal{A}_\mathrm{p}$. Pick arbitrary sequences $T_n, \beta_n \to \infty$, and consider the sequence of rescaled eternal solutions

$$\nu_t^{(n)}(\mathrm{d}x) = \frac{1}{T_n}\nu_{t/T_n}\big(\beta_n^{-1}\,\mathrm{d}x\big), \quad t > 0. \tag{4.7}$$

Observe that $\nu_t^{(n)}(E) = t^{-1}$, therefore,

$$F_{T_n}^{(n)}(\beta_n x) = \nu_1(x) = \hat{F}(x)$$

for every x. Thus, $\hat{F} \in \mathcal{A}_\mathrm{p}$ by Definition 2.3.

Conversely, suppose $\hat{F} \in \mathcal{A}_\mathrm{p}$. We shall show that $\hat{F} = \nu_1$ for some eternal solution ν. Let $\hat{\varphi}$ denote the Laplace exponent of $\hat{F}$, and $\nu^{(n)}, T_n, \beta_n$ be as in Definition 2.3. Consider the rescaled measures

$$\tilde{\nu}_t^{(n)}(\mathrm{d}x) = T_n \nu_{tT_n}^{(n)}(\beta_n\,\mathrm{d}x).$$

This rescaling yields a solution that is defined for $t \geq 1/T_n$, and by hypothesis we have that $\tilde{\nu}_1^{(n)}$ converges weakly to $\hat{F}$. Then, by Theorem 4.1, for any $t > 0$ we infer that $\tilde{\nu}_t^{(n)}$ converges weakly to ν_t where $\nu_t(E) = t^{-1}$ and ν is a solution with $\nu_1 = \hat{F}$. The solution ν is eternal since it is defined for $t \geq t_0$ for every $t_0 > 0$. □

Let us now prove that $\mathcal{A}_\mathrm{p}$ is invariant (part (a) of Theorem 2.4). Suppose ν is a solution on some time interval $[t_1, \infty)$, normalized so $\nu_t(E) = t^{-1}$. Suppose $F_T \in \mathcal{A}_\mathrm{p}$ for some $T \geq t_1$. Replacing $\nu_t(\mathrm{d}x)$ by $T\nu_{Tt}(\mathrm{d}x)$, we may presume $T = 1$ without loss of generality. By Theorem 4.2, $F_T = \tilde{\nu}_1$ for some eternal solution $\tilde{\nu}$. But then $\nu_t = \tilde{\nu}_t$ for all $t \geq t_1$, meaning that ν is (the restriction of) an eternal solution. We obtain that $F_t \in \mathcal{A}_{\mathrm{p}}$ for all $t > 0$ by a similar scaling argument.

4.3 Lévy-Khintchine Representation of Eternal Solutions

Theorem 4.3

(a) *Let ν be an eternal solution to Smoluchowski's equation with $K = 2$. Then there is a divergent g-measure H such that as $t \downarrow 0$, the mass measure $G_t(\mathrm{d}x) = x\,\nu_t(\mathrm{d}x)$ converges to H.*

(b) *Conversely, given any divergent g-measure H there is a unique eternal solution with the properties in part* (a), *defined for all $t \in (0, \infty)$ via*

$$\varphi(t,q) = \frac{\Phi(q)}{1 + t\Phi(q)}, \qquad \Phi(q) = \int_{\bar{E}} \frac{1 - \mathrm{e}^{-qx}}{x} H(\mathrm{d}x). \tag{4.8}$$

Proof We first show (a). Immediately from the solution formula (4.5),

$$\lim_{t\to 0} \varphi(t,q) = \lim_{t\to 0} \frac{\varphi(1,q)}{1 + (t-1)\varphi(1,q)} = \frac{\varphi(1,q)}{1 - \varphi(1,q)} =: \Phi(q) \tag{4.9}$$

exists for all $q > 0$, with $\Phi(q) < \infty$, $\Phi(0^+) = 0$, and $\Phi(\infty) = \infty$. By Theorem 3.1, G_t converges to a $\overline{g}$-measure H with Laplace exponent Φ, and H is a divergent g-measure by the criteria in (3.5), (3.6).

Let us now prove (b). Let H be a divergent g-measure with Laplace exponent Φ. By (4.9), any eternal solution with the properties in part (a) must be determined by (4.8). Observe that the function $q/(1+tq)$ has completely monotone derivative for $t \in (0, \infty)$. It follows that $\partial_q \varphi(t,q)$ is completely monotone when $\varphi(t,q)$ is given by (4.8) [9, XIII.4]. Moreover, with ν_t determined from (4.3), $\nu_t(E) = \varphi(t, \infty) = t^{-1}$. Thus, ν_t is indeed an eternal solution. □

Remark 4.1 Observe that $G_t(E) = \int_E x\nu_t(\mathrm{d}x)$ is finite for some $t \in (0, \infty)$ if and only if it is finite for all t. However, it is not necessary that the mass be finite for a solution to be well-defined.

Theorem 4.3 establishes parts (a) and (b) of Theorem 2.8. To establish part (c), we need to show that the map $\nu_1 \mapsto H$ from $\mathcal{A}_{\mathrm{p}}$ to $\mathcal{S}_d$ is a bi-continuous bijection.

Theorem 4.4 *Let $\nu^{(n)}$ be a sequence of eternal solutions with corresponding divergent g-measures $H^{(n)}$. Fix $t > 0$. Then, taking $n \to \infty$, the following are equivalent*:

(i) *$\nu_t^{(n)}$ converges weakly to some measure $\hat{\nu}$ with $\hat{\nu}(E) = t^{-1}$.*
(ii) *$H^{(n)}$ converges to some divergent g-measure H.*

If either (*equivalently both*) *of these conditions hold, then $\hat{\nu} = \nu_t$ for an eternal solution with divergent g-measure H.*

Proof Assume (i), so $\nu_t^{(n)}$ converges to $\hat\nu$ with $\hat\nu(E)=t^{-1}$. Then $G_t^{(n)}(\mathrm{d}x)=x\nu^{(n)}(\mathrm{d}x)$ converges to $\hat G(\mathrm{d}x)=x\hat\nu(\mathrm{d}x)$ and the associated Laplace exponents converge: $\varphi^{(n)}(t,q)\to\hat\varphi(q)$ for all $q>0$. Hence

$$\Phi^{(n)}(q)=\frac{\varphi^{(n)}(t,q)}{1-t\varphi^{(n)}(t,q)} \quad\to\quad \Phi(q):=\frac{\hat\varphi(q)}{1-t\hat\varphi(q)}, \tag{4.10}$$

as $n\to\infty$ for every $q>0$. Since $\hat\varphi(0^+)=0$ and $t\hat\varphi(q)\to 1$ as $q\to\infty$, $\Phi(q)<\infty$ for every $q>0$, $\Phi(0^+)=0$, and $\lim_{q\to\infty}\Phi(q)=\infty$. By Theorem 3.1 and (3.5), (3.6), this proves (ii).

We now show that (ii) implies (i). Suppose the divergent g-measures $H^{(n)}$ converge to a divergent g-measure H. Then Theorem 3.1 with (3.5), (3.6) implies $\Phi^{(n)}(q)\to\Phi(q)$ for every $q>0$, $\Phi(0^+)=0$, and $\Phi(q)\to\infty$ as $q\to\infty$. Then,

$$\varphi^{(n)}(t,q)=\frac{\Phi^{(n)}(q)}{1+t\Phi^{(n)}(q)} \quad\to\quad \frac{\Phi(q)}{1+t\Phi(q)}=\varphi(t,q)$$

for every $q>0$. This yields weak convergence of $\nu_t^{(n)}$ to ν_t, where ν is the eternal solution with Laplace exponent Φ and divergent g-measure H. □

4.4 Scaling Limits and Initial Tails

We now prove Theorem 2.11 for the constant kernel.

Proof of Theorem 2.11 Introduce rescaled solutions $\tilde\nu_t^{(n)}(\mathrm{d}x)=T_n\nu_{tT_n}^{(n)}(\beta_n\,\mathrm{d}x)$, and let $\tilde F_t^{(n)}=t\tilde\nu_t^{(n)}$. Also let $\tilde G_t^{(n)}(\mathrm{d}x)=x\tilde\nu_t^{(n)}(\mathrm{d}x)$ and let $\tilde\varphi^{(n)}(t,q)$ be the associated Laplace exponent. Let H be the divergent g-measure corresponding to ν and Φ its Laplace exponent, and let $\varphi(q)$ be the Laplace exponent of $G(\mathrm{d}x)=x\nu_1(\mathrm{d}x)$.

Then statement (i) of Theorem 2.11 is equivalent to saying $\tilde F_1^{(n)}\to\hat F$ weakly, meaning the g-measures $\tilde G_1^{(n)}$ converge to G with $\int_E x^{-1}G(\mathrm{d}x)=1$. This is equivalent to saying

$$\tilde\varphi^{(n)}(1,q)\to\varphi(q),\qquad q>0,\ \text{where } \varphi\big(0^+\big)=0,\ \varphi(\infty)=1. \tag{4.11}$$

On the other hand, since $T_nx\nu_1^{(n)}(\beta_n\,\mathrm{d}x)=\tilde G_{1/T_n}(\mathrm{d}x)$, statement (ii) of Theorem 2.11 is equivalent to the assertion

$$\tilde\varphi^{(n)}\big(T_n^{-1},q\big)\to\Phi(q),\qquad q>0,\qquad \text{where } \Phi\big(0^+\big)=0,\ \Phi(\infty)=\infty. \tag{4.12}$$

But by the solution formulae (4.5) and (4.8), we have

$$\tilde\varphi^{(n)}\big(T_n^{-1},q\big)=\frac{\tilde\varphi^{(n)}(1,q)}{1+(T_n^{-1}-1)\tilde\varphi^{(n)}(1,q)},\qquad \Phi(q)=\frac{\varphi(q)}{1-\varphi(q)}.$$

Since evidently (4.11) is equivalent to (4.12), (i) is equivalent to (ii). □

4.5 The Representation at $+\infty$

For the additive kernel, Bertoin showed that an eternal solution can be uniquely identified by its asymptotic behavior as $t \to \infty$ also. For the constant kernel, an analogous result follows easily from (4.3) and (4.8).

Theorem 4.5 *Let ν be an eternal solution of Smoluchowski's equation with constant kernel $K = 2$, and let Φ be the Laplace exponent of the divergent g-measure associated with ν. Then as $t \to \infty$, the measure $t^2\nu_t$ converges weakly on $(0,\infty)$ to a measure Λ_+ with Laplace transform*

$$\Phi_+(q) := \int_0^\infty \mathrm{e}^{-qx}\Lambda_+(\mathrm{d}x) = \frac{1}{\Phi(q)} = \lim_{t\to\infty} t^2 \int_0^\infty \mathrm{e}^{-qx}\nu_t(\mathrm{d}x).$$

Clearly an eternal solution ν is uniquely determined from Λ_+ through $\Phi(q) = 1/\Phi_+(q)$. We see that the measure Λ_+ has a Laplace transform $\Phi_+(q)$ defined for all $q > 0$, and $\Phi_+(q) \to \infty$ as $q \to 0$ since $\Phi(0^+) = 0$. So $\int_0^1 \Lambda_+(\mathrm{d}x) < \infty$ and $\int_E \Lambda_+(\mathrm{d}x) = \infty$.

The class of measures Λ_+ which arise in this way is characterized by the property that $\eta(q) = \partial_q(1/\Phi_+(q))$ is the Laplace transform of some divergent g-measure H (i.e., η is completely monotone, locally integrable on $[0,\infty)$ and $\int_E \eta(q)\,\mathrm{d}q = \infty$). There does not appear to be a simple characterization by moment conditions.

Remark 4.2 This representation has an interesting probabilistic interpretation; (see [2, p. 74]). If $X_{\cdot}$ is a subordinator with Laplace exponent Φ, then $\Phi_+ = 1/\Phi$ is the Laplace transform of the potential measure U, defined on Borel sets $A \subset E$ by $U(A) = \mathbb{E}(\int_0^\infty \mathbf{1}_{\{X_s\in A\}}\,\mathrm{d}s)$.

5 The Additive Kernel

In this section we study the scaling dynamics for the additive kernel. Our main aims are to prove continuous dependence on initial data, establish the correspondence between points on the scaling attractor and eternal solutions, and revisit Bertoin's Lévy-Khintchine representation with convergence of g-measures in mind.

5.1 Solution by Laplace Transform

The solution of Smoluchowski's equation with kernel $K = x + y$ by the Laplace transform is classical [8], and remains the basis for rigorous work. Let $t_0 \in \mathbb{R}$ be arbitrary. We assume ν_{t_0} is a (possibly infinite) measure with $\int_E x\nu_{t_0}(\mathrm{d}x) < \infty$. Without loss of generality, we may assume $\int_E x\nu_{t_0}(\mathrm{d}x) = 1$. We have shown [20, Theorem 2.8] that (1.1) has a unique solution ν_t for $t \ge t_0$ in an appropriate weak sense, such that

$$\int_E x\nu_t(\mathrm{d}x) = 1, \quad t \ge t_0. \tag{5.1}$$

As for the constant kernel, we use the notation

$$\varphi(t,q)=\int_E \bigl(1-\mathrm{e}^{-qx}\bigr)\nu_t(\mathrm{d}x), \quad q\geq 0, \tag{5.2}$$

and set $\varphi_0(q)=\varphi(t_0,q)$. To study scaling limits we consider the mass distribution function, which is the natural probability distribution function associated to a solution. Let

$$F_t(x)=\int_{(0,x]} y\,\nu_t(\mathrm{d}y). \tag{5.3}$$

Note that the Laplace transform of F_t is

$$\int_E \mathrm{e}^{-qx}F_t(\mathrm{d}x)=\partial_q\varphi(t,q). \tag{5.4}$$

Thus, $\partial_q\varphi(t,q)$ is completely monotone and $\partial_q\varphi(t,0)=1$, $t\geq 0$. We know from [20] that if we substitute $f(x)=1-\mathrm{e}^{-qx}$ in (2.1) we find that $\varphi(t,q)$ solves the hyperbolic equation

$$\partial_t\varphi-\varphi\partial_q\varphi=-\varphi. \tag{5.5}$$

Following Bertoin, it is convenient to introduce the new variables

$$s=\mathrm{e}^t, \quad s_0=\mathrm{e}^{t_0}, \quad \psi(s,q)=\frac{q}{s}-\varphi\biggl(t,\frac{q}{s}\biggr). \tag{5.6}$$

By (5.1) and (5.2), ψ is the Laplace exponent

$$\psi(s,q)=\int_E y^{-2}\bigl(\mathrm{e}^{-qy}-1+qy\bigr)G_{\log s}(\mathrm{d}y), \tag{5.7}$$

where G_t denotes the g-measure

$$G_t(\mathrm{d}x)=x^2\nu_t\bigl(\mathrm{e}^t\,\mathrm{d}x\bigr). \tag{5.8}$$

Observe that G_t is not a finite measure in general, but if $G_t(E)<\infty$ for some t, then $G_t(E)$ is finite for every t for which the solution is defined, and is constant. We substitute (5.6) in (5.5) to see that ψ satisfies the inviscid Burgers equation

$$\partial_s\psi+\psi\partial_q\psi=0, \quad s>s_0. \tag{5.9}$$

The values of ψ, $\partial_q\psi$, and $\partial_q^2\psi$ are positive for $s\geq s_0$, $q>0$, and $\partial_q^2\psi(s,\cdot)$ is completely monotone since it is the Laplace transform of G_t. In addition, (5.1), (5.7), and (5.8) imply

$$\lim_{q\to\infty}\partial_q\psi(s,q)=\int_E x^{-1}G_{\log s}(\mathrm{d}x)=s^{-1}. \tag{5.10}$$

We may describe $\psi(s,q)$ globally for $s>s_0$ by the method of characteristics. A surprising fact is that we may always solve for ψ backwards in time, for all $s>0$, without developing singularities. The solution need not correspond to a positive measure ν_t for $t<t_0$, however. This is analogous to the situation for $K=2$.

Lemma 5.1 *Let $t_0 \in \mathbb{R}$ and $\nu_{t_0} \in \mathcal{M}$ with $\int_E x\, \nu_{t_0}(\mathrm{d}x) = 1$, and let $\psi_0(q_0) = q_0/s_0 - \varphi_0(q_0/s_0)$. There is a unique solution $\psi(s,q)$ to (5.9) defined for every $s > 0$ and $q > 0$, such that $\psi(s_0,\cdot) = \psi_0(\cdot)$.*

Proof Applying the method of characteristics as usual, the solution $\psi = \psi(s,q)$ is determined implicitly from the equation

$$h(s,q,\psi) := \psi - \psi_0\big(q - (s - s_0)\psi\big) = 0. \tag{5.11}$$

We have $h(s,q,0) < 0$, and $\partial_\psi h > s/s_0$ since $\partial_q \psi_0 < s_0^{-1}$ by (5.10). Since ψ_0 is analytic, (5.11) determines a solution of (5.9) analytic in (s,q) for all $s > 0$, $q > 0$. □

Equation (5.11) determines the solution at time s from data at time s_0 and plays the same role in the analysis here as (4.5) played in the previous section. Convergence properties of solutions will be deduced from the pointwise convergence properties of the Laplace exponent ψ using the theory from Sect. 3.

Theorem 5.2 (Continuous dependence on data) *For Smoluchowski's equation with additive kernel $K = x + y$, let $t_0 \in \mathbb{R}$ and let $\nu^{(n)}$ be a sequence of solutions defined for $t \ge t_0$ with $\int_E x\nu_t^{(n)}(\mathrm{d}x) = 1$ for all $t \ge t_0$.*

(a) *If $x\nu_{t_0}^{(n)}(\mathrm{d}x)$ converges weakly to a measure $x\hat{\nu}_0(\mathrm{d}x)$ with $\int_E x\hat{\nu}_0(\mathrm{d}x) = 1$, then for every $t \ge t_0$ we have that $x\nu_t^{(n)}(\mathrm{d}x)$ converges weakly to $x\nu_t(\mathrm{d}x)$, the time-t solution with initial data $\nu_{t_0} = \hat{\nu}_0$.*
(b) *For any $t \ge t_0$, if $x\nu_t^{(n)}(\mathrm{d}x)$ converges weakly to a measure $x\hat{\nu}(\mathrm{d}x)$ with $\int_E x\,\hat{\nu}(\mathrm{d}x) = 1$, then $x\nu_{t_0}^{(n)}(\mathrm{d}x)$ converges weakly to a measure $x\hat{\nu}_0(\mathrm{d}x)$ with $\int_E x\,\hat{\nu}_0(\mathrm{d}x) = 1$, and $\hat{\nu} = \nu_t$, the time-t solution with initial data $\nu_{t_0} = \hat{\nu}_0$.*

Proof We prove (a); the proof of (b) is similar. Let $G_t^{(n)}(\mathrm{d}x) = x^2\nu_t(\mathrm{e}^t\,\mathrm{d}x)$, and with $s = \mathrm{e}^t$ let

$$\psi^{(n)}(s,q) = \int_E y^{-2}\big(\mathrm{e}^{-qy} - 1 + qy\big)G_t^{(n)}(\mathrm{d}y). \tag{5.12}$$

The family $\psi^{(n)}(s_0,\cdot)$ is uniformly Lipschitz, since (5.10) implies

$$\psi^{(n)}(s_0,0) = 0, \qquad 0 \le \partial_q\psi^{(n)}(s_0,q) \le 1/s_0 \quad \text{for all } q > 0. \tag{5.13}$$

The hypothesis is equivalent to saying that the g-measures $G_{t_0}^{(n)}$ converge to a non-divergent g-measure $\hat{G}_0(\mathrm{d}x) = x^2\hat{\nu}_0(\mathrm{d}x)$ with $\int_E x^{-1}\hat{G}_0(\mathrm{d}x) = 1$. By Theorem 3.1 and the criteria in (3.5), (3.6), this is equivalent to the statement that for all $q > 0$, $\psi^{(n)}(s_0,q) \to \hat{\psi}_0(q)$, where $\hat{\psi}_0$ is the (second-order) Laplace exponent for $\hat{G}_0$, with $\partial_q\hat{\psi}_0(0^+) = 0$, $\partial_q\hat{\psi}_0(\infty) = 1/s_0$. (Note $\partial_q\hat{\psi}$ is the first-order Laplace exponent of $\hat{G}_0$.) As in (5.11) we have

$$\psi^{(n)}(s,q) - \psi^{(n)}\big(s_0, q - (s - s_0)\psi^{(n)}(s,q)\big) = 0. \tag{5.14}$$

For fixed s, q, the sequence $\psi^{(n)}(s,q)$ is bounded, and any subsequential limit ψ_* must satisfy

$$\psi_* - \hat{\psi}_0\big(q - (s - s_0)\psi_*\big) = 0, \tag{5.15}$$

due to the equicontinuity of the maps $\psi \mapsto \psi^{(n)}(s_0, q - (s - s_0)\psi)$. But (5.15) has the unique solution $\psi_* = \psi(s,q)$, where ψ is the solution of (5.9) with $\psi(s_0, q) = \hat{\psi}_0(q)$, $q > 0$. Hence the whole sequence $\psi^{(n)}(s,q)$ converges pointwise to $\psi(s,q)$. Moreover, differentiating (5.15) yields $\partial_q \psi_*(0) = 0$, $\partial_q \psi_*(\infty) = 1/s$, since $s_0 = 1$. Then the conclusion of (a) follows from Theorem 3.1, (3.5) and (3.6). □

5.2 The Scaling Attractor and Eternal Solutions

Theorem 5.3 *A probability measure $\hat{F}$ is an element of the scaling attractor $\mathcal{A}_\mathrm{p}$ for Smoluchowski's equation with additive kernel $K = x + y$ if and only if $\hat{F}(\mathrm{d}x) = x\nu_0(\mathrm{d}x)$ for some eternal solution ν.*

Proof Suppose $\hat{F}(\mathrm{d}x) = x\nu_0(\mathrm{d}x)$ for some eternal solution ν. We show $\hat{F} \in \mathcal{A}_\mathrm{p}$. Pick arbitrary sequences $T_n, b_n \to \infty$, and consider the sequence of rescaled eternal solutions

$$\nu_t^{(n)}(\mathrm{d}x) = b_n^{-1} \nu_{t-T_n}\big(b_n^{-1}\,\mathrm{d}x\big), \qquad t \in \mathbb{R}.$$

The corresponding distribution functions satisfy $F_{T_n}^{(n)}(b_n x) = \hat{F}(x)$ for every x. Thus, $\hat{F} \in \mathcal{A}_\mathrm{p}$ by Definition 2.3.

To prove the converse, suppose $\hat{F} \in \mathcal{A}_\mathrm{p}$. We show that $\hat{F} = F_0$ for some eternal solution ν. Let $\hat{\varphi}$ correspond to $\hat{\nu}$ as in (5.2), and $\nu^{(n)}, T_n, b_n$ be as in Definition 2.3. Consider the rescaled measures

$$\tilde{\nu}_t^{(n)}(\mathrm{d}x) = b_n \nu_{t+T_n}^{(n)}(b_n\,\mathrm{d}x).$$

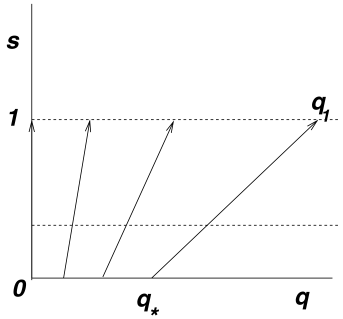
This rescaling yields a solution that is defined for $t \geq -T_n$. By assumption,

$$\tilde{F}_0^{(n)}(x) = \int_0^x y\,\tilde{\nu}_0^{(n)}(\mathrm{d}y) = F_{T_n}^{(n)}(b_n x) \to \hat{F}(x),$$

at all points of continuity. By Theorem 5.2, this implies that for any $N \in \mathbb{N}$ the solutions $\nu_t^{(n)}$ converge weakly to ν_t for all $t \geq -N$. In particular, ν_t is a solution for $t \geq -N$ for all N, thus it is an eternal solution. □

Let us now prove that $\mathcal{A}_\mathrm{p}$ is invariant (part (a) of Theorem 2.4). The proof is substantially the same as for $K = 2$. Suppose ν is a solution on some time interval $[t_1, \infty)$, normalized so $\int_E x\nu_t(\mathrm{d}x) = 1, t \geq t_1$. Suppose $F_T \in \mathcal{A}_\mathrm{p}$ for some $T \geq t_1$. We may presume $T = 0$ without loss (if not, we translate in time, replacing $\nu_t(\mathrm{d}x)$ by $\nu_{t-T}(\mathrm{d}x)$). By Theorem 5.3, $F_T = x\tilde{\nu}_0$ for some eternal solution $\tilde{\nu}$. But then $\nu_t = \tilde{\nu}_t$ for all $t \geq t_1$, meaning that ν is (the restriction of) an eternal solution. We obtain that $F_t \in \mathcal{A}_\mathrm{p}$ for every $t \in \mathbb{R}$ by a similar argument.

Fig. 1 Geometry of characteristics



5.3 Lévy-Khintchine Representation of Eternal Solutions

We now prove Bertoin's Lévy-Khintchine representation for eternal solutions. The proof mainly follows [4], and is included to stress the basic framework.

Theorem 5.4 (cf. Bertoin [4])

(a) *Let ν be an eternal solution to Smoluchowski's equation with $K = x + y$, and let $G_t(\mathrm{d}x) = x^2\nu_t(\mathrm{e}^t\,\mathrm{d}x)$ be associated g-measures. Then there is a unique divergent g-measure H such that G_t converges to H as $t \to -\infty$.*

(b) *Conversely, given a divergent g-measure H there is a unique eternal solution with the properties in part* (a), *defined as follows. Let*

$$\Psi(q) = \int_{\bar{E}} \frac{\mathrm{e}^{-qx} - 1 + qx}{x^2} H(\mathrm{d}x) \tag{5.16}$$

be the Laplace exponent of H, and let $\psi = \psi(s,q)$ be the solution to

$$\psi - \Psi(q - s\psi) = 0. \tag{5.17}$$

Then ν_t is determined by (5.7) *and* (5.8).

Proof We first prove (a). By Theorem 3.1 and (3.5), and (3.6), it is enough to show that $\Psi(q) := \lim_{s\to 0}\psi(s,q)$ exists for every $q \geq 0$, with $\partial_q\Psi(0) = 0$ and $\partial_q\Psi(\infty) = \infty$. We know $\psi \geq 0$ and $\partial_q\psi \geq 0$, so $\partial_s\psi(s,q) \leq 0$ for all q, s. Hence it suffices to show that for each $q > 0$, $\psi(s,q)$ stays bounded as $s \downarrow 0$.

1. We first show $\psi(s,q)$ stays bounded for q near 0. Choose $q_1 > 0$ such that $q_* := q_1 - \psi(1,q_1) = \varphi(0,q_1) > 0$. Then $\psi(s,q) = \psi(1,q_1)$ along the characteristic line joining $(0,q_*)$ and $(1,q_1)$, so $0 \leq \psi(s,q) \leq \psi(1,q_1)$ whenever $0 < s < 1$ and $0 < q \leq q_*$ (see Fig. 1).
2. For $q > q_*$ the complete monotonicity of $q \mapsto q^{-2}\psi(s,q)$ implies $\psi(s,q) < q^2 q_*^{-2}\psi(s,q_*)$.
3. We now show $\partial_q\Psi(0) = 0$ and $\partial_q\Psi(\infty) = \infty$. Observe that Ψ solves

$$\Psi(q) = \psi\big(1, q + \Psi(q)\big), \quad q > 0.$$

Therefore,

$$\partial_q \Psi(q) = \frac{\partial_q \psi(1, q + \Psi(q))}{1 - \partial_q \psi(1, q + \Psi(q))}. \tag{5.18}$$

Since $\psi(1,q) = q - \varphi(0,q)$, we have $\partial_q \psi(1,q) = 1 - \partial_q \varphi(0,q) \to 0$ as $q \to 0$, $\to 1$ as $q \to \infty$. Thus, $\partial_q \Psi(0) = 0$ and $\partial_q \Psi(\infty) = \infty$. This proves (a).

We now prove (b). Let H be a divergent g-measure and Ψ be defined by (5.16). Note $\partial_q \Psi(0) = 0$ and $\partial_q \Psi(\infty) = \infty$ by (3.5), (3.6). Since $\partial_q \Psi(q) > 0$, $\psi(s,q)$ is globally defined and analytic with $\psi(s,q) < q/s$, and (5.9) holds for all $s > 0$, $q > 0$. With Φ and φ defined by (5.6), (5.5) follows.

By the well-posedness theory in [20], we obtain an eternal solution through (5.4), provided we show that $\partial_q \varphi(t, \cdot)$ is completely monotone, which implies that it is the Laplace transform of a (positive) measure that we call $x\nu_t(\mathrm{d}x)$. From (5.17) we obtain that $\varphi = \varphi(t,q)$ satisfies

$$q = \varphi + \Psi(s\varphi), \tag{5.19}$$

whence

$$\partial_q \varphi = \frac{1}{1 + s\Psi'(s\varphi)}. \tag{5.20}$$

Since $q \mapsto 1 + s\Psi'(sq)$ is positive with completely monotone derivative, the map $q \mapsto (1 + s\Psi'(sq))^{-1}$ is completely monotone [9, XIII.4]. We then infer that $\partial_q \varphi(s, \cdot)$ is completely monotone by Lemma 5.5. Since $\Psi'(0) = 0$ we have the normalization (5.1), $\partial_q \varphi(t, 0) = 1$. This finishes the proof of existence.

Note that total number of clusters $\nu_t(E) = \varphi(t, \infty) = \infty$ always here.

Let us show that the eternal solution defined by this procedure is unique. Let H be a divergent g-measure and suppose $\nu, \tilde{\nu}$ are two eternal solutions with g-measures $G_t, \tilde{G}_t$ that converge to H. But this is equivalent to pointwise convergence of $\psi(s,q)$ and $\tilde{\psi}(s,q)$ to $\Psi(q)$ as $s \to 0$ where ψ and $\tilde{\psi}$ solve (5.9). But the solutions to the inviscid Burgers equation with increasing initial data are unique, thus $\psi(s,q) = \tilde{\psi}(s,q)$ and $\nu = \tilde{\nu}$. □

Lemma 5.5 *Suppose $f, g \colon E \to E$, $f' = g(f)$ and g is completely monotone. Then f' is completely monotone.*

Proof We prove by induction that the first n derivatives of $G \circ f$ alternate in sign for every completely monotone function G. For $n = 0$, $G(f) > 0$. Suppose the statement is true for some $n \geq 0$. Let G be completely monotone, and note

$$-(G \circ f)' = -G'(f)g(f) = \tilde{G}(f)$$

and $\tilde{G}$ is completely monotone since it is the product of completely monotone functions. Using the induction hypothesis, we deduce that the first $n+1$ derivatives of $G \circ f$ alternate in sign. □

To complete the proof of Theorem 2.8 for $K = x + y$, we need to check that the map $\nu_0 \mapsto H$ from $\mathcal{A}_\mathrm{p}$ to $\mathcal{S}_d$ is a bicontinuous bijection.

Theorem 5.6 *Let $\nu^{(n)}$ be a sequence of eternal solutions with corresponding divergent g-measures $H^{(n)}$. Fix $t \in \mathbb{R}$. Then, taking $n \to \infty$, the following are equivalent*:

(i) $x\nu_t^{(n)}$ *converges weakly to* $x\hat{\nu}$ *with* $\int_E x\hat{\nu}(\mathrm{d}x) = 1$.
(ii) $H^{(n)}$ *converges to a divergent g-measure H.*

If either (equivalently both) of these conditions hold, then $\hat{\nu} = \nu_t$ for an eternal solution with g-measure H.

Proof With Theorem 3.1 in hand, the proof of Theorem 5.6 is essentially the same as that of Theorem 5.2. Assume (i), so $\nu_t^{(n)}$ converges to $\hat{\nu}$ with $\int_E x\hat{\nu}(E) = 1$. Then $G_t^{(n)}(\mathrm{d}x) = x^2\nu_t^{(n)}(\mathrm{e}^t\,\mathrm{d}x)$ converges to the g-measure $\hat{G}(\mathrm{d}x) = x^2\hat{\nu}(\mathrm{e}^t\,\mathrm{d}x)$ and the associated Laplace exponents converge: $\psi^{(n)}(s,q) \to \hat{\psi}(q)$ for all $q > 0$, with $\partial_q\hat{\psi}(0) = 0$, $\partial_q\hat{\psi}(\infty) = 1/s$. Recall that $\psi^{(n)}(s,q)$ solves

$$\Psi^{(n)}\big(q - s\psi^{(n)}(s,q)\big) = \psi^{(n)}(s,q). \tag{5.21}$$

Let $M > 0$. A calculation as in (5.18) shows that $\partial_q\Psi^{(n)}(q)$ is uniformly bounded in n for $q \in [0, M]$. We claim that $\lim_{n\to\infty}\Psi^{(n)}(q - s\hat{\psi}(q))$ exists for every q. Let us restrict attention to $q \in [0, M]$. Then by (5.21)

$$\Psi^{(n)}\big(q - s\hat{\psi}(q)\big) = \psi^{(n)}(s,q) + \Big(\Psi^{(n)}\big(q - s\hat{\psi}(q)\big) - \Psi^{(n)}\big(q - s\psi^{(n)}(s,q)\big)\Big).$$

The first term converges to $\hat{\psi}(q)$ and the second to zero by the uniform estimate on $\partial_q\Psi^{(n)}(q)$ on $[0, M]$. Since $M > 0$ was arbitrary, we may use Theorem 3.1 to deduce that $\Psi^{(n)}(q)$ converges to a Laplace exponent $\Psi(q)$ that satisfies

$$\Psi\big(q - s\hat{\psi}(q)\big) = \hat{\psi}(q).$$

As with (5.18) and its sequel it follows that $\partial_q\Psi(0) = 0$ and $\partial_q\Psi(\infty) = \infty$. Thus Ψ is the Laplace exponent of a divergent g-measure H, and $H^{(n)}$ converges to H.

We now show (ii) implies (i). Suppose the divergent g-measures $H^{(n)}$ converge to a divergent g-measure H. Then Theorem 3.1 implies $\Psi^{(n)}(q) \to \Psi(q)$ for every $q > 0$, and $\partial_q\Psi(0) = 0$, $\partial_q\Psi(\infty) = \infty$. Then the characteristics emanating from $s = 0$ converge because $q + s\Psi^{(n)}(q) \to q + s\Psi(q)$. Thus, $\psi^{(n)}(s,q) \to \psi(s,q)$, which satisfies (5.17). This yields weak convergence of $x\nu_t^{(n)}$ to $x\nu_t$, where ν is the eternal solution with divergent g-measure H. □

5.4 Scaling Limits and Initial Tails

Let us now prove Theorem 2.11 for the additive kernel.

Proof of Theorem 2.11 We rescale solutions via $\tilde{\nu}_t^{(n)}(\mathrm{d}x) = \beta_n\nu_{t+T_n}^{(n)}(\beta_n\,\mathrm{d}x)$, and let $\tilde{F}_t^{(n)}(\mathrm{d}x) = \beta_n x\tilde{\nu}_t^{(n)}(\mathrm{d}x)$. Also let $\tilde{G}_t^{(n)}(\mathrm{d}x) = x^2\tilde{\nu}_t^{(n)}(\mathrm{e}^t\,\mathrm{d}x)$ and let $\tilde{\psi}^{(n)}(s,q)$ be the associated Laplace exponent as in (5.7). Observe $\tilde{G}^{(n)}$ in (2.24) is $\tilde{G}_{-T_n}^{(n)}$ and $\int_E x^{-1}\tilde{G}^{(n)}(\mathrm{d}x) = \mathrm{e}^{T_n}$. Let H be the divergent g-measure corresponding to ν and Ψ its Laplace exponent, and let $\psi(q)$ be the Laplace exponent of $G(\mathrm{d}x) = x^2\nu_0(\mathrm{d}x)$.

Then (i) is equivalent to saying $\tilde{F}_0^{(n)} \to \hat{F}$ weakly, meaning the g-measures $\tilde{G}_0^{(n)}$ converge to G with $\int_E x^{-1}G(\mathrm{d}x) = 1$. This is equivalent to saying

$$\tilde{\psi}^{(n)}(1,q) \to \psi(q), \quad q > 0, \quad \text{where } \partial_q\psi\big(0^+\big) = 0, \partial_q\psi(\infty) = 1. \tag{5.22}$$

On the other hand, since $\beta_n x^2 \nu_0^{(n)}(\mathrm{e}^{-T_n}\beta_n\,\mathrm{d}x) = \tilde{G}_{-T_n}(\mathrm{d}x)$, (ii) is equivalent to saying

$$\tilde{\psi}^{(n)}\big(\mathrm{e}^{-T_n}, q\big) \to \Psi(q), \quad q > 0, \quad \text{where } \partial_q\Psi\big(0^+\big) = 0, \quad \partial_q\Psi(\infty) = \infty. \tag{5.23}$$

For brevity, let $\tilde{\psi}^{(n)}(q)$ denote $\tilde{\psi}^{(n)}(1,q)$ and $\tilde{\Psi}^{(n)}(q)$ denote $\tilde{\psi}^{(n)}(\mathrm{e}^{-T_n}, q)$. Then the implicit solution formulas to (5.9) read

$$\psi(q) = \Psi\big(q - \psi(q)\big), \qquad \psi^{(n)}(q) = \Psi^{(n)}\big(q - \big(1 - \mathrm{e}^{-T_n}\big)\psi^{(n)}(q)\big).$$

As in the proof of Theorem 5.6 we may now deduce that (5.22) is equivalent to (5.23), implying (i) is equivalent to (ii). The details are omitted. □

6 The Multiplicative Kernel

In this section we study scaling dynamics approaching the gelation time for the kernel $K = xy$. The study of the multiplicative kernel can be reduced to the additive kernel by a simple change of variables. This trick is well-known (see [8]), and allows us to avoid separate proofs.

6.1 Solution by the Laplace Transform

The self-similar solutions for $K = xy$ have infinite number and mass, but finite second moment. However, one may develop a natural well-posedness theory using only the finiteness of the second moment [20]. We assume ν_{t_0} is a (possibly infinite) measure with $\int_E x^2\nu_{t_0}(\mathrm{d}x) < \infty$. Without loss of generality, we may scale so that $\int_E x^2\nu_{t_0}(\mathrm{d}x) = 1$ and $t_0 = -1$ as in (2.3). We define the Laplace exponent (note the change from (5.2))

$$\varphi(t,q) = \int_E \big(1 - \mathrm{e}^{-qx}\big)x\nu_t(\mathrm{d}x), \quad q \ge 0, \tag{6.1}$$

and write $\varphi_0(q) = \varphi(t_0, q)$. We may substitute (6.1) in the moment identity (2.1) to obtain

$$\partial_t\varphi - \varphi\partial_q\varphi = 0, \quad t \in (t_0, 0). \tag{6.2}$$

Equation (6.2) may be transformed to (5.5) by the following change of variables. Let $\varphi^{\mathrm{add}}(\tau, q)$, $\tau \in [0,\infty)$, denote a solution to (5.5) with initial data $\varphi_0(q)$. Then the solution to (6.2) is given by

$$\varphi(t,q) = \mathrm{e}^{\tau}\varphi^{\mathrm{add}}(\tau, q), \quad \tau = \log\big(|t|^{-1}\big), \tag{6.3}$$

which may also be written in terms of the number density as

$$x\nu_t(\mathrm{d}x) = \mathrm{e}^{\tau}\nu_\tau^{\mathrm{add}}(\mathrm{d}x). \tag{6.4}$$

Conservation of mass (5.1) is now replaced by

$$\int_E x^2\nu_t(\mathrm{d}x) = |t|^{-1}, \quad t \in [t_0, 0), \tag{6.5}$$

and the probability measure F_t associated to ν_t is defined by

$$F(t,x) = |t|\int_0^x y^2\nu_t(\mathrm{d}y). \tag{6.6}$$

As in (5.8) we define the g-measure

$$G_t(\mathrm{d}x) = x^3\nu_t\big(|t|^{-1}\,\mathrm{d}x\big) = G_\tau^{\mathrm{add}}(\mathrm{d}x), \quad t \in [t_0, 0), \tag{6.7}$$

and the associated Laplace exponent

$$\psi(t,q) = \int_{\bar{E}} y^{-2}\big(\mathrm{e}^{-qy} - 1 + qy\big)G_t(\mathrm{d}y) = \psi^{\mathrm{add}}\big(|t|^{-1}, q\big). \tag{6.8}$$

The correspondences (6.4), (6.7), and (6.8) map normalized solutions for $K = xy$ on the time interval $t \in [-1, 0)$ to normalized solutions with $K = x + y$ on the interval $\tau \in [0, \infty)$. The same change of variables may be applied to eternal solutions defined on $t \in (-\infty, 0)$. By consequence, the results established so far for the additive kernel carry over in an obvious way for the multiplicative kernel. This yields continuous dependence of solutions on data (by Theorem 5.2), the correspondence between the scaling attractor and eternal solutions (Theorem 2.4), the Lévy-Khintchine representation (Theorem 2.8), and how initial tails encode scaling limits (Theorem 2.11). For completeness, we make explicit the map from divergent g-measures to eternal solutions implicit in Theorem 2.8(b).

Theorem 6.1 *Given a divergent g-measure G there is a unique eternal solution defined as follows. Let*

$$\Psi(q) = \int_{\bar{E}} \frac{\mathrm{e}^{-qx} - 1 + qx}{x^2} H(\mathrm{d}x), \tag{6.9}$$

and $\psi(t,q)$, $t \in (-\infty, 0)$ be the solution to

$$\psi - \Psi\big(q + t^{-1}\psi\big) = 0. \tag{6.10}$$

Then ν_t is determined by (6.7) and (6.8).

7 Doeblin Solutions

This section is inspired by Feller's treatment of Doeblin's universal laws and domains of partial attraction [9, XVII.9]. But apparently we must be content with using more words to prove fewer results. Our aim is to prove:

Theorem 7.1 *There exists an eternal solution ν whose scaling ω-limit set contains every element of the proper scaling attractor, $\mathcal{A}_{\rm p}$.*

We will show later that $\mathcal{A}$ is the closure of $\mathcal{A}_{\rm p}$ (see Corollary 9.7). Therefore, Theorem 7.1 establishes Theorem 2.12.

The proof is based on suitably "packing the tails" of the corresponding divergent g-measure. The following is adapted from Feller [9, XVII.9]. Given a g-measure G and $a, b > 0$ we define a rescaled measure $G^{a,b}$ by

$$G^{a,b}(x) = aG(bx). \tag{7.1}$$

Lemma 7.2 *Let G_k be a sequence of g-measures with*

$$\int_{\bar{E}} x^{-1} G_k(\mathrm{d}x) \le k. \tag{7.2}$$

Then there exist sequences a_k, b_k such that $a_k \to 0$, $a_k b_k \to \infty$,

$$G := \sum_{k=1}^{\infty} G_k^{a_k^{-1}, b_k^{-1}} \tag{7.3}$$

defines a g-measure, and $G^{a_k,b_k} - G_k$ converges to zero.

The growth assumption (7.2) is included only for concreteness and implies no real loss of generality. Our main purpose is to approximate divergent g-measures.

Lemma 7.3 *Let H be a divergent g-measure. There exists a sequence of g-measures G_k satisfying (7.2) such that G_k converges to H.*

Proof of Theorem 2.12

1. Let $\tilde{\nu}^{(n)}$ be an arbitrary sequence of eternal solutions with corresponding divergent g-measures $\tilde{H}^{(n)}$. Partition the integers into infinitely many subsequences, and choose $G_k \to \tilde{H}^{(n)}$ for k in the nth subsequence as in Lemma 7.3.
2. Now define a_k, b_k and G as in Lemma 7.2, and put

$$H = \delta_0 + G.$$

   H has an atom at the origin, thus is the divergent g-measure for an eternal solution. By construction, $G^{a_k,b_k} \to \tilde{H}^{(n)}$ as $k \to \infty$ along the nth subsequence. Moreover, since $a_k \to 0$, under rescaling $\delta_0^{a_k,b_k} = a_k\delta_0$ converges to zero. Thus, if we take limits along the nth subsequence, $H^{a_k,b_k} \to \tilde{H}^{(n)}$.

3. We now apply Theorem 2.8 together with (8.1). We have $H^{a_k,b_k} = \mathfrak{S}_p(F_{t_0}^{a_k,b_k})$ and $F_{t_0}^{a_k,b_k}(x) = F_{T_k}(\beta_k x)$ where

$$(T_k,\beta_k) = \begin{cases} (a_k b_k, b_k) & (K=2), \\ (\log(a_k b_k), a_k b_k^2) & (K=x+y), \\ (-(a_k b_k)^{-1}, a_k b_k^2) & (K=xy). \end{cases} \tag{7.4}$$

Observe that $T_k \to T_{\min}$ and $\beta_k \to \infty$. We take limits along the nth subsequence to obtain $F_{T_k}(\beta_k x) \to \tilde{F}^{(n)}(x)$ at every point of continuity. Hence, *given any sequence of eternal solutions $\nu^{(n)}$ there exists an eternal solution ν whose scaling ω-limit set contains each $\tilde{\nu}_1^{(n)}$.*
4. The space of divergent g-measures is separable. The g-measures which are concentrated at finitely many rational points (including 0) with rational weights form a countable set which is dense with respect to convergence of g-measures. By ordering these in a sequence $\tilde{H}^{(n)}$ and using the construction above, we see that there exist eternal solutions ν such that for *every* eternal solution $\tilde{\nu}$, $\tilde{\nu}_1$ is in the scaling ω-limit set of ν. This finishes the proof of the theorem. □

7.1 The Packing Lemma

We will need to choose a sequence c_k that grows so fast that

$$c_k \sum_{j=k+1}^{\infty} j c_j^{-1} \to 0.$$

The choice $c_k = \mathrm{e}^{k^2}$ will do. For $j \geq 2$ we have the elementary estimate

$$j\mathrm{e}^{-j^2} < \int_{j-1/2}^{j+1/2} y\mathrm{e}^{-y^2}\,\mathrm{d}y = \mathrm{e}^{-j^2-1/4}\cosh j.$$

Therefore for $k \geq 1$,

$$\mathrm{e}^{k^2} \sum_{j=k+1}^{\infty} j\mathrm{e}^{-j^2} < \mathrm{e}^{k^2} \int_{k+1/2}^{\infty} y\mathrm{e}^{-y^2}\,\mathrm{d}y = \frac{\mathrm{e}^{-k-1/4}}{2} \to 0.$$

Proof of Lemma 7.2

1. Fix $a_k b_k = c_k$. Then G defines a g-measure since

$$\int_{\bar{E}} x^{-1} G(\mathrm{d}x) = \sum_{k=1}^{\infty} c_k^{-1} \int_{\bar{E}} x^{-1} G_k(\mathrm{d}x) \leq \sum_{k=1}^{\infty} k c_k^{-1} < \infty.$$

2. Let $\Phi^{(k)}$ and Φ denote the Laplace exponents of G_k and G, respectively. We use the definition (7.1) and (7.3) to obtain $\Phi(q) = \sum_{j=1}^{\infty} c_j^{-1}\Phi^{(j)}(qb_j)$. Observe that $G^{a_k,b_k} - G_k$ is a positive measure with Laplace exponent

$$c_k\Phi\big(qb_k^{-1}\big) - \Phi^{(k)}(q) = c_k\sum_{j\neq k} c_j^{-1}\Phi^{(j)}\big(qb_jb_k^{-1}\big).$$

3. To prove convergence to zero, it suffices to show that the right-hand side converges to zero for every $q > 0$. We first control the tail. Since $\int_{\bar{E}} x^{-1}G_j(\mathrm{d}x) \le j$,

$$c_k \sum_{j=k+1}^{\infty} c_j^{-1}\Phi^{(j)}\big(qb_jb_k^{-1}\big) \le c_k \sum_{j=k+1}^{\infty} jc_j^{-1} \to 0.$$

4. We now choose b_k inductively to control the first $k-1$ terms in the range $0 \le q \le k$. Suppose $b_1, \dots, b_{k-1}$ have been chosen. Since $\Phi^{(j)}(q) \to 0$ as $q \to 0$, we choose b_k so large that

$$a_k = c_kb_k^{-1} \le \frac{1}{k}, \quad c_k\sum_{j=1}^{k-1} c_j^{-1}\Phi^{(j)}\big(kb_jb_k^{-1}\big) \le \frac{1}{k}.$$

□

7.2 Proof of Lemma 7.3

First, suppose H has no atom at the origin. Since $\int_x^{\infty} y^{-1}H(\mathrm{d}y) \to 0$ as $x \to \infty$, we may choose a decreasing sequence ε_k such that $\int_{\varepsilon_k}^{\infty} y^{-1}H(\mathrm{d}y) \le k$. Let $G_k(\mathrm{d}y) = \mathbf{1}_{y>\varepsilon_k}H(\mathrm{d}y)$. Clearly, G_k satisfies both conditions of Definition 2.6.

Next, let $H = \delta_0$. In this case we choose $G_k(\mathrm{d}x) = (x\log k)^{-1}\mathbf{1}_{x\ge k^{-1}}dx$. Then G_k satisfies (7.2) as

$$\int_{\bar{E}} x^{-1}G_k(\mathrm{d}x) = (\log k)^{-1}\int_{k^{-1}}^{\infty} x^{-2}\,\mathrm{d}x = k(\log k)^{-1} \le k,$$

and

$$\Phi^{(k)}(q) = q(\log k)^{-1}\int_{qk^{-1}}^{\infty} \frac{1-\mathrm{e}^{-x}}{x^2}\,\mathrm{d}x \to q = \Phi(q).$$

The general case follows by superposition of these two special cases.

8 Scaling-Periodic Solutions

In this section we characterize scaling-periodic solutions and show that they are dense in the scaling attractor. That is, we prove Theorems 2.14 and 2.15.

8.1 Characterization

Proof of Theorem 2.14

1. Given a scaling-periodic solution, a solution satisfying (2.25), we can scale it as in (2.16), (2.17) so that t_0 is as in (2.6). Then, under the map $F \mapsto F^{a,b}$ given by

$$F_t^{a,b}(x) = \begin{cases} F_{abt}(bx) & (K=2), \\ F_{t+\log(ab)}(ab^2x) & (K=x+y), \\ F_{t/ab}(ab^2x) & (K=xy), \end{cases} \tag{8.1}$$

for some $a, b > 0$ we have $F_t = F_t^{a,b}$ for all $t \in [t_0, T_{\max})$. Explicitly,

$$(ab, b) = \begin{cases} (t_1, \beta) & (K=2), \\ (e^{t_1}, \beta e^{-t_1}) & (K=x+y), \\ (-t_1^{-1}, -\beta t_1) & (K=xy). \end{cases} \tag{8.2}$$

Observe that $ab > 1$ in all three cases. Iterating the map, we get that the solution must be (the restriction of) an eternal solution. By Theorem 2.9 and (8.1), $F = F^{a,b}$ is equivalent to $H = H^{a,b}$, that is,

$$H(x) = aH(bx), \quad x > 0. \tag{8.3}$$

Without loss of generality we may suppose $b > 1$ since (8.3) is equivalent to $a^{-1}H(b^{-1}x) = H(x)$.

2. Equation (8.3) implies $H(0_+) = aH(0_+)$. If H has an atom at the origin, this forces $a = 1$. Then $H(x) = H(bx)$ for every $x > 0$, and since $b > 1$ and $H(x)$ is nondecreasing, it follows $H(x) = c = H(0_+)$ for all $x > 0$. Therefore, if H has an atom at the origin, then $H = c\delta_0$ for some $c > 0$.
3. Suppose H does not have an atom at the origin. We iterate (8.3) to find that

$$\int_1^{b_-} H(dx) = a^j \int_{b^j}^{b_-^{j+1}} H(dx), \quad \int_1^{b_-} \frac{H(dx)}{x} = (ab)^j \int_{b^j}^{b_-^{j+1}} \frac{H(dx)}{x}.$$

In order that H is a g-measure we require

$$\int_E (1 \wedge x^{-1}) H(dx) = \sum_{j<0} a^{-j} \int_1^{b_-} H(dx) + \sum_{j\ge 0} (ab)^{-j} \int_1^{b_-} x^{-1} H(dx) < \infty.$$

Thus, $a < 1$ and $ab > 1$. Given $x > 0$ let $k = \max\{j : b^j \le x\}$. A similar calculation yields

$$H(x) = \sum_{j<k} a^{-j} \int_1^{b_-} H(dx) + a^{-k} \int_1^{b^{-k}x_-} H(dx).$$

This shows H is determined by its restriction to $[1, b)$.

4. Conversely, suppose $H = H^{a,b}$ and (i) or (ii) hold. Notice that H is automatically divergent since it either has an atom at the origin or

$$\int_E x^{-1} H(\mathrm{d}x) = \int_1^{b_-} x^{-1} H(\mathrm{d}x) \sum_{j=-\infty}^{\infty} (ab)^{-j} = \infty.$$

Thus, it determines an eternal solution, which by (8.1) satisfies $F = F^{a,b}$. □

8.2 Self-Similar Solutions

As remarked in Sect. 2.6, the case (i) is simple but important. The associated divergent g-measure is scale-invariant for every $b > 1$ and the scaling-periodic solutions are the classical self-similar solutions with exponential tails. If a scaling-periodic solution satisfies (2.25) for every $t_1 > t_0$ (with changing β), it follows that for some fixed a and b, $H(x) = a^r H(b^r x)$ for all rational and hence all real r. The fundamental rigidity lemma for scaling limits [9, VIII.8] then implies $H(x) = C_\theta x^\theta$ for some $\theta \in \mathbb{R}$. The finiteness condition $\int_E (1 \wedge x^{-1}) H(\mathrm{d}x) < \infty$ then implies $\theta = 1 - \rho$, $\rho \in (0, 1]$. If $\rho = 1$, H is an atom at the origin corresponding to (i) above. The self-similar profiles and their domains of attraction are discussed further in Sect. 10.2.

8.3 Density of Scaling-Periodic Solutions

To prove Theorem 2.15 and establish density of scaling-periodic solutions in the full scaling attractor $\mathcal{A}$, it will suffice to prove such solutions are dense in the proper scaling attractor $\mathcal{A}_\mathrm{p}$ (see Corollary 9.7).

Theorem 8.1 *Scaling-periodic solutions are dense in* $\mathcal{A}_\mathrm{p}$.

Proof

1. Let $\hat{F} \in \mathcal{A}$ be arbitrary. Let $a_n \downarrow 0$, $b_n \uparrow \infty$ be sequences such that $a_n b_n^{1/2} \to 0$ and $a_n b_n \to \infty$. We claim that there exist scaling-periodic solutions $\nu^{(n)}$ with scale parameters (a_n, b_n) such that $F_{t_0}^{(n)}(\mathrm{d}x) = x^\gamma \nu_{t_0}^{(n)}(\mathrm{d}x)$ converges weakly to $\hat{F}$ as $n \to \infty$. Let H denote the divergent g-measure associated with ν. By Theorems 2.8 and 2.9 it suffices to construct divergent g-measures $H^{(n)}$ such that $H^{(n)} = a_n H^{(n)}(b_n \cdot)$ and $H^{(n)}$ converges to H.
2. Consider first the case where H has no atom at the origin. In this case we define $H^{(n)}$ to be the scaling-invariant extension of H restricted to the interval $I_n := [b_n^{-1/2}, b_n^{1/2})$. Then for any $x > 0$ that is a point of continuity of H, for n large we have $x \in I_n$ and

$$H^{(n)}(x) = \int_{b_n^{-1}}^{x} H(\mathrm{d}x) + \sum_{j<0} (a_n)^{-j} \int_{b_n^{-1}}^{1} H(\mathrm{d}x) \to H(x)$$

as $n \to \infty$. Moreover,

$$\int_x^\infty \frac{H^{(n)}(\mathrm{d}y)}{y} = \int_x^{b_n} \frac{H(\mathrm{d}y)}{y} + \sum_{j \geq 1} (a_n b_n)^{-j} \int_1^{b_n} \frac{H(\mathrm{d}y)}{y} \to \int_x^\infty \frac{H(\mathrm{d}y)}{y}.$$

This establishes the desired convergence of g-measures.

3. In case $H=\delta_0$, we let $H^{(n)}$ be a sum of delta masses $\delta_j^{(n)}$, $j\in\mathbb{Z}$ concentrated at points $\beta_j=b_n^{j-1/2}$, so that $H^{(n)}=\sum_j(a_nb_n)^j\delta_j^{(n)}$. Observe that there is no mass in $(b_n^{-1/2},b_n^{1/2})$; thus for any $x>0$, for n large we have

$$H^{(n)}(x)=\sum_{j\le 0}(a_nb_n)^j=\frac{1}{1-a_nb_n}\to 1,$$

and

$$\int_x^\infty y^{-1}H^{(n)}(\mathrm{d}y)=b_n^{1/2}\sum_{j>0}a_n^j=\frac{a_nb_n^{1/2}}{1-a_n}\to 0.$$

Hence the g-measures $H^{(n)}$ converge to δ_0.

4. In the general case, we simply superpose the separate constructions. Observe that the restriction $a_nb_n^{1/2}\to 0$ is only needed in the critical case when H has an atom at the origin. □

9 Extended Solutions, with Dust and Gel

9.1 Extended Solutions

A proper solution to Smoluchowski's equation satisfies $\int_E x^\gamma\nu_t(\mathrm{d}x)=m_\gamma(t)$ with $m_\gamma(t)$ normalized as in (2.4). However, a sequence of proper solutions may lose mass in the limit. We append atoms at 0 and ∞ to account for these defects, considering measures on $\bar{E}=[0,\infty]$ of the form

$$\mu_t=a_0(t)\delta_0+x^\gamma\nu_t+a_\infty(t)\delta_\infty, \tag{9.1}$$

where ν_t is a size-distribution measure on E. We call the atoms a_0 and a_∞ the dust and gel, respectively. An associated probability measure on $\bar{E}$ is defined as in (2.5), by

$$\bar{F}_t(\mathrm{d}x)=\frac{\mu_t(\mathrm{d}x)}{\mu_t(\bar{E})}=\frac{a_0(t)\delta_0(\mathrm{d}x)+x^\gamma\nu_t(\mathrm{d}x)+a_\infty(t)\delta_\infty(\mathrm{d}x)}{a_0(t)+\int_E x^\gamma\nu_t(\mathrm{d}x)+a_\infty(t)}. \tag{9.2}$$

The g-measure associated to a solution in (2.14) is replaced by the $\overline{g}$-measure

$$\bar{G}_t(\mathrm{d}x)=x^{\gamma+1}\nu_t\big(\lambda(t)\,\mathrm{d}x\big)+g_\infty(t)\delta_\infty(\mathrm{d}x),\qquad g_\infty(t)=\frac{a_\infty(t)}{\lambda(t)^\gamma}. \tag{9.3}$$

The measures μ_t define Laplace exponents by evident modification of (4.3) for $K=2$, namely

$$\varphi(t,q)=\int_{\bar{E}}\big(1-\mathrm{e}^{-qx}\big)\mu_t(\mathrm{d}x)=a_\infty(t)+\int_E\big(1-\mathrm{e}^{-qx}\big)\nu_t(\mathrm{d}x), \tag{9.4}$$

and of (5.2) and (5.7) for $K = x + y$ and (6.1) for $K = xy$, both yielding

$$\varphi(t,q) = \int_{\bar{E}} \frac{1-\mathrm{e}^{-qx}}{x} \mu_t(\mathrm{d}x) = a_0(t)q + \int_E \left(1-\mathrm{e}^{-qx}\right)\nu_t(\mathrm{d}x). \tag{9.5}$$

The evolution equations for these exponents remain

$$\partial_t\varphi = \begin{cases} -\varphi^2 & (K=2), \\ \varphi\partial_q\varphi - \varphi & (K = x+y), \\ \varphi\partial_q\varphi & (K = xy). \end{cases} \tag{9.6}$$

This motivates the following definition.

Definition 9.1 A family of triples $(\nu_t, a_0(t), a_\infty(t))$, $t \in [t_0, T_{\max})$, defines an *extended solution* to Smoluchowski's equation for the kernels $K = 2$, $x + y$ and xy with initial data $(\hat{\nu}, \hat{a}_0, \hat{a}_\infty)$, if

(a) The measures μ_t in (9.1) satisfy $\mu_t(\bar{E}) = m_\gamma(t)$ with $m_\gamma(t)$ as in (2.4), for $t \in [t_0, T_{\max})$.
(b) Equation (9.6) holds for $q > 0$ and $t \in (t_0, T_{\max})$.
(c) $\mu_t \to \hat{\mu} = \hat{a}_0\delta_0 + x^\gamma\hat{\nu} + \hat{a}_\infty\delta_\infty$ weakly as $t \downarrow t_0$.

Due to the normalization in (a), we regard extended solutions as determined by the associated probability distributions $\bar{F}$ in (9.2). We will usually denote an extended solution with values $(\nu_t, a_0(t), a_\infty(t))$ simply by ν.

Extended solutions provide the correct compactification in light of the following theorem. Since every proper solution also defines an extended solution, the theorem applies in particular to sequences of proper solutions.

Theorem 9.2 *Let $\bar{F}_t^{(n)}$, $t \in [t_0, T_{\max})$, be probability measures associated with a sequence of extended solutions $\nu^{(n)}$. Then there exists a sequence $n_j \to \infty$ and probability measures $\bar{F}_t$ associated with an extended solution ν, such that $\bar{F}_t^{(n_j)}$ converges weakly to $\bar{F}_t$ for every $t \in [t_0, T_{\max})$.*

Proof Consider the sequence of probability measures $\bar{F}_{t_0}^{(n)}$ on $\bar{E}$. Then there exists a subsequence n_j and a probability measure $\hat{\bar{F}}_0$ such that $\bar{F}_{t_0}^{(n_j)}$ converges weakly to $\hat{\bar{F}}_0$. We use $\hat{\bar{F}}_0$ to determine initial data to define an extended solution ν for $t \in [t_0, T_{\max})$. Continuous dependence on initial data as in Theorem 9.3 below implies the weak convergence of $\bar{F}_t^{(n_j)}$ to $\bar{F}_t$ for every $t \in [t_0, T_{\max})$. □

We state the following result without proof, as it is an easy consequence of Definition 9.1, and Theorems 4.1 and 5.2. The notion of extended solution allows us to simplify matters, as it is no longer necessary to assume that $\hat{\mu}_0(\bar{E}) = m_\gamma(t_0)$, or $\hat{\mu}(\bar{E}) = m_\gamma(t)$ as in parts (a) and (b) of Theorem 4.1 and 5.2.

Theorem 9.3 (Continuous dependence on data) *For Smoluchowski's equation with kernels $K = 2$, $x + y$ or xy, let $t_0 \in (T_{\min}, T_{\max})$ and let $\bar{F}^{(n)}$ determine a sequence of extended solutions defined for $t \in I = [t_0, T_{\max})$.*

(a) *If* $\bar{F}_{t_0}^{(n)}$ *converges weakly to a measure* $\hat{\bar{F}}_0$, *then for every* $t \in I$, $\bar{F}_t^{(n)}$ *converges weakly to* $\bar{F}_t$, *associated with the time-*t *extended solution with initial data determined by* $\bar{F}_{t_0} = \hat{\bar{F}}_0$.

(b) *For any* $t \in I$, *if* $\bar{F}_t^{(n)}$ *converges weakly to a measure* $\hat{\bar{F}}$, *then* $\bar{F}_{t_0}^{(n)}$ *converges weakly to a probability measure* $\hat{\bar{F}}_0$ *and* $\hat{\bar{F}} = \bar{F}_t$, *associated with the time-*t *solution with initial data determined by* $\bar{F}_{t_0} = \hat{\bar{F}}_0$.

9.2 Transformation to Proper Solutions

Clusters of “zero” or “infinite” size interact with other clusters in simple ways. The invariances of the evolution equations (9.6) allow us to relate all extended solutions (except pure dust and gel) to proper solutions. Let us consider the constant and additive kernels in turn.

9.2.1 The Constant Kernel

The dust and gel are recovered as limits as $q \to 0$ and ∞, respectively:

$$a_\infty(t) = \varphi\bigl(t, 0^+\bigr), \qquad a_0(t) = \mu_t\bigl(\bar{E}\bigr) - \varphi\bigl(t, \infty^-\bigr). \tag{9.7}$$

Since $\mu_t(\bar{E}) = t^{-1}$, we take limits in (9.6) to see that the dust and gel satisfy

$$\frac{\mathrm{d}a_\infty}{\mathrm{d}t} = -a_\infty^2, \qquad \frac{\mathrm{d}(t^{-1} - a_0)}{\mathrm{d}t} = -\bigl(t^{-1} - a_0\bigr)^2. \tag{9.8}$$

The extended solution corresponds to purely dust and gel when $\mu_t(E) = 0$, so that $a_0(t) + a_\infty(t) = t^{-1}$. We may exploit (9.6) to show that every extended solution that is not purely dust and gel is in correspondence with a proper solution after a simple change of scale. Suppose $\varphi(t, q)$ is the Laplace exponent of an extended solution. If $a_\infty(t_0) > 0$, let

$$\hat{\varphi}\bigl(\hat{t}, q\bigr) = \alpha(t)^{-2}\bigl(\varphi(t, q) - a_\infty(t)\bigr), \tag{9.9}$$

where

$$\hat{t}^{-1} = \alpha(t)^{-2}\bigl(t^{-1} - a_0(t) - a_\infty(t)\bigr), \quad \alpha(t) = \frac{a_\infty(t)}{a_\infty(t_0)}. \tag{9.10}$$

Then we find $\hat{\varphi}(\hat{t}, 0^+) = 0$, $\hat{\varphi}(\hat{t}, \infty^-) = \hat{t}^{-1}$, and $\partial_{\hat{t}}\hat{\varphi} = -\hat{\varphi}^2$. Thus, $\hat{\varphi}(\hat{t}, q)$ is the Laplace exponent of a proper solution defined on $[\hat{t}_0, \infty)$.

For vanishing gel ($a_\infty(t_0) \to 0$) the transformation above simplifies, yielding $\alpha = 1$, $\hat{t} - \hat{t}_0 = t - t_0$, $\hat{\varphi} = \varphi$. Zero-size clusters combine trivially with other clusters, so the presence of dust only shifts time in accord with our normalization of total number. Observe that if gel is present ($a_\infty(t_0) > 0$), the probability of being gel approaches one ($a_\infty(t)/\mu_t(\bar{E}) \to 1$) and the relative distribution of finite-size clusters approaches a state reached by the proper solution at a finite time; we have $\hat{t} \to \hat{t}_0 + 1/a_\infty(t_0)$ as $t \to \infty$.

9.2.2 The Additive Kernel

In this case, $\partial_q \varphi(t,q) = \int_{\bar{E}} \mathrm{e}^{-qx} \mu_t(\mathrm{d}x)$ and $\mu_t(\bar{E}) = 1$, so the dust and gel are given by

$$a_0(t) = \partial_q \varphi(t,\infty), \qquad a_\infty(t) = 1 - \partial_q \varphi\bigl(t, 0^+\bigr). \tag{9.11}$$

The similarity with the constant kernel is clear if we use the time scale $s = e^t$ and the Laplace exponent $\psi(s,q)$ defined in (5.6) and (5.7). Let

$$b_0(s) = \frac{1}{s} - \partial_q \psi(s,\infty) = \frac{a_0(t)}{s}, \qquad b_\infty(s) = \partial_q \psi(s,0) = \frac{a_\infty(t)}{s}. \tag{9.12}$$

We then take limits in (5.11) to see that

$$\frac{\mathrm{d}(s^{-1} - b_0)}{\mathrm{d}s} = -\bigl(s^{-1} - b_0\bigr)^2, \qquad \frac{\mathrm{d}b_\infty}{\mathrm{d}s} = -b_\infty^2, \tag{9.13}$$

which is equivalent to the following closed equations for the dust and gel:

$$\frac{\mathrm{d}a_0}{\mathrm{d}t} = -a_0(1-a_0), \qquad \frac{\mathrm{d}a_\infty}{\mathrm{d}t} = a_\infty(1-a_\infty). \tag{9.14}$$

The extended solution is purely dust and gel when $a_0(t) + a_\infty(t) = 1$. If it is not, we exploit the invariances of the inviscid Burgers equation (5.9) to reduce extended solutions to proper solutions by a change of scale. Given initial data ψ_0 with $\partial_q \psi_0(0) = b_\infty(s_0) \geq 0$ and $\partial_q \psi_0(\infty) = s_0^{-1} - b_0(s_0) > b_\infty(s_0) > 0$, we define a proper solution via the change of variables

$$\hat{\psi}(\hat{s},\hat{q}) = \alpha(s)^{-1}\bigl(\psi(s,q) - b_\infty(s)q\bigr), \tag{9.15}$$

where

$$\hat{s}^{-1} = \alpha(s)^{-2}\bigl(s^{-1} - b_0(s) - b_\infty(s)\bigr), \qquad \hat{q} = \alpha(s)q, \quad \alpha(s) = \frac{b_\infty(s)}{b_\infty(s_0)}.$$

This ensures $\partial_{\hat{q}} \hat{\psi}(s,0) = 0$, $\partial_{\hat{q}} \hat{\psi}(s,\infty) = s^{-1}$, and $\partial_{\hat{s}} \hat{\psi} + \hat{\psi}\partial_{\hat{q}} \hat{\psi} = 0$ for $\hat{s} > \hat{s}_0$.

9.3 Lévy-Khintchine Representation

Definition 9.4 An extended solution to Smoluchowski's equation that is defined for all $t \in (T_{\min}, T_{\max})$ is called an *eternal extended solution*.

The following representation theorem is the completion of Theorem 2.8. We establish a bijection between the set of eternal extended solutions and the space $\mathcal{S}$ consisting of all $\overline{g}$-measures together with a point at infinity. The point at infinity corresponds to all measures such that $\int_{\bar{E}} (1 \wedge x^{-1}) H(\mathrm{d}x) = \infty$. These measures give rise to the (unique) Laplace exponents $\Phi(q) = \Psi(q) = \infty$, $q > 0$. We say a sequence of $\overline{g}$-measures converges to the point at infinity if $\Phi^{(n)}(q) \to \infty$, $q > 0$ for the associated Laplace exponents. This special case corresponds to the eternal extended solution that is pure gel. It is the counterpoint to the Laplace exponents $\Phi(q) = \Psi(q) = 0$, $q > 0$ which generate the eternal extended solution that is pure dust.

Theorem 9.5

(a) *Let ν be an eternal extended solution for Smoluchowski's equation with* $K = 2, x+y$ *or* xy. *If ν is not pure gel, there is a $\overline{g}$-measure H such that $\bar{G}_t$ converges to H as $t \downarrow T_{\min}$. If ν is pure gel, $\bar{G}_t$ converges to the point at infinity in $\mathcal{S}$.*

(b) *Conversely, for every $\overline{g}$-measure H, there is a unique eternal extended solution ν such that $\bar{G}_t$ converges to H as $t \downarrow T_{\min}$. The point at infinity generates the extended solution ν that is pure gel.*

(c) *Let $\mathfrak{S}: \mathcal{A} \to \mathcal{S}$ map the (full) scaling attractor $\mathcal{A}$ to the set $\mathcal{S}$ of $\overline{g}$-measures by $\mathfrak{S}(\hat{\bar{F}}) = H$, where H is the $\overline{g}$-measure associated to the eternal extended solution ν such that $\hat{\bar{F}} = \bar{F}_{t_0}$ with t_0 as in (2.3). Then $\mathfrak{S}$ is a bicontinuous bijection. Moreover, $\mathfrak{S}_p: \mathcal{A}_{\mathrm{p}} \to \mathcal{S}_d$ is the restriction of $\mathfrak{S}$ to $\mathcal{A}_{\mathrm{p}}$.*

The map $\mathfrak{S}$ is defined in terms of Laplace exponents by the same formulas as for proper solutions: Equation (4.8) for $K = 2$, (5.11) for $K = x + y$, and (6.10) for $K = xy$. Parts (a) and (b) of the theorem are then proven just as in Theorems 4.3 and 5.4. The proof here is simpler, since we no longer need verify the divergence conditions on the $\overline{g}$-measure.

The proof of part (c) relies on two separate arguments. It is easy to show as in Theorems 4.4 and 5.6 that the map $H \mapsto \bar{F}_{t_0}$ is a bicontinuous bijection. However, we must also identify such $\bar{F}_{t_0}$ as belonging to the attractor. Here the arguments deviate slightly from those of Sects. 4 and 5. We use parts (a) and (b) of Theorem 9.5 in the proof of part (c) via the following intermediate theorem.

Theorem 9.6

(a) *The scaling attractor $\mathcal{A}$ is invariant: If ν is an extended solution of Smoluchowski's equation, and $\bar{F}_t \in \mathcal{A}$ for some t, then ν is eternal and $\bar{F}_t \in \mathcal{A}$ for all $t \in (T_{\min}, T_{\max})$.*

(b) *A probability measure $\hat{\bar{F}}$ on $\bar{E}$ belongs to $\mathcal{A}$ if and only if $\hat{\bar{F}} = \bar{F}_{t_0}$ for some extended eternal solution ν.*

Proof The proof differs from earlier arguments only in the first part of Theorems 4.2 and 5.3 (the assertion that $\bar{F}_{t_0} \in \mathcal{A}$ if ν is eternal). In order to prove this, let us suppose ν is an extended eternal solution with associated $\overline{g}$-measure $\bar{H} = (H, g_\infty)$ where H is a g-measure and g_∞ is the charge of $y^{-1}H(\mathrm{d}y)$ at ∞. To show that $\hat{\bar{F}} := \bar{F}_{t_0}$ is in the scaling attractor, we must find $T_n \uparrow T_{\max}$, $\beta_n \to \infty$ and a sequence of *proper* solutions such that $F^{(n)}_{T_n}(\beta_n x) \to \hat{\bar{F}}(x)$ at points of continuity. We use the Lévy-Khintchine formula to find such solutions. We approximate $\bar{H}$ by the sequence of *divergent* g-measures $H^{(n)} = n^{-1}\delta_0 + H + g_\infty n\delta_n$. It follows that for the corresponding (proper) eternal solutions $\nu^{(n)}$, the probability measures $\bar{F}^{(n)}_t$ converge to $\bar{F}_t$ for every $t > T_{\min}$. Given any sequence $T_n \uparrow T_{\max}$, $\beta_n \to \infty$ we consider a sequence of rescaled solutions determined as in (2.17), by

$$\tilde{F}_t^{(n)}(x) = \begin{cases} F_{t/T_n}^{(n)}(\beta_n^{-1}x) & (K=2), \\ F_{t-T_n}^{(n)}(\beta_n^{-1}x) & (K=x+y), \\ F_{t/|T_n|}^{(n)}(\beta_n^{-1}(x)) & (K=xy). \end{cases}$$

We then have $\tilde{F}_{T_n}^{(n)}(\beta_n x) = F_{t_0}^{(n)}(x) \to \hat{\bar{F}}(x)$ at all points of continuity.

The converse implication and part (a) are proven exactly as in Theorems 4.2 and 5.3 and the sequel. □

This also proves a property alluded to several times before.

Corollary 9.7 *$\mathcal{A}$ is the closure of $\mathcal{A}_{\mathrm{p}}$.*

Proof If $\hat{\bar{F}} \in \mathcal{A}$ has $\overline{g}$-measure $\bar{H}$, we approximate $\bar{H}$ by a sequence of divergent g-measures as above. □

9.4 Scaling Limits and Initial Tails

We now state the natural extension of Theorem 2.11 to eternal extended solutions. The proof is almost identical to that of Theorem 2.11 except that we no longer need verify divergence of the g-measure.

Theorem 9.8 *Let $\hat{\bar{F}} \in \mathcal{A}$ with associated $\overline{g}$-measure H. Let $\nu^{(n)}$ be any sequence of proper solutions defined for $t \in [t_0, T_{\max})$, with associated initial g-measures given by $G^{(n)}(\mathrm{d}x) = x^{\gamma+1}\nu_{t_0}^{(n)}(\mathrm{d}x)$. Let $T_n \to T_{\max}$, $\beta_n \to \infty$. Then the following are equivalent*:

(i) $F_{T_n}^{(n)}(\beta_n x) \to \hat{\bar{F}}(x)$ *as $n \to \infty$, at every point of continuity.*

(ii) *The rescaled initial g-measures $\tilde{G}^{(n)}$ defined by (2.24) converge to the $\overline{g}$-measure H as $n \to \infty$.*

10 Initial Tails and Ultimate Scaling Dynamics

In this section, we present two applications of the principle that ultimate scaling dynamics are encoded in the initial tails (as formalized in Theorems 2.11 and 9.8). The first is a proof of the shadowing Theorem 2.16. The second is a streamlined proof of the classification of domains of attraction in [20] that avoids the use of Karamata's Tauberian theorem.

10.1 Initial Tails and Shadowing

Proof of Theorem 2.16 1. As in Sect. 2, we let dist$(\cdot,\cdot)$ denote any metric on $\bar{\mathcal{P}}$ which induces the weak topology. Suppose that for Smoluchowski's equation with kernel $K=2, x+y$ or xy, ν and $\bar{\nu}$ are two solutions defined on $[t_0, T_{\max})$, and make the assumptions stated in the theorem. Suppose that (2.30) fails, i.e., that

$$\operatorname{dist}\big(F_t\big(b(t)\,\mathrm{d}x\big), \bar{F}_{\bar{t}}\big(\bar{b}(t)\,\mathrm{d}x\big)\big) \not\to 0 \quad \text{as } t \to T_{\max}. \tag{10.1}$$

Then since $\bar{\mathcal{P}}$ is compact, by passing to subsequences we can find sequences $T_n \uparrow T_{\max}$ and $\beta_n = b(T_n)$ and *different* probability measures $\hat{F}$, $\check{F} \in \bar{\mathcal{P}}$, such that as $n \to \infty$ we have

$$F_{T_n}(\beta_n x) \to \hat{F}(x), \qquad \bar{F}_{\bar{T}_n}\big(\bar{\beta}_n x\big) \to \check{F}(x), \tag{10.2}$$

at every point of continuity of the limit. Here the values $\bar{T}_n$, $\bar{\beta}_n$ are those that correspond via the map $(t,b) \mapsto (\bar{t}, \bar{b})$ stated in the theorem. Relabeling if necessary, we may assume $0 < \hat{F}(x)$ for some finite x, i.e., $\hat{F}$ does not represent pure gel. Therefore, according to the extended Lévy-Khintchine representation Theorem 9.5, there exists a $\overline{g}$-measure H that corresponds to $\hat{F}$.

2. Let

$$\alpha_n = \begin{cases} \beta_n & (K=2), \\ \beta_n \mathrm{e}^{-T_n} & (K=x+y), \\ \beta_n |T_n| & (K=xy) \end{cases} \qquad \lambda_n = \begin{cases} T_n & (K=2), \\ \mathrm{e}^{T_n} & (K=x+y), \\ |T_n|^{-1} & (K=xy) \end{cases} \tag{10.3}$$

and similarly define $\bar{\alpha}_n$, $\bar{\lambda}_n$ in terms of $\bar{\beta}_n$, $\bar{T}_n$. Note that $\bar{\alpha}_n = \alpha_n$. We claim that $\alpha_n \to \infty$. This is evident for $K=2$, and once we prove it for $K=x+y$ it follows for $K=xy$ by the transformation formula (6.3). For $K=x+y$, one can prove $\alpha_n \to \infty$ by following the beginning of the proof of Theorem 7.1 in [20] up to (7.8) using only subsequential convergence. From (7.8) one deduces $\lambda \mathrm{e}^{-t} \to \infty$, which corresponds here to $\alpha_n \to \infty$.

3. Define rescaled initial g-measures (see (2.24)) by

$$G^{(n)}(x) = \lambda_n \alpha_n^{-1} G(\alpha_n x), \qquad \bar{G}^{(n)}(x) = \bar{\lambda}_n \alpha_n^{-1} G(\alpha_n x). \tag{10.4}$$

According to the extended encoding Theorem 9.8 the g-measures $G^{(n)}$ converge to H. Let $\varphi^{(n)}$, $\bar{\varphi}^{(n)}$ and Φ be the first-order Laplace exponents associated to $G^{(n)}$, $\bar{G}^{(n)}$, and H, respectively, as in (3.1). We have $\bar{\lambda}_n = \lambda_n / L(\alpha_n)$, and with $\hat{L}(1/q) = \bar{\varphi}(q)/\varphi(q)$, the hypothesis (2.28) ensures $\hat{L}$ is slowly varying and $\hat{L} \sim L$. Hence, for any $q \in (0,\infty)$ we have

$$\bar{\varphi}^{(n)}(q) = \bar{\lambda}_n \bar{\varphi}(q/\alpha_n) = \varphi^{(n)}(q) \frac{\hat{L}(\alpha_n/q)}{\hat{L}(\alpha_n)} \frac{\hat{L}(\alpha_n)}{L(\alpha_n)} \to \Phi(q)$$

as $n \to \infty$. By Theorem 3.1, it follows $\bar{G}^{(n)}$ converges to H, and by the extended Lévy-Khintchine representation theorem, this yields $\hat{F} = \check{F}$ in (10.2). This contradicts (10.1) and finishes the proof. □

10.2 Self-Similar Solutions and Domains of Attraction

The self-similar solutions are the simplest examples of eternal solutions. All self-similar solutions are generated by the g-measures $H(x) = Cx^{1-\rho}$ with $\rho \in (0, 1]$, $C > 0$, with corresponding Laplace exponents

$$\Phi(q) = Cq^{\rho}\frac{\Gamma(2-\rho)}{\rho}, \qquad \Psi(q) = Cq^{1+\rho}\frac{\Gamma(2-\rho)}{\rho(1+\rho)}. \tag{10.5}$$

Thus, there is a one-parameter family up to trivial scalings. By (4.8) and (5.19), for appropriate C, the Laplace exponent $\varphi = \varphi(t,q)$ of the solution satisfies

$$\varphi = \frac{\Phi}{1+t\Phi} = \frac{q^{\rho}}{1+tq^{\rho}} \quad (K = 2), \tag{10.6}$$

$$q = \varphi + \Psi\big(\mathrm{e}^t\varphi\big) = \varphi + \big(\mathrm{e}^t\varphi\big)^{1+\rho} \quad (K = x + y). \tag{10.7}$$

The self-similar solutions were described in [20], and can all be captured by expressing the associated probability distribution in the form

$$F(t,x) = F_{\rho,\gamma}\big(x/a_{\rho,\gamma}(t)\big), \tag{10.8}$$

for $\gamma = 0, 1, 2$, where the scale factors are

$$a_{\rho,0}(t) = t^{1/\rho}, \qquad a_{\rho,1}(t) = \mathrm{e}^{t/\beta}, \qquad a_{\rho,2}(t) = |t|^{-1/\beta}, \tag{10.9}$$

with $\beta = \rho/(1+\rho)$, and the probability distributions $F_{\rho,\gamma}$ are explicitly

$$F_{\rho,0}(x) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{\rho k}}{\Gamma(1+\rho k)}, \tag{10.10}$$

$$F_{\rho,1}(x) = F_{\rho,2}(x) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^{k\beta}}{k!}\Gamma(1+k-k\beta)\frac{\sin k\pi\beta}{k\pi\beta}. \tag{10.11}$$

We now restate the characterization of the domains of attraction of these self-similar solutions obtained in [20]. We say a probability measure on E is nontrivial if it is not concentrated at the origin.

Theorem 10.1 *Let F_t denote the probability measure associated to a solution to Smoluchowski's coagulation equations with $K = 2, x + y$, or xy.*

(a) *Assume there is a rescaling $b(t) \to \infty$ and a nontrivial probability measure $\hat{F}$ on E such that $F_t(b(t)x) \to \hat{F}(x)$ at all points of continuity. Then there is $\rho \in (0, 1]$, and a function L slowly varying at infinity such that*

$$G_{t_0}(x) = \int_{(0,x]} y^{\gamma+1}\nu_{t_0}(\mathrm{d}y) \sim x^{1-\rho}L(x), \quad x \to \infty. \tag{10.12}$$

(b) *Conversely, assume* (10.12) *holds. Then there is a rescaling* $b(t) \to \infty$ *such that*

$$\lim_{t\to\infty} \operatorname{dist}\big(F_t\big(b(t)\,\mathrm{d}x\big), F_{\rho,\gamma}(\mathrm{d}x)\big) = 0. \tag{10.13}$$

Theorem 10.1 illustrates the *rigidity of scaling limits*. If we insist on the existence of a proper limit as $t \to \infty$ (as opposed to subsequential limits), the only possibility is that $\hat{F}(x) = F_{\rho,\gamma}(ax)$ for some $\rho \in (0, 1]$ and $a \in (0, \infty)$. (For degenerate limits, see Remark 10.1.) Theorems 2.11 and 2.16 shed more light on this result as they clarify the main hypothesis (see (10.12)) and allow us to avoid the use of Karamata's Tauberian theorem in the proof.

Proof Let us first prove (a). Suppose there is a (possibly discontinuous) rescaling $b(t) \to \infty$ such that $\lim_{t\to\infty} F_t(b(t)x) = \hat{F}(x)$ at all points of continuity of $\hat{F}$. Then $\hat{F} \in \mathcal{A}_{\mathrm{p}}$, so it is associated to a divergent g-measure H. Theorem 2.11 (ii) now implies the convergence of the g-measures $\tilde{G}^{(t)} \to H$ where

$$\tilde{G}^(x) = \tilde{\lambda}\alpha^{-1} G_{t_0}(\alpha x), \tag{10.14}$$

$$\alpha(t) = \begin{cases} b(t) & (K = 2), \\ b(t)\mathrm{e}^{-t} & (K = x + y), \\ b(t)|t| & (K = xy); \end{cases} \qquad \tilde{\lambda}(t) = \begin{cases} t & (K = 2), \\ \mathrm{e}^{t} & (K = x + y), \\ |t|^{-1} & (K = xy). \end{cases} \tag{10.15}$$

As we have seen in the proof of Theorem 2.16, $\alpha(t)$ diverges as $t \to T_{\max}$ in each case. Then by (10.14), the Laplace exponent φ_0 for G_{t_0} satisfies

$$\tilde{\lambda}\varphi_0(q/\alpha) \to \Phi(q) \tag{10.16}$$

as $t \to T_{\max}$, where Φ is the Laplace exponent of H. Taking $t \to T_{\max}$ along a sequence t_n such that $\tilde{\lambda}(t_{n+1})/\tilde{\lambda}(t_n) \to 1$, by a fundamental rigidity lemma [9, VIII.8.3], we infer that the only possible limits are power laws, meaning $\Phi(q) = cq^{\rho}$ for some $\rho \ge 0$. Since H is a nontrivial g-measure, we must have $0 < \rho \le 1$ and $c > 0$. Moreover we infer φ_0 is regularly varying at 0, meaning $\varphi_0(q) = q^{\rho}\hat{L}(q)$, where $\hat{L}(aq)/\hat{L}(q) \to 1$ as $q \to 0$ for every $a > 0$. Note that by (10.16),

$$\tilde{\lambda} \sim c\alpha^{\rho}/\hat{L}(1/\alpha), \qquad c_n = \tilde{\lambda}(t_n)\varphi_0\big(1/\alpha(t_n)\big) \to c. \tag{10.17}$$

With t_n as described and $\alpha_n = \alpha(t_n)$, we claim $\alpha_{n+1}/\alpha_n \to 1$ as $n \to \infty$. Let $a > 1$ and suppose $\alpha_{n+1}/\alpha_n > a$ for infinitely many n. Then since φ_0 is strictly increasing, along this subsequence we have

$$\frac{c_{n+1}}{\tilde{\lambda}(t_{n+1})}\frac{\tilde{\lambda}(t_n)}{c_n} = \frac{\varphi_0(1/\alpha_{n+1})}{\varphi_0(1/\alpha_n)} \le \frac{\varphi_0(a^{-1}/\alpha_n)}{\varphi_0(1/\alpha_n)} \quad \to \quad a^{-\rho} < 1.$$

But the left-hand side converges to 1. Hence $\limsup \alpha_{n+1}/\alpha_n \le 1$. Similarly we deduce $\liminf \alpha_{n+1}/\alpha_n \ge 1$, establishing the claim.

We may now apply the rigidity lemma [9, VIII.8.3] to (10.14) to infer that G_{t_0} is regularly varying at ∞, meaning (10.12) holds. (The value of ρ must be the same here, due to (10.17) and (10.5).) This proves part (a).

To prove the converse, we assume that (10.12) holds. Since (10.12) holds we may choose increasing rescalings $\alpha(t) \to \infty$ and $\tilde{\lambda}(t)$ such that the g-measures $G(x) = \tilde{\lambda}\alpha^{-1}G_{t_0}(\alpha x)$ converge to $H = x^{1-\rho}$. Let $b(t)$ be defined by (10.15) for the various kernels. It then follows that $F_t(b(t)x) \to F_{\rho,\gamma}(x)$ for every $x > 0$. Since the metric is equivalent to weak convergence we also have (10.13). □

Remark 10.1 A remaining nontrivial possibility discussed in [20] is that of a defective limit on E, which we may now take to mean that $F_t(b(t)x) \to \hat{F}(x)$ where $\hat{F}$ is a probability measure on $\bar{E} = [0, \infty]$, with $0 < \hat{F}(\infty^-) < 1$, meaning that gel appears in the limit. If this is the case, then $\hat{F}$ is an element of the full scaling attractor $\mathcal{A}$, and by Theorem 9.8, the rescaled g-measures $\tilde{G}^{(t)} \to H$, the $\overline{g}$-measure associated to $\hat{F}$. Moreover, $y^{-1}H(\mathrm{d}y)$ must have nonzero charge h_∞ at ∞, and hence $\Phi(0+) > 0$. This means that in the proof above, the rigidity lemma must yield $\rho = 0$, i.e., we must have $\Phi(q) = c > 0$, corresponding to an eternal extended solution consisting of a pure dust/gel mixture. By the discussion in [20] (see Remarks 5.4 and 7.4) a necessary and sufficient condition for this to occur is that

$$\int_{[x,\infty)} y^{-1} G_{t_0}(\mathrm{d}y) \sim L(x), \quad x \to \infty, \tag{10.18}$$

where L is slowly varying at ∞.

11 Discussion

The results of this article together with [20] complete a description of dynamic scaling behavior for Smoluchowski's coagulation equations with solvable kernels that bears a striking resemblance to classical probability theory, as we have emphasized. When initial data is well-localized, with a finite $\gamma + 1$st moment, there is a universal scaling limit in the weak topology, analogous to the central limit theorem. For many physical applications this is the main case of interest. Our previous paper [21] dealt with this situation, providing proofs of uniform convergence of scaled densities to the classical self-similar forms under near-optimal conditions on moments and regularity of initial data $n(t_0, x)$. See [21] for a discussion of related literature as well.

In this paper, by contrast, the natural well-posedness theory has allowed us to develop a rather comprehensive theory of scaling dynamics for heavy-tailed solutions, and here we have found a rich family of limit points and sensitive dependence on initial tails, by building upon Bertoin's Lévy-Khintchine representation for eternal solutions with $K = x + y$. In view of the great range of applications of Smoluchowski's equations, we believe that heavy-tailed solutions should not be dismissed lightly as unphysical. In probability theory, heavy-tailed distributions have found numerous applications in recent years.

As we indicated in the introduction, the solvable kernels themselves hold substantial interest for applications. For example, in [19] we applied the results of [20] to a problem in Burgers turbulence, specifically the inviscid Burgers equation with Lévy-process initial data (random-walk data with stationary independent increments) having only downward jumps. Carraro and Duchon [6] and Bertoin [3] showed that this class of data is closed under Cole-Hopf dynamics, and it is implicit in these works that mean-field theory for the dynamics of the shock size distribution is exact. In particular, the dynamics turns out to be governed by Smoluchowski's coagulation equation with $K = x + y$. Based on this fact, in [19] we characterized all domains of attraction, i.e., all universality classes for scaling limits, for this class of random initial data.

Now, it is natural to ask to what extent refined results for solvable kernels might extend to the kinds of homogeneous or asymptotically homogeneous kernels that appear in a much wider range of applications. Here there has been encouraging recent progress on two fronts: First, the long-outstanding question of existence of self-similar solutions with exponential decay was settled for a general class of homogeneous kernels by Fournier and Laurençot [11], and Escobedo, Mischler and Rodriguez Ricard [17]. And second, for a general class of kernels of homegeneity γ, well-posedness of the initial value problem for measure solutions with initial data having just a finite γth moment has been established by Fournier and Laurençot [12].

This result raises the question whether the scaling dynamics of heavy-tailed solutions for general homogeneous kernels is like that for the solvable kernels. Some features of the proofs in the present paper offer hope that the present results may generalize. In particular, the correspondence between points of the scaling attractor, eternal solutions, and associated g-measures in the limit $t \downarrow T_{\min}$, depends only on continuity properties of the solution map that perhaps could be established without using the Laplace transform as was done here. Then the linearization of the ultimate dynamics in terms of g-measures might follow directly by scaling as in the proof of Theorems 2.9 and 2.10.

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