Approach to Self-Similarity
in Smoluchowski's Coagulation Equations

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Abstract

We consider the approach to self-similarity (or dynamical scaling) in Smoluchowski's equations of coagulation for the solvable kernels $K(x, y) = 2, x + y$, and xy. In addition to the known self-similar solutions with exponential tails, there are one-parameter families of solutions with algebraic decay whose form is related to heavy-tailed distributions well-known in probability theory. For $K = 2$ the size distribution is Mittag-Leffler, and for $K = x + y$ and $K = xy$ it is a power-law rescaling of a maximally skewed α-stable Lévy distribution. We characterize completely the domains of attraction of all self-similar solutions under weak convergence of measures. Our results are analogous to the classical characterization of stable distributions in probability theory. The proofs are simple, relying on the Laplace transform and a fundamental rigidity lemma for scaling limits. © 2004 Wiley Periodicals, Inc.

1 Introduction

Smoluchowski's coagulation equations provide a mean field description of several processes of mass aggregation in nature. We study the evolution of $n(t, x)$, the number of clusters of mass x per unit volume at time t. Clusters of mass x and y coalesce by binary collisions at a rate governed by a symmetric kernel $K(x, y)$, whence

$$
\frac{\partial n}{\partial t}(t, x) = \frac{1}{2} \int_0^x K(x - y, y) n(t, x - y) n(t, y) dy
$$

(1.1)

$$
- \int_0^\infty K(x, y) n(t, x) n(t, y) dy .
$$

All microscopic interactions are subsumed into the agglomeration rate kernel K, and the process is assumed to be stationary in space. A broad survey of applications, especially in physical chemistry, may be found in the article by Drake [14]. Equation (1.1) has been used in an amazingly diverse range of applications, such as the formation of clouds and smog [17], the clustering of planets, stars, and galaxies [32], the kinetics of polymerization [35], and even the schooling of fishes [29]

and the formation of "marine snow" (see [21]). In the past few years, there has been a resurgence of mathematical interest in the field, largely due to the work of probabilists. An influential survey article by Aldous summarizes the recent state of affairs [2].

An issue of importance for homogeneous kernels, which are kernels that satisfy $K(\alpha x, \alpha y) = \alpha^\gamma K(x, y)$, is the phenomenon of *dynamical scaling* for all initial data in a *universality class*. Mathematically, this corresponds to the problem of existence of *scaling* or *self-similar solutions* and characterization of their *domains of attraction*. In the aerosols community, the relevant rubric is the theory of self-preserving spectra, which is treated at length in Friedlander's book [17] and the extensive survey of Drake [14]. For a large class of kernels with $\gamma \leq 1$, there is numerical evidence that solutions evolve to a self-similar form [23]. There are also physical self-consistency arguments that have been used to derive asymptotics for scaling solutions [34]. In the case $\gamma > 1$, for a general class of kernels it is known that solutions must lose mass (presumably to infinite-mass clusters) after some finite time T_{gel} [15, 19], but there is no general rigorous result on the precise nature of this blowup in mass transport. In several instances the known solutions have unphysical divergences such as infinite mass. Thus, a general existence theory for finite-mass self-similar solutions, for example, would be of some value.

The kernels $K(x, y) = 2$, $x + y$, and xy play a special role, as (1.1) can then be solved by the Laplace transform. It is widely known that each of these kernels admits a self-similar solution with exponential decay (see table 2 in [2]). These kernels are also special since certain solutions to (1.1) can be viewed as ergodic averages in beautiful probabilistic constructions, involving thinning of renewal processes ($K = 2$) and tree-valued Markov processes and their self-similar limits ($K = x + y$, xy) [3]. The additive kernel also figures in interesting recent applications given by Bertoin. It provides a natural probabilistic interpretation of a sticky particle model related to Zeldovich's model of gravitational clustering [8]. Also, the known self-similar solution appears in a simple model of turbulence, the inviscid Burgers equation with Brownian motion initial data, as the characteristic measure for a Poisson point process that describes the shock strengths [6].

In short, aside from heuristics and numerics, there are no rigorous mathematical proofs of the existence of self-similar solutions and the approach to self-similar form for general kernels (see [2, sec. 2]). And there are only a few partial results for the solvable kernels: For $K = 2$, Kreer and Penrose [22] proved local uniform convergence to the scaling solution under some technical hypotheses on initial data. (Also see [11] regarding the discrete case.) A simple weak convergence theorem in this case follows from a classical result on the thinning of renewal processes [2]. In a recent article, Deaconu and Tanré proved a weak convergence result for all three kernels but under restrictive hypotheses on initial data [12]. Aldous and Pitman have studied the "eternal additive coalescent," and Bertoin has characterized "eternal solutions" to the Smoluchowski equation with additive kernel, solutions

defined globally for $-\infty < t < \infty$ [4, 7]. Bertoin showed these solutions correspond to the Lévy measure of a first-passage process related to Lévy processes with no positive jumps, and as a particular consequence he derived a new family of self-similar solutions related to the Lévy stable laws of probability.

In this article we find new families of self-similar solutions for the constant kernel, rederive the self-similar solutions for the additive and multiplicative kernels by analytical means, and characterize all possible domains of attraction under weak convergence for all the solvable kernels. We show the following:

(1) For each of the solvable kernels, Smoluchowski's equation admits a one-parameter family of scaling solutions parametrized by a number $\rho \in (0, 1]$ that characterizes the rate of divergence of the $(\gamma + 1)^{\text{th}}$ moment of the number density. For $\rho = 1$ these solutions reduce to the known solutions with exponential tails, while for $0 < \rho < 1$ the number density has algebraic decay ("fat tails"). For $K = 2$ ($\gamma = 0$) the normalized size distribution is a Mittag-Leffler distribution as studied by Pillai [31]. For $K = x + y$ ($\gamma = 1$) and xy ($\gamma = 2$), the γth moment distributions are transformed by power-law rescaling to the Lévy stable laws of probability theory (see (6.5) and [7]).

(2) The domains of attraction (under weak convergence of measures) for any scaling solution is determined by a condition on the tails of the initial data—the algebraic rate of divergence of the $(\gamma + 1)^{\text{th}}$ moment. A precise characterization is given via Karamata's notion of regular variation. All solutions with finite $(\gamma + 1)^{\text{th}}$ moment are attracted to self-similar solutions with exponential tail. However, with suitably diverging $(\gamma + 1)^{\text{th}}$ moment, there are initial data for which there is no convergence to any self-similar solution.

The self-similar solutions can all be captured by expressing their γ^{th} moment distribution in the general form

$$(1.2) \qquad x^{\gamma} n(t, x) = m_{\gamma}(t) \lambda_{\gamma}(t)^{-1} f_{\rho, \gamma}\left(x \lambda_{\gamma}(t)^{-1}\right),$$

where explicitly, with $\beta = \rho/(1 + \rho)$,

$$(1.3) \qquad m_0(t) = t^{-1}, \quad m_1(t) = 1, \quad m_2(t) = (1 - t)^{-1},$$

$$(1.4) \qquad \lambda_0(t) = t^{\frac{1}{\rho}}, \quad \lambda_1(t) = e^{\frac{t}{\beta}}, \quad \lambda_2(t) = (1 - t)^{-\frac{1}{\beta}},$$

and the $f_{\rho, \gamma}$ are probability densities given by

$$(1.5) \qquad f_{\rho, 0}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{\rho k - 1}}{\Gamma(\rho k)},$$

$$(1.6) \qquad f_{\rho, 1}(x) = f_{\rho, 2}(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{k\beta - 1}}{k!} \Gamma(1 + k - k\beta) \sin k\pi\beta.$$

We work with measure-valued solutions to (1.1) denoted ν_t, where $\nu_t((a, b])$ denotes the number of clusters with size $x \in (a, b]$: $\nu_t((a, b]) = \int_a^b n(t, x)\, dx$ if ν_t has integrable density $n(t, x)$. For each of the solvable kernels, there is a unique

solution for any initial data ν_0 with finite γ^{th} moment, and we associate a natural probability distribution function $F(t, x)$ to the solution

$$(1.7) \qquad F(t, x) = \frac{\displaystyle\int_0^x y^\gamma \nu_t(dy)}{\displaystyle\int_0^\infty y^\gamma \nu_t(dy)} \, .$$

This is the size-biased distribution for $K = 2$, the mass distribution for $K = x + y$, and the second moment distribution for $K = xy$. We are interested in necessary and sufficient conditions for the convergence of a rescaling $F(t, \lambda(t)x)$ to a nontrivial limit $F_*(x)$. Our results may be summarized in the following.

METATHEOREM *For the kernels $K(x, y) = 2$, $x + y$, and xy with degree of homogeneity $\gamma = 0, 1, 2$, respectively, let $T_\gamma = \infty$ for $\gamma = 0, 1$ and $T_\gamma = T_{\text{gel}}$ for $\gamma = 2$. Then for any solution of Smoluchowski's coagulation equation, there is a rescaling $\lambda(t)$ and a nontrivial probability distribution function F_* such that*

$$\lim_{t \to T_\gamma} F(t, \lambda(t)x) = F_*(x) \quad \textit{at all points of continuity of } F_*$$

if and only if

$$\int_0^x y^{\gamma+1} \nu_0(dy) \sim x^{1-\rho} L(x) \quad \textit{as } x \to \infty,$$

where $\rho \in (0, 1]$ and $L(x)$ is a function slowly varying at infinity. In the converse implication, F_* must be a rescaling of $F_{\rho,\gamma}(x) = \int_0^x f_{\rho,\gamma}(y) \, dy$.

Precise statements are deferred to Theorem 5.1, Theorem 7.1, and Theorem 8.1. We write the results in this form to stress the analogy with the classical characterization of the Lévy stable distributions in probability theory [16]. For the additive and multiplicative kernels, the analogy is an intimate relation with distributions for asymmetric Lévy flights: *the self-similar solutions can be transformed by a power-law rescaling into maximally skewed α-stable Lévy distributions.* A deeper understanding of our results is obtained from Bertoin's study of eternal solutions for the additive kernel [7]. The eternal solutions are analogous to infinitely divisible distributions of probability theory. Let ν_t be the value of an eternal solution at time t. Then, loosely speaking, for any $s < t$ the measure ν_s decomposes ν_t such that ν_t is reconstituted from ν_s under coagulation. This heuristic statement is made precise by Bertoin's characterization of Lévy pairs (σ^2, Λ) for the eternal solutions. This is the analogue of the classical Lévy-Khintchine characterization of the infinitely divisible distributions [16]. Among the infinitely divisible distributions, the stable distributions are of special interest, and their Lévy canonical measures are pure power laws. And indeed, the self-similar solutions to Smoluchowski's equation (1.1) have Lévy pairs corresponding to pure power laws: $\sigma^2 = 0$ and $\Lambda(dx) = cx^{-\alpha-1} dx$ for $1 < \alpha < 2$, and $\sigma^2 = 1$ and $\Lambda = 0$ for $\alpha = 2$, exactly as in the classical characterization. When viewed in this context, our theorems are entirely natural.

For $K = 2$ the main theorem may be interpreted probabilistically as a stability result for renewal processes on the line under uniform thinning (see [24]). For $K = x + y$, the results are related to Burgers turbulence for solutions of the inviscid Burgers equation when the initial velocity is given by a Lévy process with no positive jumps [6].

We exploit the analogy with the classical limit theorems of probability to obtain simple proofs of optimal theorems. The proofs involve little more than the solution for the Laplace transform and a fundamental rigidity lemma that characterizes scaling limits via functions of regular variation [16, VIII.8.3]. And the analogy extends much further. The central limit theorem is perhaps the most intensively studied result in probability theory. Thus, we can demand stronger forms of convergence as in expansions related to the central limit theorem. In companion articles we plan to study

  (a) uniform convergence of densities to the self-similar solutions with exponential tails (in analogy with the uniform convergence of densities in the central limit theorem) [27],
  (b) metric estimates (in analogy with the Berry-Esseen theorem), and
  (c) large deviation estimates.

We have found proofs for (a) and partial results for (b) and (c) that follow easily from a combination of the solution formula and the classical method of characteristic functions outlined in Feller [16].

It is worth remarking that the folklore in the applied literature is that the scaling solutions are unique (e.g., see [17, 34]). This is false in general (though the solutions with exponential tails are indeed special—they attract all solutions with finite $(\gamma + 1)^{\text{th}}$ moment). With hindsight, this nonuniqueness is not surprising. The existence of a one-parameter family of scaling solutions is well-known in physically related mean field models that show coarsening, such as the Lifshits-Slyozov-Wagner model [28], one-dimensional models for the coalescence of droplets [13], and cut-and-paste models of coarsening [18]. From the mathematical point of view the fundamental role of regular variation in branching processes is well established [9], and it is only natural that it should reappear in the "dual process" of coalescence. We conjecture that analogous results hold for general homogeneous kernels, but these lie beyond our techniques based on the Laplace transform.

2 Well-Posedness for Measure-Valued Solutions

2.1 Desingularized Laplace Transforms

Smoluchowski's equations determine a process of mass transport, and measure-valued solutions are an appropriate mathematical abstraction that contains solutions to the discrete and continuous coagulation equations within a unified framework. Norris recently proved several strong results for well-posedness of measure-valued solutions, but these do not apply with quite the generality we prefer for

$K = x + y$ and $K = xy$ [30]. We work with a somewhat different notion of solution motivated by the explicit solution obtained with the Laplace transform. The use of the Laplace transform for these kernels is classical [14], and aside from trivial changes of notation, many of the equations below may be found in Bertoin's article [7]. While our primary aim in writing this article is not to tackle the question of well-posedness, we show below that the divergence of the self-similar solutions necessitates some care in the definition of solutions. The impatient reader may skim through this section making a note of the main theorems and the explicit solution formulas. Our primary source for background on the Laplace transform is Feller [16].

Let E denote the open interval $(0, \infty)$, and let $\mathcal{M}_+$ denote the space of positive Radon measures on $(0, \infty)$. We interpret the number of clusters of size $x \in (a, b]$ per unit volume as $\nu((a, b])$ for $\nu \in \mathcal{M}_+$. We use the same letter to denote the distribution function of the measure, writing $\nu(x) = \nu((0, x])$ if this quantity is finite. We let $m_p = \int_E x^p \nu(dx)$ denote the pth moment of the measure, so m_0 is the total number of clusters and m_1 is the total mass.

We let $\eta(s)$ be the Laplace transform of $\nu(x)$ defined by

$$\eta(s) = \int_E e^{-sx} \nu(dx) = \int_E e^{-sx} n(x) dx \,.$$

The last equation holds when ν has a density n. In what follows we need to work with time-dependent measures ν_t for which the total number of clusters and/or total mass may be infinite. Consequently, it is more convenient to work with the variables (the "desingularized Laplace transform") given by

$$(2.1) \qquad \varphi(t, s) = \int_E (1 - e^{-sx}) \nu_t(dx) \,, \qquad \psi(t, s) = \int_E (1 - e^{-sx}) x \nu_t(dx) \,.$$

The variable $u = \partial_s \varphi$ has the important physical interpretation that it is the Laplace transform of the mass measure. Probabilists will recognize the obvious similarity to the Lévy-Khintchine representation. The equations of evolution in terms of these variables are extremely simple:

$$(2.2) \qquad\qquad\qquad \partial_t \varphi = -\varphi^2 \quad \text{for } K = 2,$$

$$(2.3) \qquad\qquad\qquad \partial_t \varphi - \varphi \partial_s \varphi = -\varphi \quad \text{for } K = x + y,$$

$$(2.4) \qquad\qquad\qquad \partial_t \psi - \psi \partial_s \psi = 0 \quad \text{for } K = xy.$$

We will construct measures using these equations and establish that they are solutions of Smoluchowski's equation in an appropriate weak sense.

The function $1 - e^{-sx}$ does not have compact support in E but has finite limits at 0 and ∞. A simple way to treat these limits is to consider $\bar{E} = [0, \infty]$ and consider continuous functions on $\bar{E}$, where $f(\infty)$ always means $\lim_{x \to \infty} f(x)$.

DEFINITION 2.1

- $C(\bar{E})$ is the space of continuous maps $f : \bar{E} \to \mathbb{R}$ equipped with the norm $\|f\|_{C(\bar{E})} = \sup_x |f(x)|$.
- $C^k(\bar{E})$ consists of k times continuously differentiable functions on E such that $f, \ldots, f^{(k)} \in C(\bar{E})$. It is equipped with the norm $\|f\|_{C^k(\bar{E})} = \|f\|_{C(\bar{E})} + \cdots + \|f^{(k)}\|_{C(\bar{E})}$.
- $C_c^k(E)$ is the subspace of $C^k(E)$ with compact support in E.
- $\mathcal{E}_k$ denotes the subspace of $C^k(\bar{E})$ of functions whose derivatives up to order k decay exponentially, that is,

$$\sum_{j=1}^{k} |f^{(j)}(x)| \le C_f e^{-\alpha x} \quad \text{for some } \alpha > 0,$$

and whose derivatives up to order $k - 1$ vanish at 0, that is, $f^{(j)}(0) = 0$ for $j < k$.

The following classical approximation lemma shows that the functions $1 - e^{-sx}$ span a dense set in $C(\bar{E})$.

LEMMA 2.2

(i) *Let $f \in C(\bar{E})$. Then for every $s > 0$ there is a sequence $P_n(x; s) = \sum_{k=1}^{n} a_{n,k}(s)(1 - e^{-sx})^k$ such that*

$$\lim_{n \to \infty} \|P_n(x; s) - f(x)\|_{C(\bar{E})} = 0.$$

(ii) *If $f \in \mathcal{E}_k$, then we also have $\lim_{n \to \infty} \|P_n(x; s) - f(x)\|_{C^k(\bar{E})} = 0$ for sufficiently small s.*

PROOF: The problem may be reduced to polynomial approximation on the unit interval by the transformation $y = 1 - e^{-sx}$ and $g(y) = f(x)$. Then $g \in C([0, 1])$ for $f \in C(\bar{E})$. Assertion (i) now follows from Weierstrass's approximation theorem. A particularly useful choice are the Bernstein polynomials $B_{n,g}(y)$ of g and $P_n(x; s) = B_{n,g}(y)$. Suppose $g \in C^k[0, 1]$. It is then classical that $\lim_{n \to \infty} \|g(y) - B_{n,g}(y)\|_{C^k[0,1]} = 0$ [25, p. 25]. Thus, in order to obtain (b) it suffices to show that $g \in C^k[0, 1]$. A decay assumption on f is warranted, because by the chain rule

$$g'(y) = f'(x)\frac{e^{sx}}{s}, \quad g''(y) = \frac{e^{2sx}}{s^2}(f''(x) + sf'(x)), \quad \text{etc.}$$

But $f \in \mathcal{E}_k$, so that $\sum_{j=1}^{k} |f^{(j)}(x)| \le C_f e^{-\alpha x}$. Hence, for $s < \alpha/k$ we have $\lim_{y \to 1} g'(y) = \cdots = \lim_{y \to 1} g^{(k)}(y) = 0$ and $g \in C^k[0, 1]$. Thus, given any $\varepsilon > 0$, there is an $n(\varepsilon)$ such that $|g'(y) - B'_{n,g}(y)| < \varepsilon$. The change of variables now works to our advantage, for we have $|f'(x) - P'_n(x)| < \varepsilon s e^{-sx}$. A similar calculation holds for all k derivatives proving (ii). $\qquad \square$

2.2 A Weak Formulation for Measure-Valued Solutions

Following Norris, we will generalize Smoluchowski's equation as follows. To every *finite*, positive measure ν we associate the measure $L(\nu)$ defined by duality with continuous functions with compact support.

$$(2.5) \qquad \langle f, L(\nu) \rangle = \frac{1}{2} \int_E \int_E [f(x+y) - f(x) - f(y)] K(x, y) \nu(dx) \nu(dy).$$

It is then natural to consider the weak formulation

$$(2.6) \qquad \langle f, \nu_t \rangle = \langle f, \nu_0 \rangle + \int_0^t \langle f, L(\nu_\tau) \rangle d\tau \quad \text{for every } f \in C_c(E).$$

This suffices for the case $K = 2$, but it is insufficient for $K = x + y$ and $K = xy$. The self-similar solutions for these kernels are not finite measures, and consequently they are not solutions in the sense of Norris, since they fail condition (3) in his definition [30, p. 80].

The basic obstruction is that $L(\nu_t)$ is not a measure in general, since $\langle f, L(\nu_t) \rangle$ may not be finite for all continuous functions. The reason is that even though f may have compact support in E, the function $Tf(x, y) := f(x+y) - f(x) - f(y)$ does not have compact support in $E \times E$ and may not be integrable with respect to the product measure $\nu \otimes \nu$.

Here is a counterexample for $K = x + y$. Let $\chi(x)$ be the indicator function for the interval $(0, 1)$, and let $\nu(dx) = x^{-3/2} \chi(x) dx + \delta(x - 2)$. Let f be a continuous function with support in $[a, b] = [2, 5]$. Then the values of Tf are as shown in Figure 2.1. Notice that $\nu((x, \infty))$ diverges like $O(x^{-1/2})$ as $x \to 0$, but ν has finite mass. Thus, in order for the integrals in the definition of L to converge, it is necessary that there be suitably rapid cancellations as we approach the boundaries. One may show that the integral is finite on all regions except near the axes in the shaded regions. However, we explicitly compute that for any small $\delta > 0$,

$$(2.7) \qquad \int_0^\delta \int_0^\infty [f(x+y) - f(x) - f(y)](x+y)\nu(dx)\nu(dy) =$$

$$\int_0^\delta f(2+y)(2+y)y^{-\frac{3}{2}}\, dy.$$

This is evidently infinite if $f(x)$ rises sufficiently steeply for $x > 2$. Therefore $L(\nu)$ is not a measure.

This means that the space of continuous functions is not appropriate as a space of test functions in (2.6). The smaller spaces $\mathcal{E}_\gamma$ serve as a suitable substitute.

DEFINITION 2.3 For each kernel $K(x, y) = 2$, $x + y$, and xy with degree of homogeneity $\gamma = 0, 1, 2$, respectively, let $T_\gamma = \infty$ for $\gamma = 0, 1$ and $T_\gamma = T_{\text{gel}}$ for $\gamma = 2$. We say that a map $t \mapsto \nu_t : [0, T_\gamma) \mapsto \mathcal{M}_+$ is a solution to Smoluchowski's coagulation equation if

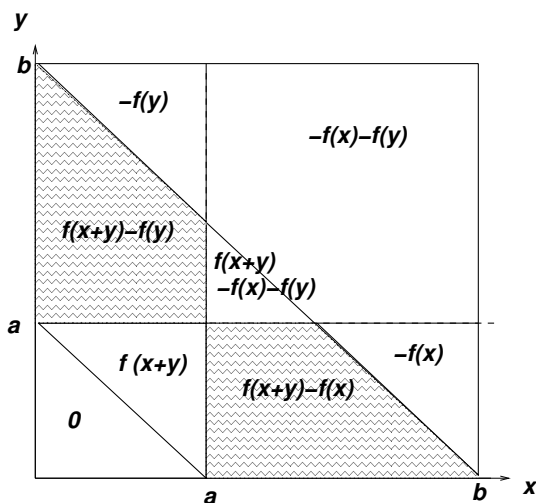   (i) $m_\gamma(0) = \int_E x^\gamma \nu_0(dx) < \infty$,

FIGURE 2.1. Cancellations in $Tf(x, y)$. Integrals over the shaded regions diverge absolutely unless f has a suitable modulus of continuity.

(ii) for all compact sets $B \subset E$, the map $t \mapsto \nu_t(B)$ is measurable,
(iii) $\int_0^t m_\gamma(\tau)^2 \, d\tau < \infty$ for all $t \in (0, T_\gamma)$, and
(iv) for all $f \in \mathcal{E}_\gamma$ and $t \in [0, T_\gamma)$, we have

$$(2.8) \qquad \langle f, \nu_t \rangle = \langle f, \nu_0 \rangle + \int_0^t \langle f, L(\nu_\tau) \rangle d\tau .$$

2.3 Existence and Uniqueness for the Constant Kernel

We will set $K = 2$ instead of the usual convention of setting $K = 1$, since it simplifies several calculations (we actually revert to an older convention; see, for example, equation (459) in [10]).

THEOREM 2.4 *Let $\nu_0 \in \mathcal{M}_+$ be a finite measure. Then Smoluchowski's coagulation equation with kernel $K = 2$ has a unique solution with initial data ν_0, and this solution is determined by the solution of* (2.2).

Theorem 2.4 is a consequence of [30, theorem 2.1]. We will prove it anew with the Laplace transform, as the explicit solution formula is needed later. Let ν_t denote the number distribution at time t, and $\varphi(t, \cdot)$ be determined from $\nu = \nu_t$ by (2.1) for each t. Then formally φ should solve the simple equation (2.2). For fixed $s > 0$, (2.2) is an ordinary differential equation with the solution

$$(2.9) \qquad \varphi(t, s) = \frac{\varphi_0(s)}{1 + \varphi_0(s)t} .$$

LEMMA 2.5 *Assume $\nu_0 \in \mathcal{M}_+$ is finite. Formula (2.9) determines a weakly continuous map $[0, \infty) \ni t \mapsto \nu_t \in \mathcal{M}_+$ with decreasing total number $m_0(t) = m_0(0)/(1 + m_0(0)t)$.*

PROOF: The solution $\varphi(t, s)$ has the important property that its derivative is completely monotone for $t \geq 0$. This is because it may be written as a composition of positive functions with completely monotone derivative

$$\varphi(t, s) = \frac{p}{1 + tp} \circ (\varphi_0(s)) .$$

Recall that the derivative of $\varphi(t, s)$ is $u(t, s)$, the Laplace transform of the mass measure, say $\mu_t(dx)$. Because $u(t, s)$ is completely monotone, it follows that $\mu_t(dx) \in \mathcal{M}_+$. Since $u(t, s)$ is analytic in t, we see that the measures μ_t are weakly continuous by the classical duality between pointwise convergence of the Laplace transform and weak convergence of measures. That is, for any continuous function f with compact support in E, we have $\langle f, \mu_\tau \rangle \to \langle f, \mu_t \rangle$ as $\tau \to t$. It follows that $\langle f, \nu_\tau \rangle \to \langle f, \nu_t \rangle$ as $\tau \to t$ where $\nu_t(dx) = x^{-1}\mu_t(dx)$ is the number measure. The statement regarding $m_0(t)$ follows by taking $s \to \infty$ in (2.9). □

Remark 2.6. It is strange at first to consider the desingularized Laplace transform when the initial data is finite, and indeed the usual Laplace transform suffices. But (2.9) shows us that the solution is instantly regularizing in the following sense: If $\nu_0(\infty) = \varphi_0(\infty) = \infty$, the solution satisfies

$$\lim_{s \to \infty} \varphi(t, s) = \lim_{s \to \infty} \frac{\varphi_0(s)}{1 + t\varphi_0(s)} = \frac{1}{t} .$$

Thus, the number of clusters is finite for $t > 0$. Thus, $\varphi(t, s)$ defines a natural solution even for an initially infinite measure. However, it is hard to verify (2.8) in this case (even for $f \in C_c^1(E)$), and we restrict ourselves to finite measures in what follows.

PROOF OF THEOREM 2.4: Let us first show that the measures ν_t determined by the lemma form a solution to Smoluchowski's equation in the sense of Definition 2.3. Condition (i) has been assumed. It is easy to check condition (ii). The measures ν_t are weakly continuous. Thus for a fixed compact set $B \subset E$, the function $t \mapsto \nu_t(B)$ is semicontinuous. Condition (iii) follows from Lemma 2.5.

It is not a priori obvious that $\langle 1 - e^{-sx}, L(\nu_t) \rangle$ is indeed $-\varphi(t, s)^2$. But the measures are finite since $\nu_t(E) \leq \nu_0(E) < \infty$, and thus we may set $f = 1 - e^{-sx}$ in the definition of $\langle f, L(\nu_t) \rangle$ to recover $-\varphi(t, s)^2$. In particular, this shows that (2.8) holds for $f = 1 - e^{-sx}$. This equation also holds if f is a monomial $(1 - e^{-sx})^k$ because

$$(1 - e^{-sx})^k = \sum_{j=0}^{k} \binom{k}{j}(-1)^j e^{-jsx} = -\sum_{j=0}^{k} \binom{k}{j}(-1)^j(1 - e^{-jsx}) .$$

Given $f \in C(\bar{E})$ and $\varepsilon > 0$, Lemma 2.2 guarantees an approximation P_n with $\|f - P_n\|_{C(\bar{E})} < \varepsilon$. Then, $|\langle f - P_n, L(v_t) \rangle| \leq 3\varepsilon(v_t(E))^2 \leq 3\varepsilon(v_0(E))^2$. Thus,

$$\left| \langle f, v_t \rangle - \langle f, v_0 \rangle - \int_0^t \langle f, L(v_\tau) \rangle d\tau \right|$$

$$\leq \langle |f - P_n|, v_t \rangle + \langle |f - P_n|, v_0 \rangle + \int_0^t \left| \langle |f - P_n|, L(v_\tau) \rangle \right| d\tau$$

$$\leq \varepsilon \left(v_0(E) + v_0(E) + 3t(v_0(E))^2 \right).$$

This shows that the measures v_t define a solution.

Suppose v_t and $\tilde{v}_t$ are two solutions with the same initial data. Since $f = 1 - e^{-sx} \in \mathcal{E}_0 = C(\bar{E})$ and $\varphi(t, s) = \langle f, v_t \rangle \leq m_0(t)$ for a.e. t, we can use (2.8) and condition (iii) of Definition 2.3 to obtain (2.3) in time-integrated form for each solution. It follows easily that for fixed $s > 0$, each $\varphi(t, s)$ is C^1 in t and satisfies (2.3). But this equation has a unique solution $\varphi(t, s)$ as in (2.9). As we have noted in Lemma 2.5, $\varphi(t, s)$ determines the measure v_t. Thus $v_t = \tilde{v}_t$. $\qquad\square$

2.4 Existence and Uniqueness for the Additive Kernel

We always work with solutions of finite mass, but we do not assume that the number of clusters is finite. Therefore, $L(v)$ will not be a measure. Nevertheless, it does define a bounded linear functional on the space of Lipschitz functions on E.

LEMMA 2.7 *Let $v \in \mathcal{M}_+$ with $m_1 := \int_E x v(dx) < \infty$, and let $L(v)$ be defined by (2.5) with $K = x + y$. Suppose f is Lipschitz and $f(0) = 0$. Then*

$$(2.10) \qquad\qquad |\langle f, L(v) \rangle| \leq 2m_1^2 \operatorname{Lip}(f).$$

PROOF: By the symmetry of the integral in (2.5), we see that

$$\left| \langle f, L(v) \rangle \right| \leq \int_E \int_E |f(x+y) - f(x) - f(y)| \, y v(dy) v(dx).$$

The integrand is controlled by

$$|f(x+y) - f(x) - f(y)| \leq |f(x+y) - f(y)| + |f(x)| \leq 2\operatorname{Lip}(f)x.$$

Thus we obtain

$$\left| \langle f, L(v) \rangle \right| \leq 2\operatorname{Lip}(f) \int_E \int_E xy v(dy) v(dx) = 2m_1^2 \operatorname{Lip}(f).$$

$\qquad\square$

THEOREM 2.8 *Let $v_0 \in \mathcal{M}_+$ satisfy $\int_E x v_0(dx) = m_1 < \infty$. Then Smoluchowski's coagulation equation with kernel $K = x + y$ has a unique solution with initial data v_0 such that $\int_E x v_t(dx) = m_1$ for all $t \in [0, \infty)$.*

We will construct a solution using the desingularized Laplace transform and then prove its uniqueness. The evolution equation for φ is

$$\partial_t \varphi - \varphi \partial_s \varphi = -m_1 \varphi.$$

We may always normalize initial data such that $m_1 = 1$, and we assume this in all that follows. Thus, we have

(2.11) $$\partial_t \varphi - \varphi \partial_s \varphi = -\varphi.$$

It is striking that (2.11) is simply the inviscid Burgers equation with linear damping. However, there is no shock formation, since the initial data are analytic with a completely monotone derivative satisfying $\partial_s \varphi_0(s) \leq 1$. This can be seen in the explicit solution below, which is valid for all time. Since $u = \partial_s \varphi$, differentiating (2.11) we have

(2.12) $$\partial_t u - \varphi \partial_s u = -u(1 - u).$$

We solve (2.11) and (2.12) globally by the method of characteristics. Let $s(t, \sigma)$ denote the characteristic that originates at σ at $t = 0$. Then we have

(2.13) $$\frac{ds}{dt} = -\varphi, \quad \frac{d\varphi}{dt} = -\varphi, \quad \frac{du}{dt} = -u(1 - u),$$

on a characteristic. We integrate (2.13) along the characteristics to obtain

(2.14) $$\varphi(t, s) = e^{-t} \varphi_0(\sigma),$$

(2.15) $$s(t, \sigma) - \varphi(t, s) = \sigma - \varphi_0(\sigma),$$

(2.16) $$u(t, s) = \frac{e^{-t} u_0(\sigma)}{1 - (1 - e^{-t}) u_0(\sigma)}.$$

LEMMA 2.9 *Suppose $v_0(x) \in \mathcal{M}_+$ with $\int_E x \, v_0(dx) = 1$. Then equation (2.11) determines a map $[0, \infty) \ni t \mapsto v_t \in \mathcal{M}_+$ such that*

   (i) *$\int_E x v_t(dx) = 1$ for all t and*
   (ii) *$\mu_t = x v_t$ is weakly continuous.*

PROOF: Observe that

$$\sigma - \varphi_0(\sigma) = \int_E (e^{-sx} - 1 + sx) v_0(dx) > 0, \quad \sigma > 0.$$

Thus, by (2.14) and (2.15)

(2.17) $$s(t, \sigma) = \sigma - \varphi_0(\sigma)(1 - e^{-t}) > 0, \quad \sigma > 0.$$

The right-hand side is a strictly increasing function of σ for all $t \geq 0$; thus, the inverse map $\sigma(t, s)$ is well-defined. Differentiating (2.17) with respect to s, we find that

(2.18) $$\frac{d\sigma}{ds} = \frac{1}{1 - (1 - e^{-t}) u_0(\sigma(s))} \quad \text{whence} \quad u(t, s) = e^{-t} u_0(\sigma) \frac{d\sigma}{ds}.$$

Since u_0 is the Laplace transform of a positive measure, it is a completely monotone function of σ.

In order to show that $u(t, s)$ is completely monotone in s, it suffices to show that $d\sigma/ds$ is completely monotone in s (see criteria 1 and 2 in [16, XIII.4]). We prove this as follows: Let us consider the sequence of iterates

$$\sigma_0(s) = s \quad \text{and} \quad \sigma_{n+1}(s) = s + (1 - e^{-t})\varphi_0(\sigma_n(s)), \quad n \geq 0.$$

Clearly, $|\sigma_{n+2}(s) - \sigma_{n+1}(s)| < |\sigma_{n+1}(s) - \sigma_n(s)|$ so that $\sigma_n(s) \to \sigma(s)$, the unique solution to (2.17). Moreover, we have

$$\frac{d\sigma_{n+1}(s)}{ds} = 1 + (1 - e^{-t})u_0(\sigma_n(s))\frac{d\sigma_n(s)}{ds}.$$

Thus if $d\sigma_n/ds$ is completely monotone, then so is $d\sigma_{n+1}/ds$. But $d\sigma_0/ds = 1$ is completely monotone. By induction, $d\sigma_n/ds$ is completely monotone for $n \geq 1$ and so is the limit $d\sigma/ds$.

We may now conclude that the solution $\varphi(t, s)$ to (2.11) defined by (2.14) exists for all $t \geq 0$, is unique, and has a completely monotone derivative $u(t, s)$. Thus $u(t, s)$ defines a unique mass measure, say $\mu_t(dx)$. We see from the solution (2.16) that $u(t, 0) = u_0(0) = 1$. Thus, the total mass $\int_E \mu_t(dx) = u(t, 0) = 1$ for all $t \geq 0$. The measures μ_t are weakly continuous since $u(t, s)$ is analytic in time. $\quad \square$

PROOF OF THEOREM 2.8: Let us first check that the measures ν_t determined by (2.11) solve Smoluchowski's equation in the sense of Definition 2.3. Conditions (i) and (ii) in Definition 2.3 are verified as in the proof of Theorem 2.4. Since ν_t has constant mass, it follows from Lemma 2.7 that the functionals $L(\nu_t)$ are uniformly bounded on $\mathcal{E}_1$. In particular,

(2.19) $$\langle 1 - e^{-sx}, L(\nu_t) \rangle = -\varphi + \varphi\partial_s\varphi,$$

as desired. This shows that (2.8) holds for $f = 1 - e^{-sx}$ and thus for monomials $(1 - e^{-sx})^k$. Given any $f \in \mathcal{E}_1$ and $\varepsilon > 0$, we choose an approximation $P_n(x; s)$ as in Lemma 2.2(ii) so that $\|f - P_n\|_{C^1(\bar{E})} < \varepsilon$. Then by Lemma 2.7

$$\left|\langle f - P_n, L(\nu_t) \rangle\right| \leq 2\,\mathrm{Lip}(f - P_n) < 2\varepsilon.$$

Similarly,

$$|\langle f - P_n, \nu_t \rangle| \leq \int_E |f(x) - P_n(x; s)|\nu_t(dx) \leq \mathrm{Lip}(f - P_n)\int_E x\nu_t(dx) < \varepsilon.$$

This shows that ν_t is a solution in the sense of Definition 2.3.

Now suppose only that ν_t is a solution in the sense of Definition 2.3. The function $f = 1 - e^{-sx} \in \mathcal{E}_1$ and $f \leq sx$. For a.e. $t > 0$ we have $m_1(t) < \infty$, and from $\varphi = \langle f, \nu_\tau \rangle \leq s^{-1}m_1(t)$ it follows that $\varphi(t, s)$ is analytic in s with $|\partial_s^k\varphi| \leq s^{1-k}m_1(t)$ for $k = 1, 2, \ldots$. Condition (iii) of Definition 2.3 ensures that

for fixed $s > 0$, $m_1(t)$ is locally square-integrable on $[0, \infty)$, and (2.8) gives (2.11) in time-integrated form, since (2.19) holds for a.e. t and

$$|\langle f, L(v_t)\rangle| = |\varphi \partial_s \varphi - \varphi| \le s m_1(t)(m_1(t) + 1).$$

It follows that φ is continuous in t, uniformly for s in compact sets in E. Moreover, one can justify differentiating (2.8) in s and infer that $\partial_s \varphi$ is continuous. Then φ is a C^1 solution of (2.11), whence $\varphi(t, s)$ is uniquely determined by initial data. But as we have noted in Lemma 2.9, $\varphi(t, s)$ uniquely determines the measure v_t. Thus the solution is unique. $\square$

2.5 Existence and Uniqueness for the Multiplicative Kernel

The multiplicative kernel differs from the constant and additive kernels, since it is not well-posed for all time. But, the analysis can be formally reduced to the additive case by a change of variables. This is well-known [14], but we include it for completeness.

The divergence of the classical self-similar solution is $O(x^{-5/2})$ as $x \to 0$. The total number and mass are infinite, but the second moment is finite. Therefore, we consider the following desingularized Laplace transform:

$$(2.20) \qquad \phi(t, s) = \int_E (e^{-sx} - 1 + sx) v_t(dx).$$

We substitute $f = e^{-sx} - 1 + sx$ in the equation of evolution (2.5) to find

$$(2.21) \qquad \partial_t \phi = \langle f, L(v_t)\rangle = \frac{1}{2}(\partial_s \phi)^2.$$

Equation (2.21) is the Hamilton-Jacobi equation associated to the inviscid Burgers equation. Thus, we let

$$(2.22) \qquad \psi(t, s) = \partial_s \phi = \int_E (1 - e^{-sx}) x v_t(dx).$$

Then from (2.21) we have

$$(2.23) \qquad \partial_t \psi - \psi \partial_s \psi = 0.$$

The exact solution to (2.23) with initial data $\psi_0(s)$ may be found by the method of characteristics. The characteristic originating at s_0 is denoted

$$s(t, s_0) = s_0 - \psi_0(s_0)t.$$

Let $t(s_0, s_1)$ denote the time for two characteristics originating at $s_0 < s_1$ to intersect. Then, if this is the first intersection

$$\frac{1}{t} = \frac{\psi_0(s_1) - \psi_0(s_0)}{s_1 - s_0} \quad \text{whence} \quad \frac{1}{\partial_s \psi_0(s)} < t(s_0, s_1) < \frac{1}{\partial_s \psi_0(s_1)},$$

where the inequalities follows from the mean value theorem and the complete monotonicity of $\partial_s \psi_0$. Thus, letting $s_0 = 0$ and $s_1 \to 0$, we see that the least time taken for characteristics to intersect is given by

$$T_{\text{gel}}^{-1} = \partial_s \psi_0(0) = \int_E x^2 v_0(dx) = m_2(0) .$$

Without loss of generality, we may assume that the initial data is normalized so that $m_2(0) = 1 = T_{\text{gel}}$ and thus $0 \leq t < 1$. This normalization assumption is analogous to the assumption that $m_1 = 1$ for the additive kernel. Equation (2.23) should be compared with equation (2.3). In fact, given initial data $\psi_0(s)$, by changing the time scale in (2.3) it is easy to check that $\psi(t, s)$ is the solution to (2.23) if and only if

$$(2.24) \qquad \psi(t, s) = \frac{1}{1 - t} \varphi(-\log(1 - t), s) ,$$

where $\varphi(t, s)$ is the unique solution to (2.3) with initial data ψ_0. The next lemma follows immediately from Lemma 2.9.

LEMMA 2.10 *Suppose* $v_0(x) \in \mathcal{M}_+$ *with* $\int_E x^2 v_0(dx) = 1$. *Then equation* (2.23) *determines a map* $[0, 1) \ni t \mapsto v_t \in \mathcal{M}_+$ *such that*

  (i) $m_2(t) = \int_E x^2 v_t(dx) = (1 - t)^{-1}$ *and*
  (ii) $x^2 v_t$ *is weakly continuous on* $[0, 1)$.

It is natural to term the measure v_t the solution to Smoluchowski's coagulation equation with kernel $K = xy$. However, it is harder to formulate a completely natural well-posedness theory in this case, and we will settle for a reasonable compromise.

DEFINITION 2.11 Define the norm

$$(2.25) \qquad \sup_{x, y > 0} \frac{|f(x + y) - f(x) - f(y)|}{xy} := \|f\|_V$$

and the associated Banach space $V = \{f \in C_0(E) : \|f\|_V < \infty\}$.

It is clear that V is a Banach space. The norm $\| \cdot \|_V$ is natural in the following sense.

LEMMA 2.12 *Let* $v \in \mathcal{M}_+$ *such that* $m_2 < \infty$, *and let* $L(v)$ *be defined by* (2.5) *with* $K = xy$. *Then* $L(v)$ *defines a bounded linear functional on* V *with norm* $\leq m_2^2/2$.

PROOF: Since $|f(x + y) - f(x) - f(y)| \leq xy\|f\|_V$, we have

$$|\langle f, L(v) \rangle| \leq \frac{1}{2} \int_E \int_E \|f\|_V x^2 y^2 v(dx) v(dy) = \frac{m_2^2}{2} \|f\|_V .$$

$\square$

It is easy to check that finite sums $f(x) = \sum_{k=1}^{n} a_k(1 - e^{-s_k x})$ are in V. We would like to believe that these functions are dense in V, but this seems hard to prove, since the norm above is unwieldy. Instead we will work with C^2 functions and use the following, whose easy proof we omit.

LEMMA 2.13 *Let f be a C^1 function such that $f(0) = 0$ and f' is Lipschitz. Then $\|f\|_V \leq 2 \operatorname{Lip}(f')$.*

THEOREM 2.14 *Let $\nu_0 \in \mathcal{M}_+$ satisfy $m_2(0) < \infty$. Then Smoluchowski's coagulation equation with kernel $K = xy$ has a unique solution with initial data ν_0 on the time interval $[0, m_2(0)^{-1})$.*

PROOF: Without loss of generality we may suppose that $m_2(0) = 1$. The measures ν_t of Lemma 2.10 are a candidate solution, and it is easy to check that conditions (i) and (ii) of Definition 2.3 are satisfied. Since $m_2(t) = (1 - t)^{-1}$ by Lemma 2.12, one sees that $L(\nu_t)$ is a bounded linear operator on V; in particular,

$$\langle 1 - sx - e^{-sx}, L(\nu_t) \rangle = -\frac{1}{2}(\phi_s)^2.$$

Thus, $\phi(t, s)$ solves (2.21). Yet, some care is needed in checking that (2.8) holds in full generality. Let $f \in \mathcal{E}_2$. We apply Lemma 2.2 to f' (notice, *not* f) to obtain an approximation $P_n(x; s)$ with $\sup_x |P_n - f'| < \varepsilon$ and $\sup_x |P'_n - f''| < \varepsilon$. Observe that we may rewrite

$$P_n(x; s) = \sum_{k=1}^{n} a_{n,k}(1 - e^{-sx})^k = \sum_{k=1}^{n} b_{n,k}(1 - e^{-skx})$$

by expanding $(1 - e^{-sx})^k$ with the binomial formula and defining $b_{n,k}$ as the corresponding linear combinations of $a_{n,k}$. We integrate P_n to obtain the approximation

$$Q_n(x; s) = \sum_{k=1}^{n} \frac{b_{n,k}}{sk}(e^{-skx} - 1 + skx).$$

Notice that $P_n(0; s) = 0$. Therefore, by the fundamental theorem of calculus, we also have

$$|P_n(x; s) - f'(x)| \leq \int_0^x |P'_n(z; s) - f''(z)| dz < \varepsilon x,$$

and upon integration again,

$$|f(x) - Q_n(x; s)| < \frac{\varepsilon x^2}{2}.$$

But we then have

$$|\langle f - Q_n, \nu_\tau \rangle| \leq \frac{\varepsilon}{2} \int_E x^2 \nu_\tau(dx) = \frac{\varepsilon}{2} m_2(\tau), \quad \tau \in [0, 1).$$

Since $\sup_x |f'' - Q_n''| < \varepsilon$, we apply Lemma 2.12 and Lemma 2.13 to obtain

$$|\langle f - Q_n, L(\nu_\tau)\rangle| < \varepsilon m_2^2(\tau), \quad \tau \in [0, 1).$$

This proves that ν_t is a solution.

It is slightly harder to prove uniqueness in this case. As in Theorem 2.4 and Theorem 2.8, it suffices to deduce uniqueness of the measure-valued solution via uniqueness of solutions to (2.21). The obstruction is that it is not clear from the definition of the weak solution that (2.21) holds, since the test functions $1 - sx - e^{-sx}$ do not lie in $\mathcal{E}_2$. This can be overcome with an approximation argument that we only sketch. We consider $f_n(x) = (1 - sx - e^{-sx})\chi_n(x)$ where χ_n is a C^∞ cutoff function such that $\chi_n = 1$, $x \le n$, $\chi_n = 0$, and $x \ge n + 1$. By the monotone convergence theorem, $\lim_{n\to\infty}\langle f_n, \nu_t\rangle = \langle 1 - sx - e^{-sx}, \nu_t\rangle$. By Lemma 2.12 and Lemma 2.13, $\langle f_n, L(\nu_t)\rangle$ is well-defined and uniformly bounded by $C_s m_2(t)^2$. We may then use the dominated convergence theorem to deduce that (2.21) holds in the limit $n \to \infty$. Uniqueness of ν_t follows. $\qquad\square$

3 Regular Variation

Several formal calculations by physicists working on Smoluchowski's equations take the following form: (1) assume that the number density $n(x) \sim x^\alpha$ for some scaling exponent α, and (2) conclude based on physical arguments that α takes a particular value. The theory of regular variation helps us makes these formal calculations precise and lays bare the mechanism controlling the approach to scaling form. Our primary source is Feller's book, and we restate below useful results from [16, VIII.8]. The theory of regular variation has many applications in analysis and probability, and an authoritative text, rich in examples, is [9].

3.1 Rigidity of Scaling Limits

Loosely speaking, a function is *slowly varying* if it is asymptotically flat under changes of scale. Precisely, we say that a positive function $L(x)$ is *slowly varying at infinity* if

$$(3.1) \qquad \lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \text{for all } t > 0.$$

For example, all powers and iterates of $\log x$ are slowly varying at infinity. If we consider the limit $x \to 0$ instead, we obtain functions that are slowly varying at 0.

A function $N(x)$ is *regularly varying at infinity* with index $\rho \in \mathbb{R}$ if there is a slowly varying function $L(x)$ such that

$$(3.2) \qquad N(x) \sim x^\rho L(x) \quad \text{as } x \to \infty.$$

The notation $\sim$ means $\lim_{x\to\infty}(N(x)/x^\rho)L(x) = 1$.

The notion of regular variation is intimately related to necessary and sufficient conditions for the existence of scaling limits. This is reflected in the following

classical "rigidity" lemma [16, lemma VIII.8.3], which will be one of our principal tools.

LEMMA 3.1 *Suppose that $a_{n+1}/a_n \to 1$ and $\lambda_n \to \infty$ as $n \to \infty$. If φ is a positive, monotone function such that*

$$\lim_{n\to\infty} a_n\varphi\left(\frac{s}{\lambda_n}\right) = g(s) \leq \infty$$

exists for s in a dense subset of $(0, \infty)$, and g is finite and positive on some interval, then φ varies regularly at 0 and $g(s) = cs^\rho$ for $-\infty < \rho < \infty$ and some $c > 0$.

3.2 Tauberian Theorems

We will rigorously deduce the asymptotics of ν by the beautiful Hardy-Little-wood-Karamata Tauberian theorem [16, XIII.5].

THEOREM 3.2 *If L is slowly varying at infinity and $0 \leq \alpha < \infty$, then the following are equivalent:*

$$\nu(x) \sim x^\alpha L(x) \ \ as \ x \to \infty \quad and \quad \eta(s) \sim s^{-\alpha} L\left(\frac{1}{s}\right)\Gamma(1+\alpha) \ \ as \ s \to 0.$$

Moreover, this equivalence remains true when we interchange the roles of the origin and infinity, namely, when $s \to \infty$ and $x \to 0$.

We will use the following lemma to show that there is no loss of generality in working with φ instead of η.

LEMMA 3.3 *Suppose $\partial_s\psi$ is the Laplace transform of a positive measure. Let $\alpha < 1$ and L be a function slowly varying at 0. The following are equivalent:*

  (i) *$\psi(s) - \psi(0) \sim s^{1-\alpha}L(s)$ as $s \to 0$ and*
  (ii) *$\partial_s\psi(s) \sim (1-\alpha)s^{-\alpha}L(s)$ as $s \to 0$.*

PROOF: Suppose (i). Since $\psi(s) - \psi(0) = s^\alpha L(s)h(s)$ with $\lim_{s\to 0} h(s) = 1$, without loss of generality we may write $\psi(s) - \psi(0) = s^\alpha L(s)$. Fix $a > 1$. Then by the mean value theorem and the complete monotonicity of $\partial_s\psi$ we have

$$s(a-1)\partial_s\psi(s) \geq \psi(as) - \psi(s) = s^{1-\alpha}L(s)\left(a^{1-\alpha}\frac{L(as)}{L(s)} - 1\right).$$

Thus, letting $s \to 0$ and using (3.1) we have

$$\liminf_{s\to 0} \frac{s^\alpha\partial_s\psi(s)}{L(s)} \geq \frac{a^{1-\alpha} - 1}{a - 1}.$$

Since $a > 1$ is arbitrary, we may maximize the right-hand side to obtain

$$\liminf_{s\to 0} \frac{s^\alpha\partial_s\psi(s)}{L(s)} \geq 1 - \alpha.$$

Choosing $a < 1$ and using a similar argument yields

$$\limsup_{s \to 0} \frac{s^{\alpha} \partial_s \psi(s)}{L(s)} \leq 1 - \alpha.$$

Thus, $\partial_s \psi(s) \sim (1 - \alpha)s^{-\alpha}L(s)$.

Conversely, assume (ii). Then we have

$$\psi(s) - \psi(0) = (1 - \alpha)\int_0^s t^{-\alpha}L(t)dt = (1 - \alpha)s^{1-\alpha}L(s)\int_0^1 t^{-\alpha}\frac{L(st)}{L(s)}\,dt.$$

Since L is slowly varying at 0, then for any constants $A > 1$ and $\delta > 0$ there exists s_0 such that for $0 < s \leq s_0$ and $0 < t \leq 1$, we have $L(st)/L(s) \leq At^{-\delta}$ (this is not hard to show, but see theorem 1.5.6 in [9]). Then (i) follows by dominated convergence. $\square$

4 Scaling Solutions for the Constant Kernel

4.1 Mittag-Leffler Distributions

The scaling solution

$$(4.1) \qquad\qquad n(t, x) = t^{-2}\exp\left(\frac{-x}{t}\right), \quad t > 0,$$

is the continuous limit of a special solution found by Smoluchowski [2]. Kreer and Penrose proved that the rescaled number density $t^2n(t, xt)$ converges uniformly to $e^{-\alpha x}$ on compact sets, under the assumption that the initial number density $n_0(x)$ be C^2 and have exponential decay in x [22]. The constant α is determined by the initial mass.

In this section we show that the solution (4.1) is just one of a one-parameter family of scaling solutions given by

$$(4.2) \qquad\qquad n(t, x) = t^{-1-\frac{1}{\rho}}n_\rho\left(xt^{-\frac{1}{\rho}}\right), \quad t > 0,\ \rho \in (0, 1],$$

where $n_\rho(x) = F_\rho'(x)$ is the density, and F_ρ the distribution function for the Mittag-Leffler distribution

$$(4.3) \qquad\qquad F_\rho(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{\rho k}}{\Gamma(1 + \rho k)}, \quad \rho \in (0, 1].$$

Of these solutions, only the solution (4.1) for $\rho = 1$ has finite mass, and the others have fat tails. The Mittag-Leffler distribution was studied by Pillai [31], who showed that these distributions are infinitely divisible and geometrically infinitely divisible for $\rho \in (0, 1]$.

For our purposes, it is especially relevant that the Mittag-Leffler distribution has Laplace transform

$$(4.4) \qquad\qquad \int_0^{\infty} e^{-sx}n_\rho(x)dx = \int_0^{\infty} e^{-sx}F_\rho(dx) = \frac{1}{1 + s^{\rho}}.$$

In terms of the Mittag-Leffler function

$$E_\rho(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + \rho k)},$$

one has $F_\rho(x) = 1 - E_\rho(-x^\rho)$. It is interesting and useful to note the following recent calculation of Tsoukatos [33].

LEMMA 4.1 *For* $0 < \rho < 1$ *we have*

$$(4.5) \qquad E_\rho(-x^\rho) = \frac{1}{\pi} \int_0^\infty e^{-rx} \frac{r^{\rho-1} \sin \pi\rho}{(r^\rho + \cos \pi\rho)^2 + (\sin \pi\rho)^2} \, dr.$$

Hence $E_\rho(-x^\rho)$ *and* $n_\rho(x) = -\partial_x E_\rho(-x^\rho)$ *are completely monotone.*

PROOF: We sketch the argument of Tsoukatos [33]. Since (4.4) implies

$$(4.6) \qquad \int_0^\infty e^{-sx} E_\rho(-x^\rho) dx = \frac{1}{s}\left(1 - \frac{1}{1 + s^\rho}\right),$$

one can invoke the Laplace inversion formula and evaluate it by deforming the contour to fold along the negative real axis to obtain, for any $\sigma > 0$,

$$E_\rho(-x^\rho) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{sx} \frac{s^\rho}{1 + s^\rho} \frac{ds}{s} = \frac{1}{\pi} \operatorname{Im} \int_0^\infty e^{-rx} \frac{r^\rho e^{i\pi\rho}}{1 + r^\rho e^{i\pi\rho}} \frac{dr}{r},$$

and the result follows. $\qquad\square$

Note that the complete monotonicity of $E_\rho(-x)$ (conjectured by Feller and proven by Pollard in 1948) also implies the complete monotonicity above, since x^ρ is positive with completely monotone derivative. A point of confusion in the literature is that the term "Mittag-Leffler law" is used by some for the distribution whose Laplace transform is $E_\rho(-s)$ [9].

4.2 Scaling Solutions

Let us now check that $n_\rho(t, x)$ defined by (4.2) is indeed a solution to (1.1) when $K = 2$. We take the Laplace transform of (1.1) and its limit at $s = 0$ to obtain

$$(4.7) \qquad \frac{\partial \eta}{\partial t} = \eta^2 - 2\eta(t, 0)\eta, \qquad \frac{\partial \eta(t, 0)}{\partial t} = -\eta(t, 0)^2.$$

In analogy with (4.1) we make the ansatz $\eta(t, s) = t^{-1}\eta_\rho(s\lambda(t))$, $\eta_\rho(0) = 1$, in equation (4.7). Letting $\xi = s\lambda$, we have

$$(4.8) \qquad \left(t\frac{\dot\lambda}{\lambda}\right)\xi\eta_\rho' = -\eta_\rho(1 - \eta_\rho).$$

Equation (4.8) may be simplified by separating variables. We let ρ denote the separation constant to obtain

$$(4.9) \qquad \xi\eta_\rho' = -\rho\eta_\rho(1 - \eta_\rho), \qquad \frac{\dot\lambda}{\lambda} = \frac{1}{\rho t}.$$

The general solution to (4.9) is

$$(4.10) \qquad \eta_\rho(\xi) = \frac{1}{1 + c_1 \xi^\rho}, \qquad \lambda(t) = (c_2 t)^{\frac{1}{\rho}},$$

where $c_1, c_2 > 0$ are arbitrary constants. We combine the two solutions to find for each ρ a family of solutions related by scaling in s and t,

$$(4.11) \qquad \eta(t, s) = t^{-1} \frac{1}{1 + c_1 c_2 s^\rho t}, \qquad t > 0.$$

$\eta(t, s)$ is completely monotone if and only if $\rho \in (0, 1]$ [31]; thus it is only for $\rho \in (0, 1]$ that we obtain positive solutions to equation (1.1). By a trivial scaling we may achieve $c_1 = c_2 = 1$, and then $n(t, x)$ is given by (4.2). The scaling solutions have finite mass only when $\rho = 1$, and in this case the mass is conserved.

4.3 Asymptotics of Scaling Solutions

We may use equation (4.3) to obtain the convergent expansion

$$(4.12) \qquad n_\rho(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{\rho k - 1}}{\Gamma(\rho k)}, \qquad x > 0,$$

which implies the divergence $n_\rho(x) \sim x^{\rho - 1}/\Gamma(\rho)$ as $x \to 0^+$ for $\rho \in (0, 1)$.

Since $n_\rho(x)$ is completely monotone for $\rho \in (0, 1)$, its asymptotic properties as $x \to \infty$ may be obtained rigorously by differentiating the formula in Lemma 4.1 and using the Tauberian theorem. We obtain

$$(4.13) \qquad n_\rho(x) \sim x^{-\rho-1} \Gamma(2 + \rho) \frac{\sin \pi \rho}{\pi(1 + \rho)} = \frac{x^{-\rho-1}}{-\Gamma(-\rho)} \qquad \text{as } x \to \infty,$$

using $y = 1 + \rho$ in the identity

$$(4.14) \qquad \Gamma(1 + y)\Gamma(1 - y) \frac{\sin \pi y}{\pi y} = 1.$$

From (4.13), or because $-\eta'_\rho(s) \sim \rho s^{\rho - 1}$ as $s \to 0$, we find

$$(4.15) \qquad \int_0^x y n_\rho(y) dy \sim \frac{\rho x^{1-\rho}}{\Gamma(2 - \rho)} \qquad \text{as } x \to \infty.$$

The case $\rho = \frac{1}{2}$ curiously admits the exact solution (see [1, chap. 29])

$$n_\rho(x) = \frac{1}{\sqrt{\pi x}} - e^x \mathrm{erfc} \sqrt{x}.$$

5 Weak Convergence for the Constant Kernel

Let ν_t be the measure-valued solution of Smoluchowski's equation with kernel $K = 2$ obtained in Section 2.2, given initial size distribution ν_0 that is a finite measure. We normalize to define a *probability* distribution function (the size-biased distribution)

$$(5.1) \qquad F(t, x) = \frac{\nu_t((0, x])}{\nu_t((0, \infty))}, \qquad t > 0.$$

We then have the following characterization of permissible limits under rescaling and their domains of attraction. Below, we call a probability distribution function $F_*(x)$ nontrivial if $F_*(x) < 1$ for some $x > 0$, meaning the distribution is proper $(\lim_{x \to \infty} F(x) = 1)$ and not concentrated at 0.

THEOREM 5.1

(i) *Suppose there is a rescaling function $\lambda(t) \to \infty$ and a nontrivial probability distribution function $F_*(x)$ such that*

$$(5.2) \qquad \lim_{t \to \infty} F(t, \lambda(t)x) = F_*(x)$$

at all points of continuity of F_*. Then there exists $\rho \in (0, 1]$ and a function L slowly varying at infinity such that

$$(5.3) \qquad \int_0^x y\nu_0(dy) \sim x^{1-\rho} L(x) \quad \text{as } x \to \infty.$$

(ii) *Conversely, suppose there exists $\rho \in (0, 1]$ and a function L slowly varying at infinity such that (5.3) holds. Then it follows that there is a strictly increasing rescaling $\lambda(t) \to \infty$ such that*

$$\lim_{t \to \infty} F(t, \lambda(t)x) = F_\rho(x), \qquad x \in (0, \infty),$$

where F_ρ is the Mittag-Leffler distribution function defined in (4.3).

PROOF: Our proof is based on the Laplace transform. We first reformulate (5.2) and (5.3) in terms of $\varphi_0(s)$. First, by the well-known characterization of weak convergence by the Laplace transform [16], (5.2) is equivalent to the assertion that the Laplace transforms converge pointwise, i.e., that

$$(5.4) \qquad \lim_{t \to \infty} \frac{\eta(t, s\lambda^{-1})}{\eta(t, 0)} \to \eta_*(s) := \int_0^\infty e^{-sx} F_*(dx), \qquad s \in [0, \infty).$$

The assumption $F_*(x) < 1$ for some $x > 0$ ensures that $0 < \eta_*(s) < 1$ for all $s > 0$. Since $\eta(t, s) = \varphi(t, \infty) - \varphi(t, s)$ and $\varphi(t, \infty) = \eta(t, 0)$, by the solution formula (2.9) we have

$$(5.5) \qquad \frac{\eta(t, s\lambda^{-1})}{\eta(t, 0)} = \frac{1 + \varphi_0(s\lambda^{-1})\varphi_0(\infty)^{-1}}{1 + t\varphi_0(s\lambda^{-1})}.$$

Because $\varphi_0(0) = 0$, existence of the limit in (5.4) with $0 < \eta_*(s) < 1$ is equivalent to the existence of

$$(5.6) \qquad g(s) := \lim_{t \to \infty} t\varphi_0(s\lambda(t)^{-1})$$

with $0 < g(s) < \infty$ for all $s > 0$.

The behavior in (5.3) may also be reformulated in terms of the Laplace transform. Applying Theorem 3.2 and Lemma 3.3 to the mass distribution function appearing on the left-hand side, we find (5.3) equivalent to

$$(5.7) \qquad \varphi_0(s) \sim \frac{\Gamma(2 - \rho)}{\rho} s^\rho L\left(\frac{1}{s}\right) \quad \text{as } s \to 0.$$

Now to prove the first part of the theorem, we take $t = 1, 2, \ldots$, in (5.6) and apply Lemma 3.1 to conclude that $\varphi_0(s)$ is regularly varying at 0 and $g(s) = cs^\rho$ for some $c > 0$ and $\rho \geq 0$. In fact, $\rho > 0$ since $\eta_*(s) = (1 + cs^\rho)^{-1}$ must satisfy $\eta_*(0) = 1$. Hence (5.7) holds, and it remains to show that $\rho \in (0, 1]$. This will follow from complete monotonicity of the limit. Since $\eta(t, s)$ is completely monotone, it follows that $\eta_*(s)$ is completely monotone since it is the limit of a sequence of completely monotone functions. This is possible only if $\rho \in (0, 1]$, since the second derivative of $(1 + cs^\rho)^{-1}$ is not ultimately positive if $\rho > 1$ [31].

Conversely, we prove the second part by showing that (5.7) implies (5.4) with $F_* = F_\rho$. We define $\lambda(t)$ for sufficiently large t by

$$(5.8) \qquad t\varphi_0(\lambda(t)^{-1}) = 1.$$

$\lambda(t)$ is strictly increasing because $\varphi_0(s)$ is strictly increasing. Moreover, we have $\lim_{t \to \infty} \lambda(t) = \infty$ since $\varphi(0) = 0$. Since φ_0 is regularly varying with index ρ, we have

$$\lim_{t \to \infty} t\varphi_0(s\lambda^{-1}) = \lim_{t \to \infty} \frac{\varphi_0(s\lambda^{-1})}{\varphi_0(\lambda^{-1})} = s^\rho.$$

But then (5.5) yields (5.4) with $\eta_*(s) = (1 + s^\rho)^{-1}$. $\qquad \square$

Remark 5.2. Let $\varphi_0(s) = s^\rho L(1/s)$. Then equation (5.8) shows that

$$\lambda(t)L(\lambda(t))^{\frac{1}{\rho}} = t^{\frac{1}{\rho}}.$$

Comparison with the time scaling $\lambda(t) = t^{1/\rho}$ for the self-similar solution (4.10) shows that $\lambda(t)$ chosen in the proof is essentially the time scaling of the self-similar solution, possibly modified by a slowly varying correction.

Remark 5.3. When $\rho = 1$, the condition for being attracted to the exponential distribution is $\int_0^x yv_0(dy) \sim L(x)$ as $x \to \infty$. Thus, all solutions with initially finite mass are attracted to the finite-mass exponential distribution, but it is not necessary for this that the initial mass be finite. It suffices that the mass distribution function diverge sufficiently weakly.

Remark 5.4. A remaining nontrivial possibility to discuss is that a nonzero limit in
(5.2) may exist where the function F_* is a *defective* probability distribution satis-
fying $F_*(\infty) < 1$. If this is the case, then since $\eta_*(0^+) = F_*(\infty) < 1$, it follows
$g(s) = cs^\rho$ with $\rho = 0$ and that $\varphi_0(s) \sim L(1/s)$ is slowly varying at 0. We cannot
ensure (5.3) in this case. Instead we note that $\varphi_0(s)/s = \int_0^\infty e^{-sx} \int_x^\infty v_0(dy)\, dx$,
and it follows from the Tauberian theorem and the fact that $x \mapsto \int_x^\infty v_0(dy)$ is
monotone [16, XIII.5.4] that the tail distribution function is slowly varying at ∞
with

$$(5.9) \qquad\qquad v_0((x, \infty)) = \int_x^\infty v_0(dx) \sim L(x).$$

Conversely, if (5.9) holds, then $\varphi_0(s)$ is a function slowly varying at 0 that
strictly increases. For any $c \in (0, \infty)$, we can choose $\lambda(t)$ strictly increasing such
that $t\varphi(\lambda(t)^{-1}) = c$. Then it follows that (5.4) holds with $\eta_*(s) = (1 + c)^{-1}$
for $s > 0$, so (5.2) holds with the defective distribution function $F_*(x) = (1 +
c)^{-1}$. This means that under such scalings, an arbitrary fraction of the particle sizes
concentrate at 0 and the rest escapes to infinity.

6 Scaling Solutions for the Additive Kernel

6.1 A One-Parameter Family of Solutions

Golovin found an exact solution to Smoluchowski's equations with monodis-
perse initial condition for $K = x + y$. One may take limits in his solution to obtain
the scaling solution [2]

$$(6.1) \qquad\qquad n(t, x) = \frac{1}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-t} \exp\left(-e^{-2t}\frac{x}{2}\right).$$

This solution has sometimes been criticized as unphysical, since the number of
clusters is infinite. However, recently Deaconu and Tanré [12] proved a result
equivalent to weak convergence to this solution under restrictive assumptions on
the initial data (the existence of all moments and their domination by the moments
of a Gaussian random variable).

We will consider only solutions of finite mass, normalized to 1. The solution
(6.1) is but one of a one-parameter family of solutions (independent of the trivial
scaling $c^2 n(t, cx)$). For each $\rho \in (0, 1]$, in this section we derive finite-mass
scaling solutions in the following form with $\beta = \rho/(1 + \rho)$:

$$(6.2) \qquad\qquad n(t, x) = e^{-\frac{2t}{\beta}} n_\rho\left(e^{-\frac{t}{\beta}}x\right)$$

where

$$(6.3) \qquad\quad n_\rho(x) = \frac{1}{\pi} \sum_{k=1}^\infty \frac{(-1)^{k-1} x^{k\beta-2}}{k!} \Gamma(1 + k - k\beta) \sin \pi k\beta.$$

The associated mass distribution function is given by

$$(6.4) \qquad M_\rho(x) = \int_0^x y n_\rho(y) dy = \sum_{k=1}^\infty \frac{(-1)^{k-1} x^{k\beta}}{k!} \Gamma(1 + k - k\beta) \frac{\sin \pi k\beta}{\pi k\beta}.$$

Remark 6.1. It is an interesting fact that these scaling solutions are related by a nonlinear scaling to the *Lévy stable laws* in probability theory. Feller [16, XVII.7] gives the formula

$$(6.5) \qquad p(x; \alpha, \gamma) = \frac{1}{\pi x} \sum_{k=1}^\infty \frac{(-x)^k}{k!} \Gamma\left(1 + \frac{k}{\alpha}\right) \sin \frac{k\pi}{2\alpha} (\gamma - \alpha)$$

for a family of stable densities, $1 < \alpha < 2$, $|\gamma| \leq 2 - \alpha$. Taking $\alpha = (1 - \beta)^{-1}$, $\gamma = 2 - \alpha$, we find that the mass density from (6.3) satisfies

$$(6.6) \qquad x n_\rho(x) = x^{\beta-1} p(x^\beta; 1 + \rho, 1 - \rho).$$

These remarkable self-similar solutions were first discovered by Bertoin [7] (then independently by us). Bertoin's derivation explains the nonlinear rescaling formula in terms of a scaling property of Lévy stable processes, and he writes the self-similar solution in the form

$$(6.7) \qquad n(t, x) = e^{-t} x^{\beta-2} p(e^{-t} x^\beta; 1 + \rho, 1 - \rho).$$

It is quite remarkable that there are *two* scaling limits associated to this solution. One of them is more transparent in (6.7): we find that $e^t n(t, x) dx$ converges vaguely towards the measure $x^{\beta-2} dx$. This is the scaling limit alluded to in corollary 1 of Bertoin's article [7], and it suffices to uniquely identify the self-similar solution in the class of eternal solutions. On the other hand, (6.2) reflects more clearly the self-similar nature of the solution relative to the mean cluster size $e^{t/\beta}$.

Note that the stable densities are defined on the whole line $(-\infty, \infty)$. We obtain total mass 1 on $(0, \infty)$ through the nonlinear rescaling. Also, if $F(x; \alpha, \gamma)$ denotes the distribution function for the stable law with density $p(x; \alpha, \gamma)$, then the tail of the mass distribution corresponds to this through

$$(6.8) \qquad 1 - M_\rho(x) = \beta^{-1}(1 - F(x^\beta; 1 + \rho, 1 - \rho)).$$

The total number of clusters diverges for all the solutions in (6.2). This is caused by the predominance of small clusters, and it may be desingularized by working with the variable φ introduced in Section 2.1. The scaling solutions above are given implicitly in terms of $\varphi_\rho = \varphi_\rho(s)$ satisfying

$$(6.9) \qquad s = \varphi_\rho + \varphi_\rho^{1+\rho}.$$

Since $u_\rho(s) = \partial_s \varphi_\rho$ is the Laplace transform of the mass distribution, differentiating (6.9) we have

$$(6.10) \qquad u_\rho = \frac{1}{1 + (1 + \rho)\varphi_\rho^\rho},$$

which exhibits a connection to the Mittag-Leffler distribution (see (4.4)). We use this to show below that the mass distribution is infinitely divisible. When $\rho = 1$, equation (6.9) is a quadratic equation with two solutions, one of which is $\varphi(s) = \sqrt{1 + 4s} - 1$, which corresponds to (6.1). For $\rho \in (0, 1)$ we will solve (6.9) by Lagrange's inversion formula as an infinite series to obtain (6.3).

6.2 Scaling Solutions

We may derive mass-preserving scaling solutions to (2.3) as follows: Let $\lambda(t)$ be a rescaling to be determined, and let $\xi = s\lambda$. We substitute the ansatz $\varphi(t, s) = \lambda^{-1}\varphi_\rho(s\lambda) = \lambda^{-1}\varphi_\rho(\xi)$ in (2.3) to obtain

$$(6.11) \qquad -\frac{\dot{\lambda}}{\lambda}(\varphi_\rho - \xi\partial_\xi\varphi_\rho) - \varphi_\rho\partial_\xi\varphi_\rho = -\varphi_\rho.$$

We separate variables by letting $\dot{\lambda}/\lambda = a$ or $\lambda = c_1 e^{at}$. Then by (6.11) we have

$$(6.12) \qquad (a\xi - \varphi_\rho)\partial_\xi\varphi_\rho + (1 - a)\varphi_\rho = 0.$$

Equation (6.12) is not separable, but it may be solved implicitly by rewriting it as the linear equation

$$(6.13) \qquad \frac{d\xi}{d\varphi_\rho} - \frac{a}{a-1}\frac{\xi}{\varphi_\rho} = \frac{1}{1-a}.$$

Put $\rho = (a - 1)^{-1}$ so $a = (1 + \rho)/\rho = 1/\beta$. Integrating, we find a family of nontrivial solutions determined by

$$(6.14) \qquad \xi = \varphi_\rho + c_2\varphi_\rho^{1+\rho}, \quad c_2 > 0.$$

The range of admissible ρ is narrowed by requiring that $\lim_{\xi \to 0} \varphi_\rho/\xi = 1$ (finite mass), which implies $\rho > 0$. Without loss of generality we may take $c_2 = 1$, since we can recover all other solutions by a trivial scaling.

We now show that $\rho > 1$ is inadmissible. Let $U(\xi) = \varphi(\xi)/\xi$. Integrating (2.1) by parts, we see that $U(\xi) = \int_0^\infty e^{-\xi x}N(x)dx$ where $N(x) = \nu((x, \infty))$ is the tail distribution. In particular, $U(0) = 1$ and U is completely monotone. Dividing (6.14) by ξ and differentiating, we see that

$$U'(\xi) = \frac{-c\xi^{\rho-1}U^{1+\rho}}{1 + c(\rho + 1)\xi^\rho U^\rho} \to 0 \quad \text{as } \xi \to 0,$$

which is impossible if U is completely monotone. Thus, the admissible range of nontrivial solutions is restricted to $\rho \in (0, 1]$.

6.3 Series Expansion and Asymptotics of Scaling Solutions

The asymptotic properties of the scaling solutions for $\rho \in (0, 1)$ may be rigorously obtained from Theorem 3.2. By (6.9), $\varphi_\rho'' \sim \rho(\rho + 1)s^{\rho-1}$ as $s \to 0$. But $\varphi_\rho'' = \int_0^\infty e^{-sx}x^2 n_\rho(x)dx$, and Theorem 3.2 implies that

$$(6.15) \qquad \int_0^x y^2 n_\rho(y)dy \sim \frac{\rho(\rho + 1)}{\Gamma(2 - \rho)}x^{1-\rho} \quad \text{as } x \to \infty.$$

Thus the second moment is finite only for $\rho = 1$. For $0 < \rho < 1$ the mass distribution has fat tails. Equation (6.15) is a weak version of the pointwise behavior

$$(6.16) \qquad n_\rho(x) \sim \frac{\rho + 1}{|\Gamma(-\rho)|} x^{-(2+\rho)} \quad \text{as } x \to \infty, \ \rho \in (0, 1),$$

which follows from (6.6) due to the known power law asymptotics of the stable densities [5].

The behavior as $x \to 0$ is described completely by the series (6.3), derived as follows. We rewrite (6.9) in terms of $U = \varphi_\rho/s$ and $\beta = \rho/(1 + \rho)$ as

$$U(s) = s^{-\beta}(1 - U)^{1-\beta}.$$

We solve for U using Lagrange's inversion formula (see, e.g., [20, sec. 6.3] for a similar calculation), obtaining

$$U(s) = \sum_{k=1}^{\infty} \frac{s^{-k\beta}}{k!} \frac{d^{k-1}}{dx^{k-1}} (F(x))^k \bigg|_{x=0} \quad \text{with} \quad F(x) = (1-x)^{1-\beta}.$$

We evaluate the derivatives and find that

$$(6.17) \qquad \partial_s \varphi_\rho(s) = \partial_s(sU) = \sum_{k=1}^{\infty} \frac{s^{-k\beta}}{k!}(-1)^{k-1} \prod_{j=1}^{k}(j - k\beta).$$

This is the Laplace transform of the mass distribution function given through term-by-term Laplace inversion as

$$(6.18) \qquad M_\rho(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \frac{x^{k\beta}}{\Gamma(1 + k\beta)} \prod_{j=1}^{k}(j - k\beta).$$

We then deduce (6.4) using $y = k\beta$ in the gamma function identity (4.14). By differentiating (6.4) we obtain the number density in (6.3).

It is straightforward to check that when $\rho = 1$, the even terms vanish and (6.3) reduces to the function $(4\pi)^{-1/2} x^{-3/2} e^{-x/4}$, which is a scaled version of (6.1). Correspondingly, $M_1(x) = \text{erf}(\frac{1}{2}\sqrt{x})$. One may also check that the series solution above is absolutely convergent for $x \in (0, \infty)$. Thus, n_ρ is analytic.

7 Weak Convergence for the Additive Kernel

We let ν_t be a solution of Smoluchowski's equation with kernel $K = x + y$ given initial data ν_0 with finite mass normalized to $\int_0^\infty x\nu_0(dx) = 1$. Then at all times the mass distribution is a probability distribution, with the distribution function denoted as

$$M(t, x) = \int_0^x y\nu_t(dy).$$

It follows from the explicit solution (2.16) that the Laplace transform of the mass distribution satisfies $u(t, 0) = 1$ for all $t \geq 0$ and $\lim_{t\to\infty} u(t, s) = 0$ for $s > 0$. This phenomenon of concentration is equivalent to the assertion that asymptotically

all the mass escapes to infinity. As earlier, we may hope that suitable rescaling in s will give convergence to a nontrivial limit. Precisely, we have the following characterization.

THEOREM 7.1

(i) *Suppose there is a rescaling function $\lambda(t) \to \infty$ as $t \to \infty$ and a nontrivial probability distribution function $M_*(x)$ such that*

$$(7.1) \qquad \lim_{t \to \infty} M(t, \lambda(t)x) = M_*(x)$$

at all points of continuity of M_*. Then there exists $\rho \in (0, 1]$ and a function L slowly varying at infinity such that

$$(7.2) \qquad \int_0^x y^2 \nu_0(dy) \sim x^{1-\rho} L(x) \quad \text{as } x \to \infty.$$

(ii) *Conversely, assume that there exists $\rho \in (0, 1]$ and a function L slowly varying at infinity such that* (7.2) *holds. Then there is a strictly increasing rescaling $\lambda(t) \to \infty$ such that*

$$\lim_{t \to \infty} M(t, \lambda(t)x) = M_\rho(x), \quad 0 \le x < \infty,$$

where M_ρ from (6.4) *is the mass distribution function for a scaling solution.*

PROOF: We will prove the theorem after reformulating (i) and (ii) as equivalent assertions using the Laplace transform. First, the weak convergence of the mass distribution $M(t, \lambda x)$ is equivalent to the pointwise convergence of its Laplace transform

$$(7.3) \qquad \lim_{t \to \infty} u(t, s\lambda^{-1}) = u_*(s) := \int_0^\infty e^{-sx} M_*(dx), \quad 0 \le s < \infty.$$

The assumption $M_*(x) < 1$ for some $x > 0$ ensures that $0 < u_*(s) < 1$ for $s > 0$. Second, (7.2) is equivalent to $-\partial_s u \sim s^{\rho-1} L(1/s) \Gamma(2 - \rho)$ as $s \to 0$ by Theorem 3.2. Since $\rho \in (0, 1]$, by Lemma 3.3 this is equivalent to

$$(7.4) \qquad 1 - u_0(s) \sim s^\rho L\left(\frac{1}{s}\right) \frac{\Gamma(2 - \rho)}{\rho} \quad \text{as } s \to 0.$$

We prove the first part of the theorem by showing that (7.3) implies (7.4). Since $u_*(s)$ is a limit of completely monotone functions, it is itself completely monotone. Moreover, since $u(t, s) = \partial_s \varphi$, we also have the convergence

$$(7.5) \qquad \lim_{t \to \infty} \lambda \varphi(t, s\lambda^{-1}) = \varphi_*(s) = \int_0^s u_*(s')ds'.$$

Clearly, φ_* is strictly increasing.

In what follows, we consider $\sigma(t, s)$ defined by replacing s with $s/\lambda(t)$ in (2.15), i.e., by

$$(7.6) \qquad \sigma - \varphi_0(\sigma) = s\lambda^{-1} - \varphi(t, s\lambda^{-1}).$$

It then follows from (7.6) that as $t \to \infty$ with s fixed, we have $\sigma \to 0$ and

(7.7)
$$\lim_{t\to\infty} \lambda(\sigma - \varphi_0(\sigma)) = s - \varphi_*(s).$$

From (2.14) and (7.5) we then have

(7.8)
$$\lim_{t\to\infty} \lambda e^{-t}\varphi_0(\sigma) = \lim_{t\to\infty} \lambda\varphi(t, s\lambda^{-1}) = \varphi_*(s).$$

Replacing s by $s\lambda^{-1}$ in the exact solution (2.16), we also have

$$\lim_{t\to\infty} \frac{u_0(\sigma)}{e^t(1 - u_0(\sigma)) + u_0(\sigma)} = u_*(s).$$

Since $\sigma \to 0$ and $u_0(0) = 1$, we deduce that

(7.9)
$$\lim_{t\to\infty} e^t(1 - u_0(\sigma)) = \frac{1 - u_*(s)}{u_*(s)}.$$

We now show that $\sigma \to 0$ at the rate $a(t) := e^t\lambda(t)^{-1}$ ($\lim_{t\to\infty} a(t) = 0$ by (7.8)). We may rewrite (7.6) as

$$\sigma = a(t)[\varphi_*(s) + se^{-t} + \lambda e^{-t}\varphi_0(\sigma)(1 - e^{-t}) - \varphi_*(s)]$$
$$= a(t)[\varphi_*(s) + r(t, s)],$$

where the error term $r(t, s) \to 0$ by (7.8). Therefore, σ is asymptotically a scaling of $\varphi_*(s)$. We now claim that for all $s > 0$,

(7.10)
$$\lim_{t\to\infty} e^t\big(1 - u_0(a(t)\varphi_*(s))\big) = \frac{1 - u_*(s)}{u_*(s)}.$$

Fix $s > 0$ and let $\delta > 0$ be sufficiently small. Since φ_* is strictly increasing, for sufficiently large t (depending on s and δ), we have

$$\varphi_*(s - \delta) + r(t, s - \delta) < \varphi_*(s) < \varphi_*(s + \delta) + r(t, s + \delta),$$

whence

$$1 - u_0(\sigma(t, s - \delta)) < 1 - u_0(a(t)\varphi_*) < 1 - u_0(\sigma(t, s + \delta)).$$

Multiply by e^t and take $t \to \infty$; then $\delta \to 0$. The claim (7.10) then follows from equation (7.9).

It follows directly from (7.10) and Lemma 3.1 that $1 - u_0$ is regularly varying at 0 with some exponent $\rho \in \mathbb{R}$, and the limit in (7.10) has the form $c\varphi_*^\rho$ for some positive constant c. Clearly $0 < \rho \le 1$ since u_0 is bounded and completely monotone and $u_*(0) = 1$ by the hypothesis that M_* is a probability distribution. This finishes the proof of the first part. Note furthermore that (6.10) holds after scaling s.

We prove the converse statement by showing that (7.4) implies (7.3) with $u_* = u_\rho$. By the explicit solution formula (2.16), it suffices to show that as $t \to \infty$, we have

(7.11)
$$e^t\big(1 - u_0(\sigma(t, s\lambda^{-1}))\big) = u(t, s\lambda^{-1})^{-1} - 1 \to (1 + \rho)\varphi_\rho(s)^\rho$$

for all $s > 0$. We write $a(t) = e^t\lambda(t)^{-1}$ as earlier. We choose $\lambda(t)$ to satisfy $\lambda(0) = 0$ and

$$(7.12) \qquad e^t\big(1 - u_0(a(t))\big) = 1 + \rho\,, \quad t > 0\,.$$

Then $a(t) \to 0$, $\lambda(t)$ is strictly increasing, and $\lambda(t) \to \infty$. It follows from (7.4) that for fixed $\varphi_* > 0$, as $t \to \infty$ we have

$$(7.13) \qquad e^t\big(1 - u_0(a(t)\varphi_*)\big) = (1 + \rho)\frac{1 - u_0(a(t)\varphi_*)}{1 - u_0(a(t))} \to (1 + \rho)\varphi_*^\rho\,.$$

For fixed $\varphi_* > 0$ we define $s(t, \varphi_*)$ as the value of s determined from (7.6) using $\sigma = a(t)\varphi_*$. Then it follows that, for all $\varphi_* > 0$,

$$(7.14) \qquad \lim_{t \to \infty} u(t, s(t, \varphi_*)\lambda^{-1})^{-1} - 1 = (1 + \rho)\varphi_*^\rho\,.$$

Using (2.14) with (7.6), we have

$$s(t, \varphi_*) = \lambda(a\varphi_* - \varphi_0(a\varphi_*)) + \lambda e^{-t}\varphi_0(a\varphi_*)$$

$$= (1 + \rho)\frac{a\varphi_* - \varphi_0(a\varphi_*)}{a(1 - u_0(a))} + \frac{1}{a}\varphi_0(a\varphi_*)\,.$$

Since $1 - u_0 \sim s^\rho L(1/s)$, the proof of Lemma 3.3 shows that

$$s - \varphi_0(s) \sim (1 + \rho)^{-1}s^{\rho+1}L\left(\frac{1}{s}\right)\,.$$

Therefore, as $t \to \infty$, we have

$$(7.15) \qquad s(t, \varphi_*) \sim \frac{(a\varphi_*)^{1+\rho}L(1/(a\varphi_*))}{a^{1+\rho}L(1/a)} + \varphi_* \to \varphi_*^{1+\rho} + \varphi_*\,.$$

Now to prove (7.11), fix $s_0 > 0$. Then there is a unique $\varphi_* = \varphi_\rho(s_0) > 0$ so that $s_0 = \varphi_*(\varphi_*^\rho + 1)$. Since $\varphi_* \mapsto s(t, \varphi_*)$ is strictly increasing in φ_* for all t, by substituting $\varphi_* \pm \delta$ for φ_* in (7.14) we easily deduce (7.11). $\square$

Remark 7.2. When $1 - u_0 = s^\rho$, the choice of time scaling $\lambda(t)$ in (7.12) gives $\lambda(t) = e^{(1+1/\rho)t}$ in accordance with (6.12). More generally, when $1 - u_0 = s^\rho L(1/s)$, the rescaling $\lambda(t)$ is modified by a slowly varying correction. The choice of time scale when $\rho = 1$ and the second moment is finite deserves special comment. In this case we find $\lambda(t) = e^{2t}$. In the applied literature it is common to define mean cluster size as a ratio of moments. Let $m_k(t) = \int_0^\infty x^k v_t(dx)$. It is clear that any ratio of the form $m_{k+1}(t)/m_k(t)$ has the dimensions of length, and two distinct but natural definitions of mean cluster size are (see [34])

$$c_1(t) = \frac{m_1(t)}{m_0(t)} \quad \text{and} \quad c_2(t) = \frac{m_2(t)}{m_1(t)}\,.$$

For Golovin's solution the initial data are monodisperse, and $m_0(0) = m_1(0) = m_2(0) = 1$. It is easy to calculate explicitly that $c_1(t) = e^t$, but $c_2(t) = e^{2t}$. Thus the two notions of cluster size differ, and only e^{2t} is the correct scaling for convergence to self-similar form. More generally, e^{2t} is the only choice of time scaling that fixes *both* m_1 and m_2 as required for convergence to self-similar form.

Remark 7.3. Theorem 7.1 with $\rho = 1$ shows that the mass distribution is attracted to the classical self-similar solution for all initial data v_0 with finite first and second moment. But for this behavior it is not necessary for the initial data to have finite second moment. It suffices that it diverge sufficiently weakly, with $\int_0^x y^2 v_0(dy) \sim L(x)$ slowly varying at infinity.

Remark 7.4. A remaining nontrivial possibility is that a nonzero limit in (7.1) may exist with M_* defective, satisfying $M_*(\infty) < 1$. If this is true, then most of the proof of the first part of the theorem carries through. The limit in (7.10) must have the form $c\varphi_*^\rho$ with $c > 0$, but we must have $\rho = 0$, since $u_*(0) < 1$. Moreover, it follows that $1 - u_0(s) \sim L(1/s)$ is slowly varying. We do not obtain (7.2) in this case. Instead, we note

$$(7.16) \qquad \frac{1 - u_0(s)}{s} = \int_0^\infty e^{-sx} \int_x^\infty y v_0(dy) dx .$$

As for the constant kernel, it follows from the Tauberian theorem and monotonicity that the tail of the mass distribution is slowly varying at infinity, with

$$(7.17) \qquad \int_x^\infty y v_0(dy) \sim L(x) .$$

In the converse direction, if (7.17) holds, then $1 - u_0(s)$ is a strictly increasing function that is slowly varying at 0. For any $c \in (0, \infty)$ we can choose $\lambda(t)$ strictly increasing so that $e^t(1 - u_0(a(t))) = c$. Then it follows as in the proof of the second part of the theorem that (7.3) holds with $u_*(s) = (1 + c)^{-1}$ for $s > 0$, so (7.1) holds with the defective distribution function $M_*(x) = (1+c)^{-1}$. This means that under such scalings, an arbitrary fraction of the mass concentrates at 0 and the rest escapes to infinity.

We conclude this section with a useful observation about the self-similar solutions.

THEOREM 7.5 *For each $\rho \in (0, 1]$, the probability distribution M_ρ is infinitely divisible.*

PROOF: It suffices to show that the Laplace transform $u_\rho = e^{-\psi_\rho}$, where $\psi_\rho(0) = 0$ and ψ_ρ has completely monotone derivative [16, XIII.7.1]. By (6.10), $\psi_\rho = \log(1 + (1 + \rho)\varphi_\rho^\rho)$. Clearly, $\psi_\rho(0) = 0$. Moreover,

$$\partial_s \psi_\rho = \frac{(1 + \rho)\rho \varphi_\rho^{\rho-1} u_\rho}{1 + (1 + \rho)\varphi_\rho^\rho} .$$

By Theorem 7.1, u_ρ is completely monotone. The other factor can be written as a composition with the Mittag-Leffler distribution

$$\frac{\varphi_\rho^{\rho-1}}{1 + \varphi_\rho^\rho} = \frac{s^{\rho-1}}{1 + s^\rho} \circ \varphi_\rho .$$

The function $s^{\rho-1}$ is completely monotone, as is $(1+s^\rho)^{-1}$. Thus, their product is completely monotone. Thus, the composed function above is completely monotone since it is the composition of a completely monotone function with a function that has a completely monotone derivative [16, XIII.4.2]. Finally, $\partial_s \psi_\rho$ is the product of two completely monotone functions and is hence completely monotone. $\quad\square$

8 Approach to Self-Similar Gelation for the Multiplicative Kernel

8.1 McLeod's Solution

McLeod found the following explicit solution to the discrete Smoluchowski equation (1.1) for $K = xy$ and monodisperse initial data $\nu_0 = \delta(x-1)$ [2, 26]:

$$(8.1) \qquad \nu_t = \sum_{k=1}^{\infty} n_k(t)\delta(x-k), \quad n_k(t) = \frac{t^{k-1}k^{k-2}}{k!e^{tk}}.$$

A beautiful probabilistic interpretation of this solution in terms of a Poisson-Galton-Watson branching process may be found in [2]. The solution is valid only for $0 \le t < 1$. When $t = 1$, $n_k(t)$ has only algebraic decay, and the second moment $m_2(t) = \infty$. Moreover, mass can no longer be conserved for $t > 1$. At a microscopic level, this is commonly ascribed to the formation of a cluster of infinite mass (the gel).

The formal scaling limit of (8.1) is obtained by considering the large k limit as $t \to 1$. By Stirling's approximation $k! \sim \sqrt{2\pi k}e^{-k}k^k$, as $t \to 1$ we find

$$n_k(t) \sim \frac{1}{\sqrt{2\pi}}k^{-\frac{5}{2}}e^{k(1-t+\log t)}$$

$$\sim \frac{1}{\sqrt{2\pi}}k^{-\frac{5}{2}} \exp\left(-k\left(\frac{(1-t)^2}{2} + \frac{(1-t)^3}{3} + \dots\right)\right).$$

Let $x = k(1-t)^2$ and consider the limit $k \to \infty, t \to 1$, such that x is held fixed. Thus, we find

$$\lim_{\substack{t \to 1 \\ k \to \infty}} (1-t)^{-5}n_k(t) = \frac{1}{\sqrt{2\pi}} x^{-\frac{5}{2}}e^{-\frac{x}{2}}.$$

This shows convergence of the discrete solution to the scaling solution [2]

$$(8.2) \qquad n(t,x) = \frac{1}{\sqrt{2\pi}} x^{-\frac{5}{2}}e^{-(1-t)^2\frac{x}{2}}, \quad x \in (0,\infty), \ t \in (-\infty, 1).$$

We will see below that this scaling solution emerges coherently from the scaling solutions to the additive kernel and is just one of a one-parameter family of scaling solutions.

8.2 Scaling Solutions and Weak Convergence

The scaling solutions for the multiplicative kernel can be obtained by our knowledge of the scaling solutions to the additive kernel via a general relation between solutions for the two kernels. Recall from Section 2.5 that the initial data are normalized so the initial second moment $m_2(0) = 1$. Then $m_2(t) = (1 - t)^{-1}$, and the gelation time $T_{\text{gel}} = 1$. The second-moment probability distribution function

$$(8.3) \qquad V(t, x) = \frac{\displaystyle\int_0^x y^2 v_t(dy)}{\displaystyle\int_0^\infty y^2 v_t(dy)}$$

is the analogue of $M(t, x)$ for the additive kernel. From (2.22) we see that the Laplace transform of $V(t, x)$ is $(1 - t)\partial_s \psi(t, s)$. We differentiate equation (2.24) with respect to s to obtain

$$(8.4) \qquad (1 - t)\partial_s \psi(t, s) = \partial_s \varphi(-\log(1 - t), s) = u(\tau, s)$$

where $\tau(t) := \log(1 - t)^{-1}$. As a consequence,

$$(8.5) \qquad V(t, x) = \tilde{M}(\tau, x),$$

where $\tilde{M}(\tau, x)$ is the mass distribution function for the corresponding solution with additive kernel. For solutions with densities $n(t, x)$ and $\tilde{n}(\tau, x)$ for the multiplicative and additive kernels, respectively, this means

$$(8.6) \qquad x^2 n(t, x) = (1 - t)^{-1} x \tilde{n}(\tau, x).$$

From this relation we obtain the scaling solutions for the multiplicative kernel as described in the introduction. Explicitly

$$(8.7) \qquad n(t, x) = (1 - t)^{-1 + \frac{3}{\beta}} n_\rho \big(x(1 - t)^{\frac{1}{\beta}} \big),$$

where $\beta = \rho/(1 + \rho)$ and

$$(8.8) \qquad n_\rho(x) = \frac{1}{\pi} \sum_{k=1}^\infty \frac{(-1)^{k-1} x^{k\beta - 3}}{k!} \Gamma(1 + k - k\beta) \sin \pi k\beta.$$

Notice that these scaling solutions do *not* preserve mass—in fact, all of them have infinite mass! Instead, they have a finite second moment for $t < 1$, which blows up as $t \to 1$. For $0 < \rho < 1$ the third moment is infinite. When $\rho = 1$, the scaling solution reduces to the exponentially decaying solution in (8.2) after a trivial scaling. Finally, we note that though we have assumed $t \in [0, 1)$, these solutions are well-defined for $t \in (-\infty, 1)$.

Theorem 7.1 characterizes the convergence of $\tilde{M}(\tau, \lambda x)$, and it is easy to adapt to characterize convergence to self-similar form approaching the gelation time.

THEOREM 8.1

(i) *Suppose there is a rescaling function* $\lambda(t) \to \infty$ *as* $t \to 1$ *and a nontrivial probability distribution function* $V_*(x)$ *such that*

$$(8.9) \qquad \lim_{t \to 1} V(t, \lambda(t)x) = V_*(x)$$

at all points of continuity of V_*. *Then there exists* $\rho \in (0, 1]$ *and a function L slowly varying at infinity such that*

$$(8.10) \qquad \int_0^x y^3 v_0(dy) \sim x^{1-\rho} L(x) \quad as \; x \to \infty.$$

(ii) *Conversely, assume that there exists* $\rho \in (0, 1]$ *and a function L slowly varying at infinity such that* (8.9) *holds. Then there is a strictly increasing rescaling* $\lambda(t) \to \infty$ *such that*

$$\lim_{t \to 1} V(t, \lambda(t)x) = V_\rho(x), \quad 0 \le x < \infty,$$

where V_ρ *is the second moment distribution function for a scaling solution given by* $V_\rho = M_\rho$ *from* (6.4).

It is worth pointing out explicitly that the domain of attraction of the scaling solution in (8.2) includes all initial data with finite second and third moments, as well as data whose third moment diverges sufficiently weakly (the case $\rho = 1$ above). Each of the infinite-mass self-similar solutions, however, attracts finite-mass solutions whose third moment diverges at the appropriate rate detailed in the theorem.

The behavior of the rescaling function $\lambda(t)$ and the characterization of possibly defective limits can be easily deduced from the corresponding results for the additive case that appear in the remarks following Theorem 7.1.

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