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Šilnikov manifolds in coupled nonlinear Schrödinger equations

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Abstract

We consider two coupled nonlinear Schrödinger equations with even, periodic boundary conditions, that are damped and quasiperiodically forced. We prove the existence of invariant manifolds with Šilnikov-type dynamics that are homoclinic to a spatially independent invariant torus. Such manifolds appear to induce complex behavior in numerical experiments. © 1999 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

This paper is concerned with dynamical behavior in the coupled system of nonlinear Schrödinger equations (NLS)

$$i\partial_t u_1 = \partial_x^2 u_1 + 2|u_1|^2 u_1 + i\epsilon\left[\hat{D}_1 u_1 - \Gamma_1 e^{-i2\Omega_1^2 t} + \gamma u_1 |u_2|^2\right],$$

$$i\partial_t u_2 = \partial_x^2 u_2 + 2|u_2|^2 u_2 + i\epsilon\left[\hat{D}_2 u_2 - \Gamma_2 e^{-i2\Omega_2^2 t} + \gamma u_2 |u_1|^2\right], \quad (1)$$

which contains damping in the form of a bounded negative operator $\hat{D}$, and quasiperiodic forcing with amplitudes Γ_k and frequencies Ω_k. Both $u_1(x,t)$ and $u_2(x,t)$ are even and periodic functions of x with period $L = 2\pi$. The time dependence of the equations can be eliminated through the transformation $u_k \to u_k e^{-i2\Omega_k^2 t}$, which gives

$$\partial_t u_k = -i\partial_x^2 u_k - 2i\left[|u_k|^2 - \Omega_k^2\right]u_k + \epsilon\left[\hat{D}_k u_k - \Gamma_k + \gamma u_k |u_j|^2\right], \quad k=1,2, \quad (2)$$

with $j = 1,2,\ j \neq k$. For the purposes of this paper, we select $\hat{D}_k = -\alpha_k u + \beta_k \hat{B}_K u_k$, where $\alpha,\beta > 0$, and the operator $\hat{B}_K$ is a smoothed 'model' of the diffusion operator ∂_x^2. In particular, if $\tilde{b}(l)$ denotes the Fourier transform of $\hat{B}_K u(x)$, and $\tilde{u}(l)$ denotes the Fourier transform of $u(x)$, then

$$\tilde{b}(l) = \begin{cases} -l^2 \tilde{u}(l), & \text{if} \quad l < K, \\ 0, & \text{if} \quad l \geq K, \end{cases}$$

with some fixed, large integer $K > 0$.

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With the above choice of $\hat{D}$, (2) is a coupled version of the single perturbed NLS equation that has been studied extensively in recent years (see, e.g., McLaughlin and Overman [9], Li et al. [7], McLaughlin and Shatah [10], Haller [4], and the references therein), preceded by a number of studies on its finite-dimensional approximations (see, e.g., Kovačič and Wiggins [6], Li and McLaughlin [8], Haller [3], Rothos [11], or Haller [5] for a detailed survey). The goal of this paper is to understand how the complicated dynamics found in the single perturbed NLS equation manifests itself in a coupled system of such equations. Our emphasis will be on the description of the infinite-dimensional phase space geometry and not on detailed proofs. The proofs of the technical results we use for near-integrable, NLS-type systems of partial differential equations can be found in detail in Haller [5].

Our main result is the existence of invariant manifolds in the phase space of (1) whose geometry resembles those of certain orbits in finite-dimensional ODEs, first studied by Šilnikov [12]. The exact dynamical implications of such sets are not known yet; in fact, we believe that this is the first example in which they are identified. We demonstrate numerically the existence of spatio-temporally complicated dynamics for the approximate parameter values obtained from our analysis. The important feature of Šilnikov manifolds is that the shape of individual homoclinic orbits in them can be quite different; some of them involve significant motion only in u_1, others only in u_2. This agrees with the pulse localization phenomenon observed numerically for coupled NLS equations describing pulse propagation in optical fiber arrays (see, e.g., Aceves et al. [1,2]). The methods we use in this paper are general enough to bear on the equations of coupled optical fibers, and hence we expect to identify similar behavior for those equations in future work.

2. The integrable limit

In this section we describe the $\epsilon = 0$ limit of system (2). Since in this limit the system decouples to two integrable NLS equations, we can use the integrable geometry of the NLS, as discussed, e.g., in Li et al. [7], to understand the integrable geometry of coupled NLS equations.

2.1. The resonant two-torus and its stability

Solutions with $\partial_x u_1 = \partial_x u_2 \equiv 0$ at $t = 0$ remain spatially independent under the flow, therefore the set $\Pi = \Pi_1 \times \Pi_2$ with

$$\Pi_k = \{ u_k \mid \partial_x u_k \equiv 0 \}, \quad k = 1,2,$$

is a four-dimensional invariant space. An important subset of Π is the two-dimensional invariant torus $\mathscr{C} = \mathscr{C}_1 \times \mathscr{C}_2$, the product of the two circles

$$\mathscr{C}_k = \{ u_k \mid |u_k| \equiv \Omega_k \}, \quad k = 1,2.$$

This two-torus is completely filled with equilibria of for $\epsilon = 0$, which correspond to periodic or quasiperiodic solutions of the original system (1). Both frequencies of the latter solutions are equal to those of the forcing, thus we will refer to $\mathscr{C}$ as a *resonant torus*.

Since the coupled NLS system decouples into two integrable NLS equations for $\epsilon = 0$, the stability analysis of Li et al. [7] for a single NLS equation can be used to analyze the stability of the torus $\mathscr{C}$. In particular, for $1/2 < \Omega_k < 1$, at any fixed point on $\mathscr{C}$ the coupled NLS equations admit two pairs of nonzero real eigenvalues $\pm \lambda_k = \pm \sqrt{4\Omega_k^2 - 1}$, a zero eigenvalue with multiplicity four and with a trivial Jordan block, and infinitely many purely imaginary pairs of eigenvalues $i\nu_{k,j}$ with $k = 1,2$ and $j = 1,2,\ldots$. Therefore, at any point of the torus $\mathscr{C}$, the unperturbed system admits two-dimensional stable and unstable subspaces E^s and E^u, and an infinite-dimensional center subspace. The center subspace is the direct sum of the space $E^0 \equiv \Pi$ (corresponding to the zero eigenvalues) and an infinite dimensional subspace E^c corresponding to the purely imaginary eigenvalues.

2.2. Homoclinic orbits

As earlier studies showed, each individual Schrödinger equations in the coupled NLS system admits two one-parameter families of orbits homo-

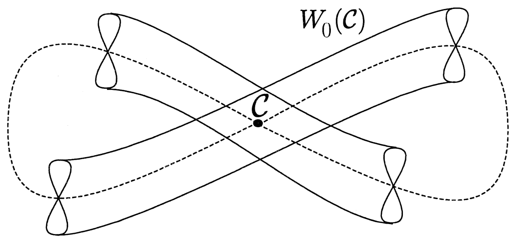

Fig. 1. The geometry of the homoclinic set $W_0(\mathscr{C})$.

clinic to the circle $\mathscr{C}_k$ for $\epsilon = 0$. We label these two families as $W_0^+(\mathscr{C})$ and $W_0^-(\mathscr{C})$.

It follows that a four-dimensional manifold $W_0(\mathscr{C})$ of homoclinic orbits exists for the torus $\mathscr{C}$ in the phase space of the coupled NLS system. The u_k component of a homoclinic orbit lies in $W_0^+(\mathscr{C}_k)$, $W_0^-(\mathscr{C}_k)$, or $\mathscr{C}_k$, where the $\pm$ signs refer to different families of orbits, i.e., different components of the stable manifold of $\mathscr{C}_k$. The homoclinic set $W_0(\mathscr{C})$ is therefore the union of eight manifolds of the form $\mathscr{A}_1 \times \mathscr{A}_2$ where

$$\mathscr{A}_k \in \{W_0^-(\mathscr{C}_k), \mathscr{C}_k, W_0^+(\mathscr{C}_k)\}.$$

Here we excluded the set $\mathscr{C}_1 \times \mathscr{C}_2$ as it gives $\mathscr{C}$ itself. Introducing the multi-index $P = (P^1, P^2)$ with $P^k \in \{-1, 0, +1\}$ and letting $W_0^0(\mathscr{C}_k) \equiv \mathscr{C}_k$, the homoclinic set $W_0(\mathscr{C})$ can be written as

$$W_0(\mathscr{C}) = \cup_{P \neq 0} W^P(\mathscr{C}),$$

$$W^P(\mathscr{C}) = W_0^{P^1}(\mathscr{C}_1) \times W_0^{P^2}(\mathscr{C}_2).$$

The geometry of this homoclinic set is sketched in Fig. 1. Individual solutions homoclinic to $\mathscr{C}$ are contained in one of the components of $W_0(\mathscr{C})$, and are of the form

$$u^{hP}(t) = \left(u_1^{hP^1}(t), u_2^{hP^2}(t)\right). \tag{3}$$

Here $u_k^{hP^k}(t)$ is a homoclinic orbit for the kth unperturbed NLS equation given by (see, e.g., Li et al. [7])

$$u_k^{h\pm}(x,t) = \Omega_k e^{i\phi_{k0}} \times \frac{\cos 2p_k - i \sin 2p_k \tanh \tau_k \pm \sin p_k \operatorname{sech} \tau_k \cos x}{1 \mp \sin p_k \operatorname{sech} \tau \cos x}, \tag{4}$$

with

$$p_k = \tan^{-1}\sqrt{4\Omega_k^2 - 1}, \quad \tau_k = \sqrt{4\Omega_k^2 - 1}\,(t + t_0).$$

The $\pm$ index of the solution reflects the sign of P^k. As an example, the pointwise norm of a homoclinic solution $\left(u_1^{h+}(x,t), u_2^{h+}(x,t)\right)$ is shown in Fig. 2. Geometrically, the orbits in the homoclinic manifold $W_0(\mathscr{C})$ are typically heteroclinic connections between different points of the torus $\mathscr{C}$. These endpoints have the same modulus, but their complex phases differ by

$$\Delta\phi = (\Delta\phi^1, \Delta\phi^2) = -4(p_1, p_2) = -4\left(\tan^{-1}\sqrt{4\Omega_1^2 - 1}, \tan^{-1}\sqrt{4\Omega_2^2 - 1}\right).$$

2.3. N-chains

In what follows, we will show the existence of solutions of the coupled NLS system that are doubly asymptotic to the space Π of spatially independent solutions. The solutions will be constructed as continuations of chains of unperturbed heteroclinic connections and will make several pulses, i.e., departures and approaches, relative to Π. A precise definition of an N-chain of unperturbed homoclinic orbits is the following.

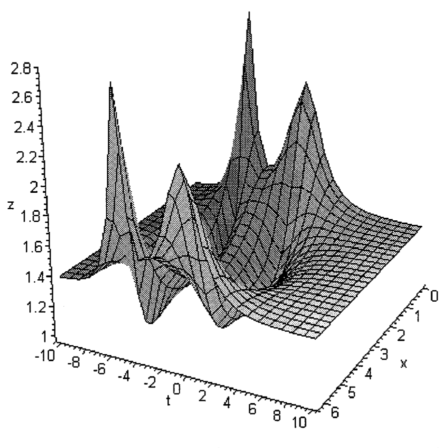

Fig. 2. The graph of $z(x,t) = \left|\left(u_1^{h+}(x,t), u_2^{h+}(x,t)\right)\right|$ for the coupled NLS system with $t_{01} = 0$, $t_{02} = 5$, $\Omega_1 = 0.6$, $\Omega_2 = 0.8$.

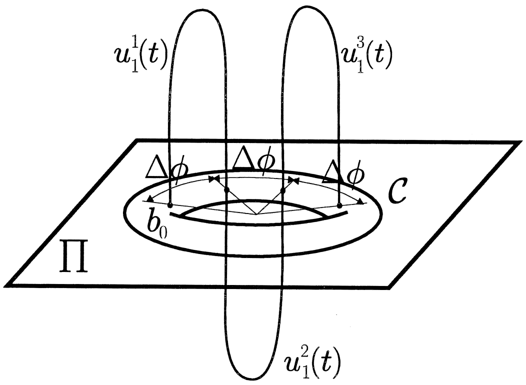

Fig. 3. The u_1 component of a 3-chain with $P_1^1=+1$, $P_2^1=-1$, and $P_3^1=+1$.

Let us consider a set of unperturbed solutions $\{u^j(t)\}_{j=1}^N$ of the type (3), each homoclinic to the torus $\mathscr{C}$, with the property

$$\lim_{t\to-\infty} u^1(t)=b_0,$$

$$\lim_{t\to+\infty} u^{j-1}(t)=\lim_{t\to-\infty} u^j(t),\, j=2,\dots,N.$$

Let $\{P_j\}_{j=1}^N$ be a sequence of vectors $P_j=\left(P_j^1,P_j^2\right)$ with $P_j^k\in\{-1,0,+1\}$. The solutions $u^1(t),\dots,u^N(t)$ are said to form an *N-chain* with basepoint b_0 and jump sequence $\{P_j\}_{j=1}^N$ if for all $t\in\mathbb{R}$, we have

$$u_k^j(t)\in\begin{cases} W_0^+(\mathscr{C}_k), & \text{if} \quad P_j^k=+1,\\ \mathscr{C}_k, & \text{if} \quad P_j^k=0,\\ W_0^-(\mathscr{C}_k), & \text{if} \quad P_j^k=-1.\end{cases}$$

We denote an N-chain with basepoint b_0 by $X^N(b_0)$.

As an example, we show the geometry of an 3-chain in Fig. 3.

3. General formulation

In this section we recall some results from Haller [5] on the existence of multi-pulse solutions for near-integrable evolution equations. We give a semi-informal description of the results for the specific case of two coupled NLS equations to avoid the introduction of further notation and terminology. For a more general formulation we refer the reader to [5]. Throughout this section, we will use the shorthand notation $\boldsymbol{H}^1$ for the Sobolev space $\boldsymbol{H}^1[0,2\pi]$ of complex-valued L^2 functions of x that admit a bounded distributional derivative in L^2.

We rewrite the coupled NLS system (1) as a system of evolution equations

$$\partial_t u_k = i\nabla_{\bar{u}_k} H_0^k(u_k,\bar{u}_k)+\epsilon g_k(u,\bar{u}),\quad k=1,2, \tag{5}$$

where

$$H_0^k(u,\bar{u})=\frac{1}{2\pi}\int_0^{2\pi}|\partial_x u_k|^2+2\Omega_k^2|u_k|^2-|u_k|^4\,dx,$$

$$g_k(u,\bar{u})=i\nabla_{\bar{u}_k}H_1(u,\bar{u})+\hat{g}_k(u,\bar{u}), \tag{6}$$

with

$$H_1(u,\bar{u})=H_1^1(u_1,\bar{u}_1)+H_1^2(u_2,\bar{u}_2)+H_1^c(u,\bar{u}),$$

$$H_1^k(u_k,\bar{u}_k)=\frac{i\Gamma_k}{2\pi}\int_0^{2\pi}\bar{u}_k-u_k\,dx,$$

$$H_1^c(u,\bar{u})=\frac{\gamma}{2\pi}\int_0^{2\pi}|u_1|^2|u_2|^2\,dx,$$

$$\hat{g}_k(u,\bar{u})=\hat{D}_k u_k. \tag{7}$$

The following features of this system of PDEs enable one to apply general results from Haller [5] on the existence of solutions close to N-chains for $\epsilon>0$:

1. The Hamiltonian H_0^k splits as $H_0^k=H_{00}^k+H_{01}^k$, such that H_{01}^k is smooth and bounded on bounded subsets of $\boldsymbol{H}^1$, while the terms in H_{00}^k are not bounded and generate the linear terms in (2) with second derivatives. More precisely, $i\nabla_{\bar{u}_k}H_{00}^k(u,\bar{u})\equiv M_0^k u$, where the linear operator M_0^k has a dense domain in $\boldsymbol{H}^1$ and $\|M_0^k u_k\|_{\boldsymbol{H}^{-1}}\le K\|u_k\|_{\boldsymbol{H}^1}$ for an appropriate positive constant K.
2. The perturbation terms in g_k smooth and bounded on bounded sets in $\boldsymbol{H}^1$.
3. For any $\epsilon\ge 0$, system (2) admits a flow $F^t:\boldsymbol{H}^1\times\boldsymbol{H}^1\to\boldsymbol{H}^1\times\boldsymbol{H}^1$ that is continuous in t and of class C^r in u and ϵ for any fixed t.
4. In the finite-dimensional invariant subspace Π, there exists an invariant torus $\mathscr{C}$ of fixed points for $\epsilon=0$. Any fixed point on $\mathscr{C}$ admits two stable, two unstable and infinitely many center directions.

5. If $E(p)$ denote the subspace of the center subspace that corresponds to purely imaginary eigenvalues, then $E(p) \equiv E$ is independent of p and the restriction of the operator $M_0 = \left(M_0^1, M_0^2\right)$ to this subspace generates a uniformly bounded group on E. More specifically, for $A = M_0 | E$,

   $\| e^{At} u \|_{\boldsymbol{H}^1} \le \sqrt{2} \, \| u \|_{\boldsymbol{H}^1}$

   holds.
6. The torus $\mathscr{C}$ admits a four-dimensional homoclinic manifold $W_0(\mathscr{C})$ with the properties described above.

3.1. N-pulse orbits

Under the above conditions, the following results apply from Haller [5]. Let us define the Nth-order energy function

$$\Delta^N \mathscr{H} = \left(\Delta^N \mathscr{H}^1, \Delta^N \mathscr{H}^2 \right)$$

with

$$\Delta^N \mathscr{H}^k(\phi_0) = - \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left\langle \nabla H_0^k, G_k \right\rangle\big|_{u^l(t)} \, dt, \qquad (8)$$

where $G_k = (g_k, \bar{g}_k)$, and $u^1(t), \dots, u^N(t)$ form an N-chain with basepoint b_0. As we will see below, one can think of the energy function as a generalized Melnikov function: Its transverse zeros will correspond to families of N-pulse orbits homoclinic to a neighborhood of $\mathscr{C}$.

Theorem 1. *For $\epsilon > 0$ small enough, the following hold*:
(i) *There exists a codimension-four invariant manifold $\mathscr{M}_\epsilon$ in the phase space of (2) that contains a neighborhood of the unperturbed torus $\mathscr{C}$. $\mathscr{M}_\epsilon$ also contains the invariant plane Π.*
(ii) *Suppose that for some positive integer N, the phase vector $\phi_0 \in \mathbb{T}^2$ is a transverse zero of the function $\Delta^N \mathscr{H}$, i.e.,*

$$\Delta^N \mathscr{H}(\phi_0) = 0, \quad \det\left[D \Delta^N \mathscr{H}(\phi_0) \right] \neq 0.$$

Suppose further that for k = 1,2,

$$\Delta^j \mathscr{H}^k(\phi_0) \neq 0$$

holds for all integers j = 1, ..., N − 1.

Then, for any choice of the vector $P_1 \in \mathbb{R}^2$ with $P_1^k \in \{-1, +\}$ there exist infinitely many orbits that are homoclinic to the manifold $\mathscr{M}_\epsilon$. The homoclinic orbits are close to N-chains with jump sequence

$$P_{j+1}^k = \operatorname{sign}\left(\Delta^j \mathscr{H}^k(\phi_0) \right) P_j^k, \quad j = 1, \dots, N-1. \qquad (9)$$

There also exists a similar family of orbits with jump sequence $-\{Pj\}_{j=1}^N$.

Since the homoclinic orbits rendered by the above theorem are close to N-chains, we call them *N-pulse orbits*. We note that, in general, the two families of N-pulse orbits described above do not form a smooth set in the phase space $\boldsymbol{H}^1$. However, they can be shown to form a smooth, infinite-dimensional manifold in a higher-order Sobolev space, $\boldsymbol{H}^{4N-1}$. We also remark that the manifold $\mathscr{M}_\epsilon$ perturbs from a center manifold to the torus $\mathscr{C}$. The details of all these results can be found in Haller [5].

3.2. N-pulse orbits homoclinic to plane waves

If $\phi_0 \in \mathbb{R}^2$ satisfies the conditions of Theorem 8, then the intersection set of the plane $\phi = \phi_0$ is a two-dimensional subset of $\mathscr{C}$. It turns out to be a first-order approximation for 'takeoff' points of a four-dimensional family of N-pulse homoclinic orbits. They form a subset of the large family described in Theorem 1, and have the special property that they are asymptotic in backward time to the four-dimensional plane Π. It is feasible to try to find first their backward limit sets on Π, then investigate their exact forward-time behavior as they come back, after N-pulses, to a neighborhood of $\mathscr{C}$. We shall be interested in the case where the backward limit set for these orbits is a fixed point, and in forward time the orbits return to the same fixed point.

We first introduce action-angle variables on the invariant plane Π by letting

$$(u_1, u_2) = \left(\sqrt{I_1}\, e^{i\phi_1}, \sqrt{I_2}\, e^{i\phi_2} \right).$$

To focus on a vicinity of the resonant torus $\mathscr{C}$, we introduce the new coordinates $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ defined as

$$I_k = \sqrt{\Omega_k} + \sqrt{\epsilon}\, \eta_k.$$

A direct calculation shows that in the vicinity of $\mathscr{C}$, the flow on Π obeys the equations

$$\dot{\eta}_k = -\sqrt{\epsilon}\, D_{\phi_k}\mathscr{H}_g(\eta,\phi) + \mathscr{O}(\epsilon),$$
$$\phi_k = \sqrt{\epsilon}\, D_{\eta_k}\mathscr{H}_g(\eta,\phi) + \mathscr{O}(\epsilon), \tag{10}$$

with

$$\mathscr{H}_g(\eta,\phi) = \mathscr{H}_g^1(\eta_1,\phi_1) + \mathscr{H}_g^2(\eta_2,\phi_2),$$
$$\mathscr{H}_g^k(\eta_k,\phi_k) = -\eta_k^2 + 2\Gamma_k\Omega_k\sin\phi_k + 2\alpha_k\Omega_k^2\phi_k. \tag{11}$$

Note that the reduced system has no dependence on the parameter β at leading order. The following theorem gives conditions for fixed points of the reduced system (10) to admit manifolds of N-pulse orbits in function space.

Theorem 2. *Let $V \in \mathbb{R}^p$ be an open set in the space of all parameters of system (1). Assume that*
(i) *The Hamiltonian $\mathscr{H}_g$ has a hyperbolic equilibrium $p_0(\lambda) \in \Pi$. If $p_\epsilon(\lambda) \in \Pi$ is the corresponding equilibrium of the perturbed system, then the manifold $W^s(p_\epsilon(\lambda)) \cap \mathscr{M}_\epsilon$ is codimension-two within the manifold $\mathscr{M}_\epsilon$.*
(ii) *The size of $W^s(p_\epsilon(\lambda))$ in directions transverse to the plane Π is at least of order $\mathscr{O}(\epsilon^q)$ with some $q < \frac{3}{5}$.*
(iii) *For some positive integer N and for all $\lambda \in V$, there exists a function $\phi_0(\lambda)$ that satisfies the conditions of Theorem 1.*
(iv) *The plane $\{\phi = \phi_0(\lambda)\} \subset \Pi$ intersects transversely the unstable manifold of the fixed point $p_0(\lambda) \in \Pi$ of the Hamiltonian $\mathscr{H}_g$.*
(v) *If $(\eta_0(\lambda),\phi_0(\lambda))$ are the coordinates of this transverse intersection point, then the point $(\eta_0,\phi_0(\lambda) + N\Delta\phi(\lambda))$ crosses the stable manifold of p_0 transversely as λ is varied through λ_0.*

Then for every vector $P_1 \in \mathbb{Z}^s$ with $P_1^i \in \{-1,1\}$, there exists a codimension-two set $M^+ \subset \mathbb{R}^p \times \mathbb{R}$ near the point $(\lambda_0,0)$ such that for every parameter value $(\lambda,\epsilon) \in M$, the system (1) admits a two-dimensional manifold of N-pulse orbits homoclinic to the point $p_\epsilon(\lambda)$. The jump sequence of the orbits is given by

$$P_{j+1}^k = \operatorname{sign}\left(\Delta^j\mathscr{H}^k(\phi_0)\right)P_j^k, \quad j = 1,\dots,N-1.$$

Theorem 2 is just a special case of a more general result proved in Haller [5]. The proof is based on infinite-dimensional invariant manifold techniques, geometric singular perturbation theory, and detailed estimates on solutions.

4. The existence of Šilnikov manifolds

In this section we apply Theorem 2 to the coupled NLS system (1). Writing out the reduced Eq. (10) in detail, we obtain that for $\epsilon > 0$ the flow near the torus $\mathscr{C}$ satisfies the system of ODEs

$$\dot{\eta}_1 = -\sqrt{\epsilon}\left(2\Gamma_1\Omega_1\cos\phi_1 + 2\alpha_1\Omega_1^2\right) + \mathscr{O}(\epsilon),$$
$$\dot{\phi}_1 = -\sqrt{\epsilon}\,2\eta_1 + \mathscr{O}(\epsilon),$$
$$\dot{\eta}_2 = -\sqrt{\epsilon}\left(2\Gamma_2\Omega_2\cos\phi_2 + 2\alpha_2\Omega_2^2\right) + \mathscr{O}(\epsilon),$$
$$\dot{\phi}_2 = -\sqrt{\epsilon}\,2\eta_2 + \mathscr{O}(\epsilon). \tag{12}$$

This shows that the two NLS equations decouple form each other at leading order on Π. At that order we obtain that the phase space structure is the product of those of two forced pendula, as shown in Fig. 4.

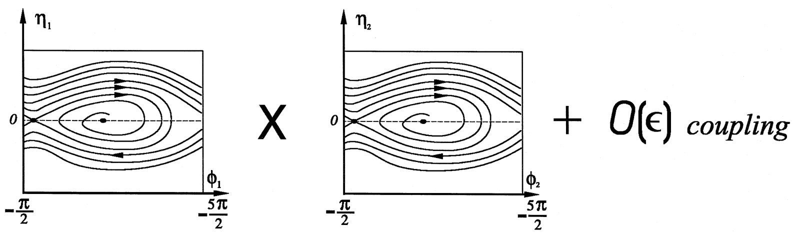

Fig. 4. Phase portrait on the invariant four-dimensional plane Π near the torus $\mathscr{C}$ for $\epsilon > 0$.

For $\epsilon > 0$ small enough, nondegenerate critical points of $\mathscr{H}_g$ give rise to $\mathscr{O}(\sqrt{\epsilon})$-close fixed points for (12). Defining $\chi_{\alpha_k} = \alpha_k/\Gamma_k$, we see that for $|\chi_{\alpha_k}\Omega_k| < 1$, $\mathscr{H}_g$ has a saddle point

$$s_0(\chi_\alpha) = \left(0,\pi + \cos^{-1}\left(-\chi_{\alpha_1}\Omega_1\right),0,\pi + \cos^{-1}\left(-\chi_{\alpha_2}\Omega_2\right)\right).$$

In view of Theorem 2, this saddle point is a candidate for a fixed point that admits manifolds of multi-pulse orbits. Below we will verify the conditions of the theorem for $s_0(\chi_\alpha)$.

Since the linear stability of the perturbed fixed point $s_\epsilon(\chi_\alpha)$ is not changed by the coupling terms, $s_\epsilon(\chi_\alpha)$ admits a codimension-four stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(s_\epsilon(\chi_\alpha))$. The intersection of this stable manifold with $\mathscr{M}_\epsilon$ is a codimension-two submanifold of $\mathscr{M}_\epsilon$. The 'height' of $W^{\mathrm{s}}(s_\epsilon(\chi_\alpha))$ is $\mathscr{O}(\epsilon^{3/4})$, as follows from the normal form derived in Li et al. [7] for individual NLS equations and from the fact that (12) decouples at leading order.

To satisfy (iii) of Theorem 2, we have to ensure that the energy function $\Delta^N\mathscr{H}$ has a transverse zero. Using (6) and (7), the expression (8) for $\Delta^N\mathscr{H}^k$ can be rewritten as

$$\Delta^N\mathscr{H}^k(\phi_0) = 2\,\Omega_k\Gamma_k\left[\sin(\phi_k + N\Delta\phi_k) - \sin\phi_k\right] - N\Omega_k\left[\alpha_k\mathscr{I}_\alpha(\Omega_k) - \beta_k\mathscr{I}_\beta(\Omega_k,K) - \gamma\mathscr{I}^k_\gamma(\Omega)\right], \tag{13}$$

where

$$\mathscr{I}_\alpha(\Omega_k) = \frac{1}{\pi\Omega_k}\mathrm{Re}\int_{-\infty}^{\infty}\int_0^{2\pi}\left(u_k\partial_x^2\bar{u}_k + 2|u_k|^2\left[|u_k|^2 - \Omega_k^2\right]\right)\Big|_{u_k^{h\pm}(t)}\,dx\,dt,$$

$$\mathscr{I}_\beta(\Omega_k,K) = \frac{1}{\pi\Omega_k}\mathrm{Re}\int_{-\infty}^{\infty}\int_0^{2\pi}\left(\partial_x^2\bar{u}_k + 2\left[|u_k|^2 - \Omega_k^2\right]\bar{u}\right)\hat{B}_K u_k\Big|_{u_k^{h\pm}(t)}\,dx\,dt,$$

$$\mathscr{I}^k_\gamma(\Omega) = \frac{1}{\pi\Omega_k}\mathrm{Re}\int_{-\infty}^{\infty}\int_0^{2\pi}\left(\partial_{xx}\bar{u}_k + 2\left[|u_k|^2 - \Omega_k^2\right]\bar{u}_k\right)u_k|u_j|^2\Big|_{u^{h\pm}(t)}\,dx\,dt.$$

Since the homoclinic solutions $u^{h\pm}(t)$ are given by $\boldsymbol{H}^\infty$ functions that decay exponentially in time as $t \to \pm\infty$, the integral

$$\mathscr{I}_0(\Omega_k) = \frac{1}{\pi\Omega_k}\mathrm{Re}\int_{-\infty}^{\infty}\int_0^{2\pi}\left(\partial_x^2\bar{u}_k + 2\left[|u_k|^2 - \Omega_k^2\right]\bar{u}_k\right)\partial^2_{xx}u_k|_{u_k^{h\pm}(t)}\,dx\,dt \tag{14}$$

converges, and we have

$$\lim_{K\to\infty}\mathscr{I}_\beta(\Omega_k,K) = \mathscr{I}_0(\Omega_k).$$

In fact, by analyticity of the integrand in (14), we can write

$$\mathscr{I}_\beta(\Omega_k,K) = \mathscr{I}_0(\Omega_k) + \mathscr{O}(\mathrm{e}^{-\nu K}). \tag{15}$$

for appropriate $\nu > 0$ and K sufficiently large.

Now, (13) shows that for

$$\Delta\phi_k \neq \frac{2j\pi}{N}, \quad j \in \mathbb{Z},$$

$$\left|\alpha_k\mathscr{I}_\alpha(\Omega_k) - \beta_k\mathscr{I}_\beta(\Omega_k,K) - \gamma\mathscr{I}^k_\gamma(\Omega)\right| < \frac{4\Gamma_k}{N}\left|\sin\frac{N\Delta\phi_k}{2}\right|, \quad k = 1,2, \tag{16}$$

the function $\Delta^N\mathscr{H}$ always has a transverse zero. Furthermore, since the phase portrait of (12) is of the type shown in Fig. 4, the two-dimensional plane $\{\phi_k = \phi_{k0},\, k = 1,2\}$ has at least one transverse intersection point (η_0,ϕ_0) with the two-dimensional unstable manifold of the saddle point $s_0(\chi_\alpha)$.

To apply Theorem 2, it remains to verify that the point $(\eta_0,\phi_0 + N\Delta\phi)$ crosses the two-dimensional homoclinic manifold of the saddle $s_0(\chi_\alpha)$ transversely at some critical value of the parameters. Upon such a crossing, both (η_{k0},ϕ_{k0}) and $(\eta_{k0},\phi_{k0} + (N\Delta\phi_k)\mathrm{mod}\,2\pi)$ must lie on the same level curve of the local Hamiltonian $\mathscr{H}^k_g$, i.e.,

$$\mathscr{H}^k_g(\eta_{k0},\phi_{k0} + (N\Delta\phi_k)\mathrm{mod}2\pi) = \mathscr{H}^k_g(\eta_{k0},\phi_{k0})$$

must hold for $k=1,2$. Using the fact that ϕ_{k0} is a zero of $\Delta^N\mathscr{H}$, this last equation can be rewritten as

$$\beta_k=\frac{\alpha_k}{\mathscr{I}_\beta(\Omega_k,K)}\left[\mathscr{I}_\alpha(\Omega_k)+\frac{2\Omega_k}{N}[N\Delta\phi_k(\Omega_k)]\bmod 2\pi\right]-\frac{\gamma I_\gamma^k(\Omega)}{\mathscr{I}_\beta(\Omega_k,K)}, \quad (17)$$

for $k=1,2$. This, however, only gives a meaningful β_k value if

$$\chi_{\alpha_k}<\frac{2}{\Omega_k}\frac{|\sin[N\Delta\phi_k(\Omega_k)/2]|}{|N\Delta\phi_k(\Omega_k)|\bmod 2\pi},$$
$$\chi_{\alpha_k}<\frac{1}{\Omega_k},\quad k=1,2. \quad (18)$$

Here the first inequality is obtained by combining the second inequality in (16) with Eq. (17), while the second inequality ensures the existence of the saddle point $s_0(\chi_\alpha)$.

Note that as long as $N\Delta\phi(\Omega)<2\pi$, N-pulse orbits coexist with single-pulse orbits. This can be seen by noting that for any $N\Delta\phi(\Omega)<2\pi$, we have $\frac{1}{N}[N\Delta\phi(\Omega)]\bmod 2\pi\equiv\Delta\phi(\Omega)$ in (17).

By (15), for large enough K we can replace $\mathscr{I}_\beta(\Omega_k,K)$ with $\mathscr{I}_0(\Omega_k)$. Since β_k is a linear function of α_k in (17), the derivative $d\beta_k/d\alpha_k$ is nonzero whenever the condition gives a nonzero β_k. As a consequence, the point $(\eta_0,\phi_0+N\Delta\phi)$ crosses the two-dimensional homoclinic manifold of the saddle $s_0(\chi_\alpha)$ transversely. Therefore, Theorem 2 and our calculations in this section imply the following result.

Theorem 3. *Let N be an arbitrary but fixed positive integer, and let M_0 be the set of points in the $(\alpha,\beta,\gamma,\Gamma,\epsilon)$ parameter space that satisfy*

$$\beta_k=\frac{\alpha_k}{\mathscr{I}_0(\Omega_k)}\left[\mathscr{I}_\alpha(\Omega_k)+\frac{2\Omega_k}{N}\times(N\Delta\phi_k(\Omega_k))\bmod 2\pi\right]-\frac{\gamma I_\gamma^k(\Omega)}{\mathscr{I}_0(\Omega_k)},$$
$$\frac{\alpha_k}{\Gamma_k}<\frac{1}{\Omega_k}\min\left\{\frac{|\sin[N\Delta\phi_k(\Omega_k)/2]|}{|N\Delta\phi_k(\Omega_k)|\bmod 2\pi},1\right\},$$
$$\tfrac{1}{2}<\Omega_k<1, \quad (19)$$

for $k=1,2$, with $\mathscr{I}_0$ defined in (14). Assume further that M_0 is a nonempty codimension two surface. Then for small enough $\epsilon>0$ and large enough $K>0$, there exists a codimension-two surface M_ϵ in the $(\alpha,\beta,\gamma,\Gamma,\epsilon)$ space with the following properties:

(i) *M_ϵ is C^0-close to the surface M_0 in the $(\alpha,\beta,\gamma,\Gamma,\epsilon)$ parameter space.*
(ii) *For every $(\alpha,\beta,\gamma,\Gamma,\epsilon)\in M_\epsilon$, system admits four four-dimensional manifolds of N-pulse homoclinic orbits which are doubly asymptotic to an invariant 2-torus of the original system (1).*
(iii) *In terms of the transformed system (2), the manifolds of homoclinic orbits are two-dimensional and connect a fixed point $s_\epsilon(\chi_\alpha)\in\Pi$ to*

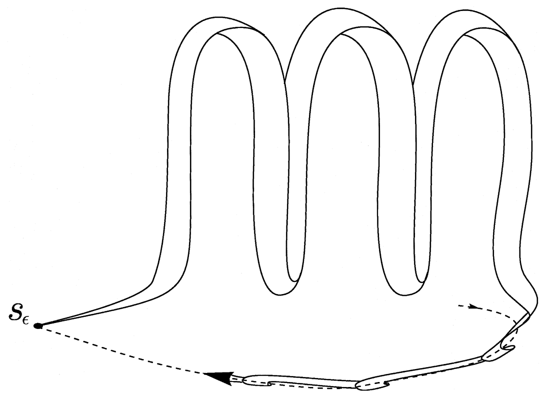

Fig. 5. Šilnikov manifold for the coupled NLS system.

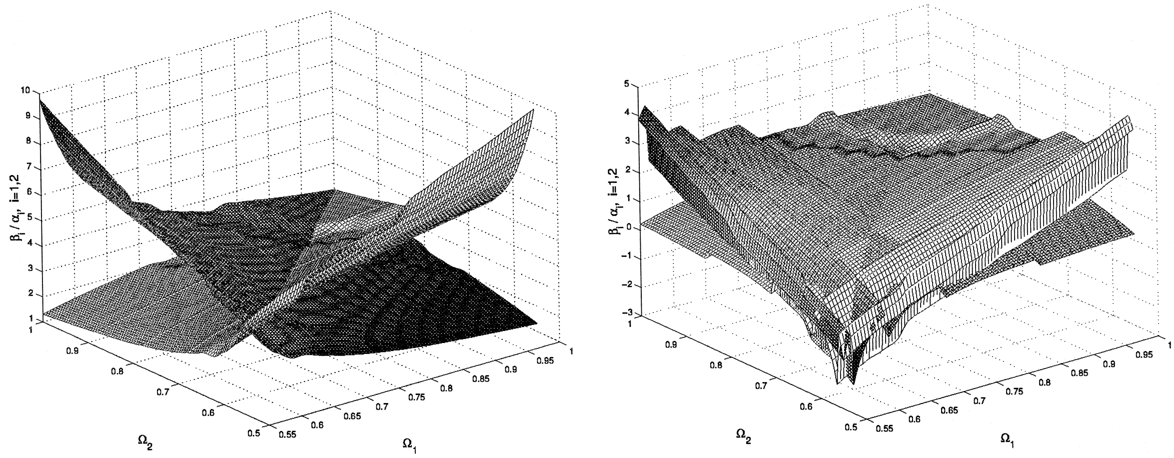

Fig. 6. Approximate parameter sets for the existence of Šilnikov manifolds. Fixed parameter ratios are $\alpha_1 = \alpha_2$, $\gamma/\alpha_k = 1$. The pulse number is $N = 1$ for the first case and $N = 10$ for the second case.

itself. Furthermore, the N-pulse orbits in (2) are close to N-chains with jump sequence

$$P_{j+1}^{k} = \mathrm{sign}\Big\{ 2\,\Omega_k \Gamma_k \big[\sin\big(\cos^{-1}(-\chi_\alpha \Omega) + j\Delta\phi\big) - \sin\big(\cos^{-1}(-\chi_\alpha \Omega)\big)\big] - j\Omega\big(\alpha_k \mathscr{I}_\alpha(\Omega) - \beta_k \mathscr{I}_0(\Omega_k) - \gamma \mathscr{I}_\gamma^k(\Omega)\big)\Big\} P_j^k,$$

$j = 1,\dots,N-1, \quad k = 1,2,$

where P_1^k can be either $+1$ or -1.

The geometry of the manifolds of N-pulse orbits obtained for the coupled NLS system (2) is shown in Fig. 5. The manifolds exhibit a dynamical behavior similar to that of Šilnikov-type homoclinic orbits in ODEs. This is due to the fact that the infinitely many purely imaginary eigenvalues in directions transverse to Π in the integrable limit turn into complex eigenvalues with negative real parts. This geometry prompts us to refer to the manifolds we obtained as *Šilnikov manifolds*.

5. Numerical experiments

To illustrate the results of Theorem 3, we plot the intersection of the two surfaces satisfying the condi-

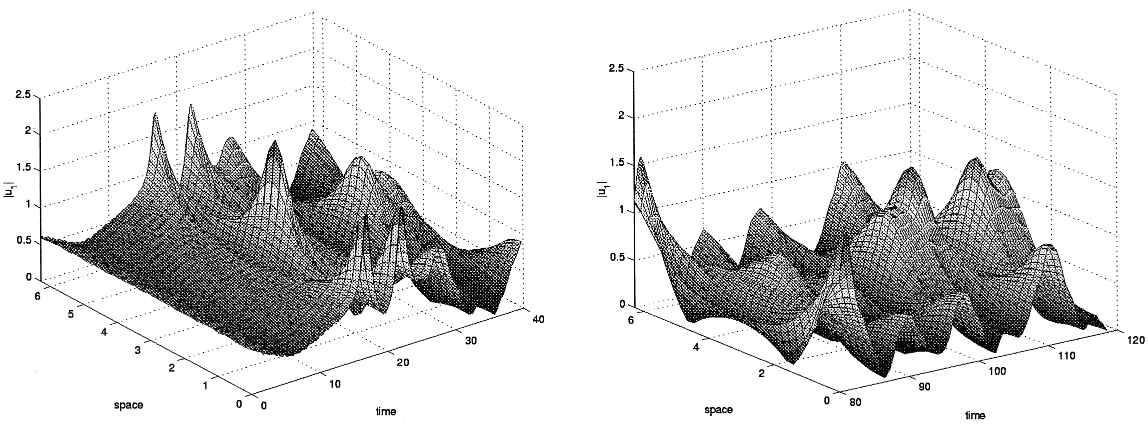

Fig. 7. The spatio-temporal behavior of $|u_1(x,t)|$ on an initial and a later time interval. The parameter values are $\alpha_1 = \alpha_2 = 0.1$, $\gamma_1 = \gamma_2 = 0.1$, $\Omega_1 = 0.6$, $\Omega_2 = 0.7$, $\Gamma_1 = \Gamma_2 = 50.0$, $\epsilon = 0.001$. The approximae β_k values, computed for these parameters, are $\beta_1 = 0.40623$ and $\beta_2 = 0.33298$.

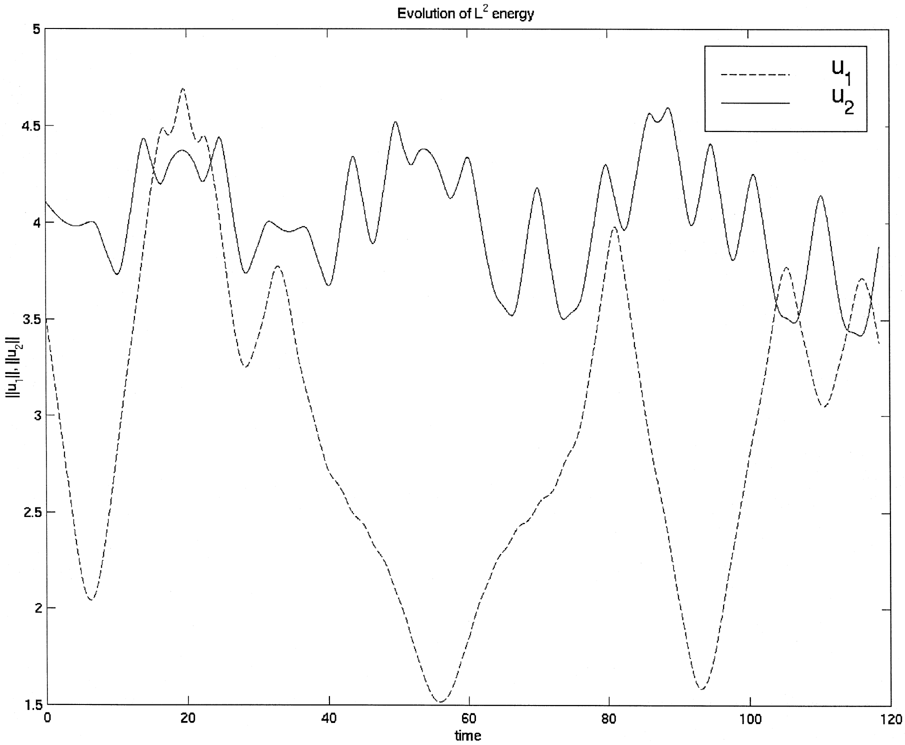

Fig. 8. The L^2 norm of u_1 and u_2 as a function of time on the solution shown in the Fig. 7.

tions (19) in Fig. 6. In order to visualize the resulting six-dimensional intersection surface in the eight-dimensional parameter space, we chose to fix the ratio of certain parameters and plotted the resulting β_k/α_k values as a function of Ω_1 and Ω_2. We recall that for any integer N with $N\Delta\phi(\Omega)<2\pi$, the first intersection surface in Fig. 6 also marks the existence of N-pulse orbits.

The dynamical implications of Šilnikov manifolds are unknown at this point, but we expect them to be the landmarks of temporally chaotic dynamics. Such a behavior is clearly seen in our numerical experiments, as we show in Fig. 7. The figure shows the time dependence of $|u_1(x,t)|$ for a typical run with parameter values satisfying (19). A nearly 'flat' initial condition was chosen by randomly selecting a point on the unperturbed resonant torus $\mathscr{C}$, then applying a small (of the size 10^{-2}) random perturbation off the torus. In our numerical experiments $u_1(x,t)$ went through periods where its energy was quiescent for a long time, then it jumped to large oscillations. This phenomenon, shown in Fig. 8, shows that the complicated behavior is not localized to a single NLS equation for the initial conditions we picked.

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