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J. Non-Newtonian Fluid Mech. xxx (2007) xxx–xxx

Journal of Non-Newtonian Fluid Mechanics

www.elsevier.com/locate/jnnfm

Mathematical approaches to dynamic scaling☆

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Received 13 December 2006; accepted 5 January 2007

Abstract

This is a brief overview of some mathematical work on dynamic scaling and coarsening in problems of phase separation.

Keywords: Dynamic scaling; Viscous fingering; Coarsening; Smoluchowski's coagulation equation

1. Introduction

Studies of phase separation lead to many interesting problems in non-equilibrium thermodynamics. The hypothesis of *dynamic scaling* plays an essential role in our understanding of such processes (see for example, Ref. [1]). This is a well-tested hypothesis based on experiments and deterministic and stochastic numerical simulations. This article is a short introduction to mathematical methods useful in the analysis of dynamic scaling. It is organized into two case studies: (1) dynamic scaling of the interface in viscous fingering; (2) mean-field models of coalescence. These studies illustrate two distinct techniques—(1) the use of estimates to analyze scaling in 'fully nonlinear regimes'; (2) the relation between scaling laws and probabilistic methods. This article is didactic. There are no new research results here (the two cases simply summarize problems I have worked on). However, the techniques used in these problems are natural and simple, and have been used in many similar models. Given the interdisciplinary nature of the IWNET 006 meeting, my hope is that an informal, but concrete, introduction to the mathematical research they represent will be of interest to the wide readership of this journal. The curious reader is also referred to a very readable survey of this area [14].

2. Scaling of the interface in viscous fingering

2.1. Introduction

The Saffman–Taylor instability of free-surfaces in a Hele–Shaw cell or porous medium is a widely studied problem in fluid mechanics. A good review is provided by Homsy [5]. The classical approach to this problem relies on special solutions (the famous Saffman–Taylor fingers [15]) and their linear stability. A basic limitation of linear stability results is that they hold for very small time regimes. Consequently, a great deal of bold handwaving is needed to assert that these special solutions have anything to do with a typical experimental situation where one has an interface between two phases with many interacting fingers. My goal is to illustrate another approach, based on bounds (inequalities) for physically natural quantities. This perspective yields useful information about such 'fully nonlinear regimes' with little fuss. It is similar in spirit to the now classical work of Howarth [6].

To fix ideas, I will focus on the simplest setting of gravity driven *miscible* fingering studied experimentally by Wooding [17]. Consider an infinite column of fluid in a porous medium with a heavy solute lying above a pure phase. For sufficiently dilute solutions, the mobility is independent of the solute concentration, and after a suitable non-dimensionalization Wooding obtained the system:

$$\partial_t s + \mathbf{u} \cdot \nabla s = \Delta s, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

☆ Partially supported by NSF awards DMS 03-05985 and DMS 06-05006.
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doi:10.1016/j.jnnfm.2007.01.003

Please cite this article in press as: G. Menon, Mathematical approaches to dynamic scaling, J. Non-Newtonian Fluid Mech. (2007), doi:10.1016/j.jnnfm.2007.01.003

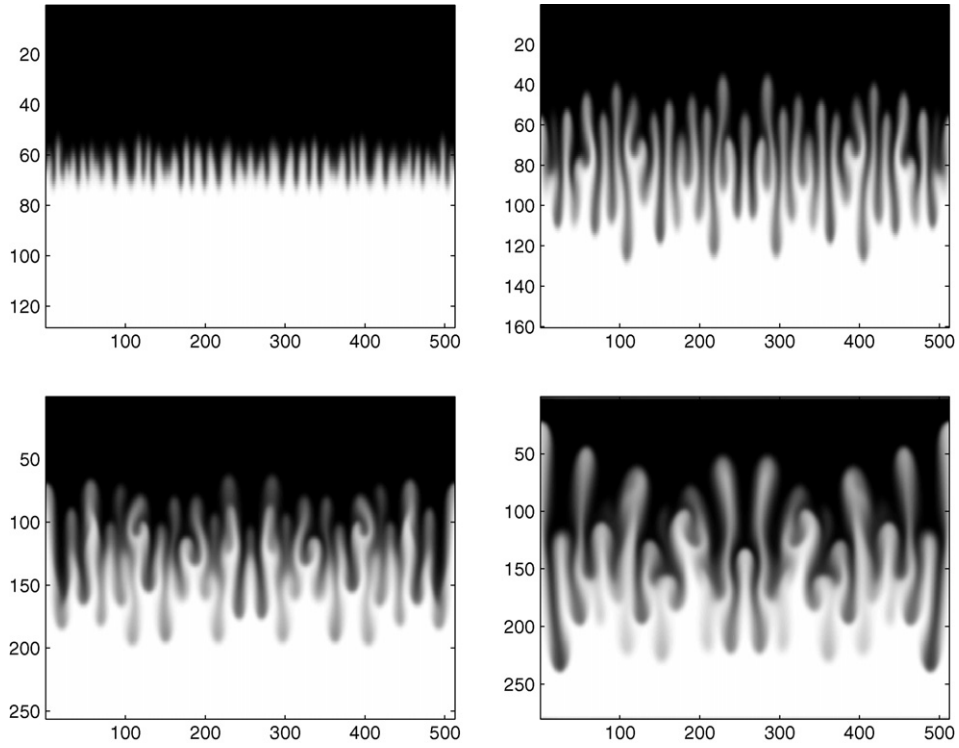

Fig. 1. Coarsening of fingers and bulk transport.

$$\mathbf{u} + \nabla p = -s\mathbf{e}_z. \tag{3}$$

Here $0 \le s \le 1$ is the normalized concentration of the solute that is transported by convection and diffusion. Eq. (3) is Darcy's law: the velocity is linearly proportional to the driving force which comprises a pressure gradient and buoyancy ($-s\mathbf{e}_z$). The domain is $x = (y, z) \in [0, L] \times \mathbb{R}$. The Peclet number, L, is a measure of the strength of diffusion. It is the only external parameter. We are interested in scaling behavior that is independent of L and boundary effects, and in particular the behavior as $L \to \infty$. Therefore, it is convenient to use periodic boundary conditions in y.

2.2. The scaling laws

Fig. 1 shows four snapshots of the evolution starting with small perturbations of the flat unstable stratification. After an initial transient, the system develops a mixing zone with an intricate network of fingers. The details of fingering are sensitive to initial data, but there is a remarkable statistical regularity observed in physical [17] and numerical experiments [5]:

(a) The end-to-end width of the mixing zone is typically t.
(b) The fingers broaden at the rate $O(\sqrt{t})$.

The only dependence on the Peclet number L is as a finite-size cut-off. The scaling regime persists until time $O(L^2)$ when the finger-width is comparable to the domain size. For large L, this regime is clearly of greater importance than the initial stages of the instability governed by linearization.

The emergence of such regularity given the initial instability is striking. In practical terms, what is most important is the size of the mixing zone (or the speed at which the upper and lower boundaries spread). A survey of scientific interest in this question may be found in a recent article of Yortsos and Salin [18]. It is also interesting to understand the internal structure of the mixing zone. For example, numerical simulations show clearly that the coarsening is driven by coalescence of fingers, not diffusive spreading.

2.3. Energy, perimeter and mixing entropies

A rigorous formulation of dynamic scaling involves a definition of vertical and horizontal length scales (denoted $a(t)$ and $b(t)$ respectively as in Fig. 2), followed by upper and lower bounds of the form:

$$1 - o(1) \le \frac{a(t)}{t} \le 1, \qquad c \le \frac{b(t)}{\sqrt{t}} \le C, \quad t \gg 1 \tag{4}$$

for some constants $C \ge c > 0$, under minimal assumptions on initial data. However, (4) is false in such generality: the unstable stratification s_0 (defined in (5) below), evolves diffusively without fingering. Therefore, for this solution $a(t) \sim \sqrt{t}$ and there is no coarsening since there are no fingers. Nevertheless, by focusing on physically meaningful quantities such as the potential energy, mean perimeter, and mixing entropies we can prove one-sided bounds that scale in the natural way with time. Though we obtain only one-sided estimates, these are simple and robust and free of any ansatz on the structure of the flow.

Let Q denote the spatial domain $x{:=}(y, z) \in [0, L] \times \mathbb{R}$. We consider periodic boundary conditions in y. The unstable strati-

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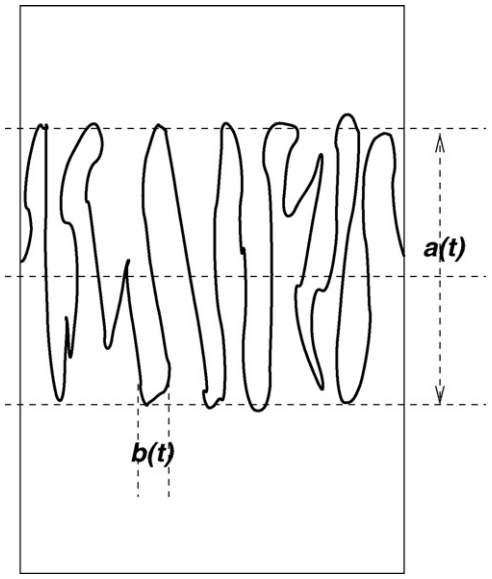

Fig. 2. Caricature of $a(t)$ and $b(t)$.

fication:

$$s_0(z) = \begin{cases} 0, & z < 0 \\ 1, & z \geq 0 \end{cases} \tag{5}$$

will serve as the main reference configuration. To obtain estimates independent of the Peclet number L, we consider normalized integrals of the form:

$$⨍ f \,\mathrm{d}x := \int_{\mathbb{R}} \frac{1}{L} \int_0^L f(y, z) \,\mathrm{d}y \,\mathrm{d}z := \int_{\mathbb{R}} \bar{f} \,\mathrm{d}z. \tag{6}$$

The (negative) gravitational potential energy of $s(t, x)$ is defined by

$$E(t) = ⨍ (s_0(z) - s(t, x))z \,\mathrm{d}x = \int_{\mathbb{R}} (s_0(z) - \bar{s}(t, z))z \,\mathrm{d}z. \tag{7}$$

Observe that since $s \in [0, 1]$ we have $E = ⨍ |z||s_0 - \bar{s}| \,\mathrm{d}z$. Therefore, $\sqrt{E}$ is a length scale, and we define the vertical length scale as $a = 2\sqrt{6E}$ (the choice of constant is explained in [9]). We also consider the mean perimeter:

$$P(t) = ⨍ |\nabla s(t, x)| \,\mathrm{d}x. \tag{8}$$

If there were no diffusion, $P(t)$ would measure the mean length of the interface between the regions $s = 0$ and $s = 1$. One effect of diffusion is to smooth sharp transitions and create "mushy zones" where $0 < s < 1$. The size of these mixing zones can be measured by "mixing entropies" that vanish in the pure phases where s is 0 or 1. We will work mainly with the entropies:

$$H(t) = ⨍ s(1 - s) \,\mathrm{d}x,$$

$$S(t) = -⨍ (s \log s + (1 - s) \log (1 - s)) \,\mathrm{d}x. \tag{9}$$

The word "entropy" is used here in the same sense that it is used in the mathematical theory of hyperbolic conservation laws. Aside from the fact that the functions $s(1 - s)$ and $-(s \log s + (1 - s) \log (1 - s))$ are concave for $s \in [0, 1]$ and vanish when $s = 0$ or $s = 1$, there is no deeper physical motivation in the choice of these mixing entropies.

2.4. Scale-invariant estimates and their interpretation

The following estimates provide an upper bound on $a(t)$ and a lower bound on $b(t)$ independent of the Peclet number L.

Theorem 1. *Let $s(t, x)$ be a smooth solution to (1)–(3), with energy $E(t)$, mixing entropy $H(t)$, and perimeter $P(t)$. Then*

$$\limsup_{t\to\infty} \frac{E(t)}{t^2} \leq \frac{1}{6}, \qquad \limsup_{t\to\infty} \frac{H(t)}{t} \leq \frac{1}{3}, \tag{10}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_0^t P^2(\tau) \,\mathrm{d}\tau \leq \frac{\pi}{9}. \tag{11}$$

The first inequality is essentially the assertion that $a(t) \leq 2t$ asymptotically. The crux of this problem is really to prove the sharp bound $a(t) \leq t$ (as done in [9]), but I will focus on this 'naive' bound here. Inequality (11) can be viewed as a lower bound on the finger width of the form $b(t) \geq C\sqrt{t}$. This is seen as follows. Firstly, we note that (11) is an integrated version of the (unproven) pointwise inequality:

$$P(t) \leq \frac{\sqrt{2\pi t}}{3}. \tag{12}$$

More precisely, the largest C and α in a scaling ansatz $P(t) = Ct^\alpha$ compatible with (11) are the values in (12). If we assume the typical form of s is as shown in Fig. 2, we see that

$$P(t) \approx ⨍ |\partial_y s| \,\mathrm{d}x \approx \int_{|z|\leq a/2} N(z) \,\mathrm{d}z = a\bar{N} = \frac{a}{b}, \tag{13}$$

where $N(z)$ is the number of fingers per unit width on any horizontal level $z = \text{constant}$, $\bar{N}$ the mean number of fingers, and $b = 1/\bar{N}$ is the mean wavelength of fingers. The upper estimate (12) now yields:

$$b(t) \geq \frac{a(t)}{P(t)} \geq \frac{3\sqrt{t}}{\sqrt{2\pi}} \tag{14}$$

if $a(t) = t$. It is in this averaged (but also robust) sense, that (11) is an estimate on coarsening. We should note that quantities like the perimeter are more basic to the problem than a somewhat artificial notion of fingers based on special solutions. All we see in experiments and simulations is a jagged interface between two phases.

2.5. The back-of-the-envelope proof

Theorem 1 is based on energy balance, control of gradients using mixing entropies, and an interpolation argument linking the mixing entropies and energy. One may integrate by parts to

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find that the decay of the energy is given by

$$\dot{E} = \fint |u|^2 \,\mathrm{d}x + 1 = \fint (s-\bar{s})^2 - \fint |\nabla p|^2 \,\mathrm{d}x + 1$$
$$= \int_{\mathbb{R}} \bar{s}(1-\bar{s})\,\mathrm{d}z - H(t) - \fint |\nabla p|^2 \,\mathrm{d}x + 1. \tag{15}$$

Since the entropy functions are concave, the mixing entropies grow because of diffusion as follows

$$\dot{H} = 2\fint |\nabla s|^2 \,\mathrm{d}x, \qquad \dot{S} = \fint \frac{|\nabla s|^2}{s(1-s)} \,\mathrm{d}x. \tag{16}$$

Finally, the energy and mixing entropies are linked by the following inequalities that are purely calculus facts, and have nothing to do with the equations:

$$H \le \int_{\mathbb{R}} \bar{s}(1-\bar{s})\,\mathrm{d}z \le \sqrt{\frac{2E}{3}}, \qquad S \le \pi\sqrt{\frac{2E}{3}}. \tag{17}$$

We combine these relations to find

$$\dot{E} \le \int_{\mathbb{R}} \bar{s}(1-\bar{s})\,\mathrm{d}z + 1 \le \sqrt{\frac{2E}{3}} + 1, \tag{18}$$

which may be integrated to yield the first inequality in (10). Then (17) immediately yields the second inequality in (10). To prove (11) we apply the Cauchy–Schwarz inequality and (16) to obtain

$$P(t) = \fint |\nabla s| \le \left(\fint s(1-s)\right)^{1/2} \left(\fint \frac{|\nabla s|^2}{s(1-s)}\right)^{1/2}$$
$$= H^{1/2}(\dot{S})^{1/2}. \tag{19}$$

We integrate in time to obtain

$$\int_0^t P^2(\tau)\,\mathrm{d}\tau \le \int_0^t H(\tau)\dot{S}(\tau)\,\mathrm{d}\tau \le H(t)S(t) \le \frac{2\pi}{3}E(t). \tag{20}$$

In the second inequality we have used the monotonicity of H and S. In the third inequality we used (17). We combine (20) and (10) to obtain (11). This completes the proof of Theorem 1.

2.6. Outlook

Mathematical approaches based on bounds should be contrasted with the older approach that relies on exact solutions and linear stability. There is little doubt that the newer methods provide a more solid foundation for our understanding in regimes of physical interest. The most appealing aspect of the approach here is that it mimics back-of-the-envelope calculations that underlie many dynamic scaling laws, yet it is completely rigorous. The method is robust: for example, it applies with no change in higher dimensions, and simplifying assumptions such as uniform mobility are easily removed [10]. The basic argument used here was introduced by Kohn and Otto in the context of spinodal decomposition of alloys [7]. It has since been used to prove bounds on coarsening in epitaxial growth and thin fluid films. A key aspect is to find the right physical quantities to estimate, and the right notion of length scale (this can be quite sophisticated). There is also a closely related body of work on bounds for turbulent flows and combustion problems [2,3].

3. Mean-field models of clustering

3.1. The LSW model

In the example above, we analyzed dynamic scaling starting with the basic field equations. There are many situations where this is too complicated and simpler models are called for. For example, a fundamental problem in phase separation is to understand the growth of a solid phase in an undercooled liquid. This is described mathematically by a Stefan problem with surface tension at the boundary between phases. This formulation has a sound thermodynamic basis, and may be used to prove one-sided bounds on the coarsening rates [7]. However, a finer analysis is possible under simplifying assumptions. What follows is an outline of a basic mean-field model of this process, following [13].

In the late stages of solidification, the solid phase often consists of many spherical particles of different sizes. In this stage, mass is transported from small to large particles via a quasistatic diffusion in the liquid phase. Since we are most interested in the statistics of the particle sizes, it is reasonable to focus on $f(t, v)$, a function that describes the number of particles of volume v at time t. A mean-field description of the evolution of f is the following model derived independently by Lifshitz and Slyozov [8] and Wagner [16]. We assume all particles interact with the same diffusion field, so that the particle volume changes according to the equation:

$$\partial_t v = v^{1/3}\theta(t) - 1 := \Lambda(v, \theta(t)). \tag{21}$$

Here $\theta(t)$ is the same for all particles, and the critical particle volume is $\theta(t)^{-3} := 4\pi/3r_c^3$, where r_c is the critical radius. Particles with radius larger than r_c grow at the expense of those with radius smaller than r_c. The evolution of f is determined by conservation of mass:

$$\partial_t f(t, r) + \partial_v(\Lambda)(v, \theta(t))f) = 0. \tag{22}$$

Finally, $\theta(t)$ is determined by the self-consistency condition:

$$\theta(t) = \frac{\int_0^\infty f(t, v)\,\mathrm{d}v}{\int_0^\infty v^{1/3} f(t, v)\,\mathrm{d}v}. \tag{23}$$

Lifshitz, Slyozov and Wagner predicted that (a) the critical radius $r_c(t) \sim 4t/9$ as $t \to \infty$, (b) f approaches a self-similar form after rescaling by the critical volume, and (c) that this self-similar form is universal, has compact support, and may be computed explicitly. Observe that we are no longer satisfied with only a coarse relation such as $r_c(t) \sim 4t/9$, instead we also want to understand the 'universality of scaling' ((b) and (c)).

A short summary of the vast scientific literature on this model, is that predictions (a) and (b) have been experimentally verified, but (c) has not. The basic obstruction is that there is not just one 'universal' self-similar form with compact support, but in fact infinitely many self-similar solutions that differ delicately at the

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outer tip of their support. An interesting mathematical contribution to this problem was made by Niethammer and Pego [13]. Given initial data f_0 with compact support in $[0, v_0]$ they showed that if $f_0 \sim (v_0 - v)^p$ near v_0 for any $p < \infty$, then the convergence to the solution predicted by Lifshitz, Slyozov and Wagner is impossible. Their work makes two distinct contributions. The first is to make precise the obstructions to 'universality' implicit in many scientific papers on the LSW model. The second was to recognize the utility of methods from probability theory to study mean-field models such as (21)–(23). The analysis of the LSW model is quite tricky, and as often happens, the model where these probabilistic methods are most transparent was studied later. This is another fundamental model of clustering: *Smoluchowski's coagulation equation*. But first, here is a quick introduction to the basic method.

3.2. The analogy with the central limit theorem

The fundamental probabilistic result on 'universality' is the central limit theorem, and its extension to the stable laws discovered by Lévy. These theorems deal with fluctuations in the summation of independent, identically distributed random variables (for example, a coin-toss experiment). Let X_n be a sequence of independent, identically distributed random variables with mean zero. To be concrete, let us restrict attention to symmetric random variables, that is $P(X_n > a) = P(X_n < -a)$ for every $a \geq 0$. In a coin toss, we would have $X_n = \pm 1$ with probability 1/2. The central limit theorem asserts that if $E(X_n^2) = 1$ then the normalized sums $S_n = (X_1 + \cdots + X_n)/\sqrt{n}$ converge in distribution to the Gaussian with unit variance. This is simply the assertion that the characteristic function of $S_n/\sqrt{n}$ converges to that of the unit Gaussian, that is

$$\lim_{n\to\infty} E(\mathrm{e}^{\mathrm{i}ks_n/\sqrt{n}}) = \mathrm{e}^{-k^2/2}, \quad k \in \mathbb{R}. \tag{24}$$

This is proved as follows. Consider the characteristic function $\varphi(k) = E(\mathrm{e}^{\mathrm{i}kX_n})$. For example, if $X_n = \pm 1$ with probability 1/2, $\varphi(k) = \cos k$. The essence of universality is that the precise form of $\varphi(k)$ is of little relevance–what matters is that X_n has mean zero $E(X_n) = 0$) and bounded fluctuations $E(X_n^2) = 1$). This ensures $\varphi(k) \approx 1 - k^2/2$ as $k \to 0$. Since the X_n are iid we now have

$$E(\mathrm{e}^{\mathrm{i}ks_n/\sqrt{n}}) = \left(\varphi\left(\frac{k}{\sqrt{n}}\right)\right)^n \approx \left(1 - \frac{k^2}{2n}\right)^n \to \mathrm{e}^{-k^2/2}.$$

What if $E(X_n^2)$ is divergent? A non-trivial limit is still possible if we assume $\varphi(k) \sim 1 - |k|^\alpha$ as $k \to 0$ for some $\alpha \in (0, 2]$. We now obtain

$$\frac{S_n}{n^{1/\alpha}} = \frac{X_1 + \cdots + X_n}{n^{1/\alpha}} \to Y_\alpha, \tag{25}$$

where Y_α is the symmetric, stable distribution with exponent α. Y_α is defined completely by its characteristic function:

$$E(\mathrm{e}^{\mathrm{i}kY_\alpha}) = \mathrm{e}^{-|k|^\alpha}, \quad 0 < \alpha \leq 2. \tag{26}$$

The proof is simple. As before, the characteristic functions converge for fixed k as $n \to \infty$:

$$E(\mathrm{e}^{\mathrm{i}kn^{-1/\alpha}S_n}) = (\varphi(kn^{-1/\alpha}))^n \approx \left(1 - \frac{|k|^\alpha}{n}\right)^n \to \mathrm{e}^{-|k|^\alpha}.$$

To summarize, we see that (a) there is a one-parameter family of self-similar distributions, of which only one has finite variance; (b) the domains of attraction of the self-similar distributions are determined by the tails of the initial distribution (the behavior of the characteristic function near $k = 0$ is determined by the tails of the distribution of X_n). Let me now indicate how this method translates immediately to Smoluchowski's coagulation equation.

3.3. Universality in Smoluchowski's coagulation equation

The following coagulation equation was proposed by Smoluchowski to study the clustering of colloids in suspension. Assume the suspension is dilute so that only binary collisions occur, and that particles of volume v and w meet at a rate $K(v, w)$. We sum over the gain and loss of particles of size v in this process to obtain the rate equation:

$$\partial_t f(t, v) = \frac{1}{2}\int_0^v K(v - w, w) f(t, v - w) f(t, w)\,\mathrm{d}w - \int_0^\infty K(v, w) f(t, w) f(t, v)\,\mathrm{d}w. \tag{27}$$

All knowledge of the underlying mechanisms is contained in the rate kernel K. For example, for the coagulation of colloids by Brownian motion, Smoluchowski derived (after non-dimensionalization):

$$K(v, w) = (v^{1/3} + w^{1/3})(v^{-1/3} + w^{-1/3}). \tag{28}$$

Note that $v^{1/3}$ is the length scale of a spherical particle. Roughly speaking, the term $(v^{1/3} + w^{1/3})$ measures the collision cross-section, and $(v^{-1/3} + w^{-1/3})$ measures the rate of diffusion given by the Stokes–Einstein relation.

The simplicity of this model ensures that it has been used in a vast array of applications ranging from problems in physical chemistry (coagulation of colloids, formation of aerosols, kinetics of polymerization), environmental science (formation of smoke, dust and haze), astrophysics (gravitational clustering of asteroids, planets, stars and galaxies), computer science (random graphs, hashing algorithms) and mathematical biology (schooling of fishes). A good, if slightly dated, review is provided by Drake [4].

Most interesting kernels are homogeneous; that is, $K(av, aw) = a^\gamma K(v, w)$ for some fixed γ and every $a, v, w > 0$. This may be viewed as a modeling assumption needed to capture the dynamic scaling seen in experiments. Our goal is to deduce dynamic scaling, once this modeling assumption has been made. The most direct approach (largely worked out by van Dongen and Ernst) is to make the ansatz:

$$f(t, v) = \lambda(t)^{-2} f_*\left(\frac{v}{\lambda(t)}\right) \tag{29}$$

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in Eq. (27). This scaling is chosen so that solutions conserve mass (normalized initially to unity). That is

$$\int_0^\infty v f(t, v)\,\mathrm{d}v = \int_0^\infty v f_*(v)\,\mathrm{d}v = 1, \quad t > 0. \tag{30}$$

Mass-preservation holds only when there is no runaway growth caused by the interaction of large-large clusters (*gelation*). This is known to hold in the range $\gamma \le 1$ and we find an integral equation for f_* and an ordinary differential equation for $\lambda(t)$ with the solution $\lambda(t) = t^{1/(1-\gamma)}$, $\gamma < 1$ or $\lambda(t) = \mathrm{e}^{ct}$, $\gamma = 1$. Except for very special cases, we cannot solve for f_* explicitly. However, the folklore in the scientific literature (based on formal asymptotics and numerics) is that (a) f_* is unique, (b) f_* decays exponentially as $v \to \infty$, (c) f_* is universal, that is all solutions approach this distribution as $t \to \infty$ after suitable rescaling.

Let me contrast this folklore, with precise mathematical results for three *solvable kernels* $K(v, w) = 2, v + w$ and vw. To explain the ideas most simply, I will only consider $K(v, w) = 2$. This is a crude truncation of the kernel in (28), since $K(v, w) = 2 + v^{1/3}w^{-1/3} + v^{-1/3}w^{1/3}$. For these three kernels, one may use the Laplace transform to solve (27). If we define

$$\varphi(q, t) = \int_0^\infty (1 - \mathrm{e}^{-qv}) f(t, v)\,\mathrm{d}v, \tag{31}$$

we may use (27) to find for $K = 2$:

$$\partial_t \varphi = -\varphi^2 \tag{32}$$

with explicit solution:

$$\varphi(q, t) = \frac{\varphi(q, 0)}{1 + t\varphi(q, 0)}. \tag{33}$$

It is now easy to guess self-similar solutions of the form:

$$\varphi(q, t) = t^{-1}\frac{tq^\rho}{1 + tq^\rho} := t^{-1}\varphi_\rho(qt^{1/\rho}). \tag{34}$$

It turns out that these do define *positive* self-similar solutions to (32) for every $\rho \in (0, 1]$ which have the form:

$$f(t, v) = \lambda_\rho(t)^{-2} f_\rho\left(\frac{v}{\lambda(t)}\right), \quad \lambda_\rho(t) = t^{1/\rho}. \tag{35}$$

When $\rho = 1$, $\lambda(t) = t$ and $f_* = f_1(v) = \mathrm{e}^{-v}$ is the 'universal' profile with an exponential tail. However, for $0 < \rho < 1$ the solutions have algebraic ('fat') tails, and do not have finite mass.

The domains of attraction ('universality classes') of these self-similar solutions are characterized precisely as follows. Initial data f_0 is in the domain of attraction of f_ρ if and only if there is a function L, slowly varying at infinity, such that

$$\int_0^v w f_0(w)\,\mathrm{d}w \sim v^{1-\rho} L(v), \quad v \to \infty. \tag{36}$$

Roughly speaking, the assertion is that f_0 is in the domain of attraction of the fat-tailed solutions f_ρ, $0 < \rho < 1$ if and only if it has an algebraic tail that is almost identical to that of f_ρ. Typical examples of slowly varying functions are $\log v$, $\log\log v$, $(\log v)^2$ (i.e., all iterates and powers of $\log v$), so the term L should be thought of as a correction to the dominant power-law $v^{1-\rho}$. The endpoint $\rho = 1$ is special. In order to be attracted to solution f_1 it is sufficient that $\int_0^\infty v f_0(v)\,\mathrm{d}v < \infty$. Similar results hold for the other solvable kernels, and in all likelihood for all homogeneous kernels.

3.4. Outlook

To summarize, the folklore on uniqueness of the self-similar profile in Smoluchowski's coagulation equation is false. There is a one-parameter family of self-similar profiles with algebraic tails and their domains of attraction can be classified completely in terms of the tails of initial data. As in the LSW model, what is crucial for universality is that the distribution of large clusters should be suitably regular.

The basic probabilistic analogy is simply the starting point for a deeper analysis that exploit connections between such scaling dynamical systems and probabilistic methods [11,12]. At a fundamental level, both the addition of independent random variables, the LSW model and Smoluchowski's coagulation equation are mechanisms for transporting mass from small to large scales with no cut-off scales. Thus, the same degeneracies appear in both problems. The fat-tailed solutions are consequences of the modeling assumptions we make, and if these are considered pathological, we should view this as a modeling defect, to be resolved by the incorporation of additional scale-breaking effects such as small or large volume cut-offs not present in the original model.

References

[1] A.J. Bray, Theory of phase-ordering kinetics, Adv. Phys. 51 (2002) 481–587.
[2] P. Constantin, A. Kiselev, A. Oberman, L. Ryzhik, Bulk burning rate in passive-reactive diffusion, Arch. Ration. Mech. Anal. 154 (2000) 53–91.
[3] C.R. Doering, P. Constantin, Bounds for heat transport in a porous layer, J. Fluid Mech. 376 (1998) 263–296.
[4] R.L. Drake, A general mathematical survey of the coagulation equation, in: G.M. Hidy, J.R. Brock (Eds.), Topics in Current Aerosol Research, vol. 2, International reviews in Aerosol Physics and Chemistry, Pergammon, 1972, pp. 201–376.
[5] G.M. Homsy, Viscous fingering in porous media, Ann. Rev. Fluid Mech. 19 (1987) 271–311.
[6] L.N. Howarth, Bounds on flow quantities, Ann. Rev. Fluid. Mech. 4 (1972).
[7] R.V. Kohn, F. Otto, Upper bounds for coarsening rates, Commun. Math. Phys. 229 (2002) 375–395.
[8] I.M. Lifshitz, V.V. Slyozov, The kinetics of precipitation from supersaturated solid solutions, J. Phys. Chem. Solids 19 (1961) 35–50.
[9] G. Menon, F. Otto, Dynamic scaling in miscible viscous fingering, Commun. Math. Phys. 257 (2005) 303–317.
[10] G. Menon, F. Otto, Diffusive slowdown in miscible viscous fingering, Commun. Math. Sci. 4 (2006) 267–273.
[11] G. Menon, R.L. Pego, The scaling attractor and ultimate scaling dynamics for Smoluchowski's coagulation equations, September 2006. Online at http://www.arxiv.org/nlin.AO/0609026.
[12] G. Menon, R.L. Pego, Universality classes in Burgers turbulence. Commun. Math. Phys., in press.
[13] B. Niethammer, R.L. Pego, Non-self-similar behavior in the LSW theory of Ostwald ripening, J. Stat. Phys. 95 (1999) 867–902.
[14] R.L. Pego, Lectures on dynamics in models of coarsening and coagulation. Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, in press.

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[15] P.G. Saffman, G. Taylor, The penetration of a fluid into a porous medium or Hele–Shaw cell containing a more viscous liquid, Proc. Roy. Soc. London. Ser. A 245 (1958) 312–329 (2 plates).
[16] C. Wagner, Theorie der Alterung von Niederschldgen durc Umlosen, Z. Elektrochem. 65 (1961) 581–594.
[17] R.A. Wooding, Growth of fingers at an unstable diffusing interface in a porous medium or Hele–Shaw cell, J. Fluid Mech. 39 (1969) 477–495.
[18] Y. Yortsos, D. Salin, On the selection principle for viscous fingering in porous media, J. Fluid Mech. 557 (2006) 225–236.

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