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The Airy function is a Fredholm determinant

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Abstract Let G be the Green's function for the Airy operator

$$L\varphi := -\varphi'' + x\varphi, \quad 0 < x < \infty, \quad \varphi(0) = 0.$$

We show that the integral operator defined by G is Hilbert-Schmidt and that the 2-modified Fredholm determinant

$$\det_2(1 + zG) = \frac{\mathrm{Ai}(z)}{\mathrm{Ai}(0)}, \quad z \in \mathbb{C}.$$

Keywords Airy function · Fredholm determinant · Hilbert-Schmidt operators

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1 Introduction

Let L denote the Airy operator on the half-line $\mathbb{R}_+$ with Dirichlet boundary condition

$$L\varphi := -\varphi'' + x\varphi, \quad 0 < x < \infty, \quad \varphi(0) = 0. \tag{1}$$

Recall that the differential equation $\varphi'' = x\varphi$ admits two linearly independent solutions, denoted Ai and Bi [1]. As $x \to \infty$, $\mathrm{Ai}(x)$ decays, and $\mathrm{Bi}(x)$ grows, at the super-exponential rate $x^{-1/4}e^{-2x^{3/2}/3}$. For $z \in \mathbb{C}$, both $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are entire functions of exponential type $3/2$. The zeros of Ai lie on the negative real axis and are denoted

$$-\infty < \ldots < -a_n < -a_{n-1} < \ldots < -a_1 < 0. \tag{2}$$

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For each n, $\mathrm{Ai}(x-a_n)$ is an eigenfunction of L with eigenvalue a_n. Further, these eigenfunctions form a complete basis for $L^2(\mathbb{R}_+)$ (see e.g. [7, §4.12]).

The spectral problem for L may also be approached via the Hilbert-Schmidt theory of integral equations. Let $f_\pm$ denote linearly independent solutions to the Airy equation $y''-xy=0$, with $f_+(x)=\mathrm{Ai}(x)$, and

$$f_-(x)=\frac{\pi}{\mathrm{Ai}(0)}\left(\mathrm{Ai}(0)\mathrm{Bi}(x)-\mathrm{Ai}(x)\mathrm{Bi}(0)\right). \tag{3}$$

Then $f_-(0)=0$ and

$$W(f_+,f_-):=f_+(x)f_-'(x)-f_-(x)f_+'(x)=1. \tag{4}$$

The Green's function for L is

$$G(x,y)=f_-(\min(x,y))f_+(\max(x,y)), \quad x,y\in\mathbb{R}_+, \tag{5}$$

and the integral equation

$$\varphi(x)=z\int_0^\infty G(x,y)\varphi(y)\,dy \tag{6}$$

is equivalent to the spectral problem

$$L\varphi=z\varphi, \quad z\in\mathbb{C}. \tag{7}$$

Theorem 1 *The kernel G defines a Hilbert-Schmidt operator (also denoted G) on $L^2(\mathbb{R}_+)$. The 2-modified Fredholm determinant*

$$\det{}_2(1+zG)=\frac{\mathrm{Ai}(z)}{\mathrm{Ai}(0)}=3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)\mathrm{Ai}(z), \quad z\in\mathbb{C}. \tag{8}$$

Let L_α, $\alpha\in(-a_1,\infty)$, denote the Airy operator on $[\alpha,\infty)$ with Dirichlet boundary condition at α, and let G_α be the associated Green's function analogous to (5).

Corollary 1

$$\det{}_2(1+zG_\alpha)=\frac{\mathrm{Ai}(z+\alpha)}{\mathrm{Ai}(\alpha)}, \quad z\in\mathbb{C}. \tag{9}$$

The point here is the explicit expression for $\det_2(1+zG)$. The asymptotics of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ imply that G is Hilbert-Schmidt, but not trace class, on $L^2(\mathbb{R}_+)$. As a consequence, the 2-modified Fredholm determinant $\det_2(1+zG)$ exists, and is an entire function whose zeros coincide with minus the eigenvalues of L. However, this only tells us that $\det_2(1+zG)/\mathrm{Ai}(z)$ is an entire function with no zeros, not that it is a constant. Thus, (8) is not obvious.

Much of the current interest in Airy functions and Fredholm determinants stems from their importance in probability theory; for example, in the description of the Tracy-Widom distribution [8], and of Brownian motion with parabolic drift [4]. Theorem 1 was motivated by a surprising formula in Burgers turbulence [5, equation (19)]. While these probabilistic connections will be presented elsewhere, the identity (8) strikes me as a calculation of independent interest. In an area this classical, the odds that it is new would seem to be rather low. However, I could not find (8) in the literature, even after a thorough search. Thus, Theorem 1, if not new, is certainly hard to find, and I hope the reader will find some value in this exposition. It is rare that one can compute Fredholm determinants explicitly, and despite the fact that there are no new methods here, the calculation is quite pleasing.

It is a pleasure to dedicate this article to John Mallet-Paret, in respect for his deep knowledge of differential equations and functional analysis.

2 Proof of Theorem 1

2.1 A remark on trace ideals

Recall that the notion of a Fredholm determinant may be generalized to the trace ideals $\mathcal{I}_p$, $1 \leq p \leq \infty$ [6]. The trace-class operators form the ideal $\mathcal{I}_1$ and the Hilbert-Schmidt operators form $\mathcal{I}_2$. The 2-modified determinant for Hilbert-Schmidt operators is defined by extending the following formula

$$\det_2(1+A) = \det_1(1+A)e^{-\mathrm{Tr}(A)}, \quad A \in \mathcal{I}_1, \tag{10}$$

to operators in $\mathcal{I}_2$. The factor $e^{-\mathrm{Tr}(A)}$ provides a suitable "renormalization" of the divergent factor $\mathrm{Tr}(A)$ when $A \in \mathcal{I}_2$ is not trace-class [6, Ch. 9].

The proof of Theorem 1 proceeds as follows. First, we show that $G \in \mathcal{I}_2$. Next we approximate G by a sequence of trace-class operators G^b obtained by restricting the Airy operator to the interval $[0, b]$. The Fredholm determinant $\det_1(1 + zG^b)$ may be computed by the methods of the Gohberg-Krein school. Finally, the identity (8) is obtained by applying (10) with $A = zG^b$ and passage to the limit $b \to \infty$.

2.2 The operator G is Hilbert-Schmidt

In all that follows, the asymptotics of $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ as $z \to \infty$ play an important role. As $z \to \infty$ in the sector $|\arg(z)| < \pi/3$ we have [1]

$$\mathrm{Ai}(z) \sim \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}}, \quad \mathrm{Bi}(z) \sim \frac{e^{\zeta}}{\sqrt{\pi}z^{1/4}}, \quad \zeta = \frac{2}{3}z^{3/2}. \tag{11}$$

These asymptotics imply rather easily that G is Hilbert-Schmidt, but not trace-class. If G were trace-class, its trace would be

$$\int_0^\infty G(x,x)\,dx = \int_0^\infty f_-(x)f_+(x)\,dx. \tag{12}$$

Since $\int_0^\infty \mathrm{Ai}^2(s)\,ds < \infty$, it is clear that $\int_0^\infty G(x,x)\,dx$ is finite if and only if $\int_0^\infty \mathrm{Ai}(x)\mathrm{Bi}(x)\,dx < \infty$. But as a consequence of (11)

$$\mathrm{Ai}(x)\mathrm{Bi}(x) \sim \frac{1}{2\pi\sqrt{x}}, \quad x \to \infty, \tag{13}$$

and $\mathrm{Ai}(x)\mathrm{Bi}(x)$ is not summable. Thus, G is not trace-class.

However, a similar calculation shows that G is Hilbert-Schmidt. By definition,

$$\|G\|^2_{\mathrm{HS}(L^2(\mathbb{R}_+))} = \int_0^\infty \int_0^\infty G^2(x,y)\,dx\,dy.$$

In order to show that the above integral is finite, it suffices to show that

$$\int_0^\infty \int_0^\infty \left(\mathrm{Ai}(x)\mathrm{Bi}(y)\mathbf{1}_{y<x} + \mathrm{Bi}(x)\mathrm{Ai}(y)\mathbf{1}_{x<y}\right)^2 \, dx\, dy \tag{14}$$
$$= \int_0^\infty \mathrm{Ai}^2(x) \int_0^x \mathrm{Bi}^2(y)\, dy + \int_0^\infty \mathrm{Bi}^2(x) \int_x^\infty \mathrm{Ai}^2(y)\, dy < \infty.$$

The asymptotic formulae (11) imply that these integrals are finite if and only if

$$\int_0^\infty \frac{e^{-4x^{3/2}/3}}{\sqrt{x}} \int_0^x \frac{e^{4y^{3/2}/3}}{\sqrt{y}}\, dy < \infty \quad \text{and} \quad \int_0^\infty \frac{e^{4x^{3/2}/3}}{\sqrt{x}} \int_x^\infty \frac{e^{-4y^{3/2}/3}}{\sqrt{y}}\, dy < \infty. \tag{15}$$

This is easily checked by making the change of variables $x^{3/2} = u$, $y^{3/2} = v$. (A similar calculation is presented in section 2.5).

2.3 The Fredholm determinant on a finite interval $[0, b]$, $b < \infty$

Fix $0 < b < \infty$ and consider the operator L^b defined by

$$L^b\varphi := -\varphi'' + x\varphi, \quad 0 < x < b, \quad \varphi(0) = 0, \varphi(b) = 0. \tag{16}$$

The Green's function for L^b is given by

$$G^b(x,y) = f_-(\min(x,y))f_+^b(\max(x,y)) \tag{17}$$

where $f_-(x)$ denotes the solution to $-\varphi''+x\varphi = 0$ given by (3) (it does not depend on b), and $f_+^b(x)$ is the solution that satisfies the boundary condition $f_+^b(b) = 0$, as well as the normalization $W(f_-, f^b) = 1$. We find

$$f_+^b(x) = c_b\left(\mathrm{Ai}(x) - \mathrm{Bi}(x)\frac{\mathrm{Ai}(b)}{\mathrm{Bi}(b)}\right), \quad c_b = \frac{1}{1 + \frac{\mathrm{Ai}(b)\mathrm{Bi}(0)}{\mathrm{Bi}(b)\mathrm{Ai}(0)}}. \tag{18}$$

Both f_- and f_+^b lie in $L^2(0,b)$ and G^b is trace-class. Its Fredholm determinant may be computed explicitly.

Proposition 1

$$\det{}_1(1 + zG^b) = \tag{19}$$
$$1 - \frac{z\pi}{\mathrm{Ai}(0)} \int_0^b \left(\mathrm{Ai}(x) - \mathrm{Bi}(x)\frac{\mathrm{Ai}(b)}{\mathrm{Bi}(b)}\right)\left(\mathrm{Bi}(z)\mathrm{Ai}(x+z) - \mathrm{Ai}(z)\mathrm{Bi}(x+z)\right)\, dx.$$

Proof There are two steps in the proof.

1. Theorem 3.1 [3, Ch. IX] yields the following formula for the Fredholm determinant:

$$\det{}_1(1 + zG^b) = U_{22}(b,z), \tag{20}$$

where the 2×2 matrix $U(x,z)$ is the fundamental solution to the canonical system

$$\frac{\partial U}{\partial x} = -zJA(x)U, \quad U(0,z) = I, \tag{21}$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A(x) = \begin{pmatrix} f_+^b(x) \\ f_-(x) \end{pmatrix} \begin{pmatrix} f_+^b(x) & f_-(x) \end{pmatrix}. \tag{22}$$

2. Canonical systems such as (21) may be solved as follows (we follow [2]). Let $\hat{f^b}_+(x,z)$ and $\hat{f}_-(x,z)$ denote the solutions to the Volterra equations

$$\begin{aligned} \hat{f^b}_+(x,z) &= f_+^b(x) - z\int_0^x G^b(x,y)\hat{f^b}_+(y,z)\,dy, \\ \hat{f}_-(x,z) &= f_-(x) - z\int_0^x G^b(x,y)\hat{f}_-(y,z)\,dy. \end{aligned} \tag{23}$$

We may then check by substitution that

$$U(x,z) = I - z\begin{pmatrix} \int_0^x f_-(y)\hat{f^b}_+(y,z)\,dy & \int_0^x f_-(y)\hat{f}_-(y,z)\,dy \\ -\int_0^x f_+(y)\hat{f^b}_+(y,z)\,dy & -\int_0^x f_+^b(y)\hat{f}_-(y,z)\,dy \end{pmatrix} \tag{24}$$

is the unique solution to (21).

The calculation so far applies to all semi-separable kernels. Since G^b is the Green's function of L^b, we may go further. We apply L^b to (23) to find that

$$L^b\hat{f^b}_+ = -z\hat{f^b}_+, \quad L^b\hat{f}_- = -z\hat{f}_-. \tag{25}$$

Thus, $\hat{f^b}$ and $\hat{f}_-$ are linear combinations of $\mathrm{Ai}(x+z)$ and $\mathrm{Bi}(x+z)$. Moreover, the initial conditions on $\hat{f^b}_+$ and $\hat{f}_-$ are determined by

$$\begin{pmatrix} \hat{f^b}_+(0) \\ \hat{f^b}'_+(0) \end{pmatrix} = \begin{pmatrix} f_+^b(0) \\ f_+^{b\,\prime}(0) \end{pmatrix}, \quad \begin{pmatrix} \hat{f}_-(0) \\ \hat{f}'_-(0) \end{pmatrix} = \begin{pmatrix} f_-(0) \\ f'_-(0) \end{pmatrix}. \tag{26}$$

We solve for $f_\pm^b$ and substitute in (24) to find that $U_{22}(b,z)$ is given by (19).

2.4 The limit $b \to \infty$ and the proof of Theorem 1

The kernel G^b may be trivially extended to an integral operator on $L^2(R)$ by setting $G^b(x,y) = 0$ if either x or y is greater than b. We abuse notation and continue to denote this extension by G^b. The proof of Theorem 1 now rests on the following assertions.

Lemma 1

$$\lim_{b\to\infty} \det{}_1(1+zG^b)e^{-z\mathrm{Tr}G^b} = \frac{\mathrm{Ai}(z)}{\mathrm{Ai}(0)}, \quad \mathrm{Re}(z) > 0. \tag{27}$$

Lemma 2

$$\lim_{b\to\infty} \|G^b - G\|_{\mathrm{HS}(\mathrm{L}^2(\mathbb{R}_+))} = 0. \tag{28}$$

Proof (of Theorem 1) For each z in the right-half plane

$$\frac{\mathrm{Ai}(z)}{\mathrm{Ai}(0)} = \lim_{b\to\infty} \det{}_1(1+zG^b)e^{-z\mathrm{Tr}G^b} = \lim_{b\to\infty} \det{}_2(1+zG^b) = \det{}_2(1+zG).$$

The first equality follows from Lemma 1, the second equality from Lemma 2. This established the identity (8) for $\mathrm{Re}(z) > 0$. Since $\det_2(1+zG)$ is an entire function of z, the identity holds for all $z \in \mathbb{C}$.

2.5 Proof of Lemma 1

Lemma 3

$$\lim_{b\to\infty} \frac{\mathrm{Tr}(G^b)}{\sqrt{b}} = 1. \tag{29}$$

Proof We must consider the asymptotics of the integral $\mathrm{Tr}(G^b) = \int_0^b f_-(x) f^b(x)\,dx$. It is clear from formulas (3) and (18) for f_- and f_+^b that the dominant term in this limit must involve $\mathrm{Bi}(x)$ as $b \to \infty$. Moreover, $c_b \to 1$. Thus,

$$\mathrm{Tr}(G^b) \sim \pi \int_0^b \left(\mathrm{Ai}(x) - \mathrm{Bi}(x)\frac{\mathrm{Ai}(b)}{\mathrm{Bi}(b)} \right) \mathrm{Bi}(x).$$

Equation (13) shows that the integral $\int_0^b \mathrm{Ai}(x)\mathrm{Bi}(x)$ is divergent, whereas as we show below, as a consequence of (11) and Lemma 4

$$\lim_{b\to\infty} \frac{\mathrm{Ai}(b)}{\mathrm{Bi}(b)} \int_0^b \mathrm{Bi}^2(x)\,dx = 0. \tag{30}$$

Hence we find

$$\lim_{b\to\infty} \frac{\mathrm{Tr}(G^b)}{\sqrt{b}} = \lim_{b\to\infty} \frac{\pi}{\sqrt{b}} \int_0^b \mathrm{Ai}(x)\mathrm{Bi}(x)\,dx = 1. \tag{31}$$

Proof (of Lemma 1) Fix z such that $\mathrm{Re}(z) > 0$. Lemma 3 allows us to ignore all the terms in (19) that remain finite as $b \to \infty$, since these are weighted by the decaying factor $e^{-z\sqrt{b}}$ as $b \to \infty$. An inspection of the terms in (19), equation (30) and Lemma 3 imply that

$$\lim_{b\to\infty} \det{}_2(1 + zG^b) = \frac{\pi z \mathrm{Ai}(z)}{\mathrm{Ai}(0)} \lim_{b\to\infty} e^{-z\sqrt{b}} \int_0^b \mathrm{Ai}(x)\mathrm{Bi}(x+z)\,dx. \tag{32}$$

As $x \to \infty$, equation (11) yields the leading order asymptotics

$$\mathrm{Ai}(x)\mathrm{Bi}(x+z) \sim \frac{1}{2\pi\sqrt{x}} e^{z\sqrt{x}}.$$

Thus, to leading order

$$\int_0^b \mathrm{Ai}(x)\mathrm{Bi}(x+z)\,dx \sim \frac{1}{2\pi} \int_0^b \frac{e^{b\sqrt{x}}}{\sqrt{x}}\,dx = \frac{e^{z\sqrt{b}}}{\pi z},$$

which combines with the right hand side of (32) to yield (27). These asymptotics can be rigorously justified without much effort (for example, by rescaling so that the integrals are over a fixed domain $[0, 1]$ and using the dominated convergence theorem to justify taking pointwise limits).

2.6 Proof of Lemma 2

Since G^b vanishes when $x > b$ or $y > b$ we have

$$\|G^b - G\|_{\mathrm{HS}}^2 = \int_0^b \int_0^b \left(G^b(x,y) - G(x,y)\right)^2 \, dx\, dy + \|G\|_{\mathrm{HS}}^2 - \int_0^b \int_0^b G^2(x,y)\, dx\, dy.$$

Since $\|G\|_{HS} < \infty$,

$$\lim_{b\to\infty} \|G\|_{\mathrm{HS}}^2 - \int_0^b \int_0^b G^2(x,y)\, dx\, dy = 0.$$

Therefore, to prove Lemma 2, it is enough to show that

$$\lim_{b\to\infty} \int_0^b \int_0^b \left(G^b(x,y) - G(x,y)\right)^2 \, dx\, dy = 0. \tag{33}$$

We use the definition of G and G^b in equations (5) and (17) to obtain

$$G^b(x,y) - G(x,y) = f_-(\min(x,y)) \left(f_+^b(\max(x,y) - f_+(\max(x,y)\right).$$

We use the definition of f_+^b in (18) and recall that $f_+(s) = \mathrm{Ai}(s)$ to obtain

$$\left(f_+^b(s) - f_+(s)\right)^2 = (1-c_b)^2 \mathrm{Ai}^2(s) + 2c_b(1-c_b)\frac{\mathrm{Ai}(b)}{\mathrm{Bi}(b)}\mathrm{Bi}(s) + c_b^2 \frac{\mathrm{Ai}^2(b)}{\mathrm{Bi}^2(b)}\mathrm{Bi}^2(s).$$

We must show that the contribution of each of these terms vanishes in the limit. Since $c_b \to 1$ it is easy to see that the first term gives a vanishing contribution. The calculation for the second and third term is similar, and we present the calculation only for the third term

$$\frac{\mathrm{Ai}^2(b)}{\mathrm{Bi}^2(b)} \int_0^b \int_0^b f_-^2(x,y)\mathrm{Bi}^2(\max(x,y))\, dx\, dy.$$

Since f_- is a linear combination of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, and $\mathrm{Ai}(x)$ decays fast, it will suffice to show that the "worst" term

$$\frac{\mathrm{Ai}^2(b)}{\mathrm{Bi}^2(b)} \int_0^b \int_0^b \mathrm{Bi}^2(\min(x,y))\mathrm{Bi}^2(\max(x,y))\, dx\, dy = \frac{\mathrm{Ai}^2(b)}{\mathrm{Bi}^2(b)} \left(\int_0^b \mathrm{Bi}^2(x)\, dx\right)^2$$

vanishes in the limit. Applying the asymptotic relations (11), we see that we must prove that

$$\lim_{b\to\infty} \frac{\mathrm{Ai}(b)}{\mathrm{Bi}(b)} \int_0^b \mathrm{Bi}^2(x)\, dx = \lim_{b\to\infty} e^{-4b^{2/3}/3} \int_0^b x^{-1/2} e^{4x^{2/3}/3}\, dx = 0.$$

We define $u = 4x^{2/3}/3$ and note that the limit above is (upto a constant)

$$\lim_{b\to\infty} e^{-4b^{2/3}/3} \int_0^{4b^{2/3}/3} u^{-2/3} e^u \, du,$$

which vanishes by the following lemma.

Lemma 4 *Fix $0 < \alpha < 1$. Then*

$$\lim_{M\to\infty} e^{-M} \int_0^M u^{-\alpha} e^u \, du = 0. \tag{34}$$

Proof Fix $a > 0$. We separate the integral into an integral over two intervals: $(0, M-a)$ and $(M-a, M)$. First,

$$e^{-M} \int_0^{M-a} u^{-\alpha} e^u \, du \leq \frac{1}{1-\alpha} e^{-a} M^{1-\alpha}.$$

Similarly,

$$e^{-M} \int_{M-a}^{M} u^{-\alpha} e^u \, du \leq \frac{a}{(M-a)^\alpha}.$$

We now choose $a(M) \to \infty$ in a such a way that $e^{-a} M^{1-\alpha} \to 0$ and $aM^{-\alpha} \to 0$ ($a = M^{\alpha/2}$ will do).

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