Solutions to HW 1

Problem 1. It is enough to suppose f is C^1. Since $|f'(x_*)| < 1$ there is an interval J containing x_* such that $\sup_{x\in J} |f'(x)| = m < 1$. We then have

$$|x_{n+1} - x_*| = |f(x_n) - f(x_*)| \leq m\, |x_n - x_*| \, .$$

Thus, every orbit with initial condition $x_0 \in J$ is contained in J and approaches x_* at least as fast as $|x_n - x_*| \leq m^n |x_0 - x_*|$. A similar argument also shows that x_* is repelling if $|f'(x_*)| > 1$. ☐

Problem 2. We may treat the logistic map as a dynamical system on $\mathbb{R}$. The fixed points solve $x = rx(1-x)$ and are given by $x = 0$ and $x = 1-1/r$. The two fixed points coalesce at $r = 1$, and exchange stability (eg. $f'(0) = r$, thus it is attracting for $r < 1$ and repelling for $r > 1$). For $r > 1$ both fixed points are contained within $[0, 1]$. It is easy to see that this is a transcritical bifurcation by plotting the graphs of $y = rx(1-x)$ for r near 1 and the line $y = x$ and looking for intersections. ☐

Problem 3. The linearization at the fixed point $x = 1-1/r$ is $f'(x) = 2-r$. Thus it is stable in the range $1 < r < 3$ and has characteristic multiplier -1 at $r = 3$. This suggests a period-doubling bifurcation. To verify this, we compute the period-2 orbits. These solve the fixed point equation $x = f^2(x)$, that is

$$x = r^2 x(1-x)\,(1 - rx(1-x))\, .$$

To obtain the roots of this quartic equation, note that the fixed points from Problem 2 ($x = f(x)$), are also solutions to $x = f^2(x)$. These may be factored out by long-division, and we obtain a quadratic equation with a pair of roots

$$p, q = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}.$$

The roots are real for $r > 3$.

In order to save some effort in computing the second bifurcation point proceed as follows. We have $f(p) = q$, $f(q) = p$. Thus,

$$(f^2)'(p) = f'(p)f'(q) = r^2(1-2p)(1-2q) = r^2(1 - 2(p+q) + 4pq).$$

Since p and q solve the quadratic above,

$$p + q = \frac{r+1}{r}, \quad pq = \frac{r+1}{r^2}.$$

Thus, $(f^2)'(p) = 4 + 2r - r^2$. The region of stability is $-1 < 4 + 2r - r^2 < 1$. The solution of this quadratic inequality gives $3 < r < 1 + \sqrt{6}$. ☐

Problem 4. To find R_0 we must find r such that $f(x) = x$ and $f'(x) = 0$. This yields $x = 1 - 1/r$ and $1 - 2x = 0$. Therefore, $R_0 = 2$. Similarly, to find R_1 we must find R_1 such that $(f^2)'(p) = 0$. Using Problem 3 we find $R_1 = 1 + \sqrt{5}$. $\square$

Problem 5. The distance between the images of two points x, y is $|f(x) - f(y)|$. In particular, if $x = 1/2$ is the critical point, then $|f(x) - f(y)| \approx 4r(x-y)^2$. Thus, points close to the critical point get mapped very close to one another. This shows up as a high density of points on the orbit diagram. The curves are the images $f^k(1/2, r)$. Compare Figure 0.1 with the orbit diagram. $\square$

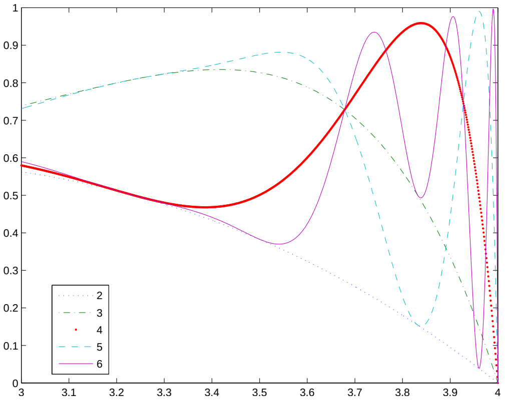

Figure 0.1: Plot of $f^k(1/2)$, k = 2,3, 4, 5,6

Problem 6. We must solve the equation $g(x) = -\alpha g^2(x/\alpha)$. Substitute $g(x) = 1 + c_2x^2$ on both sides to obtain

$$1 + c_2x^2 = -\alpha\left(1 + c_2 g(\frac{x}{\alpha})\right)^2 = -\alpha\left(1 + c_2\left(1 + 2c_2\frac{x^2}{\alpha^2} + c_2^2\frac{x^4}{\alpha^4}\right)\right).$$

Of course, the equality doesn't make sense. The approximation is to neglect the x^4 term. This yields

$$1 = -\alpha(1 + c_2), \quad c_2 = -\frac{2c_2^2}{\alpha}.$$

Thus, α satisfies the quadratic equation $\alpha^2 - 2\alpha - 2 = 0$ with positive root $1 + \sqrt{3}$. $\square$