1 Introduction

We shall consider continuous maps f of an interval I into itself and the discrete dynamical system

$$x_{n+1} = f(x_n), \quad n \geq 0. \tag{1.1}$$

We apply (1.1) to an initial condition x_0 to generate its *orbit*, the sequence $\{x_0, x_1, x_2, \ldots\}$. A convenient way to visualize the orbit is to draw the *cobweb diagram*, bouncing between the graph of f and the line $y = x$ (see the figures below). The orbit is generated by repeated composition of functions. We denote k-fold composition by f^k, that is $f^2(x) = f(f(x))$,$f^3(x) = f(f(f(x)))$ etc. A *fixed point* or *equilibrium* is a solution to

$$x = f(x). \tag{1.2}$$

A *periodic orbit* with minimal or prime period p is an orbit such that $x_k = x_{k \bmod p}$, and $x_0, x_1, \ldots, x_{p-1}$ are distinct. Clearly, $f^p(x_k) = x_k$, $0 \leq k \leq p-1$. Graphically, fixed points are given by the intersection between the graph of f and the line $y = x$. An intersection between the graph of f^p and the line $y = x$ give points with period not greater than p.

We typically study questions of asymptotic behavior, such as convergence to a fixed point or periodic orbit. Let us note at the outset that the dynamics are trivial if f is monotone and I is bounded. In this case, the orbit is a bounded monotone sequence, and every initial condition converges to a fixed point. We thus consider unimodal maps (ie. maps with a single hump). The fundamental example is the family of *logistic maps*

$$f(x) = rx(1-x), \quad r \in [0, 4]. \tag{1.3}$$

f takes a unique maximum value $r/4$ at the critical point $x = 1/2$, thus the restriction to $r \leq 4$. The orbit diagram of the logistic map has become an icon for chaos (see Figure 1.1). This is a numerically generated picture that describes the asymptotic behavior of a typical initial condition as r is varied. The complexity of the orbit diagram for this simple example is striking.

1.1 Stability and bifurcations

Linear maps $f(x) = ax$ admit the exact solution $x_n = a^n x_0$. The origin is an attracting fixed point if $|a| < 1$ and repelling if $|a| > 1$. Observe that if $a < 0$ the dynamics are oscillatory, that is x_k and x_{k+1} have opposite sign. In particular, there are two distinct forms of neutral stability: if $a = 1$

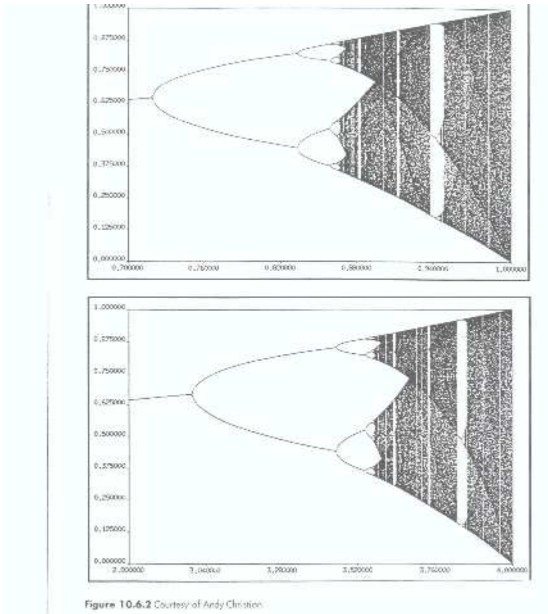

Figure 1.1: Orbit diagrams for the logistic and sine maps

every point is fixed, and if $a = -1$ every orbit is of period 2. Linear maps are used to describe the stability of fixed points. If x_* is a fixed point, the *characteristic multiplier* at x_* is $f'(x_*)$. If $|f'(x_*)| \neq 1$, the fixed point is *hyperbolic* and the characteristic multiplies determines stability (Exercise 1). We may expect bifurcations when $|f'(x_*)| = 1$.

In the logistic family, the fixed point $x_*(r) = 1 - 1/r$ loses stability to a period-2 orbit in a period-doubling bifurcation at $r = 3$. This is illustrated graphically in Figures 1.2– 1.4. When $r < 3$ trajectories spiral into the fixed point. The rate of approach is very slow as r approaches 2 (as is seen from the density of orbits in Figure 1.3). When $r > 3$ the fixed point has lost stability to period-2 orbit. This corresponds to new fixed points of f^2 seen in the lower-half of Figure 1.4.

The period-2 orbit then loses stability in another period-doubling bifurcation at $r = 1 + \sqrt{6}$ (Exercise 2). Trajectories when r is just below this bifurcation point are illustrated in Figure 1.5.

These are the first two steps in a *period-doubling cascade* at parameter values r_n. At this value, an orbit with period 2^{n-1} loses stability to an orbit with period 2^n in a period-doubling bifurcation. The first few terms in the

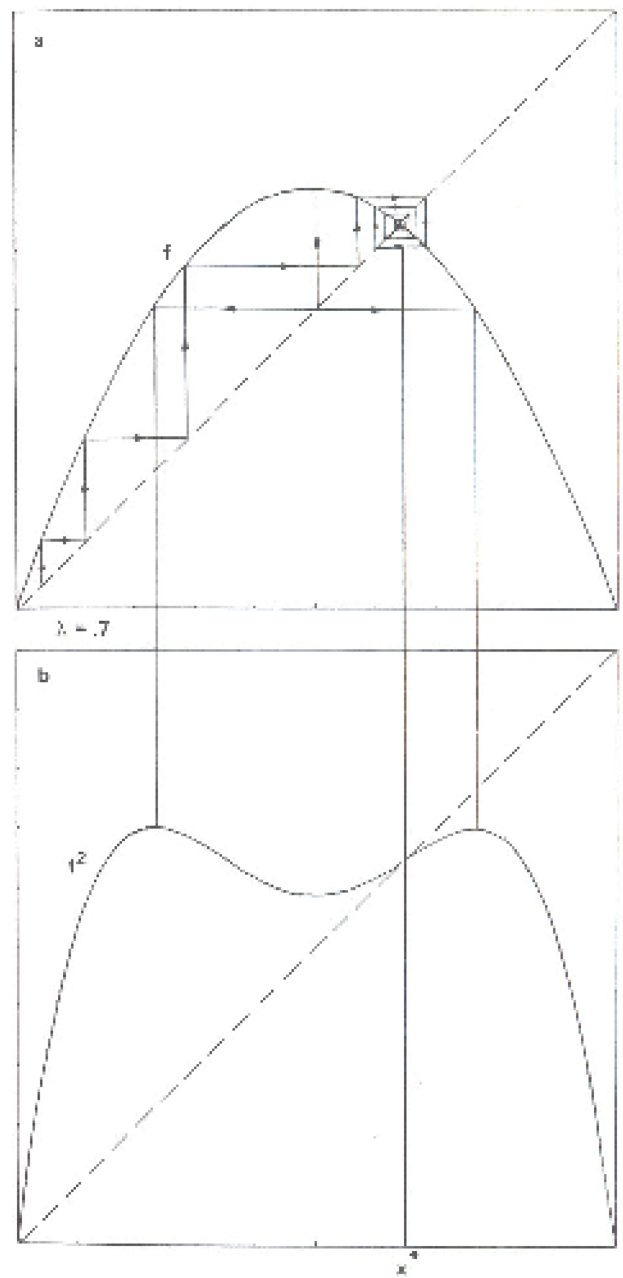

Figure 1.2: A stable fixed point with $r < 3$.

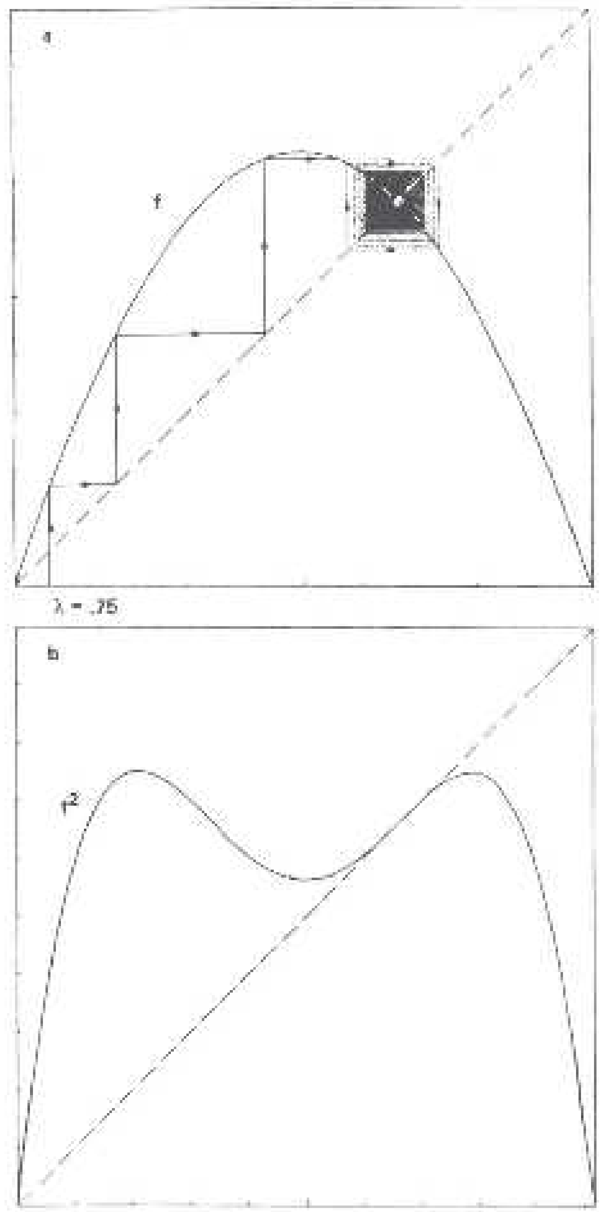

Figure 1.3: Slow approach to the fixed point as $r \to 3$.

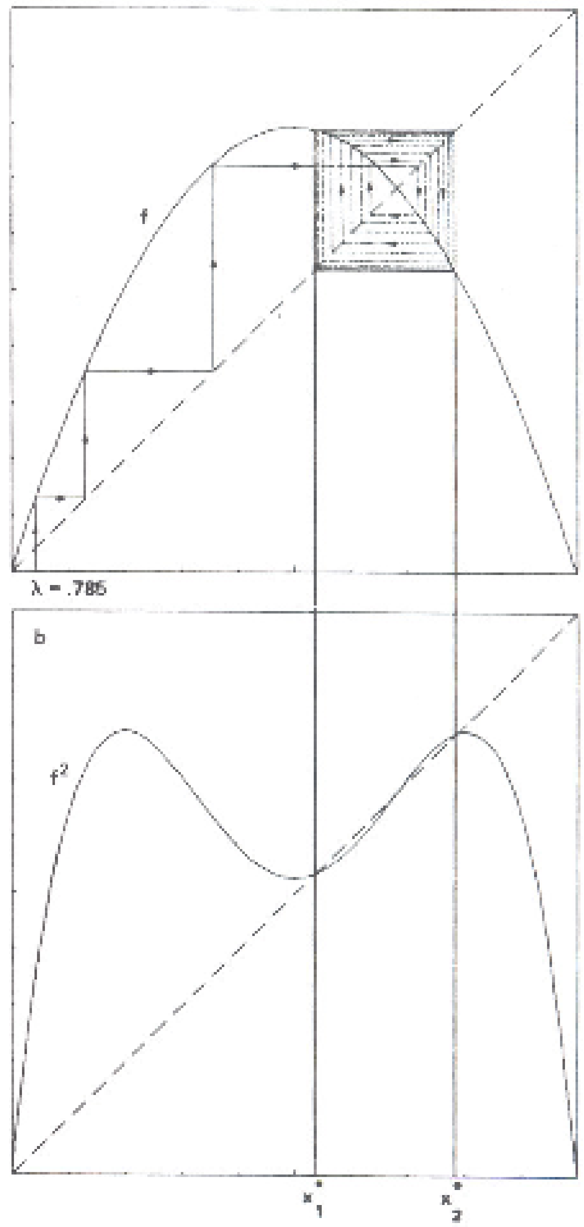

Figure 1.4: Birth of a stable period-2 orbit for $r > 3$.

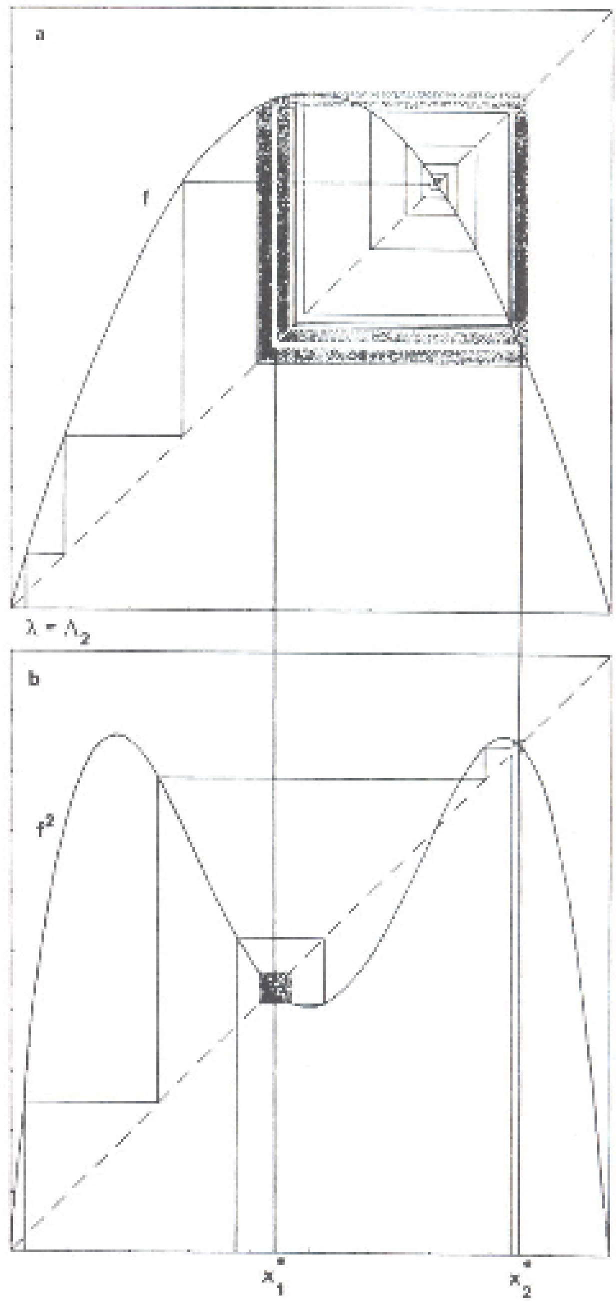

Figure 1.5: Loss of stability of period-2 orbit near $r = r_1$.

increasing sequence r_n are

$$3, 1+\sqrt{6}, \quad 3.54409\ldots, \quad 3.5644\ldots, \quad 3.568759, \ldots$$

The bifurcation values accumulate at $r_\infty = 3.566946\ldots$. The orbit diagram of the logistic map include windows of chaos and chaotic bands beyond r_∞.

1.2 Universality and renormalization

The most remarkable feature of the orbit diagram of the logistic is that its essential features depend only on 'minimal' properties. That is all families of the form $rf_1(x)$ where $f_1 : I \to I$ is unimodal with a nondegenerate maximum have similar orbit diagrams. Roughly speaking, this is what is meant by *universality.*

To illustrate this point, Figure 1.1 compares the orbit diagram for the sine family ($f(x) = \tilde{r}\sin \pi x$, $\tilde{r} \in [0,1]$) and the logistic map. In both orbit diagrams we see the same period-doubling cascade, and similar windows of order and chaos. Metropolis, Stein and Stein discovered that the ordering of these periodic orbits depends only on the fact that f is unimodal and continuous. This is an example of *qualitative* (or combinatorial) universality.

A more dramatic *quantitative* feature is the following. Let r_n and $\tilde{r}_n$ denote the values of the period-doubling bifurcations for the logistic and sine map respectively. Numerical experiments reveal the amazing fact that

$$\lim_{n\to\infty} \frac{r_{n+1} - r_n}{r_n - r_{n-1}} = \lim_{n\to\infty} \frac{\tilde{r}_{n+1} - \tilde{r}_n}{\tilde{r}_n - \tilde{r}_{n-1}} = \delta = 4.669201609102290\ldots \tag{1.4}$$

The number δ is known as *Feigenbaum's constant* in honor of his penetrating analysis of this quantitative universality.

The central feature of Feigenbaum's analysis is the notion of *renormalization*. To explain this idea, we consider a family of maps $f(r,x)$ undergoing a cascade of period-doubling bifurcation at the values r_n. We assume $f(r,\cdot)$ maps the interval $I = [-1,1]$ into itself, and has a critical point at 0. The characteristic multiplier of the orbit with period 2^n decreases from 1 at r_n to -1 at r_{n+1}. We focus on the *superstable* orbit with characteristic multiplier 0 at the value $R_n \in (r_n, r_{n+1})$. It is easy to compute the values R_0 and R_1 for the logistic map (Exercise 4).

Let us compare the graphs of $f(R_0,x)$ (Figure 1.6) and $f^2(R_1,x)$ (the lower half of Figure 1.7). The main idea is that by restricting the graph of $f^2(R_1,x)$ to a smaller interval we again obtain a unimodal map. More precisely, the assumption that the period-2 orbit of $f^2(R_1,x)$ is superstable

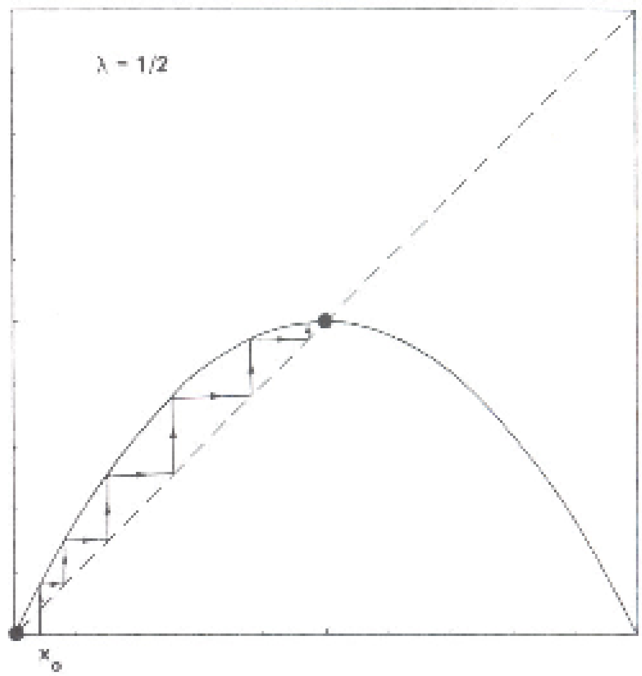

Figure 1.6: Superstable fixed point at $r = R_0$.

implies that the interval $[0, f(R_1, 0)]$ is positively invariant (consider an orbit diagram in the upper half of Figure 1.7). Let $\alpha_1 = f(R_1, 0)$ denote the length of this interval. We restrict $f^2(R_1, \cdot)$ to the interval $[-\alpha_1, \alpha_1]$ and rescale the x-axis by the factor α_1, and the y-axix by $-\alpha_1$ to obtain a new unimodal map with a superstable fixed point at the origin. This operation is renormalization, and we denote it by $\mathcal{R}$.

This procedure may now be iterated. Feigenbaum discovered that successive rescaling factors α_k converge to a universal constant

$$\alpha = 2.5029\ldots \tag{1.5}$$

Thus, to a good approximation, the operation of renormalization may be described by a shift in the parameter R, iteration, and rescaling of the axes. This yields the sequence

$$\mathcal{R}f \approx -\alpha f^2\left(R_1, \frac{x}{\alpha}\right), \ldots, \mathcal{R}^n f \approx (-\alpha)^n f^{2^n}\left(R_n, \frac{x}{\alpha^n}\right).$$

The fundamental assumption (verified by careful numerical calculations) is that there is a limiting function g_0 such that

$$\lim_{n\to\infty} (-\alpha)^n f^{2^n}\left(R_n, \frac{x}{\alpha^n}\right) = g_0(x).$$

Since the approximating sequence has a superstable fixed point at the origin, so does g_0. The effect of rescaling is to blow-up a neighborhood of the critical point. Thus, only the nature of the critical point near the origin determines the universal function g_0.

The analysis is incomplete at this point, because we would like to have an equation for g_0. In fact, g_0 is the first term in a sequence of universal functions obtained as follows. Instead of focusing on a superstable fixed point, we may focus on a superstable period-2 orbit . In this case, we begin with the unimodal map $f(R_1, x)$ (the upper half of Figure 1.7) and then consider the restriction of $f^2(R_2, x)$ to an interval on which it has a superstable period-2 orbit (inset in the upper half of Figure 1.8). That is, we seek the limit of the R-shifted sequence

$$(-\alpha)^n f^{2^n}\left(R_{n+1}, \frac{x}{\alpha^n}\right).$$

The limiting universal function, denoted g_1 has a superstable period-2 orbit. This procedure may be generalized and yields the universal hierarchy

$$g_k(x) = \lim_{n\to\infty} (-\alpha)^n f^{2^n}\left(R_{n+k}, \frac{x}{\alpha^n}\right), \tag{1.6}$$

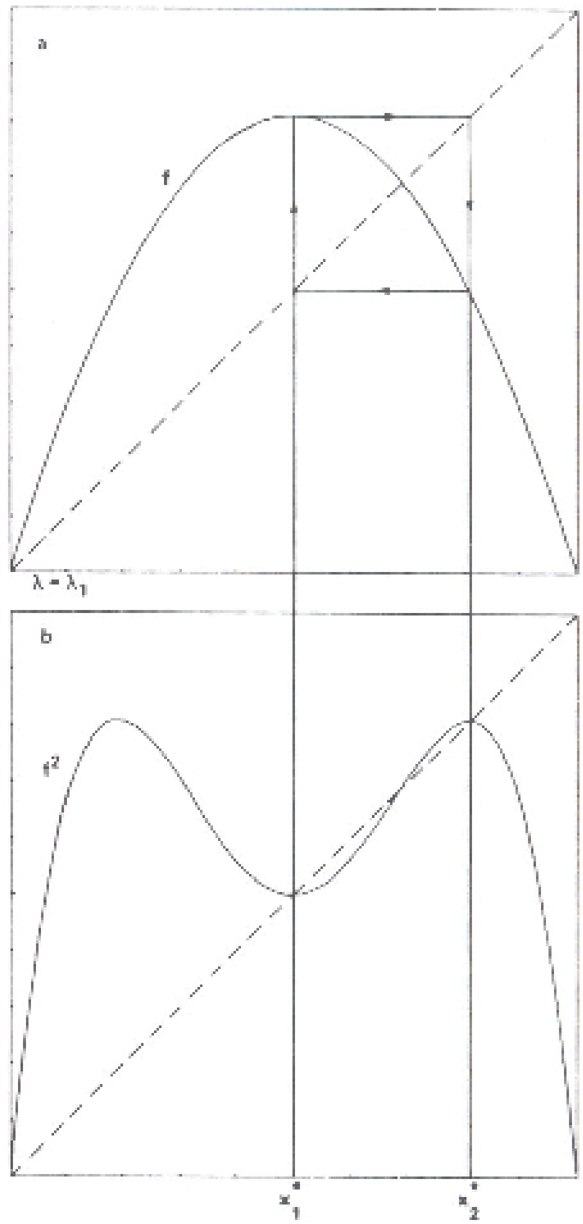

Figure 1.7: Superstable period-2 orbit at $r = R_1$.

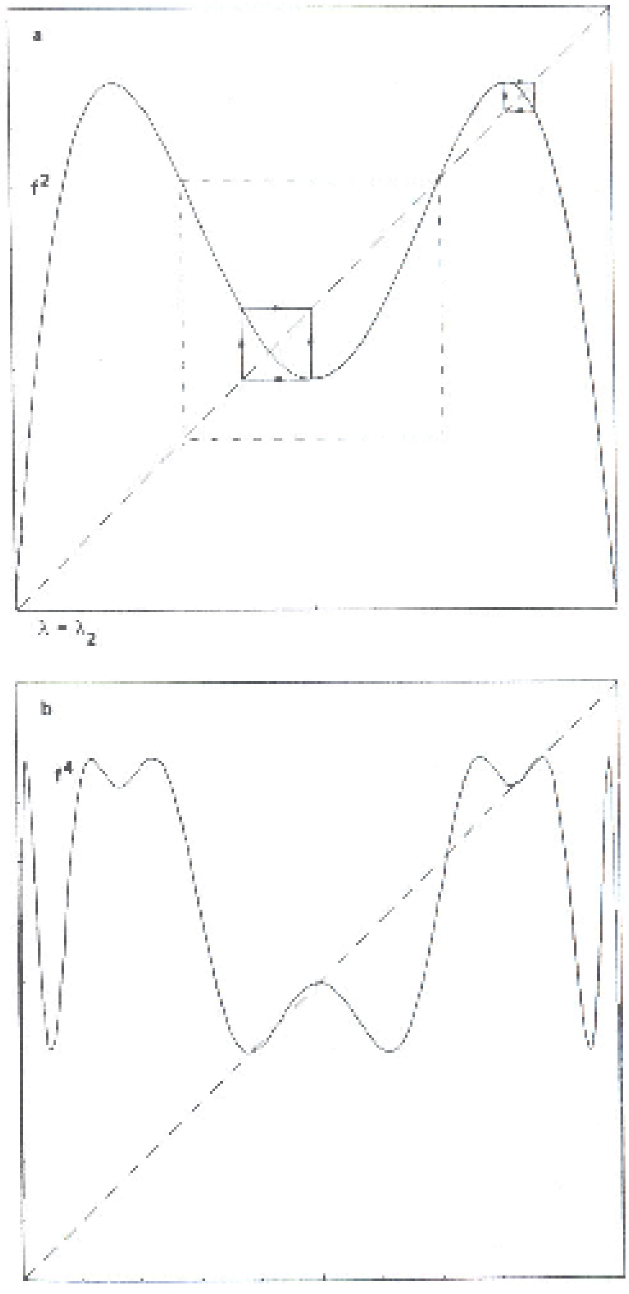

Figure 1.8: Superstable period-4 orbit at $r = R_2$. On a restricted interval, we obtain a superstable period-2 orbit as in Fig 1.7.

where g_k has a superstable orbit of period 2^k. It is easy to verify that the different levels of the hierarchy are linked by the equation

$$g_{k-1}(x) = -\alpha g_k^2(\frac{x}{\alpha}), \quad k \geq 1. \tag{1.7}$$

The index does go the right way here! This means that if we know g_m for some m we can determine g_k, $0 \leq k < m$, but not the other way around. To rectify this, we must pass to the limit $k \to \infty$ when the R-shift ceases to matter since $\lim_{k\to\infty} R_{n+k} = r_\infty$. We therefore, expect the existence of a universal function $g = \lim_{k\to\infty} g_k$. This yields the Feigenbaum-Cvitanović fixed point equation

$$g(x) = -\alpha g^2\left(\frac{x}{\alpha}\right). \tag{1.8}$$

This is a *functional equation*. The determination of α is part of the problem. This equation is scale-invariant: If g is a solution, so is $ag(x/a)$ for any $a > 0$. It is conventional to normalize by setting $g(0) = 1$. Some indication of the complicated nature of the universal hierarchy is indicated by a numerically computed plot of g_1 in Figure 1.9.

1.3 The block-spin analogue

Feigenbaum's notion of renormalization has its origins in Kadanoff and Wilson's renormalization group approach to phase transitions. Kadanoff's key idea in the study of the Ising model was to coarse grain spin interactions and replace them with an effective interaction of the same form. The following calculation is an analogue of this 'block-spin' idea. We shall consider only quadratic functions of the form

$$x_{n+1} = -(1 + r)x_n + ax_n^2. \tag{1.9}$$

The renormalization transformation consists of three steps – shift parameters, iterate and rescale. We begin with a quadratic polynomial, carry out these operations and truncate the result at second order to obtain a new quadratic polynonial. This will yield an approximate renormalization transformation. The calculation is approximate because of the truncation – iteration does *not* preserve quadratic polynomials. Nevertheless, the answers are surprisingly good.

Without loss of generality, we may set $a = 1$ by rescaling $x \mapsto x/a$. Thus, we begin with the dynamical system

$$x_{n+1} = -(1 + r)x_n + x_n^2, \tag{1.10}$$

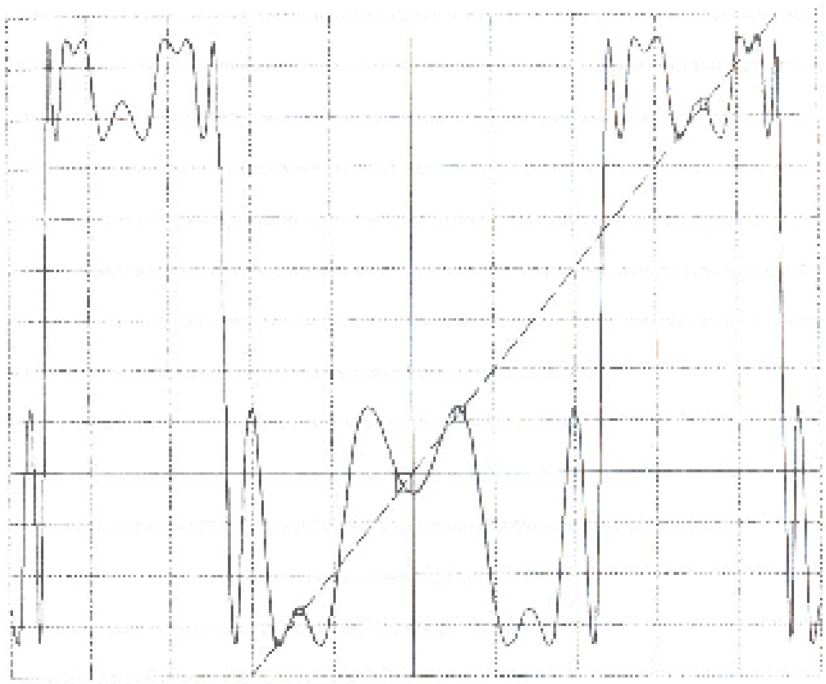

Figure 1.9: Numerical computation of g_1.

which undergoes a period-doubling bifurcation at $r = 0$. Let us denote the period-2 orbit by $p(r)$ and $q(r)$. We solve the equation $x = f^2(r, x)$ to obtain

$$p = \frac{1}{2}\left(r + \sqrt{r^2 + 4r}\right), \quad q = \frac{1}{2}\left(r - \sqrt{r^2 + 4r}\right). \tag{1.11}$$

We next consider the second order truncation of the Taylor series of $f^2(r, \cdot)$ about p. Let $x_n = p + \xi_n$. Then to second-order we have

$$\xi_{n+1} = (1 - 4r - r^2)\xi_n + a_1 \xi_n^2, \tag{1.12}$$

where

$$a_1 = 4r + r^2 - 3\sqrt{r^2 + 4r}. \tag{1.13}$$

We now rescale by setting $\eta_n = \xi_n / a_1$ to obtain the dynamical system

$$\eta_{n+1} = (1 - 4r - r^2)\eta_n + \eta_n^2. \tag{1.14}$$

If we define a new parameter $\tilde{r}$ by

$$-(1 + \tilde{r}) = 1 - 4r - r^2 \tag{1.15}$$

we again obtain a dynamical system of the form we started with (1.10). Thus, in this approximation, the renormalization transformation is completely described by a change in parameter values.

For example, since $\tilde{r} = 0$ corresponds to a period-doubling bifurcation in (1.10) we may set $\tilde{r} = 0$ in (1.15) to find r_1, the value at which the period-2 orbit loses stability to a period-4 orbit. This gives $r_1 = -2 + \sqrt{6}$ (this value is exact, and corresponds to $r = 1 + \sqrt{6}$ in the logistic family). Similarly, the next bifurcation value r_2 is found by setting $\tilde{r} = r_1$ in (1.15). Thus, the renormalization transformation is determined completely by the recursion relation

$$r_{k-1} = 1 - 4r_k - r_k^2, \quad k \geq 1, \quad r_0 = 0. \tag{1.16}$$

We may rewrite this transformation in the form

$$r_{k+1} = -2 + \sqrt{6 + r_k}, \quad k \geq 0, \quad r_0 = 0. \tag{1.17}$$

The fixed point of the recursion relation is

$$r_\infty = \frac{1}{2}\left(-3 + \sqrt{17}\right) \approx 0.56.$$

This is very close to the true value. The approximate value of δ is given by the linearization of (1.17) at r_∞ and is $1 + \sqrt{17} \approx 5.12$. The rescaling factor $\alpha = (1 + \sqrt{17})/2 \approx -2.24$.